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SOME CONCEPTS OF CONFORMAL GEOMETRY

PETER SCHERK, University of Toronto

There are several introductions to conformal geometry in the French and German literatures. However, there does not seem to exist a recent one in English. The present article cannot claim to be one. We only wish to develop a few basic notions of n-dimensional conformal geometry and to elaborate its connection with the projective geometry on the n-sphere and with $(n+2)$-dimensional pseudo-euclidean geometry.

As far as the author knows, general conformal geometry over an arbitrary field F has not yet been discussed. By means of a (generalized) stereographic projection it can be related to the geometry on an arbitrary regular n-dimensional quadric in a projective $(n+1)$-space over F. In its turn this geometry is related to a (generalized) euclidean geometry in a centro-affine $(n+2)$-space over F. The bilinear form defining this geometry will be regular but not necessarily definite. The first two geometries possess homogeneous orthogonal groups while the group of the last one is inhomogeneous. The transition to this last geometry is based on the well-known idea of providing the intersections of the quadric with the linear n-spaces with norms. The new space elements [$(n-1)$-quadrics plus norms] can be mapped one-one onto the vectors $\neq 0$ of affine $(n+2)$-space.

In the present note, only classical conformal geometry over the complex field is discussed. But the author has tried to use tools suitable for the more general situation indicated above. On the other hand he has also tried—in particular in the first sections—to motivate the introduction of conformal concepts from the point of view of elementary euclidean geometry.

The approach chosen in this paper is based on the stereographic projection of (most of) euclidean n-space E_n into the n-sphere in E_{n+1}. Insisting that this fundamental mapping be one-one, we are led in a natural way to imbed E_n into its conformal closure and E_{n+1} into projective $(n+1)$-space. The algebraic expression for this projection is obtained in several leisurely steps. An impatient reader is referred to formulas (3.21), (3.31), (3.32).

In the last sections inversions are discussed. They are shown to have most of the properties familiar from real 3-space. In particular the general conformal transformation can be decomposed into inversions. However, there are exceptions to Liouville's theorem in complex conformal spaces and not every conformal transformation will be the product of an inversion and a similarity.

1. Stereographic projection.

1.1. The stereographic mapping of the points

$$\xi = (\xi_1, \cdots, \xi_n) \tag{1.11}$$

in real euclidean n-space onto the points

$$\xi^* = (\xi_1^*, \cdots, \xi_n^*, \xi_{n+1}^*) \tag{1.12}$$

of the unit n-sphere S_n in euclidean $(n+1)$-space is well known.

Put

$$\xi_0 = \sum_1^n \xi_\lambda^2. \tag{1.13}$$

Then the image of the point ξ is the point

$$\xi^* = \frac{2}{\xi_0 + 1}(\xi_1, \cdots, \xi_n, \tfrac{1}{2}(\xi_0 - 1)). \tag{1.14}$$

Conversely, the point (1.12) of S_n has one and only one original point, *viz.*,

$$\xi = \frac{1}{1 - \xi_{n+1}^*}(\xi_1^*, \cdots, \xi_n^*). \tag{1.15}$$

However, there is one point on S_n without original points. This is the "north pole"

$$\mathfrak{v}^* = (0, \cdots, 0, 1) \tag{1.16}$$

of S_n. It is by introducing an ideal ("absolute") original point $\mathfrak{v}$ of $\mathfrak{v}^*$ that real euclidean n-space is completed to real conformal n-space.

1.2. Formula (1.14) can also be interpreted as a mapping of complex euclidean n-space E_n into the (from now on complex) unit sphere S_n. But now we have to note the following exceptions:

(i) No point of the $(n-1)$-dimensional hypersphere

$$\xi_0 + 1 = 0 \tag{1.21}$$

of radius i about the origin has an image point.

(ii) S_n intersects its tangent-hyperplane

$$\xi_{n+1}^* - 1 = 0 \tag{1.22}$$

at $\mathfrak{v}^*$ in an $(n-1)$-sphere S_{n-1}^* of radius zero ("null sphere") about $\mathfrak{v}^*$ (S_{n-1}^* can also be interpreted as a cone with the vertex $\mathfrak{v}^*$). By (1.15), no point of S_{n-1}^* has original points in E_n.

As a first step towards removing these exceptions we complete complex euclidean $(n+1)$-space to a complex projective $(n+1)$-space P_{n+1} by adding an "absolute hyperplane." This can be done by introducing homogeneous coordinates

$$\mathfrak{x}^* = (x_1, \cdots, x_n, x_{n+1}, x_{n+2}) \tag{1.23}$$

connected with the inhomogeneous coordinates (1.12) of euclidean $(n+1)$-space through

$$\xi_\mu^* = x_\mu / x_{n+2} \qquad (\mu = 1, \cdots, n+1). \tag{1.24}$$

The absolute hyperplane is then given by

$$x_{n+2} = 0 \tag{1.25}$$

and the inhomogeneous equation $\sum_1^{n+1} \xi_\mu^{*2} - 1 = 0$ of S_n can now be rewritten in a homogeneous form

$$\sum_1^n x_\lambda^2 + x_{n+1}^2 - x_{n+2}^2 = 0. \tag{1.26}$$

This way, S_n will be completed by adding the "absolute null sphere"

$$\sum_1^{n+1} x_\mu^2 = 0, \qquad x_{n+2} = 0. \tag{1.27}$$

In P_{n+1}, (1.27) is the intersection of S_n with the absolute hyperplane. It has the dimension $n-1$. By (1.24) and (1.14)

$$\begin{aligned} x_\lambda/x_{n+2} &= \xi_\lambda^* = \xi_\lambda/\{\tfrac{1}{2}(\xi_0 + 1)\} \qquad (\lambda = 1, \cdots, n). \\ x_{n+1}/x_{n+2} &= \xi_{n+1}^* = \{\tfrac{1}{2}(\xi_0 - 1)\}/\{\tfrac{1}{2}(\xi_0 + 1)\}. \end{aligned} \tag{1.28}$$

This leads to the following expression for the stereographic projection of E_n into the complex unit sphere S_n in P_{n+1}:

$$\mathbf{x}^* = \rho(\xi_1, \cdots, \xi_n, \tfrac{1}{2}(\xi_0 - 1), \tfrac{1}{2}(\xi_0 + 1)). \tag{1.29}$$

Here $\rho \neq 0$ is an arbitrary scalar. By (1.27) and (1.13), the hypersphere (1.21) is mapped onto the absolute null sphere. Thus every point of E_n now has an image point.

1.3. In order to determine the original of a point (1.23) of S_n, we introduce the quantities

$$x_0 = \sum_1^n x_\lambda^2 \tag{1.31}$$

and

$$x_{-1} = x_{n+2} - x_{n+1}, \qquad x_{-2} = x_{n+2} + x_{n+1}. \tag{1.32}$$

Thus the equation (1.26) of S_n can be rewritten

$$x_0 - x_{-1} \cdot x_{-2} = 0 \tag{1.33}'$$

and the completed tangent-hyperplane (1.22) of the north pole of S_n has the equation

$$x_{-1} = 0. \tag{1.34}$$

If the point $\mathbf{x}^*$ of S_n is the image of the point ξ of E_n, (1.29) and (1.31) yield

(1.35) $$x_0 = \rho^2\xi_0, \qquad x_{-1} = \rho, \qquad x_{-2} = \rho\xi_0.$$

From (1.28) we obtain the following version of (1.15)

(1.36) $$\xi = (1/x_{-1})(x_1, \cdots, x_n).$$

Similarly, (1.35) implies

(1.37) $$\rho = x_{-1}, \qquad \xi_0 = x_0/x_{-1}^2 = x_{-2}/x_{-1}.$$

1.4. Equation (1.36) breaks down if $x_{-1}=0$, *i.e.*, for points $\mathbf{x}^*$ on the completed null sphere S^*_{n-1} of **1.2.** As a first step towards eliminating these exceptions we note that (1.36) suggests introducing $x_1, \cdots, x_n$, and x_{-1} as homogeneous coordinates in E_n. However we add one redundant coordinate x_{-2} which is connected with the others through

(1.41) $$x_0 = \sum_1^n x_\lambda^2 = x_{-1}x_{-2};$$

cf. (1.31) and (1.33). Thus the point ξ determines its homogeneous coordinate vector

(1.42) $$\mathbf{x} = \rho(\xi_1, \cdots, \xi_n, 1, \xi_0).$$

It obviously satisfies (1.35) and (1.41); in particular

(1.43) $$x_{-1} = \rho \neq 0.$$

Conversely assume (1.41) and $x_{-1}\neq 0$. Then the vector

(1.44) $$\mathbf{x} = (x_1, \cdots, x_n, x_{-1}, x_{-2})$$

determines the point ξ in E_n through (1.36).

If homogeneous coordinates are used both in E_n and on S_n, the formulas for stereographic projection become particularly simple. Combining (1.36) with (1.29) and choosing $\rho=x_{-1}$ in the latter, we find that the point (1.44) in E_n determines the point (1.23) on S_n, where

(1.45) $$x_{n+1} = \tfrac{1}{2}(x_{-2} - x_{-1}), \qquad x_{n+2} = \tfrac{1}{2}(x_{-2} + x_{-1})$$

are obtained from (1.32).

1.5. It is now easy to enlarge E_n to complex conformal n-space C_n by interpreting each homogeneous vector $\mathbf{x}\neq 0$ which satisfies (1.41) as a point of C_n. Thus two such vectors are associated with the same point if and only if they are proportional. The "finite points" of C_n, *i.e.*, the points of E_n, are characterized by $x_{-1}\neq 0$. The "absolute points" of C_n which we have added are those with coordinate vectors

(1.51) $$\mathbf{x} = (x_1, \cdots, x_n, 0, x_{-2}).$$

By (1.41) they satisfy

$$x_0 = \sum_1^n x_\lambda^2 = 0. \tag{1.52}$$

Thus x_{-2} is no longer determined by the other coordinates. We call the set of the absolute points the absolute null sphere of C_n. Obviously it contains exactly one real point, *viz.*,

$$\mathbf{n} = (0, \cdots, 0, 0, 1). \tag{1.53}$$

Extending **1.4** we now define the stereographic projection of the point (1.44) of C_n as the point (1.23) of S_n, where x_{-1}, x_{-2} and x_{n+1}, x_{n+2} are connected by (1.32) and (1.45). We have finally disposed of the exceptions noted in **1.2** and C_n is mapped one-one onto S_n.

The image of a point (1.51) is a point of S_n satisfying

$$x_{-1} = x_{n+2} - x_{n+1} = 0. \tag{1.54}$$

Thus the absolute null sphere of C_n is mapped onto the absolute null sphere S_{n-1}^* on S_n discussed in **1.2.** In particular the image of the real absolute point $\mathbf{n}$ is the north pole of S_n

$$\mathbf{n}^* = (0, \cdots, 0, 1, 1). \tag{1.55}$$

2. Spheres in C_n.

2.1. An $(n-1)$-sphere S_{n-1} in complex euclidean n-space E_n is given by an equation

$$a_{-1}\xi_0 - 2\sum_1^n a_\lambda \xi_\lambda + a_{-2} = 0. \tag{2.11}$$

By (1.36) and (1.37), S_{n-1} has the equation

$$\sum_1^n a_\lambda \cdot x_\lambda - \tfrac{1}{2}(a_{-2}x_{-1} + a_{-1}x_{-2}) = 0 \tag{2.12}$$

in homogeneous coordinates. We call

$$\mathbf{a} = (a_1, \cdots, a_n, a_{-1}, a_{-2}) \tag{2.13}$$

the homogeneous vector of S_{n-1}. It is determined by S_{n-1} up to a proportionality factor. Every vector $\mathbf{a}$ which is not a multiple of $\mathbf{n} = (0, \cdots, 0, 0, 1)$ determines an $(n-1)$-sphere.

The $(n-1)$-planes are contained among the $(n-1)$-spheres. They are characterized by

$$a_{-1} = 0. \tag{2.14}$$

Let $a_{-1} \neq 0$. By (2.11), the center of S_{n-1} has the inhomogeneous coordinates

$$\boldsymbol{\alpha} = (1/a_{-1})(a_1, \cdots, a_n). \tag{2.15}$$

Thus by (1.42), it has the homogeneous coordinates

$$(a_1, \cdots, a_n, a_{-1}, a_0/a_{-1}), \tag{2.16}$$

where

$$a_0 = \sum_1^n a_\lambda^2. \tag{2.17}$$

By (2.11), the square of the radius of S_{n-1} is

$$r^2 = (a_0 - a_{-1}a_{-2})/a_{-1}^2. \tag{2.18}$$

Let $a_{-1} \neq 0$. Comparing (2.18), (2.16), (2.13), and (2.12) we see that the following properties of S_{n-1} are equivalent:

(i) S_{n-1} is a null sphere;
(ii) $a_0 - a_{-1}a_{-2} = 0$;
(iii) $\mathbf{a}$ is the center of S_{n-1};
(iv) $\mathbf{a} \in S_{n-1}$.

2.2. The above discussion is readily extended to conformal n-space C_n. On account of (2.14), the $(n-1)$-planes can now be characterized as the $(n-1)$-spheres through $\mathbf{n}$.

The absolute points (1.51) of C_n satisfy the equation $x_{-1} = 0$. Comparison with (2.12) shows that the absolute null sphere can actually be interpreted as an $(n-1)$-sphere with the vector $\mathbf{n}$. As it satisfies (2.14), formula (2.18) becomes meaningless. However, conditions (ii) and (iv) of **2.1** hold for $\mathbf{n}$. Multiplying (2.16) by a_{-1} before substituting $\mathbf{n}$, we see that $\mathbf{n}$ may be interpreted as the centre of the absolute null sphere.

Through the introduction of this sphere, we achieve a one-one correspondence between the $(n-1)$-spheres in C_n and the homogeneous vectors $\mathbf{a} \neq 0$.

2.3. The last remark suggests an interpretation of the $(n-1)$-spheres S_{n-1} of C_n as the points of a projective $(n+1)$-space. We readily construct an explicit mapping.

Suppose S_{n-1} is given by (2.12). Its stereographic projection S_{n-1}^* was the locus of the points

$$\mathbf{x}^* = (x_1, \cdots, x_n, x_{n+1}, x_{n+2}) \tag{2.31}$$

on the unit sphere S_n in complex projective $(n+1)$-space P_{n+1}. Here x_{n+1} and x_{n+2} were determined by (1.45) or (1.32). Thus the points (2.31) satisfy

$$\sum_1^n a_\lambda \cdot x_\lambda - \tfrac{1}{2}[a_{-2}(x_{n+2} - x_{n-1}) + a_{-1}(x_{n+2} + x_{n+1})] = 0$$

or

$$\sum_1^n a_\lambda x_\lambda + a_{n+1}x_{n+1} - a_{n+2}x_{n+2} = 0, \tag{2.32}$$

where

$$a_{n+1} = \tfrac{1}{2}(a_{-2} - a_{-1}), \qquad a_{-1} = a_{n+2} - a_{n+1},$$

(2.33) or

$$a_{n+2} = \tfrac{1}{2}(a_{-2} + a_{-1}), \qquad a_{-2} = a_{n+2} + a_{n+1}.$$

Equation (2.32) determines an n-plane in P_{n+1}. Thus its intersection S^*_{n-1} with S_n is an $(n-1)$-sphere. Obviously, the point

$$\mathbf{a}^* = (a_1, \cdots, a_n, a_{n+1}, a_{n+2}) \tag{2.34}$$

is the pole with respect to S_n of the n-plane (2.32). Thus

$$S_{n-1} \leftrightarrow \mathbf{a}^* \tag{2.35}$$

is a one-one correspondence between the $(n-1)$-spheres of C_n and the points of P_{n+1}.

By means of (2.32) and (2.33) we verify once more that the stereographic projection of the absolute null sphere of C_n is the null sphere in which S_n intersects its tangent-hyperplane at its north pole $\mathbf{n}^* = (0, \cdots, 0, 1, 1)$. Furthermore, S_{n-1} is a hyperplane if and only if

$$\mathbf{n}^* \in S^*_{n-1}. \tag{2.36}$$

S_{n-1} is a null sphere if and only if

$$\mathbf{a}^* \in S_n. \tag{2.37}$$

3. Conformal and pseudo-euclidean metrics.

3.1. We have interpreted

$$(x_1, \cdots, x_n, x_{n+1}, x_{n+2}) \tag{3.11}$$

as the homogeneous coordinate vector of a point $\mathbf{x}^*$ in complex projective $(n+1)$-space P_{n+1}. But (3.11) can also be interpreted as the inhomogeneous coordinate vector of a point $\tilde{\mathbf{x}}$ in complex affine $(n+2)$-space A_{n+2}. Then if $\tilde{\mathbf{x}}$ is distinct from the origin O of A_{n+2}, it determines a point $\mathbf{x}^*$ in P_{n+1}; and two points of A_{n+2} determine the same point in P_{n+1} if and only if they lie on the same straight line through O. Thus with each point $\mathbf{x}^* \in P_{n+1}$ all the points $\neq O$ of a line through O are associated. If $\mathbf{x}^*$ lies in S_n, they satisfy (1.26). Thus S_n corresponds to the $(n+1)$-cone K_{n+1}: $\sum_1^n x_\lambda^2 + x_{n+1}^2 - x_{n+2}^2 = 0$ in A_{n+2}.

We now map the point

$$\tilde{\mathbf{x}} = (x_1, \cdots, x_n, x_{n+1}, x_{n+2}) \neq 0 \tag{3.12}$$

of K_{n+1} onto the point

$$\mathbf{x} = (x_1, \cdots, x_n, x_{-1}, x_{-2}) \tag{3.13}$$

of C_n with the *weight* x_{-1};

$$\tilde{\mathbf{x}} \to \mathbf{x},\ x_{-1}; \tag{3.14}$$

cf. (1.32). Given a point $\mathbf{x} \in C_n$ and a weight ρ. We require a point $\tilde{\mathbf{x}}$ which is mapped by (3.14) onto $\mathbf{x}$, ρ. Unless neither or both of the numbers ρ and $x_{-1} = x_{n+2} - x_{n+1}$ vanish, this problem has no solution.

If x_{-1} and ρ are both distinct from zero, we can multiply the coordinates of $\mathbf{x}$ by exactly one factor, *viz.*, ρ/x_{-1} such that

$$\rho = x_{-1} \tag{3.15}$$

for the new coordinates of $\mathbf{x}$. Thus we have exactly one solution $\tilde{\mathbf{x}}$.

If both x_{-1} and ρ vanish, there is a straight line through O in the $(n+1)$-plane

$$x_{-1} = x_{n+2} - x_{n+1} = 0 \tag{3.16}$$

such that all the points of this line are solutions.

3.2. From now on, the same letter $\mathbf{x}^*$ will be used to denote points in P_{n+1} and in A_{n+2}. Furthermore, vectors with $n+2$ components will be considered as column vectors, with primes indicating transposition, *e.g.*,

$$\mathbf{x}' = (x_1, \cdots, x_n, x_{-1}, x_{-2}), \qquad \mathbf{x}^{*\prime} = (x_1, \cdots, x_n, x_{n+1}, x_{n+2}). \tag{3.21}$$

I_n is the $n \times n$ unit matrix. The matrix

$$J = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & 1 & 0 \\ 0 & 0 & -1 \end{array}\right) \tag{3.22}$$

defines a pseudo-euclidean scalar product

$$\mathbf{x}^{*\prime} J \mathbf{y}^* = \sum_1^n x_\lambda y_\lambda + x_{n+1} y_{n+1} - x_{n+2} y_{n+2} \tag{3.23}$$

in A_{n+2}. The locus of the points $\mathbf{x}^*$ with

$$\mathbf{a}^{*\prime} J \mathbf{x}^* = 0 \tag{3.24}$$

is an $(n+1)$-plane through O. The pseudo-euclidean norm of the vector $\mathbf{x}^*$ is

$$\|\mathbf{x}^*\| = \mathbf{x}^{*\prime} J \mathbf{x}^* = \sum_1^n x_\lambda^2 + x_{n+1}^2 - x_{n+2}^2. \tag{3.25}$$

Thus the points of K_{n+1} are characterized by

$$\|\mathbf{x}^*\| = 0. \tag{3.26}$$

In particular, $\|\mathbf{x}^*\|$ can be zero (or negative) for a real vector $\mathbf{x}^* \neq 0$.

The pseudo-euclidean distance δ between two points $\mathbf{x}^*$ and $\mathbf{y}^*$ of A_{n+2} is any root of

$$\delta^2 = \|\mathbf{x}^* - \mathbf{y}^*\| = \|\mathbf{x}^*\| + \|\mathbf{y}^*\| - 2\mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{3.27}$$

If both $\mathbf{x}^*$ and $\mathbf{y}^*$ lie in K_{n+1}, their norms vanish and we have

$$\delta^2 = -2\mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{3.28}$$

Let $\|\mathbf{a}^*\| \neq 0$, $\|\mathbf{b}^*\| \neq 0$. Then we define the pseudo-euclidean angle ϕ between the $(n+1)$-planes (3.24) and

$$\mathbf{b}^{*\prime}J\mathbf{y}^* = 0 \tag{3.29}$$

through

$$\cos\phi = (\mathbf{a}^{*\prime}J\mathbf{b}^*)/\sqrt{(\|\mathbf{a}^*\| \cdot \|\mathbf{b}^*\|)}. \tag{3.210}$$

The planes (3.24) and (3.29) or the vectors $\mathbf{a}^*$ and $\mathbf{b}^*$ are called pseudo-orthogonal if

$$\mathbf{a}^{*\prime}J\mathbf{b}^* = 0. \tag{3.211}$$

We may call $\mathbf{a}^*$ a pseudo-normal vector of the $(n+1)$-plane (3.24).

The preceding definitions can be adapted to P_{n+1} with few changes. Equation (3.24) then represents an n-plane, (3.26) characterizes the points of S_n, and (3.210) can still be used to define the angle between two n-planes.

3.3. Through (1.32), every point $\mathbf{x}^*$ in P_{n+1} (or in A_{n+2}) is mapped onto a spherical vector $\mathbf{x}$ in C_n. It is convenient to rewrite (1.32) in matrix form

$$\mathbf{x} = U\mathbf{x}^*, \tag{3.31}$$

where

$$U = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & -1 & 1 \\ 0 & 1 & 1 \end{array}\right), \qquad U^{-1} = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right). \tag{3.32}$$

Put

$$G = U^{-1}JU^{-1} = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{array}\right). \tag{3.33}$$

The conformal scalar product of the vectors $\mathbf{x}$ and $\mathbf{y}$ is the number

$$\mathbf{x}'G\mathbf{y} = \sum_1^n x_\lambda y_\lambda - \tfrac{1}{2}(x_{-1}y_{-2} + x_{-2}y_{-1}). \tag{3.34}$$

Assume (3.31) and

$$\mathbf{y} = U\mathbf{y}^*. \tag{3.35}$$

By (3.33) the conformal and pseudo-euclidean scalar products are related through

$$\mathbf{x}'G\mathbf{y} = \mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{3.36}$$

The conformal norm of $\mathbf{x}$ is

$$\|\mathbf{x}\| = \mathbf{x}'G\mathbf{x} = \sum_1^n x_\lambda^2 - x_{-1}x_{-2} = x_0 - x_{-1}x_{-2}. \tag{3.37}$$

By (3.36),

$$\|\mathbf{x}\| = \|\mathbf{x}^*\|. \tag{3.38}$$

The points (or null spheres) in C_n are characterized by

$$\|\mathbf{x}\| = 0. \tag{3.39}$$

Hence

$$\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\| - 2\mathbf{x}'G\mathbf{y} = -2\mathbf{x}'G\mathbf{y} \tag{3.310}$$

if $\mathbf{x}$ and $\mathbf{y}$ are points in C_n; *cf.* (3.28).

3.4. Given two finite points $\mathbf{x}$ and $\mathbf{y}$ in C_n with the weights x_{-1} and y_{-1}, respectively; *cf.* **3.1.** Let $\xi(\mathfrak{n})$ denote the inhomogeneous coordinates of $\mathbf{x}$ (of $\mathbf{y}$). Then by (1.13), (1.37), and (1.36)

$$\begin{aligned}(\xi - \mathfrak{n})^2 &= \sum_1^n (\xi_\lambda - \eta_\lambda)^2 = \sum_1^n \xi_\lambda^2 + \sum_1^n \eta_\lambda^2 - 2\sum_1^n \xi_\lambda\eta_\lambda \\ &= \xi_0 + \eta_0 - 2\sum_1^n \xi_\lambda\eta_\lambda = \frac{x_{-2}}{x_{-1}} + \frac{y_{-2}}{y_{-1}} - 2\sum_1^n \frac{x_\lambda y_\lambda}{x_{-1}y_{-1}} \\ &= -\frac{2}{x_{-1}y_{-1}}\left[\sum_1^n x_\lambda y_\lambda - \tfrac{1}{2}(x_{-1}y_{-2} + x_{-2}y_{-1})\right].\end{aligned}$$

Thus by (3.34) and (3.310)

$$x_{-1}y_{-1}(\xi - \mathfrak{n})^2 = -2\mathbf{x}'G\mathbf{y} = \|\mathbf{x} - \mathbf{y}\|. \tag{3.41}$$

3.5. By (2.12), the locus of the points $\mathbf{x}$ in C_n which satisfy an equation

$$\mathbf{a}'G\mathbf{x} = 0 \tag{3.51}$$

is an $(n-1)$-sphere. It is proper if and only if $a_{-1} \neq 0$. By (2.18) and (3.37), its radius r is given by

$$r^2 = (a_0 - a_{-1}a_{-2})/a_{-1}^2 = \|\mathbf{a}\|/a_{-1}^2. \tag{3.52}$$

Now let

$$\mathbf{b}'G\mathbf{x} = 0 \tag{3.53}$$

be a second proper $(n-1)$-sphere and let s be its radius. Let α and β denote the inhomogeneous coordinate vectors of the centers of the $(n-1)$-spheres (3.51) and (3.53) respectively. By (3.41), (2.16), (3.34) and (3.37)

$$\begin{aligned} a_{-1}b_{-1}(\alpha - \beta)^2 &= -2\left[\sum_1^n a_\lambda b_\lambda - \frac{1}{2}\left(a_{-1}\frac{b_0}{b_{-1}} + b_{-1}\frac{a_0}{a_{-1}}\right)\right] \\ &= -2\left[\sum_1^n a_\lambda b_\lambda - \tfrac{1}{2}(a_{-1}b_{-2} + a_{-2}b_{-1})\right] + a_{-1}\left(\frac{b_0}{b_{-1}} - b_{-2}\right) \\ &\quad + b_{-1}\left(\frac{a_0}{a_{-1}} - a_{-2}\right) \\ &= -2\mathbf{a}'G\mathbf{b} + \frac{a_{-1}}{b_{-1}}\|\mathbf{b}\| + \frac{b_{-1}}{a_{-1}}\|\mathbf{a}\|. \end{aligned}$$

Thus (3.52) leads to the following generalization of (3.41)

$$a_{-1}b_{-1}[(\alpha - \beta)^2 - r^2 - s^2] = -2\mathbf{a}'G\mathbf{b}. \tag{3.54}$$

If our two $(n-1)$-spheres are real and have a real intersection, the angle ϕ between them is determined by

$$(\beta - \alpha)^2 = r^2 + s^2 - 2rs\cos\phi.$$

Solving for ϕ and substituting (3.54) and (3.52), we obtain

$$\cos\phi = (\mathbf{a}'G\mathbf{b})/\sqrt{(\|\mathbf{a}\| \cdot \|\mathbf{b}\|)}. \tag{3.55}$$

If (3.51) is the equation of a real finite $(n-1)$-plane, the distance of a point $\mathbf{z}$ from it is equal to the absolute value of

$$(\mathbf{a}'G\mathbf{z})/(z_{-1}\cdot\|\mathbf{a}\|). \tag{3.56}$$

This readily implies that (3.55) remains valid if one (or both) of our $(n-1)$-spheres degenerate into $(n-1)$-planes. We may therefore define the angle ϕ between any two nonnull spheres through (3.55).

Two arbitrary $(n-1)$-spheres (3.51) and (3.53) will be called orthogonal if

$$\mathbf{a}'G\mathbf{b} = 0. \tag{3.57}$$

3.6. Let

$$\mathbf{a}^* = U^{-1}\mathbf{a}, \qquad \mathbf{b}^* = U^{-1}\mathbf{b}. \tag{3.61}'$$

By(3.36) and (3.38),

$$(\mathbf{a}'G\mathbf{b})/\sqrt{(\|\mathbf{a}\| \cdot \|\mathbf{b}\|)} = (\mathbf{a}^{*\prime}J\mathbf{b}^*)/\sqrt{(\|\mathbf{a}^*\| \cdot \|\mathbf{b}^*\|)}. \tag{3.62}$$

As the right-hand expression is equal to the cosine of the angles between the $(n-1)$-spheres

$$\mathbf{a}^{*\prime}J\mathbf{x}^* = 0, \qquad \mathbf{b}^{*\prime}J\mathbf{x}^* = 0, \tag{3.63}$$

on S_n, (3.62) states the well-known invariance of angles under stereographic projection.

Through $\mathbf{a}^* = U^{-1}\mathbf{a}$, the $(n-1)$-spheres $\mathbf{a}'G\mathbf{x}=0$ of C_n are also mapped one-one onto the $(n+1)$-planes $\mathbf{a}^{*\prime}J\mathbf{x}^*=0$ in A_{n+2} through the origin. Equation (3.62) also states that the pseudo-euclidean angle between two such $(n+1)$-planes is equal to the angle between the corresponding $(n-1)$-spheres in C_n.

4. The conformal group.

4.1. Let

$$\mathbf{y}^* = T^*\mathbf{x}^* \tag{4.11}$$

be any linear transformation of pseudo-euclidean $(n+2)$-space A_{n+2} into itself. Then

$$\mathbf{x} = U\mathbf{x}^*, \qquad \mathbf{y} = U\mathbf{y}^* \tag{4.12}$$

implies

$$\mathbf{y} = UT^*U^{-1}\mathbf{x}. \tag{4.13}$$

Thus T^* determines a linear transformation

$$T = UT^*U^{-1} \tag{4.14}$$

of the spherical vectors in conformal n-space C_n. By (3.32), $U = U'$. Hence (4.14) implies

$$T' = U^{-1}T^{*\prime}U. \tag{4.15}$$

4.2. The linear transformation T^* maps K^{n+1} into itself if and only if

$$T^{*\prime}JT^* = \rho J \tag{4.21}$$

is a multiple of J. This implies

$$(T^*\mathbf{x}^*)'J\cdot(T^*\mathbf{y}^*) = \rho\cdot\mathbf{x}^{*\prime}J\mathbf{y}^*. \tag{4.22}$$

Thus T^* then multiplies all the scalar products (in particular, all the norms) in A_{n+2} by the same factor. Hence the pseudo-euclidean angle is invariant under T^*. We may interpret T^* as a pseudo-euclidean similarity about the origin of A_{n+2}. The transformations T^* with $\rho\neq 0$ form a group Σ^*.

If (4.21) holds with $\rho=1$, the pseudo-euclidean scalar products, norms and distances are invariant under T^*. Conversely, if all the norms are invariant under T^*, then

$$T^{*\prime}JT^* = J. \tag{4.23}$$

These T^*'s form a subgroup Ω^* of Σ^*, the pseudo-orthogonal group of A_{n+2}.

Let T^* satisfy (4.21) with $\rho\neq 0$; thus T^* is regular. Let

$$\mathbf{a}^{*\prime} J \mathbf{x}^* = 0 \tag{4.24}$$

be the $(n+1)$-space through O in A_{n+2} with the pseudo-normal vector $\mathbf{a}^*$. By (4.24) and (4.21),

$$(T^*\mathbf{a}^*)' J (T^*\mathbf{x}^*) = \mathbf{a}^{*\prime}(T^{*\prime} J T^*)\mathbf{x}^* = \rho \mathbf{a}^{*\prime} J \mathbf{x}^* = 0.$$

Thus T^* maps (4.24) onto the $(n+1)$-space through O with the pseudo-normal vector $T^*\mathbf{a}^*$.

4.3. Through (4.14), Σ^* is mapped onto the full conformal group Σ. Since

$$T^* = U^{-1} T U, \qquad T^{*\prime} = U T' U^{-1},$$

(4.21) is equivalent to

$$(U T' U^{-1}) J (U^{-1} T U) = \rho J.$$

Thus by (3.33), Σ consists of those T which satisfy

$$T' G T = \rho G, \qquad \rho \neq 0. \tag{4.31}$$

This implies

$$(T\mathbf{a})' G (T\mathbf{b}) = \rho \cdot \mathbf{a}' G \mathbf{b}. \tag{4.32}$$

Hence T maps $(n-1)$-spheres in C_n onto $(n-1)$-spheres, null spheres onto null spheres and keeps the angle between $(n-1)$-spheres unchanged.

The image of the pseudo-orthogonal group Ω^* is the proper conformal group Ω. It consists of those $T \in \Sigma$ which keep scalar products invariant. Thus they are characterized by

$$T' G T = G \tag{4.33}$$

or by

$$\|T\mathbf{x}\| = \|\mathbf{x}\|. \tag{4.34}$$

The spherical vectors $\mathbf{a}$ and $\lambda\mathbf{a}$ determine the same $(n-1)$-sphere provided, perhaps, with different norms. Similarly, with $\mathbf{x}$, $\lambda\mathbf{x}$ is the vector of a point. The two points coincide but may have distinct weights x_{-1} and λx_{-1}. If we interpret $\mathbf{a}$ as a homogeneous spherical vector, then T and λT will induce the same transformation. It is only the weights of the points and the norms of the $(n-1)$-spheres of C_n which are transformed differently. Thus, if we abstract from norms and weights, the group of C_n will not be Σ but the factor group of Σ with respect to the invariant subgroup formed by the multiples λI_{n+2} of the identity ($\lambda \neq 0$). The complex field being quadratically closed, this factor group is isomorphic to the factor group of Ω with respect to the subgroup consisting of I_{n+2} and $-I_{n+2}$.

4.4. Put

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.41}$$

Thus

$$G = \begin{pmatrix} I_n & 0 \\ 0 & -\frac{1}{2}D \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} I_n & 0 \\ 0 & -2D \end{pmatrix}. \tag{4.42}$$

Let

$$T = \left(\begin{array}{ccc|cc} t_{11} & \cdots & t_{1n} & t_{1,-1} & t_{1,-2} \\ \vdots & & \vdots & \vdots & \vdots \\ t_{n1} & \cdots & t_{nn} & t_{n,-1} & t_{n,-2} \\ \hline t_{-11} & \cdots & t_{-1n} & t_{-1,-1} & t_{-1,-2} \\ t_{-21} & \cdots & t_{-2n} & t_{-2,-1} & t_{-2,-2} \end{array}\right) = \left(\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right) \tag{4.43}$$

be a transformation in Σ. Then by (4.31) and (4.42)

$$T^{-1} = \frac{1}{\rho} G^{-1} T' G = \frac{1}{\rho} \begin{pmatrix} I_n & 0 \\ 0 & -2D \end{pmatrix} \begin{pmatrix} T'_{11} & T'_{21} \\ T'_{12} & T'_{22} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -\frac{1}{2}D \end{pmatrix}$$

or

$$T^{-1} = \frac{1}{\rho} \begin{pmatrix} T'_{11} & -\frac{1}{2}T'_{21}D \\ -2DT'_{12} & DT'_{22}D \end{pmatrix}$$

$$= \frac{1}{\rho} \left(\begin{array}{ccc|cc} t_{11} & \cdots & t_{n1} & -\frac{1}{2}t_{-21} & -\frac{1}{2}t_{-11} \\ \vdots & & \vdots & \vdots & \vdots \\ t_{1n} & \cdots & t_{nn} & -\frac{1}{2}t_{-2n} & -\frac{1}{2}t_{-1n} \\ \hline -2t_{1,-2} & \cdots & -2t_{n,-2} & t_{-2,-2} & t_{-1,-2} \\ -2t_{1,-1} & \cdots & -2t_{n,-1} & t_{-2,-1} & t_{-1,-1} \end{array}\right). \tag{4.44}$$

The transformation T maps the spherical vector $\mathbf{x}$ onto the spherical vector

$$\mathbf{y} = T\mathbf{x}. \tag{4.45}$$

Points are mapped onto points.

If $\mathbf{x}$ and $\mathbf{y}$ are finite points, let ξ and $\mathfrak{n}$ denote their inhomogeneous coordinate vectors. Then by (4.45) and (4.43)

$$\eta_k = \frac{y_k}{y_{-1}} = \frac{\sum_1^n t_{k\lambda}x_\lambda + t_{k,-1}x_{-1} + t_{k,-2}x_{-2}}{\sum_1^n t_{-1\lambda}x_\lambda + t_{-1,-1}x_{-1} + t_{-1,-2}x_{-2}}.$$

Dividing numerator and denominator by x_{-1}, we obtain, on account of (1.36) and (1.37),

$$\eta_k = \frac{\sum_1^n t_{k\lambda}\xi_\lambda + t_{k,-1} + t_{k,-2}\xi_0}{\sum_1^n t_{-1\lambda}\xi_\lambda + t_{-1,-1} + t_{-1,-2}\xi_0}. \tag{4.46}$$

The denominator is essentially the weight of the image point $\mathbf{u}$:

$$y_{-1} = x_{-1}\left(\sum_1^n t_{-1\lambda}\xi_\lambda + t_{-1,-1} + t_{-1,-2}\xi_0\right). \tag{4.47}$$

By (4.43) or (4.47), the original of the absolute null sphere is the $(n-1)$-sphere

$$y_{-1} = \sum_1^n t_{-1,\lambda}x_\lambda + t_{-1,-1}x_{-1} + t_{-1,-2}x_{-2} = 0 \tag{4.48}$$

or

$$y_{-1}/x_{-1} = \sum_1^n t_{-1,\lambda}\xi_\lambda + t_{-1,-1} + t_{-1,-2}\xi_0 = 0. \tag{4.49}$$

Replacing T by T^{-1} and (4.43) by (4.44), we obtain the image of the absolute null sphere

$$-\tfrac{1}{2}\rho x_{-1} = \sum_1^n t_{\lambda,-2}y_\lambda - \tfrac{1}{2}(t_{-2,-2}y_{-1} + t_{-1,-2}y_{-2}) = 0 \tag{4.410}$$

or

$$-\tfrac{1}{2}\rho x_{-1}/y_{-1} = \sum_1^n t_{\lambda,-2}\eta_\lambda - \tfrac{1}{2}(t_{-2,-2} + t_{-1,-2}\eta_0) = 0. \tag{4.411}$$

4.5. By (4.48), (4.410), (4.47), the following properties of the conformal transformation

$$\mathbf{y} = T\mathbf{x} \tag{4.51}$$

are equivalent:

(4.52) The equation $x_{-1}=0$ is invariant under (4.51);
(4.53) T maps the absolute null sphere onto itself;
(4.54) $t_{-11}=t_{-12}= \cdots =t_{-1n}=t_{-1,-2}=0$;
(4.55) $t_{1,-2}=t_{2,-2}= \cdots =t_{n,-2}=t_{-1,-2}=0$;
(4.56) T multiplies the weights of all the finite points by the same factor $t_{-1,-1}$.

Every transformation in Σ multiplies the conformal norm by a constant factor. Hence (4.56) and (3.41) imply

(4.57) T is a similarity.

Suppose the transformation $T\in\Sigma$ is a projectivity. Then it maps $(n-1)$-planes into $(n-1)$-planes. Since they are determined by vectors $\mathbf{a}$ with $a_{-1}=0$, T must satisfy (4.52) and therefore also (4.57). We thus have:

(i) The only projectivities in Σ are similarities;
(ii) (4.57) is equivalent to each of (4.52)–(4.56).

Now let $T\in\Sigma$ be a similarity. By (4.54) and (4.55)

$$T=\begin{pmatrix} T_{11} & \tau & 0 \\ 0 & t_{-1,-1} & 0 \\ \mathbf{v}' & t_{-2,-1} & t_{-2,-2} \end{pmatrix}, \qquad \tau=\begin{pmatrix} t_{1,-1} \\ \cdot \\ \cdot \\ t_{n,-1} \end{pmatrix}, \qquad \mathbf{v}=\begin{pmatrix} t_{-21} \\ \cdot \\ \cdot \\ t_{-2n} \end{pmatrix}.$$

Since $T\in\Sigma$, we may apply (4.31) obtaining

$$\begin{pmatrix} T_{11}'T_{11} & T_{11}'\tau-\frac{1}{2}t_{-1,-1}\mathbf{v} & 0 \\ \tau T_{11}-\frac{1}{2}t_{-1,-1}\mathbf{v} & \tau'\tau-t_{-1,-1}t_{-2,-1} & -\frac{1}{2}t_{-1,-1}t_{-2,-2} \\ 0 & -\frac{1}{2}t_{-1,-1}t_{-2,-2} & 0 \end{pmatrix}=\rho\begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}.$$

This yields

$$T=\begin{pmatrix} T_{11} & \tau & 0 \\ 0 & t_{-1,-1} & 0 \\ 2\tau'T_{11}/t_{-1,-1} & \tau'\tau/t_{-1,-1} & \rho/t_{-1,-1} \end{pmatrix},$$

where

$$T_{11}'T_{11}=\rho I_n, \qquad \rho\neq 0, \tag{4.58}$$

or after a slight change of notation

$$T=\sigma\begin{pmatrix} T_{11} & \tau & 0 \\ 0 & 1 & 0 \\ 2\tau'T_{11} & \tau'\tau & \rho \end{pmatrix}, \tag{4.59}$$

where $\rho\sigma\neq 0$ and where T_{11} still satisfies (4.58). Thus every similarity $T\in\Sigma$ admits a representation (4.59). Conversely, if T is given by (4.59) and (4.58), it satisfies (4.31) and therefore belongs to Σ. Since it also satisfies (4.54) it is a similarity. Thus (4.59)–(4.58) is another characterization of the similarities in Σ.

Now let

$$\eta_k=\sum_1^n t_{k\lambda}\xi_\lambda+t_{k,-1} \qquad (k=1,\cdots,n) \tag{4.510}$$

be an arbitrary similarity. Put

$$T_{11}=(t_{k\lambda}), \qquad \tau'=(t_{1,-1},\cdots,t_{n,-1}), \qquad \sigma \text{ arbitrary},$$

and define T through (4.59). Thus (4.58) holds. By (4.46), (4.510) is the inhomogeneous version of $\mathbf{y} = T\mathbf{x}$ and we have

Every similarity belongs to Σ and the multiplier $t_{-1,-1}$ of (4.56) can be chosen arbitrarily.

The similarity (4.510) is an orthogonality if and only if $\rho = 1$ in (4.58)–(4.59). Hence any of the conditions (4.52)–(4.57) together with

$$t_{-1,-1} = t_{-2,-2} \tag{4.511}$$

characterizes the orthogonalities.

The transformation (4.510) is a translation if and only if $T_{11} = I_n$. Since $\rho = 1$, we obtain from (4.59): the transformation T in Σ is a translation if and only if

$$T = \sigma \begin{pmatrix} I_n & \tau & 0 \\ 0 & 1 & 0 \\ 2\tau' & \tau'\tau & 1 \end{pmatrix}. \tag{4.512}$$

5. Symmetries in A_{n+2}.

5.1. Let

$$\alpha^*: \qquad \mathbf{a}^{*\prime} J \mathbf{x}^* = 0$$

be an $(n+1)$-space through O in $(n+2)$-dimensional pseudo-euclidean space A_{n+2}. We assume

$$\|\mathbf{a}^*\| = \mathbf{a}^{*\prime} J \mathbf{a}^* \neq 0. \tag{5.11}$$

Thus the pseudo-normal vector $\mathbf{a}^*$ of α^* does not lie in $\mathbf{a}^{*\prime}$ and every vector $\mathbf{x}^*$ permits a unique decomposition of the form

$$\mathbf{x}^* = \mathbf{y}^* + \lambda \mathbf{a}^*, \qquad \mathbf{y}^* \in \alpha^*. \tag{5.12}$$

For $\mathbf{y}^* = \mathbf{x}^* - \lambda \mathbf{a}^*$ lies in α^* if and only if $\mathbf{a}^{*\prime} J \mathbf{y}^* = \mathbf{a}^{*\prime} J \mathbf{x}^* - \lambda \mathbf{a}^{*\prime} J \mathbf{a}^* = 0$, *i.e.*, if

$$\lambda = \mathbf{a}^{*\prime} J \mathbf{x}^* / \|\mathbf{a}^*\|. \tag{5.13}$$

We wish to construct a linear transformation S^* of A_{n+2} into itself which keeps each point of α^* fixed and which maps the pseudo-normal vector $\mathbf{a}^*$ of α^* onto $-\mathbf{a}^*$. Thus

$$S^*(\mathbf{y}^* + \lambda \mathbf{a}^*) = \mathbf{y}^* - \lambda \mathbf{a}^* \tag{5.14}$$

and (5.12) and (5.13) imply

$$S^*(\mathbf{x}^*) = \mathbf{x}^* - 2\lambda \mathbf{a}^* = \mathbf{x}^* - 2[(\mathbf{a}^{*\prime} J \mathbf{x}^*)/\|\mathbf{a}^*\|] \cdot \mathbf{a}^*. \tag{5.15}$$

We find that our problem has exactly one solution, *viz.*, (5.15). It has the matrix

$$S^* = I_{n+2} - 2\mathbf{a}^* \mathbf{a}^{*\prime} J / \|\mathbf{a}^*\|. \tag{5.16}$$

By (5.14)

$$S^{*2} = I_{n+2}. \tag{5.17}$$

Let $\mathbf{y}^* \in \alpha^*$. Then

$$\begin{aligned}\|\mathbf{y}^* + \lambda\mathbf{a}^*\| &= (\mathbf{y}^{*\prime} + \lambda\mathbf{a}^{*\prime})J(\mathbf{y}^* + \lambda\mathbf{a}^*) \\ &= \mathbf{y}^{*\prime}J\mathbf{y}^* + \lambda\mathbf{a}^{*\prime}J\mathbf{y}^* + \lambda\mathbf{y}^{*\prime}J\mathbf{a}^* + \lambda^2\mathbf{a}^{*\prime}J\mathbf{a}^*.\end{aligned}$$

Since the middle terms both vanish, we obtain

$$\|\mathbf{y}^* + \lambda\mathbf{a}^*\| = \|\mathbf{y}^* - \lambda\mathbf{a}^*\| = \|\mathbf{y}^*\| + \lambda^2\|\mathbf{a}^*\|. \tag{5.18}$$

In particular, the point $\mathbf{x}^* = \mathbf{y}^* + \lambda\mathbf{a}^*$ and its image (5.14) have the same norm or

$$\|S^*\mathbf{x}^*\| = \|\mathbf{x}^*\| \tag{5.19}$$

for all $\mathbf{x}^*$. Hence S^* belongs to the pseudo-orthogonal group Ω^* of A_{n+2} and we have

$$S^{*\prime}JS^* = J. \tag{5.110}$$

Obviously, S^* is uniquely determined by α^*. We call S^* the pseudo-euclidean symmetry with respect to α^*.

5.2. Suppose that we are given two distinct vectors $\mathbf{b}^*$ and $\mathbf{c}^*$ in A_{n+2} and we require an $(n+1)$-space α^* through O such that the (pseudo-euclidean) symmetry S^* with respect to α^* maps $\mathbf{b}^*$ onto $\mathbf{c}^*$. Thus

$$\mathbf{c}^* = S^*\mathbf{b}^*. \tag{5.21}$$

On account of (5.19) we have to assume

$$\|\mathbf{b}^*\| = \|\mathbf{c}^*\|, \tag{5.22}$$

i.e., $\mathbf{b}^*$ and $\mathbf{c}^*$ have the same "pseudo-euclidean length." By (5.12) we may put

$$\mathbf{b}^* = \mathbf{y}^* + \mathbf{a}^*, \qquad \mathbf{c}^* = \mathbf{y}^* - \mathbf{a}^*, \tag{5.23}$$

where $\mathbf{y}^* \in \alpha^*$ and $\mathbf{a}^*$ is pseudo-normal to α^*. Hence

$$\mathbf{a}^* = \tfrac{1}{2}(\mathbf{b}^* - \mathbf{c}^*) \tag{5.24}$$

and there is not more than one $(n+1)$-space α^* which solves our problem, *viz.*, the one determined by $\mathbf{a}^*$. It now follows from **5.1** that α^* is a solution if and only if $\mathbf{b}^*$ and $\mathbf{c}^*$ satisfy not only (5.21) but also

$$\|\mathbf{a}^*\| = \tfrac{1}{4}\|\mathbf{b}^* - \mathbf{c}^*\| \neq 0. \tag{5.25}$$

By (5.22), the last condition can be given the form

$$\mathbf{b}^{*\prime}J\mathbf{c}^* \neq \|\mathbf{b}^*\|. \tag{5.26}$$

Suppose now that (5.22) is satisfied but not (5.26). Put

$$\delta = \|\mathbf{b}^*\| = \|\mathbf{c}^*\| = \mathbf{b}^{*\prime}J\mathbf{c}^*. \tag{5.27}$$

Thus no symmetry will now map $\mathbf{b}^*$ onto $\mathbf{c}^*$. Since a symmetry maps O onto itself, no product of symmetries will map a point $\neq O$ onto O. Assume therefore

$$\mathfrak{b}^* \neq 0, \qquad \mathfrak{c}^* \neq 0. \tag{5.28}$$

J being regular, the quadric $\|\mathfrak{x}^*\| = \delta$ is either regular ($\delta \neq 0$) or once degenerate ($\delta = 0$). Thus, if $n \geqq 1$, it cannot be the union of two $(n+1)$-spaces and there is a point $\mathfrak{d}^*$ such that

$$\|\mathfrak{d}^*\| = \delta; \qquad \mathfrak{b}^{*\prime} J \mathfrak{d}^* \neq \delta, \qquad \mathfrak{c}^{*\prime} J \mathfrak{d}^* \neq \delta. \tag{5.29}$$

Hence from the above there are two symmetries S_1^* and S_2^* mapping $\mathfrak{d}^*$ onto $\mathfrak{b}^*$ respectively $\mathfrak{c}^*$. This implies: assume (5.27) and (5.28); then there are two symmetries S_1^* and S_2^* such that

$$\mathfrak{c}^* = S_2^* S_1^* \mathfrak{b}^*. \tag{5.210}$$

5.3. Given a point $\mathfrak{b}^* \in A_{n+2}$, $\mathfrak{b}^* \neq 0$, and a transformation

$$T^* \in \Sigma^*, \qquad T^* \notin \Sigma_0^*. \tag{5.31}$$

Here $\Sigma_0^* = \Sigma_0^*(\mathfrak{b}^*)$ denotes the subgroup of all the transformations of Σ^* with the fixed point $\mathfrak{b}^*$. By (4.21), there is a $\rho \neq 0$ such that

$$\|T^* \mathfrak{x}^*\| = \rho \cdot \|\mathfrak{x}^*\| \quad \text{for all } \mathfrak{x}^*. \tag{5.32}$$

We assume

$$\|T^* \mathfrak{b}^*\| = \|\mathfrak{b}^*\|. \tag{5.33}$$

Thus either $\rho = 1$, *i.e.*, $T^* \in \Omega^*$, or $\|\mathfrak{b}^*\| = 0$.

If

$$\mathfrak{b}^{*\prime} J T^* \mathfrak{b}^* \neq \|\mathfrak{b}^*\|, \tag{5.34}$$

5.2 implies that there is one and only one pseudo-euclidean symmetry S^* which interchanges $\mathfrak{b}^*$ and $T^*\mathfrak{b}^*$. Hence

$$V^* = S^* T^* \in \Sigma_0^*. \tag{5.35}$$

By (5.17), (5.35) is equivalent to

$$T^* = S^* V^*. \tag{5.36}$$

Suppose conversely that a decomposition (5.36) of T^* is given where S^* is a symmetry and $V^* \in \Sigma_0^*$. Then from (5.31)

$$\mathfrak{c}^* = S^* \mathfrak{b}^* = T^* \mathfrak{b}^* \neq \mathfrak{b}^*. \tag{5.37}$$

By **5.2**, S^* is uniquely determined and (5.35) then determines V^*. Furthermore, (5.37) and (5.22) imply (5.33), while (5.34) follows from (5.26). We thus have: Let T^* satisfy (5.31). Then there is not more than one decomposition (5.36) and such a decomposition exists if and only if (5.33) and (5.34) hold.

By (5.31), $T^{*-1} \notin \Sigma_0^*$. Further, (5.33) and (5.34) are equivalent to

$$\|T^{*-1}\mathfrak{b}^*\| = \|\mathfrak{b}^*\|$$

and

$$\mathfrak{b}^{*\prime} J T^{*-1}\mathfrak{b}^* \neq \|\mathfrak{b}^*\|,$$

respectively. Thus we may apply the above result to T^{*-1} instead of T^* obtaining: If (5.31) holds, there is at most one decomposition

$$T^* = W^* S^*, \tag{5.38}$$

where $W^* \in \Sigma_0^*$ and S^* is a symmetry; a decomposition (5.38) exists if and only if (5.33) and (5.34) are satisfied.

We now replace (5.33) and (5.34) by

$$\|T^*\mathfrak{b}^*\| = \|\mathfrak{b}^*\| = \mathfrak{b}^{*\prime} J T^* \mathfrak{b}^*, \tag{5.39}$$

and assume $n \geqq 1$. Thus (5.27) and (5.28) hold for $\mathfrak{c}^* = T^*\mathfrak{b}^*$, and by **5.2** there are two symmetries S_1^* and S_2^* such that $T^*\mathfrak{b}_2^* = S_2^* S_1^* \mathfrak{b}^*$. Hence T^* permits a decomposition

$$T^* = S_2^* S_1^* V^*, \tag{5.310}$$

where

$$V^* = S_1^* S_2^* T^* \in \Sigma_0^*. \tag{5.311}$$

We thus obtain: Suppose $T^* \in \Sigma^*$ satisfies (5.33). Then T^* can be written as the product of a transformation in Σ_0^* and two or fewer pseudo-euclidean symmetries ($n \geqq 1$).

5.4. In the case $n = 0$ the last two paragraphs of **5.2** do not apply. The conic

$$K_1\colon \|\mathfrak{x}^*\| = x_1^2 - x_2^2 = 0$$

splits into two straight lines through O and the points

$$\mathfrak{b}^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathfrak{c}^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.41}$$

respectively.

Let $T^* \in \Sigma^*$. Then by (5.32)

$$\|T^*\mathfrak{b}^*\| = \|T^*\mathfrak{c}^*\| = 0. \tag{5.42}$$

Thus $T^*\mathfrak{b}^*$ and $T^*\mathfrak{c}^*$ lie in K_1. Since

$$\mathfrak{b}^{*\prime} J \mathfrak{c}^* = (1\ 1)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2, \tag{5.43}$$

(4.22) implies

$$(T^*\mathfrak{b}^*)'J(T^*\mathfrak{c}^*) = \rho\cdot\mathfrak{b}^{*\prime}J\mathfrak{c}^* = 2\rho \neq 0. \tag{5.44}$$

Hence there are two scalars $\beta\neq 0$ and $\gamma\neq 0$ such that either

$$T^*\mathfrak{b}^* = \beta\mathfrak{b}^*, \qquad T^*\mathfrak{c}^* = \gamma\mathfrak{c}^* \tag{5.45}$$

or

$$T^*\mathfrak{b}^* = \beta\mathfrak{c}^*, \qquad T^*\mathfrak{c}^* = \gamma\mathfrak{b}^*. \tag{5.46}$$

In either case, (5.44) implies

$$\beta\gamma = \rho. \tag{5.47}$$

Obviously, J is a pseudo-euclidean symmetry which interchanges $\binom{0}{1}$ and $-\binom{0}{1}$ and which keeps the points of the x_1-axis fixed. It interchanges $\mathfrak{b}^*$ and $\mathfrak{c}^*$. Thus a transformation which satisfies (5.45) can be written as the product of J with a transformation for which (5.46) holds.

Suppose in particular $T^*\in\Omega^*$, *i.e.*, $\rho=1$ and assume (5.46). Then (5.47) implies $T^*(\mathfrak{b}^*+\beta\mathfrak{c}^*)=\beta\mathfrak{c}^*+\mathfrak{b}^*$. Thus $\mathfrak{d}^*=\mathfrak{b}^*+\beta\mathfrak{c}^*$ is invariant with respect to T^* while the vector $\mathfrak{a}^*=\mathfrak{b}^*-\beta\mathfrak{c}^*$ is mapped onto $-\mathfrak{a}^*$. As $\mathfrak{a}^*$ is pseudo-normal to $\mathfrak{d}^*$, T^* is simply the symmetry with respect to the line $\lambda\mathfrak{d}^*$. We thus have: If $n=0$, then every transformation of Ω^* is either a pseudo-euclidean symmetry or the product of two such symmetries. In the latter case one of them may be chosen to be J. (Actually, it can be arbitrarily prescribed since any symmetry in A_2 must satisfy (5.46)).

5.5. We can now show that every transformation in Ω^* can be decomposed into a product of pseudo-euclidean symmetries.

By **5.4**, this statement is true for $n=0$. Thus we may assume that it is proved up to $n-1$. Denote the pseudo-orthogonal group of A_{n+2} by Ω_n^* and let $T_n^*\in\Omega_n^*$. Put $\mathfrak{b}^{*\prime}=(1\ 0\ \cdots\ 0)$. If $\mathfrak{b}^*$ is not a fixed point of T_n^*, T_n^* permits a decomposition (5.36) or (5.310) where $V^*\mathfrak{b}^*=\mathfrak{b}^*$. Since symmetries belong to Ω_n^*, (5.35) and (5.311) imply $V^*\in\Omega_n^*$. Thus T_n^* can always be expressed as a product of a transformation of Ω_n^* with the fixed point $\mathfrak{b}_n^*$ and of symmetries. We may therefore assume

$$T_n^*\mathfrak{b}^* = \mathfrak{b}^*. \tag{5.52}$$

Put

$$J = J_n = \begin{pmatrix} I_{n+1} & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & J_{n-1} \end{pmatrix} \qquad (n = 1, 2, \cdots). \tag{5.52}$$

By (4.23) $T_n^{*-1}=J_nT_n^{*\prime}J_n$, and T_n^{*-1} has the fixed point $\mathfrak{b}^*$. Hence T_n^* has a matrix of the form

$$T_n^* = \begin{pmatrix} 1 & 0 \\ 0 & T_{n-1}^* \end{pmatrix} \tag{5.53}$$

and $T_n^{*\prime} J_n T_n^* = J_n$ implies

$$T_{n-1}^{*\prime} J_{n-1} T_{n-1}^* = J_{n-1}. \tag{5.54}$$

Thus T_{n-1}^* can be interpreted as a pseudo-euclidean transformation of the space

$$A_{n+1}\colon\ x_1 = \mathbf{b}^{*\prime} J \mathbf{x}^* = 0.$$

Let S_{n-1}^* be a symmetry in A_{n+1}. Thus S_{n-1}^* maps some vector $\mathbf{a}^* \in A_{n+1}$ onto $-\mathbf{a}^*$ and keeps the points of the n-space

$$\mathbf{a}^{*\prime} J_n \mathbf{x}^* = 0, \qquad x_1 = 0 \tag{5.55}$$

of A_{n+1} fixed. The symmetry

$$S_n^* = \begin{pmatrix} 1 & 0 \\ 0 & S_{n-1}^* \end{pmatrix} \tag{5.56}$$

of A_{n+2} will also map $\mathbf{a}^*$ onto $-\mathbf{a}^*$ and will keep the points of the $(n+1)$-space $\mathbf{a}^{*\prime} J_n \mathbf{x}^* = 0$ through the n-space (5.55) fixed. As $\mathbf{a}^* \in A_{n+1}$, this $(n+1)$-space passes through $\mathbf{b}^*$.

By the induction assumption, T_{n-1}^* can be decomposed into a product of symmetries S_{n-1}^* of A_{n+1}. Through (5.56), this determines a decomposition of T_n^* into symmetries S_n^* of A_{n+2}. This completes our proof.

6. Inversions I.

6.1. Again let

$$U = \left(\begin{array}{c|cc} I_n & 0 & \\ \hline & -1 & 1 \\ 0 & 1 & 1 \end{array}\right); \tag{6.11}$$

cf. (3.32). Through

$$\mathbf{x} = U\mathbf{x}^*, \tag{6.12}$$

the points $\mathbf{x}^* \neq 0$ of A_{n+2} were mapped onto the spherical vectors $\mathbf{x}$ of complex conformal n-space C_n. Suppose in addition

$$\mathbf{a} = U\mathbf{a}^*. \tag{6.13}$$

Then the image of the $(n+1)$-space

$$\mathbf{a}^{*\prime} J \mathbf{x}^* = 0 \tag{6.14}$$

is the linear pencil of the $(n-1)$-spheres $\mathbf{x}$ with

$$\mathbf{a}^{\prime} G \mathbf{x} = 0; \tag{6.15}$$

cf. (3.33). It consists of the $(n-1)$-spheres orthogonal to $\mathbf{a}$. A point $\mathbf{x}$ satisfies (6.15) if and only if it is the center of a null sphere orthogonal to $\mathbf{a}$, *i.e.*, if and only if it lies on $\mathbf{a}$.

Through

$$T = UT^*U^{-1}, \tag{6.16}$$

the pseudo-euclidean similarity group Σ^* (the pseudo-orthogonal group Ω^*) of A_{n+2} was mapped onto the full conformal group Σ (the proper conformal group Ω) of C_n; *cf.* **4.3.**

6.2. A pseudo-euclidean symmetry S^* of A_{n+2} is mapped by U onto an *inversion*

$$S = US^*U^{-1} \tag{6.21}$$

of C_n. By (6.21) and (5.16)

$$\begin{aligned} S &= U(I_{n+2} - 2\mathbf{a}^*\mathbf{a}^{*\prime}J/\|\mathbf{a}^*\|)U^{-1} \\ &= I_{n+2} - 2U\mathbf{a}^* \cdot (U\mathbf{a}^*)'U^{-1}JU^{-1}/\|\mathbf{a}^*\|. \end{aligned}$$

Hence by (3.33), (3.38) and (6.13)

$$S = I_{n+2} - 2\mathbf{a}\mathbf{a}'G/\|\mathbf{a}\|. \tag{6.22}$$

By (5.11)

$$\|\mathbf{a}\| \neq 0 \tag{6.23}$$

and, by (5.17) and (6.21)

$$S^2 = I_{n+2}. \tag{6.24}$$

Since $S^* \in \Omega^*$, we have $S \in \Omega$; *cf.* **4.3.** Thus

$$S'GS = G \tag{6.25}$$

and

$$\|S\mathbf{x}\| = \|\mathbf{x}\|. \tag{6.26}$$

Formula (6.22) obviously implies

$$S\mathbf{a} = -\mathbf{a} \tag{6.27}$$

and

$$S\mathbf{x} = \mathbf{x} \tag{6.28}$$

for all $\mathbf{x}$ that satisfy (6.15). Thus S keeps each point of the $(n-1)$-sphere $\mathbf{a}$ fixed and maps every $(n-1)$-sphere orthogonal to $\mathbf{a}$ onto itself. Furthermore, (6.22) shows that $S\mathbf{b}$ is a multiple of $\mathbf{b}$ if and only if either $\mathbf{b}$ is a multiple of $\mathbf{a}$ or orthogonal to $\mathbf{a}$. Thus the above $(n-1)$-spheres are the only ones which are mapped by S onto themselves.

6.3. Let

$$\mathbf{a}'G\mathbf{b} \neq 0, \qquad \mathbf{c}'G\mathbf{b} = 0. \tag{6.31}$$

Then

$$\mathbf{c}'GS\mathbf{b} = \mathbf{c}'G\left(\mathbf{b} - 2\frac{\mathbf{a}'G\mathbf{b}}{\|\mathbf{a}\|}\cdot\mathbf{a}\right) = -2\frac{\mathbf{a}'G\mathbf{b}}{\|\mathbf{a}\|}\cdot\mathbf{c}'G\mathbf{a}.$$

Hence

$$\mathbf{c}'GS\mathbf{b} = 0 \leftrightarrow \mathbf{c}'G\mathbf{a} = 0. \tag{6.32}$$

We thus have: Suppose the $(n-1)$-sphere $\mathbf{b}$ is normal to $\mathbf{c}$ but not to $\mathbf{a}$. Then the $(n-1)$-sphere $\mathbf{c}$ is normal to $\mathbf{a}$ if and only if it is normal to $S\mathbf{b}$. In the special case that $\mathbf{b}=\mathbf{x}$ is a point we obtain: Suppose $\mathbf{x}$ does not lie on the $(n-1)$-sphere $\mathbf{a}$. Then an $(n-1)$-sphere through $\mathbf{x}$ is normal to $\mathbf{a}$ if and only if it passes through $S\mathbf{x}$.

We can now readily prove: If $\mathbf{x}$ and $\mathbf{y}$ are any two points, the four points

$$\mathbf{x}, \mathbf{y}, S\mathbf{x}, S\mathbf{y} \tag{6.33}$$

are concyclic, *i.e.*, they lie on a circle.

We may assume that the points (6.33) are mutually distinct, in particular,

$$S\mathbf{x} \neq \mathbf{x}, \qquad S\mathbf{y} \neq \mathbf{y}. \tag{6.34}$$

Let $\mathbf{c}$ be any $(n-1)$-sphere through the points

$$\mathbf{x}, \mathbf{y}, S\mathbf{x}. \tag{6.35}$$

Since $\mathbf{x}$ does not lie on $\mathbf{a}$, and since $\mathbf{x}$ and $S\mathbf{x}$ lie on $\mathbf{c}$, $\mathbf{c}$ is normal to $\mathbf{a}$. Since $\mathbf{y}$ lies on $\mathbf{c}$ but not on $\mathbf{a}$, $\mathbf{c}$ must therefore contain $S\mathbf{y}$. This holds for any choice of $\mathbf{c}$. Thus $S\mathbf{y}$ lies in the intersection of all the $(n-1)$-spheres through the points (6.35), *i.e.*, on the circle through them.

We note: If the points (6.33) are mutually distinct, the circle through them is normal to $\mathbf{a}$.

6.4. Let $\mathbf{n}$, where $\mathbf{n}'=(0\ \cdots\ 0\ 1)$, denote the real absolute point of C_n; *cf.* (1.53). Let $\mathbf{a}$ be a proper $(n-1)$-sphere. By **2.1** this is equivalent to

$$a_{-1} = -2\mathbf{n}'G\mathbf{a} \neq 0. \tag{6.41}$$

By (3.52) the radius r of $\mathbf{a}$ satisfies

$$r^2 = \|\mathbf{a}\|\cdot a_{-1}^{-2}. \tag{6.42}$$

Suppose the inversion (6.22) maps $\mathbf{n}$ onto the point $\mathbf{m}$. Then

$$\mathbf{m} = S\mathbf{n} = \mathbf{n} - \frac{2\mathbf{a}'G\mathbf{n}}{\|\mathbf{a}\|}\cdot\mathbf{a} = \mathbf{n} + \frac{a_{-1}}{\|\mathbf{a}\|}\cdot\mathbf{a} = \mathbf{n} + \frac{\mathbf{a}}{a_{-1}r^2}$$

or

$$\mathbf{m}' = \{1/(a_{-1}r^2)\}(a_1, \cdots, a_n, a_{-1}, a_0/a_{-1}). \tag{6.43}$$

Thus $\mathbf{m}$ is the center of $\mathbf{a}$ with the weight r^{-2}.

Choosing $\mathbf{y}=\mathbf{n}$ in (6.33), we have: If $a_{-1}\neq 0$, the points

$$\mathbf{m},\ \mathbf{x},\ S\mathbf{x} \tag{6.44}$$

are collinear.

Now let $\mathbf{x}$ denote any finite point of C_n. Then (3.310) and (6.25) imply

$$\|S\mathbf{x}-\mathbf{m}\| = -2(S\mathbf{x})'G\mathbf{m} = -2\mathbf{x}'S'GS\mathbf{n} = -2\mathbf{x}'G\mathbf{n}$$

or

$$\|S\mathbf{x}-\mathbf{m}\| = x_{-1} = \text{weight of } \mathbf{x}. \tag{6.45}$$

Replacing $\mathbf{x}$ by $S\mathbf{x}$ and taking (6.24) into account, we obtain

$$\|\mathbf{x}-\mathbf{m}\| = \text{weight of } S\mathbf{x}. \tag{6.46}$$

Suppose in addition that $\mathbf{x}$ does not lie on the null sphere about $\mathbf{m}$. We apply (3.41) twice and make use of (6.46), (6.45), and (6.43) obtaining: The product of the squares of the distances of $\mathbf{x}$ and $S\mathbf{x}$ from $\mathbf{m}$ is equal to

$$\frac{\|\mathbf{x}-\mathbf{m}\|}{\text{weight of } \mathbf{x}\cdot\text{weight of } \mathbf{m}}\cdot\frac{\|S\mathbf{x}-\mathbf{m}\|}{\text{weight of } S\mathbf{x}\cdot\text{weight of } \mathbf{m}}$$

$$=\frac{\text{weight of } S\mathbf{x}}{\text{weight of } \mathbf{x}\cdot\text{weight of } \mathbf{m}}\cdot\frac{\text{weight of } \mathbf{x}}{\text{weight of } S\mathbf{x}\cdot\text{weight of } \mathbf{m}} = (\text{weight of } \mathbf{m})^{-2} = r^4.$$

This yields the well-known result that the product of the distances of $\mathbf{x}$ and $S\mathbf{x}$ from $\mathbf{m}$ is equal to $\pm r^2$.

6.5. The results of **6.4** can also be obtained more directly from (6.22). Put $\mathbf{y}=S\mathbf{x}$ and assume that $\mathbf{x}$ and $\mathbf{y}$ are both finite. Then

$$\frac{y_\lambda}{y_{-1}}-\frac{a_\lambda}{a_{-1}} = \frac{x_\lambda - 2\mathbf{a}'G\mathbf{x}\cdot\|\mathbf{a}\|^{-1}\cdot a_\lambda}{x_{-1}-2\mathbf{a}'G\mathbf{x}\cdot\|\mathbf{a}\|^{-1}\cdot a_{-1}}-\frac{a_\lambda}{a_{-1}}$$

readily implies

$$\frac{y_\lambda}{y_{-1}}-\frac{a_\lambda}{a_{-1}} = \frac{x_{-1}}{y_{-1}}\left(\frac{x_\lambda}{x_{-1}}-\frac{a_\lambda}{a_{-1}}\right) \qquad (\lambda=1,\cdots,n). \tag{6.51}$$

We now introduce the inhomogeneous coordinate vectors ξ of $\mathbf{x}$, $\mathfrak{n}$ of $\mathbf{y}$, and $\mathfrak{y}$ of $\mathbf{m}$; *cf.* (1.36) and (2.15). Then by (6.51)

$$\mathfrak{n}-\mathfrak{y} = (x_{-1}/y_{-1})(\xi-\mathfrak{y}). \tag{6.52}$$

In particular the points (6.44) are collinear.

From (3.41) and (6.46) we can again deduce $(\xi-\mathfrak{y})^2=y_{-1}/(x_{-1}\cdot m_{-1})$ $=(y_{-1}/x_{-1})\cdot r^2$ or

$$x_{-1}/y_{-1} = r^2/(\xi-\mathfrak{y})^2. \tag{6.53}$$

Combining this formula with (6.52), we finally obtain

$$\mathfrak{n} - \mathfrak{y} = \frac{r^2}{(\xi - \mathfrak{y})^2} \cdot (\xi - \mathfrak{y}). \tag{6.54}$$

This formula implies in particular the last result of **6.4**, *viz.*,

$$(\mathfrak{n} - \mathfrak{y})^2, (\xi - \mathfrak{y})^2 = r^4. \tag{6.55}$$

We note that (6.52)–(6.55) break down if $\mathbf{x}$ lies on the null sphere about $\mathbf{m}$.

6.6. Let $\mathbf{a}$ be a nonisotropic $(n-1)$-space. Thus

$$a_{-1} = 0 \tag{6.61}$$

and

$$\|\mathbf{a}\| = \sum_1^n a_\lambda^2 \neq 0. \tag{6.62}$$

Let S denote the inversion with respect to $\mathbf{a}$ and put $\mathbf{y} = S\mathbf{x}$; *cf.* (6.22). Formula (1.36) determines the inhomogeneous coordinates of the points $\mathbf{x}$ and $\mathbf{y}$. The inhomogeneous equation of $\mathbf{a}$ is given by (2.11) and (6.61). Thus we can verify that S is the symmetry with respect to $\mathbf{a}$.

The following proof requires no computation: By (6.61) $y_{-1} = x_{-1}$. Hence S is a similarity; *cf.* (4.56). On account of (6.24), S must therefore be an orthogonality. Since S is not the identity and keeps the points of the $(n-1)$-space $\mathbf{a}$ fixed, our result follows.

6.7. By means of **6.1**, the properties of the pseudo-euclidean symmetries obtained in **5** can be translated into properties of inversions.

Belonging to Ω, inversions preserve norms. Let $\mathbf{b}$ and $\mathbf{c}$ be two $(n-1)$-spheres with the same norms. By **5.2** there is not more than one inversion mapping $\mathbf{b}$ onto $\mathbf{c}$. There is one if and only if

$$\mathbf{b}'G\mathbf{c} \neq \|\mathbf{b}\|. \tag{6.71}$$

If (6.71) does not hold, *i.e.*, if the $(n-1)$-spheres $\mathbf{b}$ and $\mathbf{b}-\mathbf{c}$ are orthogonal, $\mathbf{b}$ can be mapped onto $\mathbf{c}$ by a product of two inversions.

We now apply **5.3.** Let $T \in \Sigma$ and let $\mathbf{b}$ be fixed; $T\mathbf{b} \neq \mathbf{b}$. Then there is not more than one decomposition

$$T = SV \tag{6.72}$$

of T into an inversion S and a transformation $V \in \Sigma$ with

$$V\mathbf{b} = \mathbf{b}. \tag{6.73}$$

Such a decomposition exists if and only if

$$\|T\mathbf{b}\| = \|\mathbf{b}\| \tag{6.74}$$

and

$$\mathbf{b}'GT\mathbf{b} \neq \|\mathbf{b}\|. \tag{6.75}$$

Condition (6.74) is satisfied if and only if either $\mathbf{b}$ is a null sphere or if $T \in \Omega$. If (6.74) holds but not (6.75), T permits a decomposition

$$T = S_1 S_2 V, \tag{6.76}$$

where V again satisfies (6.73) while S_1 and S_2 are two inversions.

Similar results hold with respect to decompositions $T = VS$.

By **5.5** every element of Ω can be decomposed into a product of inversions. Thus Ω is identical with the group generated by the inversions.

7. Inversions II.

7.1. We now specialize $\mathbf{b} = \mathbf{n}$. By **6.4**, the inversion

$$S = I_{n+2} - 2\mathbf{a}\mathbf{a}'G/\|\mathbf{a}\| \tag{7.11}$$

maps $\mathbf{n}$ onto

$$S\mathbf{n} = \mathbf{n} + (a_{-1}/\|\mathbf{a}\|)\cdot\mathbf{a}. \tag{7.12}$$

Thus the following properties of S are equivalent:

(i) $S\mathbf{n} = \mathbf{n}$,
(ii) The point $S\mathbf{n}$ lies on the absolute null sphere,
(iii) $a_{-1} = 0$, *i.e.*, $\mathbf{a}$ is an $(n-1)$-space,
(iv) S is a symmetry,
(v) S is a similarity; *cf.* **4.5**.

Suppose the point $\mathbf{m}$ does not lie on the absolute null sphere. Then by **6.7**, there is one and only one inversion S mapping the spherical vector $\mathbf{n}$ onto the spherical vector $\mathbf{m}$. The spherical vector $\mathbf{a}$ defining S may be chosen to be

$$\mathbf{a} = \mathbf{m} - \mathbf{n}. \tag{7.13}$$

Thus

$$\|\mathbf{a}\| = a_{-1} = m_{-1} \tag{7.14}$$

and

$$S = I_{n+2} - 2(\mathbf{m} - \mathbf{n})(\mathbf{m}' - \mathbf{n}')G/m_{-1}. \tag{7.15}$$

The point $\mathbf{m} = S\mathbf{n}$ is the center of the $(n-1)$-sphere (7.13). By (6.42), the square of its radius is given by

$$r^2 = 1/m_{-1}. \tag{7.16}$$

If the point $\mathbf{m} \neq \mathbf{n}$ lies on the absolute null sphere, no such inversion exists. However, $\mathbf{n}$ can be mapped onto $\mathbf{m}$ by a product of two inversions.

7.2. By **4.5** a transformation $V\in\Sigma$ is a similarity if and only if

$$V\mathbf{n} = \lambda\mathbf{n} \tag{7.21}$$

is a multiple of $\mathbf{n}$.

Let $T\in\Sigma$ and let

$$T = SV \tag{7.22}$$

be a decomposition of T into an inversion S and a similarity V. Obviously S is a similarity if and only if T itself is one. In this case, S may be an arbitrary symmetry and V is defined through

$$V = ST. \tag{7.23}$$

From now on we exclude the case that T is a similarity. Thus S is not a symmetry and the point $S\mathbf{n}$ does not lie on the absolute null sphere. By (7.21),

$$T\mathbf{n} = SV\mathbf{n} = \lambda S\mathbf{n}. \tag{7.24}$$

Hence T permits a decomposition (7.22) only if the point

$$\mathbf{m} = T\mathbf{n} \tag{7.25}$$

is finite. By **6.7** this condition is also sufficient and there exists, in particular, one and only one decomposition (7.22) with

$$V\mathbf{n} = \mathbf{n}. \tag{7.26}$$

Since then

$$S\mathbf{n} = \mathbf{m}, \tag{7.27}$$

S will be the inversion (7.15). Thus the similarity V is determined by (7.23). We wish to compute the factors by which V multiplies weights and distances.

Since $T\in\Sigma$, there is a scalar $\rho\neq 0$ such that

$$T'GT = \rho G; \tag{7.28}$$

cf. (4.31). Hence by (7.23) and (6.25)

$$V'GV = T'S'GST = T'GT = \rho G; \tag{7.29}$$

therefore

$$\|V\mathbf{x}\| = \rho\cdot\|\mathbf{x}\|. \tag{7.210}$$

Let $\mathbf{z} = V\mathbf{x}$. Then (7.26) and (7.29) imply

$$z_{-1} = -2\mathbf{n}'G\mathbf{z} = -2\mathbf{n}'V'GV\mathbf{x} = -2\rho\cdot\mathbf{n}'G\mathbf{x}$$

or

$$z_{-1} = \rho x_{-1}. \tag{7.211}$$

Thus V multiplies every weight by ρ. Incidentally, (7.211) verifies directly that V is a similarity. Formulas (7.210) and (7.211) enable us to apply (3.41) obtaining: V multiplies the squares of the distances of any two finite points by $1/\rho$.

7.3. Let $\lambda \neq 0$ be arbitrary and suppose the point (7.25) is finite. Applying **7.2** to $\lambda^{-1}T$ instead of T, we obtain: There exists exactly one decomposition

$$\lambda^{-1}T = S_\lambda \cdot \lambda^{-1}V_\lambda, \qquad i.e.,\ T = S_\lambda V_\lambda, \tag{7.31}$$

where S_λ is an inversion and $\lambda^{-1}V_\lambda$ is a similarity satisfying

$$\lambda^{-1}V_\lambda \mathbf{n} = \mathbf{n}, \qquad i.e.,\ V_\lambda \mathbf{n} = \lambda \mathbf{n}. \tag{7.32}$$

Thus V_λ itself is a similarity. By (7.32), (7.31), and (7.25).

$$S_\lambda \mathbf{n} = \lambda^{-1} \cdot S_\lambda \lambda \mathbf{n} = \lambda^{-1} \cdot S_\lambda V_\lambda \mathbf{n} = \lambda^{-1} T \mathbf{n} = \lambda^{-1} \mathbf{m}. \tag{7.33}$$

It follows from **7.1** that S_λ is the inversion with respect to the $(n-1)$-sphere about the point $\mathbf{m}$ whose radius r_λ satisfies

$$r_\lambda^2 = 1/\lambda^{-1} m_{-1} = \lambda / m_{-1} = \lambda r^2. \tag{7.34}$$

Let $\mathbf{x}$ be any finite point that does not lie on the null sphere about $\mathbf{m}$. Put

$$\mathbf{y} = S\mathbf{x}, \qquad \mathbf{z} = S_\lambda \mathbf{y} = S_\lambda S \mathbf{x}, \tag{7.35}$$

and let the points $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{m}$ have the inhomogeneous coordinate vectors $\xi, \mathfrak{n}, \zeta, \mathfrak{y}$, respectively. By (6.54)

$$\mathfrak{n} - \mathfrak{y} = \frac{r^2}{(\xi - \mathfrak{y})^2} \cdot (\xi - \mathfrak{y})$$

and therefore

$$\zeta - \mathfrak{y} = \frac{r_\lambda^2}{(\mathfrak{n} - \mathfrak{y})^2} \cdot (\mathfrak{n} - \mathfrak{y}) = \frac{r_\lambda^2}{r^2} (\xi - \mathfrak{y}).$$

Hence by (7.34)

$$\zeta - \mathfrak{y} = \lambda(\xi - \mathfrak{y}). \tag{7.36}$$

Being a similarity, $S_\lambda S$ multiplies every weight by the same factor. By (7.27) and (7.33)

$$S_\lambda S \mathbf{m} = S_\lambda \mathbf{n} = \lambda^{-1} \mathbf{m}. \tag{7.37}$$

The last two formulas completely describe $S_\lambda S$: It is a dilatation in the ratio $1:\lambda$ with the center $\mathbf{m}$. It multiplies every weight by λ^{-1}.

From (7.31) and (7.22)

$$V_\lambda = S_\lambda T = (S_\lambda S) V. \tag{7.38}$$

Thus V_λ differs from V only by the dilatation $S_\lambda S$.

The proof of (7.210) remains valid for V_λ. Thus V_λ also multiplies every norm by ρ. We note the following corollaries of (7.38) and of our discussions of V in **7.2** and of $S_\lambda S$: V_λ multiplies weights by $\lambda^{-1}\rho$ and the squares of the distances of any two finite points by λ^2/ρ. In particular, V_λ is an orthogonality if and only **if**

$$\lambda^2 = \rho. \tag{7.39}$$

This yields: The transformation T can be decomposed into a product of an orthogonality and an inversion in exactly two ways.

(We have $V_{-\lambda} = (S_{-\lambda}S_\lambda)V_\lambda$ and $S_{-\lambda}S_\lambda$ is the reflection with respect to the point **m**. It multiplies every weight by -1).

7.4. If $T \in \Sigma$ is not a similarity and if the point (7.25) lies on the absolute null sphere, no decomposition (7.22) exists. However, T will admit decompositions

$$T = S_1 S_2 V, \tag{7.41}$$

where S_1 and S_2 are inversions and V is a similarity; *cf.* **6.7.**

THE BRAUER-RADEMACHER IDENTITY

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1. Introduction. Let n and r be integers, $r>0$, and let $\phi(r)$ and $\mu(r)$ denote, respectively, the arithmetical functions of Euler and Möbius. The following remarkable identity was stated by H. Rademacher and proved by A. Brauer [1]:

$$\phi(r) \sum_{d|r,\ (d,\ n)=1} \frac{d}{\phi(d)} \mu(r/d) = \mu(r) \sum_{d|(n,\ r)} d\mu(r/d), \tag{1}$$

the summation on the left being over the divisors of r prime to n, and the one on the right over the common divisors of n and r. This formula is noteworthy simply by virtue of its arithmetical elegance; however, in another form ([3], Lemma 1), it has been shown to have interesting applications as well ([3], Sec. 5).

In this note we give a new proof of (1), as an illustration of an arithmetical inversion principle whose proof is contained in another paper [4]. This principle may be stated as follows. Let $f(n, r)$ be a (complex-valued) *even* function of $n \pmod r$; that is, $f(n, r) = f((n, r), r)$ for all values of n [2]. Then, if $r = r_1 r_2$, where r_1 and r_2 are positive integers, it follows that

$$f(n, r) = \sum_{d|(n,\ r)} g(d, r/d) \rightleftarrows g(r_1, r_2) = \sum_{d|r_1} f(r_1/d, r)\mu(d). \tag{2}$$

We also mention that (1) can be proved by means of an identity of Landau

([5], p. 88, *1). In fact, a wide generalization of the Brauer-Rademacher relation is deduced in [4] on the basis of an extension of the Landau identity.

We make a final comment concerning the method of proof. Our objective is not merely to prove an identity, but rather to express a particular function (as defined by the left of (1)) in a particular form (as on the right of (2)). The above inversion principle shows that this is *possible*, that the result is *unique*, and indicates an *explicit* procedure for obtaining the desired result.

2. Lemmas on $\phi(r)$ and $\mu(r)$. We recall first the factorability properties, $\phi(r)=\phi(r_1)\phi(r_2)$, $\mu(r)=\mu(r_1)\mu(r_2)$, where $r=r_1r_2$, $(r_1, r_2)=1$ ([5], Ch. 16). In addition, we recall the familiar evaluation ([5], Th. 6–7),

$$\phi(r) = \sum_{d|r} d\mu(r/d). \tag{3}$$

The following simple lemmas will also be needed.

LEMMA 1. *If p is a prime and l is an integer $\geqq 2$, then $\phi(p^l)=p\phi(p^{l-1})$.*

Proof. By (3), $\phi(p^t)=p^{t-1}(p-1)$ if $t>0$, and the lemma follows immediately.

We define the *core* $\gamma(r)$ of r to be 1 if $r=1$, and to be the product of the distinct prime divisors of r if $r>1$. In addition, we shall say that r is *primitive* if r contains no square divisors >1, otherwise, nonprimitive. Note that $\mu(r)\neq 0$ or $=0$ according as r is primitive or nonprimitive.

LEMMA 2. *If r is primitive and if s is an integer such that $\gamma(s)\,|\,r$, then (provided $s>1$)*

$$\sum_{d|r} \frac{d\mu(d)}{\phi(ds)} = 0. \tag{4}$$

Proof. Denote the left side of (4) by $B(s, r)$. If $r=r_1r_2$, $((r_1, r_2)=1)$, $s=s_1s_2>1$, $\gamma(s_1)\,|\,r_1$, $\gamma(s_2)\,|\,r_2$, then the factorability properties of $\phi(r)$ and $\mu(r)$ imply that $B(s, r)=B(s_1, r_1)B(s_2, r_2)$. Since either $s_1>1$ or $s_2>1$, it suffices to prove (4) in case $r=p$ (p prime), $s=p^k$, $k>0$. But $B(p, p^k)=1/\phi(p^k)-p/\phi(p^{k+1})$, which, by Lemma 1, has the value 0.

LEMMA 3. *If r is primitive and d is a divisor of r, then $\mu(r/d)=\mu(r)\mu(d)$.*

Proof. By the definition and factorability of $\mu(r)$, it follows (for r primitive) that $\mu(r/d)=\mu(r/d)\mu^2(d)=\mu(r)\mu(d)$.

LEMMA 4.

$$\sum_{d|r,\ (r/d,\ n)=1} \mu(d) = \begin{cases} \mu(r), & \text{if } r\,|\,n, \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where the summation is over divisors d of r whose conjugate divisors r/d are prime to n.

Proof. The case $n=1$ of (5) is the familiar property ([5], Th. 6-5) of $\mu(r)$,

$$\sum_{d|r} \mu(d) = \epsilon(r), \tag{5a}$$

where $\epsilon(r)$ is defined to be 1 if $r=1$ and 0 if $r>1$. We therefore have

$$\begin{aligned}\sum_{d|r,\ (r/d,\ n)=1} \mu(d) &= \sum_{d|r} \mu(d)\epsilon((r/d,\ n)) \\ &= \sum_{d|r} \mu(d) \sum_{D|(r/d,\ n)} \mu(D) = \sum_{D|(n,\ r)} \mu(D) \sum_{d|(r/D)} \mu(d) \\ &= \sum_{D|(n,\ r)} \mu(D)\epsilon(r/d),\end{aligned}$$

and the lemma results by definition of $\epsilon(r)$.

3. Proof of the Brauer-Rademacher identity (1). Denote the left member of (1) by $E(n, r)$. It is evident that $E(n, r)$ is even (mod r); hence we may apply the inversion formula (2) to obtain

$$E(n, r) = \sum_{d|(n,\ r)} g(d, r/d), \tag{6}$$

where

$$g(r_1, r_2) = \sum_{D|r_1} E(r_1/D, r)\mu(D) \qquad (r = r_1r_2). \tag{7}$$

Therefore, by definition of $E(n, r)$, we obtain from (7) and rearrangement,

$$g(r_1, r_2) = \phi(r) \sum_{d|r} \frac{d}{\phi(d)} \mu(r/d) \sum_{D|r_1,\ (r_1/D,\ d)=1} \mu(D). \tag{8}$$

By Lemma 4, the inner sum in (8) is zero unless $r_1 \mid d$, in which case it has the value $\mu(r_1)$. Placing $d=r_1e$, (8) becomes then

$$g(r_1, r_2) = r_1\phi(r)\mu(r_1) \sum_{e|r_2} \frac{e}{\phi(r_1e)} \mu(r_2/e). \tag{9}$$

In (9), the factor $\mu(r_2/e)$ is zero unless $(r_2/e) \mid \gamma(r_2)$; hence we may put $e=(r_2/\gamma(r_2))\delta$ to obtain

$$g(r_1, r_2) = \frac{r\phi(r)\mu(r_1)}{\mu(r_2)} \sum_{\delta|\gamma(r_2)} \frac{\delta\mu(\gamma(r_2)/\delta)}{\phi(r\delta/\gamma(r_2))}. \tag{10}$$

Now place $r/\gamma(r_2)=R_1R_2$, where $\gamma(R_2) \mid r_2$, $(R_1, r_2)=1$. In particular, note that $R_1 \mid r_1$. Then (10) becomes, by Lemma 3 and the factorability of $\phi(r)$,

$$g(r_1, r_2) = \frac{r\phi(r)\mu(r_1)\mu(\gamma(r_2))}{\gamma(r_2)\phi(R_1)} \sum_{\delta|\gamma(r_2)} \frac{\delta\mu(\delta)}{\phi(R_2\delta)}. \tag{11}$$

By Lemma 2, the sum in (11) is 0 unless $R_2=1$. But $R_2=1$ would imply that

$r_1r_2=r=R_1\gamma(r_2)$, so that $(r_1/R_1)r_2=\gamma(r_2)$. Hence, $g(r_1, r_2)=0$ unless $r_2=\gamma(r_2)$, $r_1=R_1$, and in particular, unless r_2 is primitive and $(r_1, r_2)=1$. In view of the factor $\mu(r_1)$ in (11), it follows then that $g(r_1, r_2)=0$, in case r is nonprimitive; restated,

$$g(r_1, r_2) = 0 \qquad \text{if } \mu(r) = 0. \tag{12}$$

Hence, in the remainder of the proof, we suppose (unless indicated otherwise) that r is *primitive*, so that $r_2=\gamma(r_2)$, $R_2=1$, $R_1=r_1$, $(r_1, r_2)=1$. Hence (11) becomes, since $\mu(r)$ and $\phi(r)$ are factorable,

$$\begin{aligned} g(r_1, r_2) &= r_1\phi(r_2)\mu(r) \sum_{\delta|r_2} \frac{\delta\mu(\delta)}{\phi(\delta)} \\ &= r_1\mu(r) \sum_{\delta|r_2} \delta\mu(\delta)\phi(r_2/\delta). \end{aligned}$$

Application of (3) and Lemma 3 yields

$$g(r_1, r_2) = r_1\mu(r_1) \sum_{\delta|r_2} \delta\mu(\delta) \sum_{D|(r_2/\delta)} D\mu(\delta D). \tag{13}$$

Placing $D\delta=E$ in (13), one obtains

$$\begin{aligned} g(r_1, r_2) &= r_1\mu(r_1) \sum_{\delta|r_2} \mu(\delta) \sum_{E|r_2,\ \delta|E} E\mu(E) \\ &= r_1\mu(r_1) \sum_{E|r_2} E\mu(E) \sum_{\delta|E} \mu(\delta). \end{aligned}$$

By (5a) it follows that

$$g(r_1, r_2) = r_1\mu(r_1) \qquad \text{if } \mu(r) \neq 0. \tag{14}$$

Formulas (12) and (14) can be combined into a single formula, valid for all r,

$$g(r_1, r_2) = \mu(r)r_1\mu(r_2). \tag{15}$$

Hence (1) results from (6) and (15).

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A CONSTRUCTION OF THE NULL SPACE OF A LINEAR TRANSFORMATION

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1. The theorem proved in this note (Sec. 2) provides a simple construction of the null space of a linear transformation of one finite-dimensional vector space into another (Sec. 3) and hence a method for finding all solutions of a system of linear equations (Sec. 4). All results apply if the underlying field is replaced by a division ring (Sec. 5). A comparison is made with known results, in order to provide a unifying point of view (Sec. 6). The details in Sections 4 and 5, while not essential for proofs, contribute to such a comparison of various methods of solution.

We assume as known the elementary results of linear algebra, as presented by Birkhoff and Mac Lane ([1], Ch. VII and VIII) and Jacobson ([3], Ch. I and II). We adopt the notation and terminology of [3] except as otherwise indicated.

By a *generalized row echelon matrix** we mean a matrix, such as the one below, whose elements belong to a field Δ and which has the property that each nonzero row precedes each zero row and that in each nonzero row after the first the number of zeros preceding the first nonzero element is greater than that number in the preceding row. The row rank of such a matrix is clearly the number of its nonzero rows.

$$\begin{pmatrix} 0 & 5 & 7 & 4 & 0 & 2 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Every matrix (α) can be transformed into a generalized row echelon matrix by a sequence S of elementary row operations ([1], pp. 172–174) or, equivalently, by multiplication on the left by the nonsingular matrix which is the product of the elementary matrices corresponding to the operations of S ([1], pp. 228–230). This does not change the row rank of (α).

2. THEOREM. *Let A be a linear transformation of rank r of an m-dimensional vector space $\mathfrak{N}_1$ into an n-dimensional vector space $\mathfrak{N}_2$ over a field Δ. Let $(e_1, \cdots, e_m)$ be an ordered basis for $\mathfrak{N}_1$, $(f_1, \cdots, f_n)$ an ordered basis for $\mathfrak{N}_2$, (α) the $m \times n$*

* *Cf.* the notions of (*row*) *echelon matrix* ([1], pp. 175–176), *row-reduced echelon matrix* ([1], pp. 173–174), and (*row*) *Hermite matrix* ([5], p. 91). From a generalized row echelon matrix one obtains a row echelon matrix by requiring that the first nonzero element of each nonzero row be 1, and a row-reduced echelon matrix by requiring in addition that in every column which contains such a first nonzero element 1 all other elements be zero. A square row-reduced echelon matrix is turned into a row Hermite matrix by permuting its rows so that all first nonzero elements 1 appear on the diagonal.

matrix of A relative to these bases, i.e.,

$$(\alpha) = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix},$$

where

$$e_jA = \sum_{k=1}^{n} \alpha_{jk} f_k, \qquad j = 1, \cdots, m. \tag{1}$$

Then any sequence of elementary row operations which takes (α) into a generalized row echelon matrix takes the $m \times m$ identity matrix 1 into an $m \times m$ matrix (μ) such that the $m-r$ vectors

$$u_i = \sum_{j=1}^{m} \mu_{ij} e_j, \qquad i = r+1, \cdots, m, \tag{2}$$

form a basis for the null space of A.

Proof. Let S be a sequence of elementary row operations which transform (α) into a generalized row echelon matrix (γ). Let (μ) be the nonsingular $m \times m$ matrix which is the product of the elementary matrices corresponding to the operations of S. These operations transform the matrix 1 into $(\mu)\,1 = (\mu)$; moreover,

$$(\mu)(\alpha) = (\gamma). \tag{3}$$

The number of nonzero rows of (γ) is the same as the row rank r of the matrix (α), and the last $m-r$ rows of (γ) are zero. Hence, if we define $m-r$ vectors u_i by (2), we have, using (1) and (3),

$$u_iA = \sum_{j,k} \mu_{ij}\alpha_{jk} f_k = \sum_{k} \gamma_{ik} f_k = 0, \qquad i = r+1, \cdots, m,$$

and the u_i belong to the null space $\mathfrak{N}$ of A. (The same conclusion can be drawn from the observation that the "row vector" $(\eta_1, \cdots, \eta_n)$ associated with $u_iA = \sum_k \eta_k f_k$ is equal to the matrix product $(\mu_{i1}, \cdots, \mu_{im})(\alpha)$, where $(\mu_{i1}, \cdots, \mu_{im})$ is the row vector associated with u_i, [3], p. 37). But $\mathfrak{N}$ has dimension $m-r$, and, since (μ) is nonsingular, the u_i are linearly independent. Hence the u_i form a basis for $\mathfrak{N}$.

3. To construct the null space of a given linear transformation A of $\mathfrak{R}_1$ into $\mathfrak{R}_2$, one writes down the matrix (α) of A relative to bases $(e_1, \cdots, e_m)$ of $\mathfrak{R}_1$, $(f, \cdots, f_n)$ of $\mathfrak{R}_2$, transforms (α) into a generalized row echelon matrix (γ) by elementary row operations, and simultaneously applies the same operations to the $m \times m$ identity matrix 1, obtaining an $m \times m$ matrix (μ). Then the vectors (2) are a basis for the null space of A. Of course, (α) and the bases may be given in advance, defining A.

Remark 1. This construction is a generalization of a well-known one of the inverse of A and of (α) ([1], pp. 230–231) in the special case in which (α) is nonsingular, $n=m=r$, $(\gamma)=(\mu)(\alpha)$ is in row-reduced echelon form, hence $(\gamma)=1$, and $(\mu)1=(\mu)$ is the matrix of the inverse of A relative to $(f_1, \cdots, f_n)$, $(e_1, \cdots, e_m)$, and is the inverse of (α). (See also the end of Sec. 4, and Sec. 6.)

Remark 2. In the above proof we have taken the dimension l of the null space of A as known. However, the proof can easily be modified to determine l. Indeed, if we do not know l, the proof still shows that $l \geqq m-r$. Also, if (μ) is the matrix, relative to $(e_1, \cdots, e_m)$, of any ordered basis of $\mathfrak{N}_1$ obtained by extending a basis of the null space of A, (μ) is nonsingular, and therefore the matrix $(\mu)(\alpha)$, which has l zero rows,* has the same rank as (α), whence $l \leqq m-r$. This shows that $l=m-r$.

4. The above construction is equivalent to finding all solutions $(\xi_1, \cdots, \xi_m)$ of the system of homogeneous linear equations

$$\begin{aligned} \xi_1\alpha_{11} + \cdots + \xi_m\alpha_{m1} &= 0, \\ \cdot \qquad\qquad \cdot \qquad\quad &\cdot \\ \cdot \qquad\qquad \cdot \qquad\quad &\cdot \\ \xi_1\alpha_{1n} + \cdots + \xi_m\alpha_{mn} &= 0, \end{aligned} \tag{4}$$

or of the corresponding vector equation in the space of n-tuples over Δ,

$$\xi_1 a_1 + \cdots + \xi_m a_m = 0, \tag{5}$$

where the $a_j, j=1, \cdots, m$, are the "row vectors of (α)," $(\alpha_{j1}, \cdots, \alpha_{jn})$, associated with the vectors (1). (Indeed, it is clear from the proof in Sec. 2 that $(\xi_1, \cdots, \xi_m)$ is a solution of (4), hence of (5), if and only if $\sum_j \xi_j e_j$ belongs to the null space of the linear transformation A considered in Secs. 2, 3. See also [3], p. 48.) Now, as is well known, one solution of the inhomogeneous system

$$\xi_1\alpha_{1k} + \cdots + \xi_m\alpha_{mk} = \delta_k, \qquad k = 1, \cdots, n, \tag{4'}$$

or of the corresponding vector equation

$$\xi_1 a_1 + \cdots + \xi_m a_m = d, \tag{5'}$$

where the a_j are as above and d is $(\delta_1, \cdots, \delta_n)$, can be found by transforming the matrix (α) into (generalized) row echelon form or into row-reduced echelon form;† and all other solutions can be found if all solutions of (4) or (5) are known. Hence we can find all solutions of (4′) or (5′) by transforming (α) into (generalized) row echelon form or row-reduced echelon form and combining the known method for finding one solution of (4′) or (5′) with the construction of the null space of A given above.

(If (γ) is the generalized row echelon matrix in (3) and the vectors c_i

* See the remark on row vectors in the proof in Section 2.

† See [1], pp. 178–179. In the procedure described there, the row echelon form could obviously be replaced by the generalized row echelon form, as is done below.

$=(\gamma_{i1}, \cdots, \gamma_{in})$, $i=1, 2, \cdots, r$, are the first r, linearly independent, row vectors of (γ), one first finds, as in [1], pp. 178–179, the only possible solution $(\eta_1, \cdots, \eta_r)$ of the vector equation

$$\eta_1 c_1 + \cdots + \eta_r c_r = d, \tag{6}$$

if any, and then substitutes for the c_i from the equations

$$c_i = \sum_{j=1}^{m} \mu_{ij} a_j, \qquad i = 1, \cdots, r,$$

derived from (3), to find a solution $(\xi_1, \cdots, \xi_m)$ of (5′), where

$$\xi_j = \sum_{i=1}^{r} \eta_i \mu_{ij}, \qquad j = 1, \cdots, m;$$

i.e., $(\xi_1, \cdots, \xi_m)$ is $\sum_{i=1}^{r} \eta_i(\mu_{i1}, \cdots, \mu_{im})$. To this vector is added the row vector corresponding to an arbitrary element of the null space of A,

$$\sum_{i=r+1}^{m} \lambda_i(\mu_{i1}, \cdots, \mu_{im}), \qquad \lambda_{r+1}, \cdots, \lambda_m \in \Delta,$$

to obtain the general solution of (5′). In matrix form this solution is

$$(\eta_1, \cdots, \eta_r, \lambda_{r+1}, \cdots, \lambda_m)(\mu), \qquad \lambda_{r+1}, \cdots, \lambda_m \in \Delta. \tag{7}$$

Here $\eta_1, \cdots, \eta_r$ can be successively determined from the "triangular" set of r equations

$$\eta_1 \gamma_{1t_i} + \cdots + \eta_i \gamma_{it_i} = \delta_{t_i}, \qquad i = 1, \cdots, r,$$

derived from (6), where t_i is the column number such that γ_{it_i} is the first non-zero element in the ith row of (γ). If (γ) is in row-reduced echelon form, these equations reduce to $\eta_i = \delta_{t_i}$, $i=1, \cdots, r$. In the special case in which (α) is nonsingular, $n=m=r$, and (γ) is 1, one has $(\mu)=(\alpha)^{-1}$ and (7) amounts to solving the matrix form of the equations (4′) by matrix inversion.)

5. The above results also apply if Δ is a division ring instead of a field and $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are left vector spaces over Δ. One only has to require that all multiplications of a row by a scalar which appear in elementary row operations, should be multiplications on the left. (See [3], pp. 19–21 for a similar argument.) To obtain these results for right vector spaces $\mathfrak{R}_{1'}$ and $\mathfrak{R}_{2'}$, one assigns to the linear transformation A with

$$e_j A = \sum_{k=1}^{n} f_k \bar{\alpha}_{kj}, \qquad j = 1, \cdots, m,$$

where $(e_1, \cdots, e_m)$ is a basis for $\mathfrak{R}_{1'}$ and $(f_1, \cdots, f_n)$ is a basis for $\mathfrak{R}_{2'}$, the $n \times m$ matrix

$$(\bar{\alpha}) = \begin{pmatrix} \bar{\alpha}_{11} & \cdots & \bar{\alpha}_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \bar{\alpha}_{n1} & \cdots & \bar{\alpha}_{nm} \end{pmatrix},$$

which has column vectors associated with the e_iA. (See [3], pp. 49–50.) One replaces, in an obvious way, elementary row operations by elementary column operations ([1], p. 233) involving multiplication by scalars on the right, row echelon forms by the corresponding column echelon forms, the $m \times m$ matrix (μ) by an $m \times m$ matrix $(\bar{\mu})$, and the second members of (2) by

$$\sum_{j=1}^{m} e_j \bar{\mu}_{ji}, \qquad i = r+1, \cdots, m,$$

where $(\bar{\mu})$ is the result of transforming 1 by a sequence of elementary column operations which takes $(\bar{\alpha})$ into a generalized column echelon matrix $(\bar{\gamma}) = (\bar{\alpha})(\bar{\mu})$. In Section 4, the first members of (4) and (4′), and of (5) and (5′), are replaced by the righthanded terms $\bar{\alpha}_{k1}\xi_1 + \cdots + \bar{\alpha}_{km}\xi_m$, $k = 1, \cdots, n$, and $\bar{a}_1\xi_1 + \cdots + \bar{a}_m\xi_m$, respectively, where the $\bar{a}_j$ are the column vectors of $(\bar{\alpha})$. Row vectors are replaced by column vectors, and (7) becomes

$$(\bar{\mu})(\bar{\eta}_1, \cdots, \bar{\eta}_r, \bar{\lambda}_{r+1}, \cdots, \bar{\lambda}_m)',$$

where the second factor is defined in the obvious way and the dash indicates transpose.

6. The above method of finding the null space of a linear transformation (of a right vector space into a right vector space, say) differs from another, known one (described, for right vector spaces, in equational form in [1], pp. 180–181, and in matrix form in [5], pp. 78–80, 93 and in [4]) by using only columns of matrices and by using only the generalized (column) echelon form as against the (row-) reduced echelon form or the (row) Hermite form. But it also requires operations both on the given matrix $(\bar{\alpha})$ and on the identity matrix, as against operations on $(\bar{\alpha})$ only. The difference is that between finding the null space of a linear transformation A and finding the space incident with the rank space of its transpose A^* ([3], p. 59). Correspondingly, the solution of an inhomogeneous vector equation or system of linear scalar equations in a right vector space, say, may be found either by column operations, using the method* sketched in Section 5, or else, if this is more appropriate for the numerical problem at hand, by row operations. (For the latter method see, for example, [1], pp. 44–47; [5], pp. 106–108, which has computations to which ours are analogous; [4], which contains similar computations, related to a notion of generalized inverse applicable to the infinite-dimensional case; and [2], pp. 93–94.)

* The general solution mentioned in Section 5 can also be obtained from [1], p. 236, Ex. 12, by taking P and D to be, respectively, the identity matrix and a matrix in one of the column echelon forms.

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A TOURNAMENT PROBLEM

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Four bridge players (or tennis players [1]) wanting to play a fair "round robin" tournament would have no difficulty at all in arranging a schedule. Using the letters A, B, C, D, to denote the players they would automatically play the three matches

A and B against C and D
A and C against B and D
A and D against B and C.

This schedule has the following features:

(a) Each player plays exactly *once with* each other player.
(b) Each player plays exactly *twice against* each other player.
(c) No one plays *against* himself.
(d) No one plays *with* himself.

It is also true that each team plays the only possible team of opponents. When there are more players, this additional feature (each team playing all possible opponents) requires a very long tournament. The question arises, "Can a schedule be arranged for v players so that at least features (a), (b), (c), (d) are satisfied?"

For $v=5$ a little effort produces a successful result such as

$$AB-CD, \quad AC-DE, \quad AD-BE, \quad AE-BC, \quad BD-CE,$$

and further effort shows that, aside from relettering of the players, no other solution is possible.

Before considering larger values of v it is helpful to deduce the number of matches necessary, *assuming* that a solution is possible. Our problem is, of course, similar to that of constructing a block design. In the usual notation for the parameters of a block design we let

v = number of players,
$\lambda = 3$ (number of matches involving a given pair),

$k = 4$ (four players to a match),
$b =$ number of matches,
$r = v - 1$ (number of times each individual plays),

and have at once the familiar

$$b = \{v(v-1)\}/4. \tag{1}$$

This makes it evident that solutions cannot exist except possibly for

$$v = 4, 5, 8, 9, 12, 13, \cdots.$$

The first two cases were covered at the outset. Some additional solutions follow, and others may be found in [4].

$v = 8$

AB – CD	BC – EH
AC – FH	BD – EG
AD – FG	BF – DH
AE – CG	BG – CF
AF – BE	CE – DF
AG – BH	CH – DG
AH – DE	EF – GH

$v = 9$

AB – CD	AH – CF	BI – EH
AC – EI	AI – FG	CG – DF
AD – HI	BC – FI	CH – DE
AE – DG	BD – GI	CI – GH
AF – BE	BF – DH	EF – DI
AG – BH	BG – CE	EG – FH

$v = 12$*

AB – HI	AL – EH	AK – DG
EF – BG	CE – LF	KI – EJ
GH – DK	AE – IC	HK – EL
BI – CG	CD – EI	AF – JD
AC – KJ	FH – BK	DE – FJ
FG – CH	DF – IG	GK – CL
LJ – FB	AD – BL	AI – FK
CJ – DH	BD – KE	BC – DL
LI – GJ	BH – IJ	HL – DI
CK – BJ	EG – HJ	KL – FI
AG – BE	AJ – GL	AH – CF

$v = 13$

BM – IF	CL – DK	EJ – GH
CA – JG	DM – EL	FK – HI
DB – KH	EA – FM	GL – IJ
EC – LI	FB – GA	HM – JK
FD – MJ	GC – HB	IA – KL
GE – AK	HD – IC	JB – LM
HF – BL	IE – JD	KC – MA
IG – CM	JF – KE	LD – AB
JH – DA	KG – LF	ME – BC
KI – EB	LH – MG	AF – CD
LJ – FC	MI – AH	BG – DE
MK – GD	AJ – BI	CH – EF
AL – HE	BK – CJ	DI – FG

$v = 16$*

AB – GN	AP – CH	AC – HP	BC – MJ	AH – CP	GH – OE
BO – LI	MP – GI	CF – EI	AE – LJ	PJ – KO	MN – EK
CD – NO	GL – CK	LN – FP	AD – KI	NJ – HI	CJ – BM
KM – EN	AG – BN	FG – DJ	EH – GO	IJ – HN	BF – HK
IO – BL	DL – HM	AM – FO	BD – PE	AO – FM	DH – LM
KL – CG	AF – MO	KN – EM	DF – JG	LP – FN	PI – GM
AN – BG	DM – HL	OC – DN	EG – OH	AK – ID	NI – HJ
DO – CN	JO – KP	NP – FL	AL – JE	DE – BP	GK – CL
BE – DP	BJ – CM	FH – BK	AI – DK	AJ – EL	MI – GP
CE – IF	PO – KJ	DG – FJ	FK – BH	EF – IC	BI – LO

* The results for $v = 12$ and $v = 16$ were obtained by D. La Dage, Boston University.

An obvious generalization of our problem involves teams of m players. With the same notation as before and $k=2m$ we replace (a) and (b) by

(a^1) Each player plays exactly $m-1$ times with each other player.

(b^1) Each player plays exactly m times against each other player.

This implies $\lambda=2m-1$ and leads at once to

$$b = \{v(v-1)\}/(2m) \tag{2}$$

Again solutions are possible only for certain sequences. The following examples were obtained by manual rearrangement of standard block designs [3].

$m=3, v=7$	$m=3, v=9$	$m=4, v=8$
ABC – DEF	ABC – DEF	ABCD – EFGH
ADE – BFG	ABE – DGH	ABEF – CDGH
AFG – BCE	ADG – CFI	ABGH – CDEF
BDG – ACF	AFH – BDI	ACEG – BDFH
BEF – CDG	AEH – CFG	ACFH – BDEG
CDF – AEG	AFI – BEG	ADEH – BCFG
CEG – ABD	ACD – EHI	ADFG – BCEH
	AGI – BCH	
	BFH – CEI	
	BDI – CEG	
	BFG – CDH	
	DEF – GHI	

It is interesting to note that the first of these is, as printed, a pair of Steiner triple systems of order seven. The second, however, cannot be so divided. Perhaps some valiant or fortunate reader will discover a design for $m=3$, $v=10$ or $m=4$, $v=9$. A few moments is sufficient, however, to try the various combinations for $m=3$, $v=6$ and discover that no solution is possible even though (2), for these values, suggests $b=5$.

This impossibility raises the general question of existence. No effort has been made to discover the answer. However, R. C. Bose has suggested that his method of Galois fields [2] can be modified to cover certain sequences; indeed, the design listed earlier for $v=13$ was found by his method. Should the military forces of any friendly nation develop a lively interest in tournaments it is very likely that many hands will become available for the task. To further these prospects this brief paper should perhaps have been titled "On the testing of individual ability in group competition."

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A GENERALIZATION OF HERMITE'S INTERPOLATION FORMULA

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1. Introduction. Hermite's interpolation formula provides an expression for a polynomial which passes through given points with given slopes. Specifically, let $x_i, f_i, f_i^{(1)}$, $i=0, 1, \cdots, n$, be given. The polynomial $f(x)$, of degree $2n+1$, for which $f(x_i)=f_i, f'(x_i)=f_i^{(1)}$ is given by ([2], p. 33)

$$f(x) = \sum_{j=0}^{n} h_j(x)f_j + \sum_{j=0}^{n} \bar{h}_j(x)f_j^{(1)},$$

where

$$h_j(x) = \left\{1 - \frac{q_n''(x_j)}{q_n'(x_j)}(x - x_j)\right\}\{L_j(x)\}^2, \qquad \bar{h}_j(x) = (x - x_j)\{L_j(x)\}^2,$$

$$q_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n), \qquad L_j(x) = \frac{q_n(x)}{(x - x_j)q_n'(x_j)}.$$

It is the purpose of this paper to generalize Hermite's formula so as to include the values of derivatives up to specified, arbitrary orders at the given points.

2. The generalized interpolation formula. We state here the general result which is to be proved.

THEOREM 1. *Let there be given* $x_j, r_j, f_j^{(k)}$, $j=0, 1, \cdots, n$; $k=0, 1, \cdots, r_j$. *Let* $p_j(x)$ *and* $g_j(x)$ *be defined by*

$$(1)\quad p_j(x) = (x - x_0)^{r_0+1}\cdots(x - x_{j-1})^{r_{j-1}+1}(x - x_{j+1})^{r_{j+1}+1}\cdots(x - x_n)^{r_n+1},$$

$$g_j(x) = [p_j(x)]^{-1}.$$

Then the polynomial $f(x)$ *of degree* $n+\sum_{j=0}^{n} r_j$ *such that*

$$f^{(k)}(x_j) = f_j^{(k)}, \qquad j = 0, 1, \cdots, n, \; k = 0, 1, \cdots, r_j,$$

is given by

$$(2)\qquad f(x) = \sum_{j=0}^{n}\sum_{k=0}^{r_j} A_{jk}(x)f_j^{(k)},$$

where

$$(3)'\qquad A_{jk}(x) = p_j(x)\frac{(x - x_j)^k}{k!}\sum_{t=0}^{r_j-k}\frac{1}{t!}g_j^{(t)}(x_j)(x - x_j)^t.$$

Before giving the proof let us see how this applies in the specific case covered by Hermite's formula. Here we have $r_j=1$ for $j=0, 1, \cdots, n$. We find

$$p_j(x) = (x - x_0)^2\cdots(x - x_{j-1})^2(x - x_{j+1})^2\cdots(x - x_n)^2.$$

Then for $k=1$ we have $A_{j1}(x)=p_j(x)(x-x_j)g_j(x_j)$. But $g_j(x_j)=[p_j(x_j)]^{-1}$ so that

$$A_{j1}(x) = (x - x_j)\frac{p_j(x)}{p_j(x_j)} = (x - x_j)\{L_j(x)\}^2,$$

which is the $\bar{h}_j(x)$ of Section 1. For $k=0$ we have

$$\begin{aligned} A_{j0}(x) &= p_j(x)[g_j(x_j) + g_j'(x_j)(x - x_j)] \\ &= p_j(x)\left[\frac{1}{p_j(x_j)} - \frac{p_j'(x_j)}{[p_j(x_j)]^2}(x - x_j)\right] \\ &= \left[1 - \frac{p_j'(x_j)}{p_j(x_j)}(x - x_j)\right]\{L_j(x)\}^2. \end{aligned}$$

But $p_j(x_j)=[q_n'(x_j)]^2$ and $p_j'(x_j)=q_n'(x_j)q_n''(x_j)$, so that $A_{j0}(x)$ turns out to be the $h_j(x)$ of Section 1, and Hermite's interpolation formula is seen to be a special case of the general formula.

3. Proof of the theorem. We seek a representation for $f(x)$ of the form (2). We must then have

$$A_{jk}^{(s)}(x_i) = 0, \quad i \neq j, \qquad A_{jk}^{(s)}(x_j) = \delta_{ks} \tag{4}$$

where δ_{ks} is the Kronecker delta, the superscript (s) indicates order of derivative, and each $A_{jk}(x)$ is a polynomial of degree $n+\sum_{j=0}^{n} r_j$. It follows from (4) that

$$A_{jk}(x) = p_j(x)(x - x_j)^k R_{jk}(x), \tag{5}$$

where $R_{jk}(x)$ is a polynomial of degree r_j-k. Let this equation be rewritten as

$$S_{jk}(x)R_{jk}(x) = A_{jk}(x)g_j(x), \tag{6}$$

where $S_{jk}(x)=(x-x_j)^k$, $g_j(x)=[p_j(x)]^{-1}$. We now differentiate (6) $k+t$ times, keeping in mind that

$$S_{jk}^{(k+t)}(x) = \begin{cases} k! \text{ for } t = 0, \\ 0 \text{ for } t > 0. \end{cases}$$

We obtain

$$\sum_{i=0}^{k}\binom{k+t}{i}S_{jk}^{(i)}(x)R_{jk}^{(k+t-i)}(x) = \sum_{i=0}^{k+t}\binom{k+t}{i}A_{jk}^{(i)}(x)g_j^{(k+t-i)}(x). \tag{7}$$

Now let $x=x_j$ in (7). Since $S_{jk}^{(s)}(x_j)=0$ for $s=0, 1, \cdots, k-1$, and also using (4), we get

$$\binom{k+t}{t}k!R_{jk}^{(t)}(x_j) = \binom{k+t}{k}g_j^{(t)}(x_j)$$

so that

$$R_{jk}^{(t)}(x_j) = \frac{1}{k!}\, g_j^{(t)}(x_j), \qquad t \leqq r_j - k.$$

We recall that $R_{jk}(x)$ is a polynomial of degree $r_j - k$. Hence

$$R_{jk}(x) = \frac{1}{k!} \sum_{t=0}^{r_j-k} \frac{1}{t!}\, g_j^{(t)}(x_j)(x - x_j)^t.$$

With this substituted in (5) the $A_{jk}(x)$ are as in the statement of the theorem, and the proof is complete.

4. Divided differences with repeated arguments. We shall give an application of the theorem. Let

$$(8) \qquad \left[\underbrace{x_0 \cdots x_0}_{r_0+1}\ \underbrace{x_1 \cdots x_1}_{r_1+1} \cdots \underbrace{x_n \cdots x_n}_{r_n+1}\right], \qquad \text{with } x_i \neq x_j \text{ for } i \neq j,$$

be the general divided difference with repeated arguments of a function $f(x)$. In ([3], p. 14) the value of this divided difference is given as

$$\frac{1}{r_0!r_1! \cdots r_n!}\, \frac{\partial^{r_0+r_1+\cdots+r_n}}{\partial x_0^{r_0}\, \partial x_1^{r_1} \cdots \partial x_n^{r_n}} [x_0x_1 \cdots x_n],$$

where

$$[x_0x_1 \cdots x_n] = \sum_{k=0}^{n} \left[f(x_k) \Big/ \prod_{\substack{i=0 \\ i\neq k}}^{n} (x_k - x_i) \right].$$

This divided difference may also be represented by the contour integral

$$\frac{1}{2\pi i} \int_C \frac{f(t)\,dt}{(t - x_0)^{r_0+1} \cdots (t - x_n)^{r_n+1}},$$

where C is a simple closed curve enclosing a simply connected region of the complex variable t which contains the points $x_0, x_1, \cdots, x_n$, provided $f(t)$ is analytic within and on C.

We may derive yet another expression for the divided difference (8).

THEOREM 2. *The divided difference* (8) *is given by*

$$\sum_{j=0}^{n} \sum_{k=0}^{r_j} \frac{1}{k!}\, \frac{1}{(r_j - k)!}\, g_j^{(r_j-k)}(x_j) f_j^{(k)},$$

where $g_j(x)$ *is given in Theorem* 1.

Proof. The divided difference in question is the coefficient of the highest degree term in the generalized interpolation formula of Section 2. From (2) and (3) this coefficient is easily seen to be given by the above double sum.

COROLLARY. *If $n=0$ we take $p_0(x)\equiv 1$, and the above theorem gives the familiar result*

$$\left[\underbrace{x_0 \cdots x_0}_{r_0+1}\right] = \frac{1}{r_0!} f_0^{(r_0)}.$$

5. Evaluation of the confluent alternant. If we consider the system of equations for the general interpolation problem of Theorem 1, we have a set of equations consisting of the r_0+1 equations

$$\begin{aligned} a_0 + a_1x_0 + a_2x_0^2 + a_3x_0^3 + \cdots &= f_0^{(0)}, \\ a_1 + 2a_2x_0 + 3a_3x_0^2 + \cdots &= f_0^{(1)}, \\ 2a_2 + 6a_3x_0 + \cdots &= f_0^{(2)}, \end{aligned}$$

and so on, followed by similar sets of r_1+1, r_2+1, $\cdots$, r_n+1 equations for each of the values $x_1, x_2, \cdots, x_n$, respectively, with $N+1\equiv n+1+\sum_{i=0}^{n} r_i$ equations in all.

The determinant of this system, which we shall denote by $\Delta_{r_n}^n$, is the confluent alternant as described in Aitken ([5], p. 119), multiplied by

$$\prod_{i=0}^{n} [r_i!(r_i-1)!\cdots 2!1!].$$

Now $\Delta_{r_n}^n$ is the denominator of the coefficient of x^N, as determined from a system of equations such as above by Cramer's rule, for the polynomial of degree N such that

$$f(x_0) = f'(x_0) = \cdots = f^{(r_0)}(x_0) = f(x_1) = \cdots = f^{(r_1)}(x_1) = \cdots = f^{(r_n-1)}(x_n) = 0,$$
$$f^{(r_n)}(x_n) = 1.$$

In view of Theorem 1, this coefficient is also the coefficient of x^N in

$$\begin{aligned} A_{n,r_n} &= p_n(x)\cdot\frac{(x-x_n)^{r_n}}{r_n!}\cdot g_n(x_n) \\ &= g_n(x_n)\cdot\frac{(x-x_n)^{r_n}}{r_n!}\cdot(x-x_0)^{r_0+1}\cdots(x-x_{n-1})^{r_{n-1}+1} \end{aligned}$$

and is therefore

$$\frac{1}{r_n!}g_n(x_n) = \frac{1}{r_n!p_n(x_n)} = \frac{1}{r_n!(x_n-x_0)^{r_0+1}\cdots(x_n-x_{n-1})^{r_{n-1}+1}}.$$

The numerator of this coefficient is the determinant formed from $\Delta_{r_n}^n$ by deleting the last row and column, hence is similar to $\Delta_{r_n}^n$, and may be denoted by $\Delta_{r_n-1}^n$. We then have

$$\Delta_{r_n}^{n} = \Delta_{r_n-1}^{n} \cdot r_n!(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1}.$$

In the same manner we obtain

$$\Delta_{r_n-1}^{n} = \Delta_{r_n-2}^{n} \cdot (r_n - 1)!(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1},$$

$$\cdots$$

$$\Delta_1^{n} = \Delta_0^{n} \cdot 1!(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1},$$

and also

$$\Delta_0^{n} = \Delta_{r_{n-1}}^{n-1}(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1},$$

where the determinant $\Delta_{r_{n-1}}^{n-1}$ is similar to $\Delta_{r_n}^{n}$, but is formed with the points $x_0, x_1, \cdots, x_{n-1}$ and is of order $n + \sum_{i=0}^{n-1} r_i$. Combining the above, we have

$$\Delta_{r_n}^{n} = \Delta_{r_{n-1}}^{n-1} \cdot r_n!(r_n - 1)! \cdots 2!1!\left[(x_n - x_0)^{r_0+1} \cdots (x_n - x_{n-1})^{r_{n-1}+1}\right]^{r_n+1}. \tag{9}$$

Similar relations then exist for $\Delta_{r_i}^{i}$, $i = n-1, \cdots, 1, 0$, with

$$\Delta_{r_0}^{0} = r_0!(r_0 - 1)! \cdots 2!1!. \tag{10}$$

A combination of relations of the type (9) for $\Delta^{i}_{r_i}$, $i = 0, 1, \cdots, n$, together with (10), gives the result

$$\Delta_{r_n}^{n} = \prod_{i=0}^{n} [r_i!(r_i - 1)! \cdots 2!1!] \cdot \prod_{i>j} (x_i - x_j)^{(r_i+1)(r_j+1)}. \tag{11}$$

The value of the confluent alternant itself is therefore the product $\prod_{i>j} (x_i - x_j)^{(r_i+1)(r_j+1)}$ which is the value given by Aitken.

In the case for which all the r_i are equal, $r_i \equiv r$, $i = 0, 1, \cdots, n$, we may denote the determinant by Δ_r, for which (11) reduces to

$$\Delta_r = [r!(r - 1)! \cdots 2!1!]^{n+1} \cdot V^{(r+1)^2},$$

where V is the Vandermonde determinant whose element in the ith row and jth column is x_{i-1}^{j-1}, $i, j = 1, \cdots, n+1$.

6. Conclusion. The main result of this paper concerns an explicit representation of the interpolation polynomial. A discussion of the remainder term may be found in [1] and [4], in both of which an interpolation is obtained in a form which involves an iterative process.

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ON THE DESIRABILITY OF PUBLISHING CLASSIFIED BIBLIOGRAPHIES OF THE MATHEMATICS LITERATURE*

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Having been engaged now in mathematical research for 60 years, I find that certain crying needs stand out very clearly as to the things required by research mathematicians in order to produce and publish more articles, particularly original ones. To my mind, first among these needs is the preparation and publication of bibliographies classified according to topics of all branches in pure and applied mathematics. Accordingly, this paper will be devoted to this topic and will be divided into two parts. The first part will consist of a discussion of a bibliography of the Theory of Numbers either with reports, as a continuation of Dickson's *History of the Theory of Numbers*, or a preparation of a bibliography consisting of titles only without reports, covering the literature since the year 1923. The second part will consist mainly of consideration of the advisability of publishing bibliographies of the various parts of mathematics without reviews.

First, we shall illustrate the kind of thing which mars a number of papers in number theory that appear at the present time, and probably we might note a great many more marred in a similar way if we knew more about the history of the topic being discussed. Some years ago two great mathematicians (whom we shall call X and Y) published an article. In their introduction they summarized, without reference, two results as being the principal ones being proved in their paper. One of these results was published without proof by a mathematician some 90 years before. The other result was stated and proved in full some 20 years before, by another mathematician. If I had written X and Y, or published a paper describing their derelictions, they could have replied that a paper of mine, which I published in 1944, contained a result and proof which was 109 years old. How can such duplications be avoided?

A number of mathematicians have expressed the hope that Dickson's *History* be brought up to date. Before doing so, it might be well to examine in detail some of the salient features of the book with a view to preserving them in the proposed extension. As far as I am concerned, I cannot do better than to quote something by Dickson himself (preface to Volume II):

> Conventional histories take for granted that each fact has been discovered by a natural series of deductions from earlier facts and devote considerable space in the attempt to trace the sequence. But men experienced in research know that at least the germs of many important results are discovered by a sudden and mysterious intuition, perhaps the result of subconscious mental effort, even though such intuitions have to be subjected later to the sorting processes of the critical faculties. What is generally wanted is a full and correct statement of the facts, not an historian's personal explanation of those facts. The more completely the historian remains in the background or the less conscious the reader is of the historian's personality, the better the history.

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I shall also quote several excerpts from a review I wrote of Dickson's Vols. I and II (*Bull. Amer. Math. Soc.*, vol. 30, 1924, pp. 65–70):

> In the reviewer's opinion, Dickson has come very close to realizing the ideals above mentioned, and this circumstance gives the work its greatest value. In particular, it appears that he has had always in mind the needs of the investigator. An endeavor is made to supply every scrap of information that might possibly aid in research on a particular problem. . . .
>
> The material shows in itself how useful the history will be in enabling investigators to avoid duplication in published results. It is to be noted in glancing over the pages how often the contents of a published article is, unknown to the writer of it, largely a repetition of previous research of some other author. . . .
>
> It often happens in the history of mathematics that a mathematician becomes a specialist in a particular topic, and, after years of experience with it, he publishes a treatise giving a harmonious and comprehensive development of the subject, the material being all arranged and presented according to his own particular point of view. This treatise may become a classic, and its readers are likely to get in the habit of ignoring, to a considerable extent, the literature that preceded its publication. In this way the points of view of the older writers are often lost sight of, as these treatises rarely, if ever, reproduce all the older material on a particular topic. It would seem that there is too great a preponderance of books of this sort in the literature and too few histories or reports of the type of Dickson's work.

In view of the characteristics of the history just discussed, we see that it is easy to verify that it is as useful now (as far as it went) as it was in 1924, the year of our review. Therefore we should try to add to his work without destroying the general plan of the history. First, the magnitude of this task must be recognized.

We have made an estimate of the number of references that it would be necessary to review between 1920 and 1956 inclusive; it is 8,500. To illustrate what that means, suppose it was possible to persuade five mature, A-1 number theorists to undertake this job, and let us assume that they had five years to complete it. This would mean that if the references were evenly distributed, each person would have to dispose of 1,640 references in the five years, which would mean that 328 reviews would have to be completed in each year by each person. That is almost one reference per day. Considering the extremely advanced and complicated papers which have been written in the interval mentioned on analytical number theory and algebraic number theory, I do not regard any number theorist I know of as being capable of carrying out his part in such a work within the time specified.

To illustrate further the magnitude of the above-mentioned task, we counted the number of reviews that Dickson made in his three volumes. The total turned out to be 8,382, and this would not include references on algebraic numbers. Also, he did not include Bernoulli and related numbers and the law of quadratic reciprocity. On the average, the papers he reviewed were far less difficult and complicated than those we are considering.

Another point in connection with this is that few, if any, number theorists would wish to work on such a project full time for perhaps more than a year at a

time, since most research mathematicians are university professors. However, they would gain the advantage of a detailed knowledge of the literature.

In view of the above remarks, it might be possible for ten collaborators working more or less in a period of ten years to bring things nearly up to date. When I say "nearly" I have in mind the fact that *during the ten years just mentioned probably at least 4,000 more papers would be published.*

Now we shall consider the advisability of publishing bibliographies of the various topics in mathematics without reviews. I think it should be obvious that we cannot hope ever to have completed and published a bibliography of any extensive branch of mathematics (*with reports* in the style of Dickson's *History*) other than in number theory. To illustrate this we will consider the subject of differential equations. To write the history (à la Dickson) of this from its beginning, would be impossible, in my opinion, in any form at this late stage, and this is more or less true for other topics not number theoretic. I think, however, that the *preparation of a nearly complete list of references might be possible.* (I use the word *nearly* here since we can hardly hope to ever locate all the existing ones.) Of course if we list references in a particular topic we should be careful, if necessary, to divide the literature on the topic into sub-topics so that a research man does not have to look up too many listed papers to decide that results he has arrived at are new. For example, we could hardly expect him to examine 1,000 references. In an issue of *Mathematical Reviews* the subject index for the year 1955 listed some 120 papers on "ordinary differential equations: existence and behavior of solutions." So this topic should be separated into quite a number of sub-topics, if humanly possible.

Now I do not expect that the collecting of a large number of references on any mathematical topic and preparing a bibliography consisting of titles only will appeal to many mathematicians. The work of a bibliophile on first glance is not attractive. Often when I have mentioned the importance of such work to a research mathematician he has registered his deep disgust at the idea of his possibly being involved in it. *But here I do not think he realizes the benefits which will probably accrue to him personally and to the science of mathematics through work of this kind. I have already mentioned the advantageous effect on a mathematician who collects references and writes reports on each. To a considerable extent this is also true when one only examines a reference and decides what topics have been treated among the separate topics initially chosen. This process requires one to read rather carefully some of the references in order to make such decisions. Consequently he will add enormously to his knowledge of the general subject involved, and it enables him in many cases to arrive at new results and publish them.* To illustrate this, the writer has collected (probably incomplete) bibliographies on a number of topics in the theory of numbers and distributed copies to various mathematicians whom he assumed would be interested. The writer is well aware of the fact that his own knowledge of each topic increased greatly thereupon, and the publication of a number of his papers was due to this.

In the first part of this paper we discussed the advantages of trying to bring up to date, or nearly up to date, Dickson's *History of the Theory of Numbers.* The first step in such a project should be the collection of *all references on number theory* starting with the year 1920 and then classifying them according to different topics. *In case it turns out that the project of continuing Dickson's work of reviewing each article falls through, then the classified list mentioned, without reports, will still be of immense value to number theorists.*

Let us now consider the physical environment under which a mathematician who undertakes a work of this kind would find most convenient. In this connection perhaps we cannot improve on the conditions surrounding Dickson while he was writing his *History.* He had an office adjoining the Mathematics Library, which fine library was separate from the Main Library at the University of Chicago. Naturally, then, his office was *very quiet,* a fact, of course, which helped him in concentrating on any matter at hand. Also, if he wished to consult or review any mathematical article, all he usually had to do was walk a few steps to locate it. If he did so, and found that he wished to study it at length, he would take the book containing the article to his office and keep it there until he had finished his work on it. If the librarian could not find some book which the records showed should have been in the stacks, he assumed it was on Dickson's desk.

For a mathematician to be in the best position to head a project for preparing a bibliography of the type we have been discussing, he should, of course, already have had considerable experience with the relevant literature. However, not all the men working with him need to be so well prepared. A young mathematician might gradually learn from a more experienced man how to arrange for the proper recognition of the topics treated in a given paper in order to attain adequate classification in the proposed bibliography.

In this MONTHLY, vol. 57, no. 7, Part II, 1950, pp. 1–110, an index of Volumes 1–56 of the MONTHLY, classified according to topics, is given. If I recall correctly, similar indices have been provided by other magazines from time to time. These partial bibliographies will, of course, help in the preparation of a complete one.

The 1958 Parliament of Science, held in March, 1958, Washington, D. C. (*Science,* vol. 127, 1958, p. 856), under the topic "Recommendations Concerning Communication Among Scientists," stated that "It is recommended that research-supporting agencies: . . . (ii) Recognize that monographs, abstracts, indices, reviews, compendia, books, *bibliographies,* and proceedings of symposia are important means of communication and that scientists should be supported in their preparation." (Italics of word in quotation are those of the author.)

E. J. McShane (*Maintaining Communication,* this MONTHLY, vol. 64, 1957, p. 313) treats various aspects of this topic, including the desirability of mathematicians engaged in work of the character done by Dickson in his *History.*

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, University of California, Berkeley 4, California.

A DEFINITION OF ORDINAL NUMBERS

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This note presents a new definition of an ordinal number which is simpler in conceptual structure than any other known to the author. The proof that it agrees with Gödel's definition [1] (which involves the concept of well-ordering) can be given in a three-line transfinite induction. However, the following development has the advantage of requiring no preparation beyond the axioms of set theory and some acquaintance with the axiomatic method, and it leads naturally to the idea of well-ordering.

A set X is said to be *transitive* if every element of X is a subset of X. We have at once

THEOREM 1. *A set X is transitive if any of the following are true. X is a union of transitive sets; or X is an intersection of transitive sets; or $X = Y \cup \{Y\}$, where Y is transitive.*

Proof. If X is a union of transitive sets S_α, each element x of X is an element of some S_α, and $x \subset S_\alpha \subset X$. In the case of an intersection x is an element of every S_α, and $x \subset \cap S_\alpha = X$. The last case is also routine.

A set X is called an *ordinal number* if every transitive proper subset of X is an element of X. Let us call a set S *untangled* if S has no element which is an element of itself.

THEOREM 2. *Every ordinal number is transitive and untangled.*

Proof. Let X be an ordinal number. Since a union of untangled sets is clearly untangled, if we let X' be the union of all transitive untangled subsets of X we see that X' itself will be transitive and untangled; in fact, X' is the largest untangled transitive subset of X. Clearly what we wish to prove is that $X = X'$. But if we suppose $X' \neq X$, then $X' \in X$ (by definition of *ordinal*), and so $X' \cup \{X'\}$ is a subset of X which is transitive (by Theorem 1) and untangled (since $X' \notin X'$). By the maximality of X' it follows that $X' \cup \{X'\} \subset X'$, which implies the contradiction $X' \in X'$.

THEOREM 3. *For any two ordinal numbers, X, Y, one is a subset of the other. In fact, if $X \neq Y$, one is an element of the other.*

Proof. $X \cap Y$ is a transitive subset of X and of Y. It is not an element of both, for this would make $X \cap Y \in X \cap Y$. Thus $X \cap Y$ is X or Y, and if it is equal to just one of them, it is an element of the other.

THEOREM 4. *Every transitive set of ordinal numbers is an ordinal number.*

Proof. Let S be a transitive set of ordinals and Y a transitive proper subset of S. Consider some element s of $S-Y$. For any $y\in Y$, $s\neq y$ and $s\notin y$ (since Y is transitive). Therefore $y\in s$ for each y, and the transitive set Y is a subset of the ordinal number s. Either $Y=s$ or $Y\in s$; in either case, $Y\in S$, as required.

THEOREM 5. *Every element of an ordinal number is an ordinal number.*

Proof. For any ordinal number X, let X' be the union of all transitive sets of ordinal numbers contained in X. X' is of course a transitive subset of X, and by Theorem 4, X' is an ordinal. Hence $X'\in X$ would imply that the transitive set $X'\cup\{X'\}$ is a subset of X', contrary to Theorem 2. Therefore X' is not a proper subset of X, $X=X'$.

THEOREM 6. *The intersection of any nonempty set of ordinal numbers is an ordinal number and an element of the set.*

Proof. The intersection of $\{X_\alpha\}$ is a transitive set of ordinals, hence an ordinal, X. If $X\neq X_\alpha$ for all α we should have $X\in X_\alpha$ for all α, whence $X\in X$.

This completes the contact with the usual treatments.

Reference

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DEVELOPMENTS OF $\sin^a x$ INTO FOURIER SERIES

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The function $\sin^a x$ is well defined and continuous for all real values of a and all x on the interval I_0: $0<x<\pi$. If $a>-1$ it satisfies the conditions of Dirichlet on $0\leqq x\leqq\pi$ for a development into a Fourier series. For that purpose we define the following two continuations of $\sin^a x$:

$$F(x+k\pi, a) = (-1)^k \sin^a x, \tag{1}$$

$$G(x+k\pi, a) = \sin^a x, \tag{2}$$

for all $x\in I_0$ and all $k=0, \pm 1, \cdots$.

For $x=0$ and all $k=0, \pm 1, \cdots$ we define $F(k\pi, a)=G(k\pi, a)=0$. Thus F and G are continuous for all x and $a>0$ and discontinuous at $x=k\pi$ if $a\leqq 0$. Clearly $F(x, a)=(-1)^k|\sin x|^a$ and $G(x, a)=|\sin x|^a$ for all x on the interval I_k: $k\pi<x<(k+1)\pi$.

Since $\sin^a(\pi/2+x)=\sin^a(\pi/2-x)$ for $0\leqq x<\pi/2$ and $F(-x, a)=-F(x, a)$, we must have

$$F(x, a) = \sum_{n=0}^{\infty} A_n(a) \sin(2n+1)x, \tag{3}$$

and, similarly,

$$G(x, a) = \frac{B_0}{2} + \sum_{n=0}^{\infty} B_n(a) \cos 2nx. \tag{4}$$

The integrals

$$A_n(a) = \frac{1}{\pi}\int_0^{2\pi} F(x, a) \sin (2n + 1)x\, dx = \frac{2}{\pi}\int_0^{\pi} \sin^a x \sin (2n + 1)x\, dx,$$

$$B_n(a) = \frac{1}{\pi}\int_0^{2\pi} G(x, a) \cos 2nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin^a x \cos 2nx\, dx,$$

may be evaluated using de Moivre's formulas*:

$$A_n(a) = \frac{2}{\pi}\sum_{m=0}^{n} (-1)^m \binom{2n + 1}{2m + 1} \int_0^{\pi} \sin^{a+2m+1} x \cos^{2(n-m)} x\, dx, \tag{5}$$

$$B_n(a) = \frac{2}{\pi}\sum_{m=0}^{n} (-1)^m \binom{2n}{2m} \int_0^{\pi} \sin^{a+2m} x \cos^{2(n-m)} x\, dx. \tag{6}$$

All the integrals are of the form $\int_0^{\pi} \sin^a x \cos^{2\lambda} x\, dx$, $\lambda = 0, 1, \cdots, \alpha > -1$, which, putting $t = \sin^2 x$, becomes

$$\int_0^{\pi} \sin^a x \cos^{2\lambda} x\, dx = 2\int_0^{\pi/2} \sin^a x \cos^{2\lambda} x\, dx$$

$$= \int_0^1 t^{(\alpha+1)/2-1}(1 - t)^{(2\lambda+1)/2-1} dt = \frac{\Gamma(\frac{1}{2}\alpha + \frac{1}{2})\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + 1 + \frac{1}{2}\alpha)}.$$

Substituting these values in (5) and (6) and using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get:

$$A_n(a) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1 + \frac{1}{2}a)}{\Gamma(n + 1 + \frac{1}{2}a + \frac{1}{2})} P_n(a), \qquad B_n(a) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a + \frac{1}{2})}{\Gamma(n + 1 + \frac{1}{2}a)} Q_n(a),$$

where

$$P_n(a) = \sum_{m=0}^{n} (-1)^m \binom{2n + 1}{2m + 1} \frac{\Gamma(n - m + \frac{1}{2})\Gamma(m + 1 + \frac{1}{2}a)}{\Gamma(\frac{1}{2})\Gamma(1 + \frac{1}{2}a)}, \tag{7}$$

$$Q_n(a) = \sum_{m=0}^{n} (-1)^m \binom{2n}{2m} \frac{\Gamma(n - m + \frac{1}{2})\Gamma(m + \frac{1}{2}a + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2})}, \tag{8}$$

are polynomials of degree n in a.

To represent these polynomials, we observe that (3) and (4) contain the inverse formulas of de Moivre:

$$\sin^{2m+1} x = \sum_{n=0}^{m} \frac{(-1)^n}{2^{2m}} \binom{2m + 1}{m + n} \sin (2n + 1)x \equiv \sum_{n=0}^{\infty} A_n(2m + 1) \sin (2n + 1)x,$$

* *I.e.*, formulas which represent $\sin kx$ and $\cos kx$ as polynomials in $\sin x$ and $\cos x$.

$$\sin^{2m} x = \sum_{n=0}^{m} \frac{(-1)^n}{2^{2m-1}} \binom{2m}{m+n} \cos 2nx \equiv \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n(2m) \cos 2nx.$$

From these formulas we have, for $n>0$, $A_n(2m+1)=0$ and $B_n(2m)=0$, $m=0, 1, \cdots, n-1$. Hence

$$P_n(a) = p(a-1)(a-3)\cdots(a-(2n-1)),$$
$$Q_n(a) = qa(a-2)\cdots(a-(2n-2)).$$

From (7) and (8) we determine p and q as

$$p = \lim_{n\to\infty} \frac{P_n(a)}{a^n} = \frac{(-1)^n}{2^n} = \lim_{n\to\infty} \frac{Q_n(a)}{a^n} = q.$$

Thus we obtain

$$P_n(a) = (n-(\tfrac{1}{2}a+\tfrac{1}{2}))(n-1-(\tfrac{1}{2}a+\tfrac{1}{2}))\cdots(1-(\tfrac{1}{2}a+\tfrac{1}{2}))$$
$$= \frac{\Gamma(n+1-(\frac{1}{2}a+\frac{1}{2}))}{\Gamma(1-(\frac{1}{2}a+\frac{1}{2}))},$$
$$Q_n(a) = (n-1-\tfrac{1}{2}a)(n-2-\tfrac{1}{2}a)\cdots(-\tfrac{1}{2}a) = \frac{\Gamma(n-\frac{1}{2}a)}{\Gamma(-\frac{1}{2}a)},$$

with $P_0(a)=Q_0(a)=1$; *i.e.*, the representation of $P_n(a)$ and $Q_n(a)$ by a quotient of Γ-functions also holds for $n=0$. Then, for $0<x<\pi$,

$$\sin^a x = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1+\frac{1}{2}a)}{\Gamma(1-(\frac{1}{2}a+\frac{1}{2}))} \sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\frac{1}{2}a+\frac{1}{2}))}{\Gamma(n+1+(\frac{1}{2}a+\frac{1}{2}))} \sin(2n+1)x, \tag{9}$$

or

$$\sin^a x = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(-\frac{1}{2}a)} \left[\frac{\Gamma(-\frac{1}{2}a)}{2\Gamma(1+\frac{1}{2}a)} + \sum_{n=1}^{\infty} \frac{\Gamma(n-\frac{1}{2}a)}{\Gamma(n+1+\frac{1}{2}a)} \right] \cos 2nx. \tag{10}$$

For all x on the interval $k\pi<x<(k+1)\pi$, where $k=0, \pm 1, \cdots$, the right sides of (9) and (10) represent $(-1)^k|\sin x|^a$ and $|\sin x|^a$ resectively.

Remark. Denoting by α and β arbitrary real numbers and considering the linear combination

$$Y(x,a) = \alpha G(x,a) + \beta F(x,a) = \begin{cases} (\alpha+\beta)\sin^a x, & 0<x<\pi, \\ (\alpha-\beta)|\sin x|^a, & -\pi<x<0 \end{cases}$$

$Y(x, a)$ represents all possible continuations of solutions of the differential equation $y'=ay \cot x$ in the interval $-\pi<x<\pi$. Especially for $\alpha+\beta=0$, an interesting representation of zero in the interval $0<x<\pi$ is obtained.

NOTE ON HAMILTON CIRCUITS*

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In a recent note, D. J. Newman† proved the following result:

Any graph with 2n vertices each of order not less than n must contain a 2n-gon.

The object of the present observations is to point out that the argument used by Newman can be used to obtain somewhat more general results on *Hamilton circuits*, that is, closed paths in graphs passing through every vertex once.

Let G be a finite graph without loops or multiple edges. The m vertices of G we arrange in some cyclic order

$$a_0, a_1, \cdots, a_{m-1}, a_0. \tag{1}$$

There is a *gap* between two vertices a_k and a_{k+1} in (1) if there is no edge connecting them. When a Hamilton circuit exists it is possible to find a cyclic order (1) without gaps.

Assume now that (1) is a cyclic arrangement of the vertices with a minimal number of gaps such that there is a gap between a_0 and a_1. Let (a_0, a_i) be one of the $\rho(a_0)$ edges from a_0. Then there can be no edge (a_1, a_{i+1}) because it would lead to a new order

$$a_0, a_i, a_{i-1}, \cdots, a_1, a_{i+1}, \cdots, a_{m-1}, a_0$$

of the vertices with one gap less.

We conclude that if (1) is an arrangement with a minimal number of gaps, none of the $\rho(a_0)$ vertices a_i connected by an edge to a_0 can be followed by any of the $\rho(a_1)$ vertices connected to a_1. Thus there are at least as many vertices not connected to a_1 as there are connected to a_0. It follows that $m-1-\rho(a_1) \geqq \rho(a_0)$. This gives:

THEOREM 1. *Let* (1) *be a cyclic order of the vertices in G with a minimal number of gaps. If there is a gap between a pair of vertices a_k and a_{k+1} then the corresponding local degrees satisfy $\rho(a_k)+\rho(a_{k+1}) \leqq m-1$.*

An immediate consequence is:

THEOREM 2. *Let G be a graph with m vertices such that for any pair of vertices a and b not connected by an edge the local degrees satisfy the condition $\rho(a)+\rho(b) \geqq m$. Then G has a Hamilton circuit.*

In particular G has a Hamilton circuit if $\rho(a) \geqq m/2$ for every vertex.

* Research supported by N.S.F. grant.

† A problem in graph theory, this MONTHLY, vol. 65, 1958, p. 611.

A BOOLEAN GEOMETRY FOR THE INTEGERS

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Metric spaces in which the distances are not real numbers have been studied by several people ([1], [2], [3], [4], [5]). The title of this note refers to the metric properties of the ring J of rational integers with respect to a distance function with values in a certain Boolean algebra $\mathfrak{B}$. This Boolean algebra is isomorphic to the algebra of subsets of a countable set. It is more convenient here, however, to define $\mathfrak{B}$ in the following way: denote by $\mathfrak{B}_2$ the Boolean algebra with two elements 0 and 1, then $\mathfrak{B}$ is the Boolean algebra whose elements are sequences $\{a_i\}$ of elements of $\mathfrak{B}_2$, with union and intersection defined componentwise.

Let p_i be the ith positive prime integer, and define a mapping $x \to \phi(x)$ of J into $\mathfrak{B}$ as follows: for $n \in J$, $\phi(n) = \{a_i\}$, where a_i is 0 or 1 according as p_i is or is not a factor of n. It is easily seen that the set of images, $\mathfrak{B}^* = \phi(J)$ is a sublattice of $\mathfrak{B}$, and that the mapping $x \to \phi(x)$ has the following properties:

$$\phi(n) = \phi(0) \text{ if and only if } n = 0, \tag{1}$$

$$\phi(n + m) \subseteq \phi(n) \cup \phi(m), \tag{2}$$

$$\phi(n \cdot m) = \phi(n) \cap \phi(m), \tag{3}$$

$$\phi(n) = \phi(1) \text{ if and only if } n = \pm 1. \tag{4}$$

Conversely, it can be shown that if ϕ is a mapping of J into a lattice $\mathfrak{L}$ satisfying these four conditions, then the set $\mathfrak{L}^* = \phi(J)$ is a sublattice of $\mathfrak{L}$ isomorphic to the lattice $\mathfrak{B}^*$ defined above.

Any ring R together with a mapping $x \to \phi(x)$ into a Boolean algebra $\mathfrak{A}$ satisfying (1), (2), and (3) is called a *Boolean-valued ring*. The mapping ϕ is called a valuation and $\mathfrak{A}$ is called a *valuation algebra*. If R is a Boolean-valued ring then the mapping ρ of $R \times R$ into $\mathfrak{A}$ defined by $\rho(x, y) = \phi(y - x)$, is called a distance function on R. It is easily seen that ρ satisfies:

$$\rho(x, y) = \phi(0) \text{ if and only if } x = y, \tag{5}$$

$$\rho(x, y) = \rho(y, x), \tag{6}$$

$$\rho(x, z) \cup \rho(z, y) \supseteq \rho(x, y). \tag{7}$$

The ring R together with this distance function is called a *Boolean space*. Thus, the ring of integers is a Boolean space with respect to the valuation ϕ defined above.

The first of the following two theorems characterizes the ring of integers as the only Boolean-valued ring with valuation algebra $\mathfrak{B}$ which is congruent to this Boolean space of the integers. The second theorem shows that the only isometries of this Boolean space of the integers are reflections and translations.

THEOREM 1. *Let R be a Boolean-valued ring with valuation algebra $\mathfrak{B}$ and valuation ψ and let ϕ be the valuation defined above for J. If $x \to g(x)$ is a* 1-1 *mapping*

of R onto J with $\psi(y-x)=\phi(g(y)-g(x))$ for all x, y in R, then R is isomorphic to J.

Proof. Let $f(x)=g(x)-g(0)$; then $f(x)$ is a 1-1 mapping of R onto J, with $f(0)=0$. Further, $\psi(y-x)=\phi(f(y)-f(x))$ for x, y in R and, in particular,

$$\psi(y) = \phi(f(y)), \qquad y \in R. \tag{8}$$

Next, note that R contains no proper divisors of zero. For if $xy=0$, then $\phi(0)=\psi(xy)=\psi(x)\cap\psi(y)=\phi(f(x))\cap\phi(f(y))=\phi(f(x)\cdot f(y))$. This implies $f(x)\cdot f(y)=0$, $f(x)=0$ or $f(y)=0$, and hence either x or y is zero. Now, choose u, v in R so that $f(u)=1$ and $f(v)=-1$. It follows from (4) and (8) that $f(u^2)=\pm 1$, $f(v^2)=\pm 1$, and $f(uv)=\pm 1$. From the first two of these last three equations, $f(u^2)=\pm f(v^2)$. If $f(u^2)=-f(v^2)$ then either $f(u^2)=f(uv)$ or $f(v^2)=f(uv)$, both of which lead to contradictions. Consequently $f(u^2)=f(v^2)=\pm 1$, and either $u^2=u$ or $v^2=v$. Thus, either u or v is an idempotent and hence an identity in R. Denote by e this identity element. Since $f(e)=\pm 1$ there is no loss in generality in assuming that $f(e)=1$. For, if $f(e)=-1$ the mapping $x\to f'(x)=-f(x)$ satisfies the hypothesis of the theorem and $f'(e)=1$. Suppose that $e+e=0$, and let y be the element of R such that $f(y)=-1$. Then $y\neq e$, but $\phi(f(y^2))=\psi(y^2)=\psi(y)=\phi(f(y))=\phi(-1)=\phi(1)$, so that $f(y^2)=\pm 1$. Thus, either $y^2=y$, a contradiction or $y^2=e=e^2$, which implies $(y-e)(y+e)=0$, also a contradiction. Hence $e+e\neq 0$, $e\neq -e$, and since $\phi(f(-e))=\psi(-e)\cap\psi(-e)=\psi(-e)=\psi(e)=\phi(1)$, it follows that $f(-e)=-1$. By two finite induction arguments it may be shown that $f(ne)=n$ and $f[(-n)e]=-n$, for every positive integer n. It then follows readily that $x\to f(x)$ is an isomorphism of R onto J.

A 1-1 mapping $x\to m(x)$ of J onto J is called an *isometry* if for all x, y in J, $\phi(y-x)=\phi(m(y)-m(x))$.

THEOREM 2. *If $x\to m(x)$ is an isometry of J, then for all x either $m(x)=x+m(0)$ or $m(x)=-x+m(0)$.*

Proof. Let $f(x)=m(x)-m(0)$, so that $f(0)=0$. Also the mapping $x\to f(x)$ is an isometry. Thus $\phi(f(1))=\phi(1)$, which implies $f(1)=\pm 1$. Suppose first that $f(1)=1$. As in the proof of Theorem 1, two finite induction arguments show that $f(n)=n$ and $f(-n)=-n$ for every positive integer n. Thus $x=f(x)=m(x)-m(0)$, for every x in J and hence $m(x)=x+m(0)$. If $f(1)=-1$, then by the preceding argument $f'(x)=x$ for every x in J, where $f'(x)=-f(x)$. Thus $-x=f(x)=m(x)-m(0)$ or $m(x)=-x+m(0)$. This completes the proof.

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THE DOUBLE DIXIE CUP PROBLEM

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The familiar childhood occupation of obtaining a complete set of pictures of baseball players, movie stars, etc., which appear on the covers of dixie cups raises some interesting questions. One, which has already been answered, is the "single dixie cup problem," that of determining the expected number, $E(n)$, of dixie cups which must be purchased before a complete set of n pictures is obtained: $E(n)=n(1+1/2+\cdots+1/n)$ ([1] p. 213).

Some time ago W. Weissblum asked how long, on the average, it would take to obtain *two* complete sets of n pictures. This corresponds to the situation observed when two tots collect cooperatively, *i.e.*, "trading" takes place.

This "double dixie cup" problem cannot be handled by the same device used for the problem of the single set and in this paper we find a new method which allows us to write down the solution, $E_m(n)$, (as an easily evaluated definite integral) for the problem of collecting m sets.

For m fixed and n large the expected number of dixie cups turns out to be $n(\log n+(m-1)\ \text{loglog}\ n+o(1))$. Thus, although the first set "costs" $n \log n$, all further sets cost n loglog n.

Suppose m sets are desired. Let p_i be the probability of failure of obtaining m sets up to and including the purchase of the ith dixie cup. Then the expected number of dixie cups $E_m(n)=\sum_{i=0}^{\infty} p_i$, by a well-known argument ([1] p. 211). Now $p_i=N_i/n^i$ where N_i is the number of ways that the purchase of i dixie cups can fail to yield m copies of each of the n pictures in the set. If we represent the pictures by $x_1, \cdots, x_n$, then N_i is simply $(x_1+\cdots+x_n)^i$ expanded and evaluated at $(1, \cdots, 1)$ after all the terms have been removed which have each exponent for each variable larger than $m-1$.

Now consider m fixed and introduce the following notation. If $P(x_1, \cdots, x_n)$ is a polynomial or power series we define $\{P(x_1, \cdots, x_n)\}$ to be the polynomial, or series, resulting when all terms having all exponents $\geqq m$ have been removed. In terms of this notation p_i is $\{(x_1+\cdots+x_n)^i\}/n^i$ evaluated at $x_1=\cdots=x_n=1$.

If we now make the definition

$$S_m(t)=\sum_{k<m}\frac{t^k}{k!} \tag{1}$$

and consider the expression

$$F=\exp(x_1+\cdots+x_n)-(e^{x_1}-S_m(x_1))\cdots(e^{x_n}-S_m(x_n)), \tag{2}$$

it is easily seen that F has no terms with all exponents $\geqq m$; but F does not have all terms of $\exp(x_1+\cdots+x_n)$ with at least one exponent $<m$. We conclude that

$$F = \{\exp(x_1 + \cdots + x_n)\} = \sum \{(x_1 + \cdots + x_n)^i\}/i!. \tag{3}$$

By contrast, we have seen that

$$E_m(n) = \sum p_i = \sum \{(x_1 + \cdots + x_n)^i\}/n^i \tag{4}$$

at $x_1 = \cdots = x_n = 1$, and so all we need now is a method for replacing $1/i!$ by $1/n^i$. This is afforded by the identity

$$n\int_0^\infty \frac{t^i}{i!} e^{-nt}dt = \frac{1}{n^i}$$

and the result is

$$n\int_0^\infty [\exp(x_1 + \cdots + x_n)t - (e^{x_1 t} - S_m(tx_1)) \cdots (e^{x_n t} - S_m(tx_n)]e^{-nt}dt. \tag{5}$$

Setting $x_1 = \cdots = x_n = 1$, finally, gives, by (4) and (5),

THEOREM 1. $n\int_0^\infty [1-(1-S_m(t)e^{-t})^n]dt = E_m(n)$.

This is the solution and it is readily integrable for small m and n. It is perhaps worthwhile to mention that if only a particular k of the pictures are desired the expected number is:

$$n\int_0^\infty 1 - \left(1 - e^{-t}\left(1 + t \cdots + \frac{t^{m-1}}{(m-1)!}\right)\right)^k dt.$$

This may also be easily generalized to the case where m_k copies of the kth pictures are desired.

For large m, by the law of large numbers, the number required is asymptotic to mn. It remains to obtain the asymptotic form for large n.

We now prove

THEOREM 2. *$E_m(n) = n[\log n + (m-1)\ \text{loglog}\ n + C_m + o(1)]$ for m fixed and $n \to \infty$.*

It suffices to prove that

$$\frac{E_m(n+1)}{n+1} - \frac{E_m(n)}{n} = \frac{1}{n+1} + \frac{m-1}{n \log n} + \lambda_n,$$

where $\sum|\lambda_n| < \infty$.

Now, by Theorem 1,

$$\frac{E_m(n+1)}{n+1} - \frac{E_m(n)}{n} = \int_0^\infty e^{-t}S_m(t)[1 - e^{-t}S_m(t)]^n dt$$

and, changing variables by

$$1 - e^{-t}S_m(t) = x, \tag{6}$$

we have $dx = e^{-t}[(t^{m-1})/(m-1)!]dt$ and so the above is equal to

$$(7)\quad \int_0^1 x^n S_m(t) \frac{(m-1)!}{t^{m-1}}\,dx = \int_0^1 x^n \left[1 + \frac{m-1}{t} + \frac{(m-1)(m-2)}{t^2} + \cdots\right] dx.$$

[$t = t(x)$ is of course defined by (6)].

We now show that

$$(A)\quad \text{For } 1 < k < m, \qquad \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{t^k}\,dx < \infty;$$

$$(B)\quad \int_0^1 \frac{x^n}{t}\,dx = \frac{1}{n \log n} + \alpha n, \quad \text{where} \quad \sum |\alpha n| < \infty;$$

and, by (7), these will suffice to prove our theorem.

Proof of (A). By (6) we have $x = 1 - e^{-t}S_m(t) \leqq 1 - e^{-t}$ so that,

$$(8)\quad t \geqq \log \frac{1}{1-x}.$$

On the other hand

$$x = 1 - e^{-t}S_m(t) = \int_0^t e^{-u} \frac{u^{m-1}}{(m-1)!}\,du \leqq \int_0^t u^{m-1}du \leqq t^m,$$

so that

$$(9)\quad t \geqq x^{1/m}.$$

Now, the infinite series given in (A) is equal to

$$\int_0^1 \frac{1}{1-x}\frac{dx}{t^k} = \int_0^{1/2} \frac{1}{1-x}\frac{dx}{t^k} + \int_{1/2}^1 \frac{1}{1-x}\frac{dx}{t^k}.$$

The first of the integrals on the right is finite by virtue of (9) and the fact that $k < m$, while the second integral is finite by (8) and the fact that $k > 1$. This proves (A).

Proof of (B). By (8) we obtain

$$t \geqq x + \frac{x^2}{2} + \cdots \frac{x^r}{r} \geqq x^r \left[1 + \frac{1}{2} + \cdots \frac{1}{r}\right] \geqq x^r \log r,$$

and so

$$(10)\quad \int_0^1 \frac{x^n}{t}\,dx \leqq \int_0^1 \frac{x^{n-r}}{\log r}\,dx = \frac{1}{(n-r+1)\log r}.$$

Now let $u \geqq 1$ be a parameter and set

$$(11)\quad a_n = 1 - S_m(u)e^{-u}.$$

We have

$$\int_0^1 \frac{x^n}{t}\,dx \geqq \int_0^a \frac{x^n}{t}\,dx \geqq \frac{1}{u}\int_0^a x^n dx$$

$$\geqq \frac{1}{u}\left[\int_0^1 x^n dx - \int_a^1 dx\right] = \frac{1}{(n+1)u} - \frac{S_m(u)e^{-u}}{u} \qquad \text{by (11).}$$

Now note that by (1) and the fact that $u \geqq 1$, we obtain $S_m(u) \leqq eu^{m-1}$. Combining this with the previous inequality gives

$$\int_0^1 \frac{x^n}{t}\,dx \geqq \frac{1}{(n+1)u} - u^{m-2}e^{1-u}. \tag{12}$$

If we now set $r = [n/\log n]$ in (10) and $u = \log n + m \operatorname{loglog} n$ in (12) we obtain

$$\frac{1}{n\log n} - \frac{C \operatorname{loglog} n}{n \log^2 n} \leqq \int_0^1 \frac{x^n}{t}\,dx \leqq \frac{1}{n\log n} + \frac{C \operatorname{loglog} n}{n\log^2 n},$$

and this completes the proof since $\sum(\operatorname{loglog} n)/(n \log^2 n) < \infty$.

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THE ANTICENTER OF A GROUP

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In contrast to the center of a group G we will define the rim of G, denoted by $R(G)$, to be the set of elements of G which permute with no elements of G except in trivial cases. We will define the anticenter of G, denoted by $AC(G)$, as the set of products of elements in $R(G)$. More precisely:

DEFINITION 1. $R(G) \equiv \{a \mid ab = ba$ *implies* $\exists\, c \in G$ *such that* $a = c^i$, $b = c^j$ *for some integers* i *and* $j.\}$

DEFINITION 2. $AC(G) \equiv \{a_1 \cdots a_n \mid a_i \in R(G).\}$

LEMMA 1. *The identity* e *of* G *belongs to* $R(G)$.

Proof. $eb = be$ implies $e = b^0$, $b = b^1$.

LEMMA 2. *If* $a \in R(G)$ *then* $a^{-1} \in R(G)$.

Proof. Suppose $a^{-1}b = ba^{-1}$. Then $ab = ba$ and hence $a = c^i$ and $b = c^j$ or $b = c^j$ and $a^{-1} = c^{-i}$.

LEMMA 3. *If* $a \in R(G)$ *then* $b^{-1}ab \in R(G)$ *for all* $b \in G$.

Proof. Let $b^{-1}abx = xb^{-1}ab$ for some x in G. Then $abxb^{-1} = bxb^{-1}a$. Hence $a = c^i$ and $bxb^{-1} = c^j$ or $b^{-1}ab = b^{-1}c^jb = (b^{-1}cb)^j$ and $x = b^{-1}c^jb = (b^{-1}cb)^j$. This proves $b^{-1}ab \in R(G)$.

V^i, R^i, I and $D_1 \neq D_2^i$ for all i. Hence $D_1 \notin R(G)$ and $D_2 \notin R(G)$. Finally $HR^2 = V = R^2H$. But $H \neq V^i$, R^i, D_1^i, D_2^i, I and $R^2 \neq H^i$ for all i. Hence $H \notin R(G)$. Thus $AC(G)$: I, R, R^2, and R^3.

The converses of Theorem 2 and Theorem 4 are false as Example 2 indicates. Finally the $AC(G)$ is, in general nontrivial, as is demonstrated by Example 3.

In a later paper, the author hopes to obtain results concerning $G/AC(G)$, some relations between the center and the anticenter, and to explore more fully the effect of isomorphism and homomorphism on the anticenter.

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CLASSROOM NOTES

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A DERIVATION OF n-DIMENSIONAL SPHERICAL COORDINATES

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An instructive example in linear algebra is the derivation of n-dimensional spherical coordinates without appealing to geometric intuition. The method of derivation is based on concepts from linear algebra; namely, bases of a vector space, scalar product, angle between vectors and projection of a vector onto a subspace. Spherical coordinates in n-dimensions are a generalization of the usual three-dimensional spherical coordinates and are particularly useful in evaluating certain integrals taken over the surface of an n-dimensional sphere. Later we shall give an example of such an integration.

Let E_n denote real n-dimensional euclidean space. Vectors in E_n will be denoted by bold-faced letters. If $\mathbf{x}$ and $\mathbf{y}$ are two vectors in E_n with components ξ_j and η_j, $j = 1, \cdots, n$, respectively, we define the scalar product of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} \xi_j \eta_j.$$

The nonnegative number $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ is called the norm of $\mathbf{x}$. The angle between $\mathbf{x}$ and $\mathbf{y}$ is defined by $\cos \phi = \mathbf{x} \cdot \mathbf{y} / \|\mathbf{x}\| \|\mathbf{y}\|$, where ϕ is restricted to the range $0 \leqq \phi \leqq \pi$. A set of vectors $\mathbf{x}_1, \cdots, \mathbf{x}_n$ is an orthonormal set in E_n if $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ or 1 accordingly as $i \neq j$ or $i = j$. Any set of n orthonormal vectors forms a basis for E_n.

Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be any orthonormal basis in E_n. Let $\mathbf{x}$ be any vector on the n-dimensional sphere of radius r about the origin, that is, $\|\mathbf{x}\| = r$. If $\mathbf{x} = \sum_{i=1}^{n} \xi_i \mathbf{e}_i$ then $\|\mathbf{x}\|^2 = \sum_{i=1}^{n} \xi_i^2$. If θ_i is the angle between $\mathbf{x}$ and $\mathbf{e}_i$ then $\xi_i = \mathbf{x} \cdot \mathbf{e}_i = r \cos \theta_i$. Hence $\mathbf{x} = \sum_{i=1}^{n} r \cos \theta_i \mathbf{e}_i$ and $\mathbf{x}$ can be specified by giving its length r and the n angles θ_i. But since $r^2 = \mathbf{x} \cdot \mathbf{x} = r^2 \sum_{i=1}^{n} \cos^2 \theta_i$ we see that the θ_i are not independent of each other. Spherical coordinates in n-dimensions show us how to pick out $n-1$ angles $\phi_1, \cdots, \phi_{n-2}, \theta$ which are independent of each other and which, when combined with the norm r, completely describe the vector $\mathbf{x}$ with respect to the given orthonormal basis.

Derivation of the coordinates. Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ and $\mathbf{x}$ be as above. Let ϕ_1 be the angle between $\mathbf{x}$ and $\mathbf{e}_1$, $0 \leqq \phi_1 \leqq \pi$. Then $\xi_1 = \mathbf{x} \cdot \mathbf{e}_1 = r \cos \phi_1$ and

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + \sum_{j=2}^{n} \xi_j \mathbf{e}_j.$$

Now

$$r^2 = \|\mathbf{x}\|^2 = r^2 \cos^2 \phi_1 + \sum_{j=2}^{n} \xi_j^2 \quad \text{or} \quad \sum_{j=2}^{n} \xi_j^2 = r^2 \sin^2 \phi_1.$$

Setting $\xi_j = \alpha_j r \sin \phi_1$, $j = 2, \cdots, n$, we have

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + r \sin \phi_1 \sum_{j=2}^{n} \alpha_j \mathbf{e}_j,$$

where $\sum_{j=2}^{n} \alpha_j^2 = 1$. (If ϕ_1 is 0 or π, then $\mathbf{x} = \pm r \mathbf{e}_1$.)

Let $\mathbf{u}_2 = \sum_{j=2}^{n} \alpha_j \mathbf{e}_j$. The vector $\mathbf{u}_2$ is a unit vector (that is $\|\mathbf{u}_2\| = 1$) in the direction of the projection of $\mathbf{x}$ onto the $(n-1)$-dimensional subspace spanned by $\mathbf{e}_2, \cdots, \mathbf{e}_n$. If ϕ_2 is the angle between $\mathbf{u}_2$ and $\mathbf{e}_2$ then $\cos \phi_2 = \mathbf{u}_2 \cdot \mathbf{e}_2 = \alpha_2$, $0 \leqq \phi_2 \leqq \pi$, and

$$\mathbf{u}_2 = \cos \phi_2 \mathbf{e}_2 + \sum_{j=3}^{n} \alpha_j \mathbf{e}_j.$$

Hence,

$$1 = \|\mathbf{u}_2\|^2 = \cos^2 \phi_2 + \sum_{j=3}^{n} \alpha_j^2 \quad \text{or} \quad \sum_{j=3}^{n} \alpha_j^2 = \sin^2 \phi_2.$$

If we set $\alpha_j = \beta_j \sin \phi_2$, $j = 3, \cdots, n$, then

$$\mathbf{u}_2 = \cos \phi_2 \mathbf{e}_2 + \sin \phi_2 \sum_{j=3}^{n} \beta_j \mathbf{e}_j,$$

where $\sum_{j=3}^{n} \beta_j^2 = 1$. Thus

$$\mathbf{x} = r \cos \phi_1 \mathbf{e}_1 + r \sin \phi_1 \cos \phi_2 \mathbf{e}_2 + r \sin \phi_1 \sin \phi_2 \sum_{j=3}^{n} \beta_j \mathbf{e}_j.$$

In general, let $\mathbf{u}_j$ be the unit vector in the direction of the projection of $\mathbf{x}$ onto the space spanned by $\mathbf{e}_j, \mathbf{e}_{j+1}, \cdots, \mathbf{e}_n$, $j=2, \cdots, n-1$ and let ϕ_{j-1} be the angle between $\mathbf{u}_j$ and $\mathbf{e}_j$, $0 \leqq \phi_j \leqq \pi$, $j=2, \cdots, n-1$. Then

$$\mathbf{x} = \sum_{j=1}^{n-2} r\left(\prod_{k=1}^{j-1} \sin\phi_k\right)\cos\phi_j \mathbf{e}_j + r\left(\prod_{k=1}^{n-2}\sin\phi_k\right)\mathbf{u}_{n-1}.$$

Now $\mathbf{u}_{n-1} = \delta_{n-1}\mathbf{e}_{n-1} + \delta_n \mathbf{e}_n$, where $1 = \|\mathbf{u}_{n-1}\|^2 = \delta_{n-1}^2 + \delta_n^2$. If now we define an angle θ by $\cos\theta = \delta_n$, $\sin\theta = \delta_{n-1}$, we see that $0 \leqq \theta \leqq \pi$ will not suffice since δ_{n-1} can be negative and $\sin\alpha \geqq 0$ for $0 \leqq \theta \leqq \pi$. In order to include all possible combinations of (δ_{n-1}, δ_n) we must have $0 \leqq \theta < 2\pi$.

Thus if $\mathbf{e}_1, \cdots, \mathbf{e}_n$ is a given orthonormal basis in E_n and $\mathbf{x}$ is a vector of norm r with components ξ_j with respect to this basis, then

$$(*)\qquad \begin{aligned} \xi_1 &= r\cos\phi_1, \\ \xi_j &= r\cos\phi_j \prod_{k=1}^{j-1}\sin\phi_k \qquad (j = 2, \cdots, n-2), \\ \xi_{n-1} &= r\sin\theta \prod_{k=1}^{n-2}\sin\phi_k, \\ \xi_n &= r\cos\theta\prod_{k=1}^{n-2}\sin\phi_k, \end{aligned}$$

where $0 \leqq \phi_j \leqq \pi$, $j=1, \cdots, n-2$; $0 \leqq \theta < 2\pi$; $0 \leqq r < \infty$.

Application to integration. Let $f(\xi_1, \cdots, \xi_n)$ be a continuous real-valued function defined in E_n which may be written in the form

$$f(\xi_1, \cdots, \xi_n) = g(\alpha_1\xi_1 + \cdots + \alpha_n\xi_n, \xi_1^2 + \cdots + \xi_n^2),$$

where the α_i are constants independent of the ξ's. We wish to compute the integral of f over the surface of the n-dimensional sphere of radius r with the origin as center. If $\mathbf{x}$ is the vector with coordinates ξ_j and $\mathbf{a}$ the vector with coordinates α_j (these coordinates being with respect to some given orthonormal basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$) then

$$\int_{\xi_1^2+\cdots+\xi_n^2=r^2} f(\xi_1, \cdots, \xi_n)dS = \int_{\|\mathbf{x}\|=r} g(\mathbf{a}\cdot\mathbf{x}, \|\mathbf{x}\|^2)dS,$$

where dS is the surface differential.

Let $\mathbf{a}_1 = \mathbf{a}/\|\mathbf{a}\|$* and choose vectors $\mathbf{a}_2, \cdots, \mathbf{a}_n$ to complete an orthonormal basis in E_n. Let the coordinates of $\mathbf{x}$ with respect to the basis $\mathbf{a}_1, \cdots, \mathbf{a}_n$ be $\zeta_1, \cdots, \zeta_n$. Then $\mathbf{a}_1\cdot\mathbf{x} = \zeta_1$. Make the spherical coordinate transformation

* If $\mathbf{a}=0$, then $\mathbf{a}_1$ may be any unit vector.

given by (*) with ξ_j replaced by ζ_j, $j=1, \cdots, n$.

The Jacobian of the transformation is

$$J = r^{n-1} \prod_{k=1}^{n-2} \sin^k \phi_{n-1-k}.$$

Also, $\mathbf{a}_1 \cdot \mathbf{x} = \zeta_1 = r \cos \phi_1$. Thus the integral becomes

$$2\pi r^{n-1}\left[\prod_{k=1}^{n-3}\int_0^{\pi} \sin^k \phi_{n-1-k} d\phi_{n-1-k}\right]\int_0^{\pi} g(\|\mathbf{a}\| r \cos \phi_1, r^2) \sin^{n-2} \phi_1 d\phi_1$$

$$= \frac{2r^{n-1}\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\int_0^{\pi} g(\|\mathbf{a}\| r \cos \phi_1, r^2) \sin^{n-2} \phi_1 d\phi_1.$$

Thus we have reduced the integral over the surface of an n-dimensional sphere to a single integral on the real line. In particular, if $f \equiv 1$, we obtain $[2\pi^{n/2}/\Gamma(n/2)]r^{n-1}$ for the surface area of an n-dimensional sphere of radius r and, integrating from 0 to r, we obtain $[2\pi^{n/2}/(n\Gamma(n/2))]r^n$ for the volume of the sphere.

MATRIX INTEGRATION OF $x^k \exp(-\beta^2 x^2)$

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Let V be the vector space of finite linear combinations of $x^k \exp(-\beta^2 x^2)$, fixed β, $k=0, 1, \cdots$, with basis $\{x^k \exp(-\beta^2 x^2)\}$. Let D be a linear transformation on V which differentiates a vector belonging to V.

Since $(x^k \exp(-\beta^2 x^2))D = kx^{k-1} \exp(-\beta^2 x^2) - 2\beta^2 x^{k+1} \exp(-\beta^2 x^2)$, the matrix of D is

$$\begin{bmatrix} 0 & -2\beta^2 & \cdot & \cdot & \cdots & 0 & \cdots \\ 1 & 0 & -2\beta^2 & \cdot & \cdots & \cdot & \cdots \\ \cdot & 2 & 0 & -2\beta^2 & \cdots & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & 0 & k & 0 & -2\beta^2 & \cdots \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

V is closed under D and the kernel of D consists of the zero vector alone. The calculation of D^{-1} may be carried out algebraically, giving an interesting equation for $\int x^k \exp(-\beta^2 x^2) dx$.

Because of the nature of D, D^{-1} may be calculated in four independent steps depending on whether k and j are even or odd, where $\|D^{-1}\| = \|a_{kj}\|$. Using $DD^{-1} = I$, we obtain the following expressions for a_{kj}.

(1) j odd, k odd: $\quad a_{kj} = 0;$

(2) j even, k odd: $$a_{kj} = \begin{cases} \dfrac{-(\frac{1}{2}k - \frac{1}{2})!}{2(\frac{1}{2}j)!\beta^{k+1-j}} & j \leqq k-1, \\ 0 & j > k-1; \end{cases}$$

(3) j odd, k even: $$a_{kj} = \begin{cases} \dfrac{(2\beta^2)^{(j-k-1)/2}}{(k+1)(k+3)\cdots j} & j \geqq k+1, \\ 0 & j < k+1; \end{cases}$$

(4) j even, k even: the elements of D^{-1} are indeterminate with one degree of freedom; choosing a_{00} determines

$$a_{kj} = \frac{\beta^j}{(\frac{1}{2}j)!} \frac{1\cdot 3\cdot 5\cdots(k-1)}{(2\beta^2)^{k/2}} a_{00}.$$

The matrix D thus has an infinite number of inverses (right and left) corresponding to different values of a_{00}. Furthermore, we see that D^{-1} is not properly defined on V since it maps a vector into an infinite linear combination of the basis vectors. We may still use D^{-1} as a computational device to obtain $\int x^k \exp(-\beta^2x^2)dx$.

For k odd, using (1) and (2), we obtain

$$\int x^k \exp(-\beta^2x^2)dx = \frac{-(\frac{1}{2}k - \frac{1}{2})!}{2\beta^{k+1}} \exp(-\beta^2x^2) \sum_{n=0}^{k/2-1/2} \frac{(\beta x)^{2n}}{n!},$$

which is equivalent to that given by Gröbner.* We also easily obtain

$$\int_0^\infty x^k \exp(-\beta^2x^2)dx = \frac{(\frac{1}{2}k - \frac{1}{2})!}{2\beta^{k+1}}, \qquad k \text{ odd}.$$

For k even, using (3) and (4), we obtain the expression†

$$\int x^k \exp(-\beta^2x^2)dx = \exp(-\beta^2x^2)(2\beta^2)^{-(k+1)/2} \sum_{j=k+1}^{\infty}{}' \frac{(2\beta^2)^{j/2}x^j}{(k+1)(k+3)\cdots j} + a_{00} \frac{1\cdot 3\cdot 5\cdots(k-1)}{(2\beta^2)^{k/2}} \exp(-\beta^2x^2) \sum_{j=0}^{\infty}{}'' \frac{x^j\beta^j}{(\frac{1}{2}j)!}.$$

Since $\sum'' = \exp(\beta^2x^2)$, the second term in the above expression for the integral reduces to $[a_{00}1\cdot 3\cdot 5\cdots(k-1)]/[(2\beta^2)^{k/2}]$, independent of x.

The choice of a_{00} is only representative of an arbitrary constant of integration, since the term containing it is independent of x. For any D^{-1}, $v \in V$, the series defined by $D^{-1}v$ converge and can be differentiated term by term, and the fundamental theorem of the calculus holds. Hence all of the inverses of D inte-

* Wolfgang Gröbner und Nikolaus Hofreiter, Integraltafel, Erster Teil, Unbestimmte Integrale, Vienna, 1949, p. 109.

† In $\sum'$, the summation is for $j = k+1, k+3, \cdots$; in $\sum''$, for $j = 0, 2, 4, \cdots$.

grate vectors in V.

We then have for k even

$$\int x^k \exp(-\beta^2x^2)dx = \frac{1\cdot3\cdot5\cdots(k-1)}{(2\beta^2)^{(k+1)/2}} \exp(-\beta^2x^2) \sum_{j=k+1}^{\infty}{}' \frac{(2\beta^2)^{j/2}x^j}{1\cdot3\cdot5\cdots j}.$$

Since $\int_0^\infty x^k \exp(-\beta^2x^2)dx$ is known, we also obtain an interesting limiting value for a series, namely,

$$\lim_{x\to\infty} \exp(-\beta^2x^2) \sum_{j=k+1}^{\infty}{}' \frac{(2\beta^2)^{j/2}x^j}{1\cdot3\cdot5\cdots j} = \frac{\sqrt{\pi}}{2\beta}, \qquad k \text{ even.}$$

NOTE ON THE CLASSICAL CANONICAL FORM OF A MATRIX

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This note concerns some aspects of the proof of the existence and the uniqueness of the classical canonical form of a square matrix without the intervention of invariant factors, elementary divisors, or modules.

As is well known, if the coefficients of A are assumed to lie in an algebraically closed field then by two simple induction proofs* one can show that A is similar to diag $(A_1, A_2, \cdots)$, where each A_i is lower triangular with only one characteristic root, say λ_i. Then one may follow Turnbull and Aitken† to show that A_i is similar to a matrix of the form λ_iI+G, where

$$G = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ b_0 & 0 & & & & & \cdot \\ b_1 & \delta_1 & 0 & & & & \cdot \\ \cdot & & \cdot & 0 & & & \cdot \\ \cdot & & & \cdot & 0 & & \cdot \\ \cdot & & & & \cdot & 0 & \cdot \\ b_{k-2} & \cdot & \cdot & \cdot & \cdot & \delta_{k-2} & 0 \end{pmatrix},$$

δ_j and b_j are 0 or 1, and $\delta_jb_j=0$.

At this point we construct§ the matrix R of a final similarity transformation such that $R^{-1}GR$ is classical, and then prove the uniqueness of the classical form of A by comparing ranks.

1. The matrix R. If q is a maximal integer $\geqq 0$ such that $\delta_{p+1}=\delta_{p+2}=\cdots=\delta_{p+q}=1$ and $b_p=1$ simultaneously, for some $p\geqq 0$, then we can partition G as follows:

* One first triangularizes the matrix and then subtriangularizes it.

† H. W. Turnbull and A. C. Aitken, Theory of Canonical Matrices, London 1950, p. 68.

§ Inspired by Turnbull and Aitken.

$$G = \begin{pmatrix} D & 0 & 0 \\ E & N & 0 \\ F & 0 & M \end{pmatrix},$$

where D is $(p+1)\times(p+1)$ and N is $(q+1)\times(q+1)$. We note that E is zero except in its upper-left corner and that:

$$G^i = \begin{pmatrix} D^i & 0 & 0 \\ N^{i-1}E & N^i & 0 \\ M^{i-1}F & 0 & M^i \end{pmatrix},$$

G^{q+2} has a zero first column.

Let

$$\Omega = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & & & & & \cdot \\ \cdot & 1 & 0 & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix},$$

and let e_i be the ith column of the k-rowed identity matrix. Then define

$$R = (e_2, e_3, \cdots, e_{p+1}, e_1, Ge_1, G^2e_1, \cdots, G^{q+1}e_1, e_{p+q+3}, \cdots, e_k) = \begin{pmatrix} \Omega & Q & 0 \\ 0 & I & 0 \\ 0 & B & I \end{pmatrix},$$

where the I's are identity matrices of the appropriate dimensions, $q+1$ and $k-p-q-2$, respectively. Since

(a)
$$R^{-1} = \begin{pmatrix} \Omega^{-1} & -\Omega^{-1}Q & 0 \\ 0 & I & 0 \\ 0 & -B & I \end{pmatrix},$$

(b)
$$\Omega^{-1}(D - QE)\Omega = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \delta_1 & 0 & & & & & \cdot \\ \cdot & \delta_2 & 0 & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & \delta_{p-1} & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \delta_p & 0 \end{pmatrix},$$

and

$$\text{(c)} \qquad F - BE = 0, \qquad MB - BN = 0, \qquad DQ - QN = 0,$$

we have

$$R^{-1}GR = \begin{pmatrix} \Omega^{-1}(D - QE)\Omega & 0 & 0 \\ E\Omega & N & 0 \\ 0 & 0 & M \end{pmatrix},$$

which is in classical form.

2. Uniqueness. If $T=\text{diag}\,(T_1, T_2, \cdots)$ and $S=\text{diag}\,(S_1, S_2, \cdots)$ are two reductions of A to classical form with T_i and S_i having the same root λ_i, then there exists a nonsingular $P=(P_{ij})$ such that $PT=SP$, hence $P_{ij}T_j=S_iP_{ij}$ for all i, j. But if $i\neq j$, it is easily shown by recursion starting from the last column of P_{ij} that $P_{ij}=0$. Hence T_i is similar to S_i for all i.

Now let T be any T_i and S the corresponding S_i. Then we may write $T=\lambda I+N$, $S=\lambda I+M$. The similarity of T and S implies that N^r and M^r have the same rank for all r and hence the same number of ones (on the subdiagonals). All chains of successive ones of length $r-1$ in N will disappear at exactly the rth power of N, and similarly for M. Also, the change in rank from N^r to N^{r+1} equals the number of chains of length $\geqq r>0$ in N, and similarly for M. Since N^r and M^r have the same rank, N and M must have the same number of chains of length r for each r.

THE TOEPLITZ-SILVERMAN THEOREM AND THE DEFINITION OF e^x

M. I. Aissen, Johns Hopkins University

The discussion of the limit of $(1+x/n)^n$ as $n\to\infty$ through integer values is often reduced to the study of the limit of $(1+1/y)^{yx}$ as $y\to\infty$ through arbitrary values. This involves the notion of arbitrary exponents. An alternative method of defining e^x is with the power series $\sum_0^\infty x^n/n!$ whose convergence can be established very early in most discussions of sequences and series. Even before the discussion of the exponential function, the student should be able to understand the statement and proof of the Toeplitz-Silverman theorem, in at least the simplest case when the matrix is nonnegative and triangular. This note is to show how this theorem can be used to establish the existence and value of $\lim_{n\to\infty}(1+x/n)^n$ in terms of the known convergent series $\sum_0^\infty x^n/n!$

We first state the Toeplitz-Silverman theorem in the simple form and then apply it.

(TS) Theorem. *Let $p_{n,k}$ be a double sequence of nonnegative real numbers satisfying $\sum_{k=1}^n p_{n,k}=1$, $n=1, 2, \cdots$, and $\lim_{n\to\infty} p_{n,k}=0$, $k=1, 2, \cdots$. If S_n is a sequence of complex numbers which converges to a number S, and if T_n is defined by $T_n=\sum_{k=1}^n p_{n,k}S_k$, then $\lim_{n\to\infty} T_n=S$.*

For our application, we define $S_n(x) = \sum_{k=0}^{n} x^k/k!$, and $T_n(x) = (1+x/n)^n$, for $n = 1, 2, \cdots$. Then

$$T_n(x) = \sum_{k=1}^{n} p_{n,k} S_k(x), \tag{1}$$

where

$$p_{n,k} = \frac{k(n!)}{n^{k+1}(n-k)!}. \tag{2}$$

(In the classroom, (1) and (2) should be derived and not merely stated.)

Since $T_n(0) = S_m(0) = 1$, we have $\sum_{k=1}^{n} p_{n,k} = 1$ and from (2) it is clear that $\lim_{n\to\infty} p_{n,k} = 0$. Hence by (TS) we have proved:

If $\sum_{k=0}^{\infty} x^k/k!$ *converges, then the sequence* $(1 + x/n)^n$ *converges to* $\sum_{k=0}^{\infty} x^k/k!$. *It is clear that the result is immediately valid for complex* x.

The importance of the Toeplitz-Silverman theorem in its more general form and of its integral analogues certainly justify the inclusion of some form of it in the curriculum soon after the notion of "limit of sequence" is introduced. This particular application might provide a reasonable place for it.

PARTICULAR SOLUTIONS FOR NONHOMOGENEOUS, LINEAR, ORDINARY DIFFERENCE EQUATIONS

LOUIS C. BARRETT AND FORREST DRISTY, South Dakota School of Mines and Technology

In his classroom note* G. E. Latta explains how Lagrange's identity may be utilized to determine particular integrals of nonhomogeneous, linear, ordinary differential equations. The present note contains a parallel treatment of nonhomogeneous, linear, ordinary difference equations.

Let the nth order linear difference operator L be defined by

$$Ly(k) \equiv p_0(k)y(k+n) + p_1(k)y(k+n-1) + \cdots + p_n(k)y(k). \tag{1}$$

Applying the formula

$$V(k)U(k+r) \equiv \Delta[V(k-1)U(k+r-1) + V(k-2)U(k+r-2) + \cdots + V(k-r)U(k)] + V(k-r)U(k)$$

to each term in the right hand member of

$$v(k)Ly(k) \equiv v(k)p_0(k)y(k+n) + v(k)p_1(k)y(k+n-1) + \cdots + v(k)p_n(k)y(k), \tag{2}$$

we obtain the identity

$$v(k)Ly(k) \equiv \Delta P[y(k), v(k)] + y(k)\bar{L}v(k), \tag{3}$$

* Particular integrals for nonhomogeneous, linear, ordinary differential equations, this MONTHLY, vol. 65, 1959, pp. 624–625.

where

$$P[y(k), v(k)] \tag{4}$$

is a bilinear form, linear in $y(k)$, $y(k+1), \cdots, y(k+n-1)$ and in $v(k-n)$, $v(k-n+1), \cdots, v(k-1)$, and where

$$\bar{L}v(k) \equiv p_0(k-n)v(k-n) + p_1(k-n+1)v(k-n+1) + \cdots + p_n(k)v(k). \tag{5}$$

From (3) it is evident that any solution of

$$\bar{L}v(k) = 0 \tag{6}$$

makes $v(k)Ly(k)$ an exact difference. We shall call $\bar{L}$ and L adjoint difference operators and say that (6) and $Ly(k)=0$ are adjoint difference equations of one another. Equation (3) and expression (4) are, respectively, the counterparts of Lagrange's identity and the bilinear concomitant of differential equation theory.

Now, suppose we are required to find a particular solution of the nonhomogeneous difference equation

$$Ly(k) = f(k). \tag{7}$$

This may be accomplished if n linearly independent solutions

$$v_i(k), \qquad i = 1, \cdots, n, \tag{8}$$

of (6) are known, and these may always be found by use of (3) if the general solution of $Ly(k)=0$ is known.

Assuming, then, that functions (8) have been found, we substitute $Ly(k)=f(k)$ and $\bar{L}v_i(k)=0$ into (3) and subsequently take indefinite sums over k. In this way we obtain the system

$$P[y(k), v_i(k)] = \sum_k v_i(k)f(k), \qquad i = 1, \cdots, n, \tag{9}$$

of n linear algebraic equations in the n unknowns $y(k), y(k+1), \cdots, y(k+n-1)$. The determinant of the coefficients of these unknowns reduces to a nonzero multiple of

$$\begin{vmatrix} v_1(k-n) & v_2(k-n) & \cdots & v_n(k-n) \\ v_1(k-n+1) & v_2(k-n+1) & \cdots & v_n(k-n+1) \\ \cdot & \cdot & \cdots & \cdot \\ v_1(k-1) & v_2(k-1) & \cdots & v_n(k-1) \end{vmatrix} \tag{10}$$

and this determinant, called Casorati's determinant, does not vanish because the functions (8) were supposed linearly independent. Hence, a particular solution of (7) may be found by solving system (9) for $y(k)$. This solution may be specialized by attributing definite values to the arbitrary constants of summation which are implicit in (9).

Example. For the equation

$$y(k+2) - y(k) = \ln (k+1)(k+2), \tag{11}$$

(3) becomes

$$v(k)[y(k+2)-y(k)]=\Delta[v(k-1)y(k+1)+v(k-2)y(k)]+y(k)[v(k-2)-v(k)]$$

and two linearly independent solutions of $v(k-2)-v(k)=0$ are $v_1(k)=1$ and $v_2(k)=(-1)^k$. Hence, equations (9) may, in this case, be particularized to

$$y(k+1)+y(k)=\sum_{n=1}^{k}\ln(n)(n+1),$$

$$(-1)^{k+1}y(k+1)+(-1)^k y(k)=\sum_{n=1}^{k}(-1)^{n+1}\ln(n)(n+1).$$

Solving for $y(k)$, we find that a particular solution of (11) is

$$y(k)=\tfrac{1}{2}\sum_{n=1}^{k}[(-1)^{n-k+1}+1]\ln(n)(n+1),$$

which simplifies to $y(k)=\ln(k!)$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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THE NATIONAL CONTEST IN HIGH SCHOOL MATHEMATICS IN UPPER NEW YORK STATE

NURA D. TURNER, State University of New York College of Education at Albany

This paper is a brief report of the experience of the Upstate Section of the Upper New York State Section of the Mathematical Association of America which first in 1958 administered the national contests sponsored by the M.A.A. and the Society of Actuaries. In this section, which includes all of New York State north of latitude 42 degrees N except for five western counties, industries over the entire area have cooperated in the program by making available cash awards that have been given in the names of those industries to top-ranking winners.

Answers have been recorded on IBM sheets. An especially designed IBM sheet was used in the 1959 contest. All answer sheets have been returned to contest headquarters at the State University of New York College of Education at Albany where the scoring and the checking of the scoring has been performed by machine under operation by qualified student help.

Scholarships have been made available by those colleges and universities in the state that have cooperated in the program. In 1959 twenty colleges and universities made twenty-one scholarships available, and three institutions let it be known that they would give careful consideration to winners identified by this contest. At the time of the writing of this paper, it is known that seven of the cooperating colleges have awarded or offered fourteen scholarships to top-ranking winners and that two colleges from outside the state have awarded three sizable scholarships to those winners.

Cash awards have been made available by industries that have cooperated in the program. In 1959 sixteen one-hundred dollar cash awards were provided, five by the IBM Corporation, four by the New York Telephone Company, two by the National Commercial Bank and Trust Company, Albany, and one each by Allegheny Ludlum Steel Corporation, Bristol Laboratories, Inc., Gannett Newspapers, State Bank of Albany, and Sterling-Winthrop Research Institute.

The one-stage program on an IBM basis in the Upstate Section has been successful during these two years from the standpoint of both finance and interest. In 1958, 3,300 students from 211 schools participated; in 1959, 5,500 students from 328 schools participated. The number of cooperating colleges increased from 11 to 23 during that time; the number of cooperating industries from 5 to 8. During the past year each participating school was charged $5.00 plus 10 cents for each test.

The chairman of the contest committee in the Upstate Section has been conducting a continuing study of the students who ranked in the top one per cent of those participating in the 1958 contest. The twenty-three students who were seniors are now attending college. The average grade of these students, over all subjects, for the first semester, was B+, in spite of the fact that the highest score made by one of them in the 1958 contest was considerably below the limiting score for Honor Roll status. Of the ten students who were not seniors, nine were juniors and one was a freshman. The freshman was aged thirteen at the time of the contest and had already passed the chemistry and physics courses offered by his school. He again ranked in the top one per cent in the 1959 contest. In fact, six of the nine who were not seniors who again competed, ranked in the top one per cent in the 1959 contest. The most frequent choice of career expressed by the 1958 group is teaching, with engineering and research next in line; the most frequent choice of career expressed by those who ranked in the top one per cent in the 1959 contest from whom there has been a response, is engineering, with teaching next in line.

The coming of the new IBM machine 9902, an Electronic Test Scoring Punch Machine, due shortly to be on the market, will make possible the responding to 150 multiple choice, five choice, questions on the regular IBM card, the scanning of the card, the performing of the scoring, the punching of holes, and the recording of grades. This machine can make possible more efficient administering and grading of the M.A.A. tests.

MU ALPHA THETA

JOSEPHINE ANDREE, Norman, Oklahoma

With the help of many mathematicians, a gift from Pi Mu Epsilon, and sponsorship by the Mathematical Association of America, Mu Alpha Theta, an honorary organization for high school students of mathematics was launched in 1957. Student members receive a membership certificate and an announcement is sent to the local newspaper. Many club sponsors tell how hard their students work to achieve membership. Clubs are expected to meet every month and are helped by suggestions for meeting topics and references.

Mu Alpha Theta publishes *The Mathematical Log*, issued thrice a year and edited by Josephine Andree. It features articles on college level mathematics written for a high school background. C. D. Olds of San Jose State College is the mathematics editor. Every issue contains a set of challenging problems provided by V. E. Hoggatt also of San Jose State College. In the *Log* are also selections from letters from various chapters telling of their activities and news items concerning such things as the Mu Alpha Theta traveling library.

The traveling library is a service offered free to chapters anywhere. About eighty from Florida to Alaska are currently taking advantage of it. A chapter needs only to send a request to the librarian, Mr. George Hunt, to receive half a dozen books selected from outstanding new publications on such topics as the calculus, set theory, geometry, integrated college mathematics, puzzles, statistics, number theory, modern algebra. After six weeks each chapter mails its library on to another school. These books give a new outlook on mathematics to many students.

Mu Alpha Theta sponsors a series of regional meetings which are often addressed by members of the Association. Students have come as far as 300 miles to attend regional meetings in Oklahoma, Northern California, Southern California, Pennsylvania, Missouri, Maryland, and Illinois.

THE MASTER OF ARTS IN THE TEACHING OF MATHEMATICS

LYLE E. MEHLENBACHER, University of Detroit

The University of Detroit has announced a program leading to the degree of Master of Arts in the Teaching of Mathematics. This degree program has the objective of preparing better teachers of mathematics in both the high school and the elementary school. The program is available to teachers who do not have the mathematical background necessary for starting the traditional Master of Arts degree in mathematics but who have the capacity to start with a concentrated course in modern concepts of calculus and to continue through courses which will provide a broad and substantial foundation for teaching in the modern curriculum in mathematics. The requirements for admission to the program include a bachelor's degree from an accredited college with a minimum of a C+

average, a teacher's certificate, at least one year of college mathematics, and an interest in teaching mathematics.

Since the applicant for this degree is likely not to have studied calculus, or to have studied the "cook book" brand of calculus many years ago, he takes a sequence of two courses entitled "Modern Concepts of Calculus." These courses cover the fundamentals of analytical geometry and calculus as a unified system from a modern rigorous point of view.

Another special course required for the degree is either "Problems in Teaching High School Mathematics" or "Problems in Teaching Elementary School Mathematics," depending upon the particular field of interest of the applicant. These are subject matter courses covering the basic mathematical concepts needed in teaching the modern courses in mathematics.

The remaining courses in mathematics include: Introduction to Modern Algebra (2 semesters), Foundations, Fundamental Concepts of Geometry, and Theory of Functions of a Complex Variable In order to meet the needs of individual programs, alternate courses may be selected from: Theory of Numbers, Theory of Probability and Statistics, Advanced Calculus, Differential Equations, Advanced Modern Algebra, Theory of Functions of a Real Variable, Point Set Topology. The completion of the program requires twenty-four semester hours of mathematics and six hours in a cognate field selected from one of the four fields of chemistry, education, philosophy or physics.

This program was first announced in the Spring of 1959. By September 1959 we had processed and accepted 69 applicants. Many of the applicants are participants in the NSF Institutes, but several are "on their own." Courses in the program are available in late afternoon, after school hours, and in the summer.

SECOND THOUGHTS ON MODERNIZING THE CURRICULUM*

D. M. MERRIELL, University of California, Santa Barbara

There is a well-known principle in history that the heresy of yesterday becomes the orthodoxy of today. The turn of events in revising the mathematics curriculum is a good illustration. To those at the College of the University of Chicago a decade ago, the mathematics now referred to as "new" had a freshness made more piquant by the awareness that it was pedagogically different and controversial. Rudiments of symbolic logic, sets, relations and functions as sets of ordered pairs, mathematical structure, the definition of a group: all of these were capable of providing a challenge to put across to secondary students and of arousing some intellectual excitement which one could communicate. Today the same material, though not yet by any stretch of the imagination entrenched in either schools or colleges, threatens to become a new orthodoxy, destined to be

* The editors of this Department believe that there is a place in the Department for expressions of individual opinion on critical issues in mathematics education. Publication of such expressions are not to be interpreted as indication of agreement on the part of the editors of the Department.

shaken down to a teachable form as dry and dusty as the present day material which it would displace.

One wonders whether the major effort which is taking place is not misdirected. First, there is the assumption that the old curriculum is obsolete. This point is made large in order to sell the new article. In a consumer society in which built-in obsolescence is accepted as normal and the new is valued as the best, this has a great appeal. Second, there is the assumption that the obsoleteness of the present curriculum accounts for our lack of mathematical power and that the new article is going to set things right. How is one to account for the fact that European countries continue to turn out more competent mathematicians per capita under an old-fashioned curriculum? And what more conservative and traditional curriculum can one discover than that of the schools in the Soviet Union? Surely the differences are accounted for not by what is taught but by the thoroughness of the teaching and the possibility of getting through a much larger quantity of preparatory material. And here, the possibility of improvement depends on some very complex social questions. It is well-known that it is very difficult to change an entrenched curriculum. But the possibility of changing social attitudes and values seems so much more remote that one suspects that the curriculum question has been singled out in order to give the illusion of progress rather than attempting to strike at the real heart of the educational problem.

One of the noticeable features about the new programs is the emphasis on language and a new point of view. Great importance is attached to the notion of structure. At the same time, large portions of Euclidean plane and solid geometry are considered expendable. Perhaps one point is being lost in the shuffle. The high school course in plane geometry, even after it had been watered down to suit the needs of mass education, was the one course which gave the students a feeling for mathematical structure. Moreover, it contained in its palmier days some genuinely interesting theorems; and in the case of those propositions horribly referred to as "originals," it gave the better students opportunities to exercise ingenuity and to develop inventive power. The very least one should ask of a new program is that it should do as well as the old in these respects. But what interesting theorems does one find in modern secondary mathematics? The University of Chicago course was conceived as a secondary level course and was more extreme than most of its successors have been. Yet even it contained no significant theorems from either set theory, group theory, or symbolic logic although the course was organized about these subjects. If it is true that more preparation is necessary than can be given to high school students in order to reach significant theorems, the suitability of such subjects should be questioned. The fact is that there are parts of mathematics, Euclidean geometry being one such, in which one can present interesting theorems at the secondary level.

The issue being raised here is not whether the new mathematics is of value. All of it is material which a mathematics student should acquire at some point

in his training. The question is whether the introduction of this material at the secondary level does in fact fulfill the needs of secondary education. Does it arouse interest? Does it stimulate students to pursue mathematics further? Does it build an appreciation for ingenuity, creativity, and elegance? College teachers are well aware of the limited interest on the part of young students in axiomatics and rigor. The fact that axiomatization is an end product, a result of an urge for orderliness and organization rather than of the disorderly creative impulse, should have some bearing on the stage at which it is suitable to introduce it. Moreover, if the result of a course based on foundation material is to convey the impression that structure and axiomatics are the principal contributions of mathematics, this is almost as one-sided and biased a presentation as the impression now conveyed by an over-emphasis on formal manipulation. It is also likely to be as sterile in developing the inventiveness which is the essence of good mathematical training.

One of the principal objectives of groups now at work on the school mathematics curriculum is to increase the number of students who elect mathematics. Material is to be selected so as to attract students capable of further study and also to communicate some reasonably approximate notion of what a mathematician does. It is hard to see how the choice of foundation material satisfies these criteria. Since one of the primary occupations of a mathematician is to solve problems, it would seem more natural to build an attractive course around a small number of interesting problems. No doubt some attention would have to be given to language and formal skills but the problems would be selected so as to emphasize ideas and methods. Distasteful though it might be to a mathematics teacher, no attempt would be made to achieve mastery of the skills by the students.

Could such a course in "mathematics appreciation" really be a success? Selecting appropriate problems would be a difficult task but there is a certain amount of literature already available. In other fields such as art, music, and literature, complex technical problems are involved but it is still considered possible to select material which gives insight into the problems and methods of the artist and which can be grasped in some sense by secondary students. Perhaps the greatest obstacle in giving a successful course of this type is the teacher. There is no really satisfactory way to measure appreciation, and measurement is very strongly built into mathematics teachers.

Nevertheless, now that the machinery for curriculum revision has been set up and the times are propitious, one should be flexible enough to investigate alternatives. It would be foolhardy to believe that there is any single best curriculum. One should also be candid enough to express doubts, match up proposals against objectives, and continually criticize and evaluate what is being done. And finally, it should be remembered that curriculum is only one part and not even the most important part of the problem of education.

Traveling Science Demonstration Lecture Program*

The Traveling Science Demonstration Lecture Program has been sponsored for three successful years by the Oak Ridge Institute of Nuclear Studies in cooperation with the National Science Foundation, Atomic Energy Commission and others.

During 1959–60, this program is being expanded by the National Science Foundation to include four separate program centers which will serve particular regions of the United States. In addition to Oklahoma State University, other centers will be located at the University of Oregon, Michigan State University and the Oak Ridge Institute of Nuclear Studies. The Oklahoma State University center will serve the region composed of the states of Oklahoma, Texas, New Mexico, Colorado, Kansas, Missouri, Arkansas and Louisiana.

How the program operates. A specially trained and equipped traveling science teacher will be scheduled in a school system for approximately one week during the 1959–60 school year. Prior to the scheduled appearance, the traveling teacher will pre-visit the school and in conference with local teachers work out the program for that school and community.

A typical program for a school would include three or four scheduled appearances per day, Monday through Friday, including (1) a general high school assembly, (2) a teachers' meeting, (3) a science club program, (4) one or two civic group appearances, and (5) ten to fifteen lecture-demonstrations on science and mathematics to particular classes or special interest groups within the school.

The program for a given school and community is very flexible and can be designed to require a minimum number of disruptions or irregularities to the regular school schedule.

Oklahoma State University is now sponsoring a traveling science teacher in Oklahoma in cooperation with the Oak Ridge program and with financial support from the Frontiers of Science Foundation of Oklahoma.

Carnegie Institute of Technology—Mathematics Education Committee

Four years ago a committee was formed at Carnegie Institute of Technology to study the coordination of high school and university curricula in mathematics. A series of luncheon meetings was held with high school administrators and teachers, representing schools from which Carnegie draws a majority of its undergraduate student body, to survey mutual problems. The committee members have fulfilled many speaking engagements before student groups and teacher organizations and institutes, conducted extended series of seminars with several school districts, and established an annual competitive scholarship examination in mathematics for high school seniors.

Under the chairmanships of Professor John H. Neelley and Professor Allen F. Strehler the committee has established mutually beneficial relations with many schools and hopes to broaden the field of its activities in the future.

Proficiency Examinations for Teachers

The 1959 Annual Meeting of the Council on Cooperation in Teacher Education (CCTE) of the American Council on Education, held in Washington in October, gave further consideration to possibilities for proficiency examinations in teacher education. The interest of CCTE in proficiency examinations was first formally expressed in a resolution approved at their 1957 Annual Meeting. During 1958 a project providing for the use of proficiency examinations as one means of certification was established in West Virginia, and planning for the development of such a program in Wisconsin was started. Reports from these states were given at a special CCTE conference held in April,

* Quoted from *Traveling Science Teacher Program*, Oklahoma State University.

1959. For more detailed information on this subject write to Howard R. Boozer, Secretary-Treasurer, CCTE, 1785 Massachusetts Avenue, N.W., Washington 6, D.C.

MAA Visiting Lecturer Program to Secondary Schools 1959–60

The Mathematical Association of America is sponsoring for a second year a regional Visiting Lecturer Program to Secondary Schools. The program is made possible by a grant to the Association from the National Science Foundation. While the grant is somewhat larger this year, the program is being conducted again on a regional basis with the plan that schools served during the academic year 1958–59 will not be served during 1959–60. Nine regions have been designated and scheduling of the visits of the lecturers in the various regions is in charge of regional representatives. The regions for 1959–60 and the respective regional representatives are as follows:

Alabama-Arkansas	Houston T. Karnes, Louisiana State University, Baton Rouge
California	Roy Dubisch, University of California, Berkeley
Connecticut-Maine-Rhode Island-Vermont	W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts
Delaware-District of Columbia-Maryland-West Virginia	John A. Brown, University of Delaware, Newark
Idaho-Oregon-Washington	Harvey M. Gelder, Western Washington College of Education, Bellingham
Illinois	Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Indiana
Kansas-Missouri-Nebraska	Russell N. Bradt, University of Kansas, Lawrence
Montana-North Dakota-South Dakota	Adrien L. Hess, Montana State College, Bozeman
North Carolina-South Carolina	Thomas D. Reynolds, Duke University, Durham, North Carolina

In general lecturers will make trips of short duration, and, in many cases, particularly in California and the Northwest, at short distances from the colleges at which the mathematicians are located. The New England region will be served by Professor Israel Rose of the University of Massachusetts, who is on a leave-of-absence during the second semester; and the North Carolina-South Carolina region will be served similarly on a leave-of-absence basis by Professor W. Norman Smith of the University of Wyoming. Other mathematicians who have agreed to give part-time to the program are as follows:

Idaho-Oregon-Washington: Bradford H. Arnold, Oregon State College; Wilfred E. Barnes, Washington State University; Z. W. Birnbaum, University of Washington; K. A. Bush, University of Idaho; Donald W. Bushaw, Washington State College; J. Richard Byrne, Portland State College; Theodore S. Chihara, Seattle University; Robert E. Gaskell, Oregon State College; George R. Johnson, Idaho State College; Anthony E. LaBarre, Jr., University of Idaho; Calvin T. Long, Washington State University; Richard D. Mayer, Idaho State College; Albert Nijenhuis, University of Washington; Thurman S. Peterson, Portland State College; Robert W. Rempfer, Portland State College; Sheldon T. Rio, Western Washington College; Joseph B. Roberts, Reed College; Hans Sagan, University of Idaho.

California: T. M. Apostol, California Institute of Technology; William Bade, University of California; Richard A. Dean, California Institute of Technology; Harley Flanders, University of California; Bernard Friedman, University of California; Charles J. A. Halberg, Jr., University of California at Riverside; Vincent C. Harris, San Diego State College; Leon Henkin, University of California; Calvin V. Holmes, San Diego State College; Paul J. Kelly, University of California at Santa Barbara; Donald H. Potts, Long Beach State College; Murray Protter, University of Cali-

fornia; Raymond M. Redheffer, University of California at Los Angeles; Judson Sanderson, Jr., University of Redlands; Elmer B. Tolsted, Pomona College.

Other regions: Richard D. Anderson, Louisiana State University; William E. Briggs, University of Colorado; Burton H. Colvin, Boeing Scientific Laboratories; J. A. Cooley, University of Tennessee; C. W. Curtis, University of Wisconsin; Mary Dolciani, Hunter College; J. C. Eaves, University of Kentucky; Wade Ellis, Oberlin College; W. T. Guy, University of Texas; Adrien Hess, Montana State College; Gerald B. Huff, University of Georgia; B. W. Jones, University of Colorado; H. S. Kaltenborn, Memphis State College; W. G. Lister, New York State College on Long Island; Robert McKelvey, University of Colorado; Herman Meyer, University of Chicago; C. O. Oakley, Haverford College; W. V. Parker, Alabama Polytechnic Institute; D. P. Richardson, University of Arkansas; Lawrence A. Ringenberg, Eastern Illinois University; P. C. Rosenbloom, University of Minnesota; A. H. Wallace, Indiana University; Eldon Whitesitt, Montana State College; Fred Young, Montana State University; J. H. Zant, Oklahoma State University.

The lecturers will be prepared not only to give lectures on mathematical topics, but to confer with students and faculty singly and in groups. They will be glad to advise students on future opportunities for study and employment, to discuss teaching problems and curriculum with members of the staff, and to throw what light they can on practices at comparable institutions. In short, the lecturers will cooperate with the schools in all possible ways toward the furtherance of the aims of the program. It is expected that, in general, the visiting lecturers will spend one day in a school. In the larger cities the lecturers may plan to spend several days visiting a number of schools both in the city and in neighboring communities. On occasion it may be desirable for teachers of a number of schools to meet jointly with the lecturer, and it may be possible for the students, or a select group of students, of several schools to meet together at one school to hear the lecturer. The experience of last year indicates that in general lecturers should not address assemblies of all high school students.

Application forms for high schools will be sent upon request addressed to the regional representative who is arranging the itinerary for the region in which the school is located.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1396. *Proposed by* (1) *Dustan Everman, Arizona State College,* (2) *A. E. Danese, Union College,* (3) *K. Venkannayah, Government College, Mercara, India*

For the Fibonacci sequence $\{f_n\}$, where $f_1=f_2=1$, $f_n=f_{n-1}+f_{n-2}$, $n\geqq 3$, show that

(1) every fifth number in the sequence is divisible by 5.

(2) $f_{n+h}f_{n+k}-f_nf_{n+h+k}=(-1)^nf_hf_k$. (All the identities in Ganis, *Notes on the Fibonacci sequence,* this MONTHLY, vol. 66, p. 129, are special cases of this.)

$$(3) \qquad H(f_1, \cdots, f_{2n-1}) = \begin{vmatrix} f_1 & f_2 & \cdots f_n \\ f_2 & f_3 & \cdots f_{n+1} \\ \cdot & \cdot & \cdots \cdot \\ f_n & f_{n+1} & \cdots f_{2n-1} \end{vmatrix} = 0, \qquad n \geqq 3.$$

E 1397. *Proposed by Leon Bankoff, Los Angeles, California*

Show that $\sum AI \leqq \sum AH$, where I is the incenter and H the orthocenter of a triangle ABC.

E 1398. *Proposed by A. N. Aheart, West Virginia State College*

If A, B, C are the angles of a triangle, show that $\cos A + \cos B + \cos C < 2$.

E 1399. *Proposed by H. S. Shapiro, New York University*

Counterfeit coins all weigh 9 grams and genuine coins all weigh 10 grams. One is given five coins of unknown composition, and an accurate scale (not a balance). How many weighings are needed to isolate the counterfeit coins?

E 1400. *Proposed by S. D. Pratico, Iona College, New Rochelle, N. Y.*

There are 6 red balls and 8 green balls in a bag. Five balls are drawn at random from the bag and placed in a red box; the remaining 9 balls are put in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number?

SOLUTIONS

Two Related Triangles

E 1366 [1959, 423]. *Proposed by V. E. Hoggatt, Jr., San Jose State College*

Show that if a, b, c form a triangle, then $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ form a triangle.

I. *Solution by Chih-yi Wang, University of Minnesota.* Since

$$|\sqrt{b} - \sqrt{c}|\,(\sqrt{b} + \sqrt{c}) < (\sqrt{a})^2 < b + c < (\sqrt{b} + \sqrt{c})^2,$$

it follows that $|\sqrt{b} - \sqrt{c}| < \sqrt{a} < \sqrt{b} + \sqrt{c}$.

II. *Solution by the proposer.* A necessary and sufficient condition [see the Seventh Polish Mathematical Olympiad, Prob. 9, First Round (*The Mathematics Teacher*, vol. LI, no. 8, Dec. 1958, p. 587)] that a, b, c form a triangle is

$$a^2b^2 + b^2c^2 + c^2a^2 > (a^4 + b^4 + c^4)/2.$$

We must therefore show that $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ satisfy

$$ab + bc + ca > (a^2 + b^2 + c^2)/2.$$

This readily follows since $b + c > a$, $c + a > b$, $a + b > c$, whence $ab + ac > a^2$, $bc + ba > b^2$, $ca + cb > c^2$.

III. *Solution by R. T. Hood, Ohio University.* We shall show that if a, b, c form a degenerate or nondegenerate triangle, then $\sqrt[n]{a}$, $\sqrt[n]{b}$, $\sqrt[n]{c}$ (n a positive integer) form a nondegenerate triangle. Assume $\sqrt[n]{a}+\sqrt[n]{b}\leqq\sqrt[n]{c}$. Raising to the nth power and omitting $n-1$ terms of the expansion, we have $a+b<c$. Hence, if $a+b\geqq c$, then $\sqrt[n]{a}+\sqrt[n]{b}>\sqrt[n]{c}$. A cyclic permutation of the letters gives corresponding conclusions when $b+c\geqq a$ and when $c+a\geqq b$. The revised statement of the problem follows immediately.

IV. *Solution by J. L. Brown, Jr., Pennsylvania State University Ordnance Research Laboratory.* We prove the more general assertion that if a, b, c form a triangle, then $f(a)$, $f(b)$, $f(c)$ form a triangle, where $f(x)$ is any nonnegative, nondecreasing, subadditive function on the domain $x>0$.

From $a\leqq b+c$, we have $f(a)\leqq f(b+c)\leqq f(b)+f(c)$. Cyclic permutation of a, b, c then yields the additional inequalities $f(b)\leqq f(c)+f(a)$ and $f(c)\leqq f(a)+f(b)$; therefore $f(a), f(b), f(c)$ form a triangle. The solution to the stated problem then follows from the observation that $f(x)=x^{1/2}$ has the properties required for the above proof.

V. *Solution by C. H. Cunkle, Cornell Aeronautical Laboratory, Inc., Buffalo, N. Y.* This property is a special case of Ex. 5, p. 26, in L. M. Blumenthal, *Theory and Applications of Distance Geometry*:

THEOREM T. *Let M be a metric space and $\phi(x)$ a monotone increasing concave (from below) function that vanishes for $x=0$. If distance pq in M be redefined as $\phi(pq)$, then the new space is metric.*

Proof. Let p, q, r be points of M so that $pq+qr\geqq pr$. Since ϕ is monotone, $\phi(pq+qr)\geqq\phi(pr)$. Since ϕ is concave,

$$\phi(pq)\geqq\{pq/(pq+qr)\}\phi(pq+qr),\qquad \phi(qr)\geqq\{qr/(pq+qr)\}\phi(pq+qr).$$

Adding we find $\phi(pq)+\phi(qr)\geqq\phi(pq+qr)\geqq\phi(pr)$. The other properties defining a metric are obvious. The hypotheses are satisfied for $\phi(x)=x^t$ if $0<t\leqq 1$.

Also solved by A. N. Aheart, Mansur Arbabi, Leon Bankoff, S. D. Beck, D. A. Breault, Mark Bridger, R. F. Brown and Joel Levy (jointly), J. H. Case, Leon Cohen, Frank Dean, Monte Dernham, C. W. Dodge, J. W. Ellis, H. B. Emerson, Irma Esrig, Hazel E. Evans, J. W. Fitzpatrick, Donald Garlock, George Glauberman, Sidney Glusman, Michael Goldberg, A. J. Goldman, L. D. Goldstone, Bernard Greenspan, Cornelius Groenewoud, Joseph Hammer, J. H. Hodges, Raymond Huck, J. Jordan, A. F. Kaupe, Jr., J. D. E. Konhauser, P. G. Kumpel, Jr., Morton Kuppreman, Leo Lapidus, D. C. B. Marsh, R. A. Melter, Morris Morduchow, L. H. Mitchell, D. L. Muench, A. A. Mullin, C. S. Ogilvy, Margaret Olmsted, D. J. Persico, C. F. Pinzka, Joseph Prieto and Max Rosenberg (jointly), B. E. Rhoades, L. A. Ringenberg, J. T. Rosenbaum, John Saccoman, C. M. Sandwick, Sr., Jack Silver, Benjamin Sims, W. B. Stovall, Jr., Anna Thompson and R. G. Thompson (jointly), Julius Vogel, T. C. Wales, Alan Wayne, R. C. Weger, and F. L. Wolf. Late solutions by R. J. Carmier, Olga D. Mitrinovic, Helge Pahus, L. J. Schneider, and E. L. Spitznagel, Jr.

An analogous problem is to show that if a, b, c, d, e, f are the edges of a tetrahedron, then $a^{1/n}$, $b^{1/n}$, $c^{1/n}$, $d^{1/n}$, $e^{1/n}$, $f^{1/n}$ ($n=1, 2, \cdots$) also form a tetrahedron.

Lapidus pointed out that essentially Theorem T quoted in Solution V is also found in G. Birkhoff, Metric foundations of Geometry, I, *Trans. Amer. Math. Soc.*, vol. 55, 1944, p. 46. An interesting question is: Are there other functions than those described in Theorem T that preserve the

metricity of the space? The answer is in the affirmative. As an example, Lapidus considered the space of points on the surface of a sphere, where the Euclidean (chord) distance between any two of the points is replaced by the shorter great circle arc through the points. If diametral points are identified, the resulting space is metric, but the transforming function is not one of those described in Theorem T.

Concerning the Alternating Harmonic Series

E 1367 [1959, 423]. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Prove that the numerator of the sum $1-1/2+1/3-\cdots$ to $[2p/3]$ terms is divisible by p when p is a prime greater than 3.

Solution by N. J. Fine, Institute for Advanced Study. The required sum is equal to

$$\begin{aligned}\sum_{1\le k<2p/3} 1/k - 2\sum_{1\le 2k<2p/3} 1/2k &= \sum_{1\le k<2p/3} 1/k - \sum_{1\le k<p/3} 1/k \\ &= \sum_{p/3<k<2p/3} 1/k = \sum_{p/3<k<p/2} 1/k + \sum_{p/3<k<p/2} 1/(p-k) \\ &= \sum_{p/3<k<p/2} \{1/k + 1/(p-k)\} = p\sum_{p/3<k<p/2} 1/k(p-k).\end{aligned}$$

Also solved by A. E. Danese, Joseph Hammer, D. C. B. Marsh, K. Venkannayah, and the proposer. Late solution by M. Sugunamma.

A Theorem about Quartics

E 1368 [1959, 423]. *Proposed by M. S. Klamkin, AVCO Research and Development*

Show that if all roots of $ax^4-bx^3+cx^2-x+1=0$ are positive, then $c\ge 80a+b$.

Solution by A. J. Goldman, National Bureau of Standards, Washington, D. C. The proof will show that 80 is the best possible constant.

(i) If $a=b=c=0$, the result is true.

(ii) If $a=b=0$, $c\neq 0$, then the roots r_1, r_2 of $cx^2-x+1=0$ are positive, so $1/c=r_1r_2>0$, hence $c>0$ and the result is true.

(iii) If $a=0$, $b\neq 0$, then the roots of $bx^3-cx^2+x-1=0$ are positive. Since $r_1r_2r_3=r_1r_2+r_2r_3+r_3r_1=1/b$, we have $1/r_1+1/r_2+1/r_3=1$, which (since all $r_i>0$) shows that all $r_i\ge 1$; hence $c/b=\sum r_i\ge 3$, so $c\ge 3b\ge b$ and the relation holds.

(iv) Suppose $a\neq 0$. Then

$$(c-b)/a = 1/2\sum_{i\neq j} r_ir_j - \sum_k r_k = \{(\sum r_i)^2 - \sum r_i^2\}/2 - \sum r_i.$$

Call this $f(r_1, r_2, r_3)$. We can use Lagrangian multipliers to determine the minimum of $f(r_1, r_2, r_3)$ in the open subset $r_1>0$, $r_2>0$, $r_3>0$, $r_4>0$ of the constraint set

$$\prod_{i=1}^{4} r_i = \sum_{j=1}^{4}\prod_{i\neq j} r_i \qquad (=1/a),$$

i.e., $\sum 1/r_i = 1$, since on this set the function is bounded below. The minimum occurs when all $r_i = 4$, $f_{\min} = 80$; that is, $(c-b)/a \geqq 80$ as desired.

Also solved by D. A. Breault, R. F. Brown and Joel Levy (jointly), Michael Goldberg, D. C. B. Marsh, Margaret Olmsted, C. M. Sandwick, Sr., Chih-yi Wang, and the proposer.

The nth Derivative of $y=(1+x^{1/2})^{2n+2}$ at $x=1$

E 1369 [1959, 423]. *Proposed by R. E. Shafer, University of California*

If $y=(1+x^{1/2})^{2n+2}$, show that $y^{(n)}=4(n+1)(n+1)!$ at $x=1$.

Solution by Chih-yi Wang, University of Minnesota. Define $z=(1-x^{1/2})^{2n+2}$. It is evident that $z^{(n)}]_{x=1}=0$. Hence

$$y^{(n)}]_{x=1} = y^{(n)} + z^{(n)}]_{x=1} = D^n\left(2\sum_{k=0}^{n+1}\binom{2n+2}{2k}x^k\right)\Bigg]_{x=1}$$
$$= 2(n+1)! + 2(n+1)(2n+1)n! = 4(n+1)(n+1)!.$$

Also solved by A. J. Goldman, J. Jordan, D. C. B. Marsh, Jack Silver, and the proposer.

Two Improper Integrals

E 1370 [1959, 424]. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

(a) Evaluate $\int_0^\infty (1-e^{-x})^n x^{-n}dx$ for integral $n>1$.

(b) Evaluate $\int_0^\infty (1-e^{-x})^2 x^{-2}\cos^2 kx dx$ for arbitrary $k>0$.

Solution by David Zeitlin, Remington Rand UNIVAC, St. Paul, Minn. All page references below are to *An Introduction to the Theory of Infinite Series*, 2nd ed. revised, by T. J. Bromwich, Macmillan and Co., 1926.

(a) On p. 473 occurs the result:

$$\int_0^\infty \left(\sum_{r=0}^n A_r e^{-a_r x}\right)x^{-n}dx = \left[-1/(n-1)!\right]\sum_{r=0}^n A_r(-a_r)^{n-1}\ln a_r,$$

where $\sum_{r=0}^n A_r a_r^k = 0$, $k=0, 1, \cdots, n-1$. Since

$$(1-e^{-x})^n = \sum_{r=0}^n (-1)^r\binom{n}{r}e^{-rx},$$

we have $A_r=(-1)^r\binom{n}{r}$, $a_r=r$, and the conditions are satisfied. Thus

$$\int_0^\infty (1-e^{-x})^n x^{-n}dx = \left[(-1)^n/(n-1)!\right]\sum_{r=2}^n (-1)^n\binom{n}{r}r^{n-1}\ln r.$$

(b) Ex. 4 on p. 488 gives the result:

$$\int_0^\infty \sum_{k=1}^n a^{-a_k x}\{A_k(1+a_k x)+B_k x\}x^{-2}dx = -\sum_{k=1}^n B_k\ln a_k - \sum_{k=1}^n A_k a_k,$$

where $\sum_{k=1}^{n} A_k = 0$, $\sum_{k=1}^{n} B_k = 0$, and the real parts of a_k are positive or zero. Since $\cos^2 kx = 1/2 + (1/2)\cos(2kx)$, the desired integral, I, becomes

$$I = \ln 2 + \operatorname{Re}\left[\int_0^\infty (1 - e^{-x})^2 e^{2ikx} x^{-2} dx\right] \Big/ 2.$$

We now apply the above result with $n = 3$, $a_1 = -2ki$, $a_2 = 1 - 2ki$, $a_3 = 2 - 2ki$, $A_1 = A_3 = 1$, $A_2 = -2$, $B_1 = 2ki$, $B_2 = 2 - 4ki$, $B_3 = -2 + 2ki$. Thus

$$I = 2 \ln 2 + k \arctan(k - 1/4k) - \arctan k - \ln[(1 + 4k^2)/(1 + k^2)]/2.$$

Also solved by George Glauberman, John Rainwater, W. F. Trench, Chih-yi Wang, and the proposer.

Alternative forms to part (b) are obtained by replacing the second term in the above answer by $-k \operatorname{arccot}(4k^3 + 3k)$ or by $2k \arctan 2k - k \arctan k - k\pi/2$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4881. *Proposed by Burton Randol, The Rice Institute, Houston, Texas*

Is there a non-trivial entire function f, such that $\int_L f(z)dz$ vanishes for all straight lines L infinite in both directions? (This is suggested by Newman's problem no. 4721 [1957, 750].)

4882. *Proposed by I. S. Gâl, Cornell University*

An (ω, ∞)-filter in a set X is a filter $\mathfrak{F}$ in X which has the countable intersection property: $F_n \in \mathfrak{F}$ implies $\bigcap F_n \in \mathfrak{F}$ $(n = 1, 2, \cdots)$. Prove that, if X is a metric space and if every (ω, ∞)-filter in X has a nonvoid adherence then the same holds for every open subspace of X.

4883. *Proposed by W. A. Schneider, Milwaukee, Wisconsin*

If

$$F = 1 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n + \cfrac{1}{a_n + \cdots \cfrac{1}{a_2 + \cfrac{1}{a_1}}}}}},$$

and P_n/Q_n is the nth convergent of F, prove $\cot^{-1} Q_{2n} = \cot^{-1} P_{2n} + \cot^{-1} P_{2n+1}$.

4884. *Proposed by Hans Schneider, The Queen's University of Belfast*

It is known that a ring whose additive group is the group of rationals (mod 1) has trivial multiplication. (C. Chevalley, *Fundamental Concepts of Algebra*, Example on Chapter 3.) Find a nontrivial ring whose additive group is the group of reals (mod 1).

4885. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research, Wilmington, Mass.*

Determine the unique solution of the integral equation

$$F(x_1, x_2, \cdots, x_n) = 1 + \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} F(y_1, y_2, \cdots y_n)dy_1dy_2\cdots dy_n.$$

(The uniqueness when $n=2$ was one of the problems in the 1958 Putnam competition.)

4886. *Proposed by Isaac Namioka, Cornell University*

Let X be a Hausdorff (T_2) topological space and let Y be its dense subset. Prove that $\kappa(X) \leqq 2^{2^{\kappa(Y)}}$, where $\kappa(A)$ denotes the cardinality of the set A. Show by an example that this inequality cannot be improved even when X is compact. Is this assertion true for T_1-spaces?

SOLUTIONS

An Invalid Inequality

4603 [1954, 571; 1956, 191]. *Proposed by H. S. Shapiro, New York University*

Given $x_i \geqq 0$, $i=1, 2, \cdots, n$. Establish

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geqq\frac{n}{2}$$

equality occurring only if all denominators are equal.

Partial Solution by M. Herschorn and J. E. L. Peck, University of Alberta, Calgary. Let S_n represent the left member. Diananda has shown the inequality true for $n \leqq 6$ [*Extensions of an inequality of H. S. Shapiro*, this MONTHLY, vol. 66, 1959, 489–491]. A counter-example has already been given with $n=20$.

We show that the inequality is false for all even $n \geqq 14$. To do this we put

$$x_i = 1 + a_i\epsilon, \text{ for } i \text{ even}; \qquad x_i = a_i\epsilon,\ a_i \geqq 0 \text{ for } i \text{ odd}.$$

Upon expanding S_n in powers of ϵ we have $S_n = n/2 + A_n\epsilon^2 + O(\epsilon^3)$, where

$$A_n = \sum_{i=1}^{n/2}(a_{2i+1} + a_{2i+2} - a_{2i})(a_{2i+1} + a_{2i+2}) - a_{2i-1}(a_{2i} + a_{2i+1}).$$

For $n=14$ the vector (0, 0, 0, 8, 3, 10, 7, 8, 10, 3, 10, -2, 5, -2) will make $A_{14} = -3$, so that for sufficiently small positive ϵ we will have $S_n < n/2 = 7$. The

vector also has three consecutive zeros. Now for any permissible vector $(a_1, \cdots, a_n)$ having three consecutive zeros we may find another in dimension $n+2$ having three consecutive zeros and for which $A_{n+2}=A_n$, thus completing an induction on the even integers starting at 14. To see this it is easiest to suppose, as we may from cyclic symmetry, that the last two coordinates of $(a_1, \cdots, a_n)$ are zero. Then in dimension $n+2$ we take the same first n coordinates and attach two more zeros. A simple calculation shows that $A_{n+2}=A_n$.

Considerations of continuity show that if there is a counter-example where some of the coordinates $a_1, \cdots, a_n$ are zero, then there is also a counter-example with coordinates not zero and thus all $x_i>0$. The cases $n=8$, 10, 12 are still unsettled, as is the case of odd $n>6$.

The above counter-example was found on the McGill University computer (IBM 650). Our thanks go to Richard V. Andree for suggesting the problem as a programming exercise.

Continuous Increasing Function

4765 [1957, 746; 1958, 637; 1959, 317]. *Proposed by J. L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let $f(x)$ be a real function defined for $x\geqq 0$. If (i) f has a finite upper bound in any finite interval, and (ii) there are two positive numbers h, k such that $x''-x'\geqq h$ implies $f(x'')-f(x')\geqq k$, then, given $\alpha>1$, there is an increasing function $g(x)$ having as many continuous derivatives as we please, such that $g(x-\alpha h)<f(x)<g(x)$ for all $x\geqq \alpha h$.

Solution by the proposer. Let $a=k/4$, $b=k/2h$,

$$g_1(x) = \sup\{a + f(\xi) + b(x-\xi) : 0 \leqq \xi \leqq x\}.$$

If $x'<x''$ we have

$$\begin{aligned} g_1(x') &= \sup\{a+f(\xi)+b(x'-\xi):0\leqq\xi\leqq x'\} \\ &= \sup\{a+f(\xi)+b(x''-\xi):0\leqq\xi\leqq x'\} - b(x''-x') \\ &\leqq \sup\{a+f(\xi)+b(x''-\xi):0\leqq\xi\leqq x''\} - b(x''-x') \\ &= g_1(x'') - b(x''-x'), \end{aligned}$$

so that g_1 is an increasing function; more precisely $g_1(x'')-g_1(x')\geqq b(x''-x')$. Obviously $g_1(x)\geqq a+f(x)>f(x)$.

Suppose now $x\geqq\alpha h$ and let $\xi(0\leqq\xi\leqq x-h)$ be so chosen that

$$g_1(x-h) < 2a + f(\xi) + b(x-h-\xi);$$

and let n be the greatest integer such that $\xi+nh\leqq x(n\geqq 1)$, whence

$$f(x) \geqq nk + f(\xi) > nk + g_1(x-h) - 2a - bnh \geqq g_1(x-h).$$

For any given integer N, we set $\alpha=1+N\epsilon$ and define the iterated mean value

$$g(x) = (\epsilon h)^{-N} \int_x^{x+\epsilon h} dt_N \cdots \int_{t_3}^{t_3+\epsilon h} dt_2 \int_{t_2}^{t_2+\epsilon h} g_1(t_1)dt_1,$$

which is obviously an increasing function with $N-1$ continuous derivatives. Since $x \leqq t_N \leqq \cdots \leqq t_2 \leqq t_1 \leqq t_2+\epsilon h \leqq \cdots \leqq x+N\epsilon h$, it follows that

$$g_1(x) \leqq g(x) \leqq g_1(x + N\epsilon h),$$

whence $g(x-\alpha h) \leqq g_1(x-\alpha h+N\epsilon h) = g_1(x-h) < f(x) < g_1(x) \leqq g(x)$.

Power Series with Binomial Coefficients

4835 [1959, 238]. *Proposed by F. H. Northover, Carleton University, Ottawa, Canada*

Prove

$$\sum_{\nu=0}^{\infty} \binom{n+\nu-1}{\nu}^2 x^\nu = (1-x)^{1-2n} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 x^k.$$

I. *Solution by R. C. Read, University College of the West Indies.* It is sufficient to show that, for each ν, the coefficients of x^ν on the two sides of the proposed equation agree, *i.e.*, that

$$\binom{n+\nu-1}{\nu}^2 = \sum_{j=0}^{n-1} \binom{n-1}{j}^2 \binom{2n-2+\nu-j}{\nu-j}.$$

But this is already, with obvious change in notation, a known identity. For reference and a simple proof see Nanjundiah, *Remark on a note by P. Turán*, this MONTHLY, v. 65, 1958, p. 354.

II. *Solution by Immanuel Marx, Purdue University.* To prove the somewhat more general formula

$$\sum_{\nu=0}^{\infty} \binom{n+\nu-1}{\nu}\binom{m+\nu-1}{\nu} x^\nu = (1-x)^{1-m-n} \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{m-1}{k} x^k,$$

one rewrites the coefficients in each of the sums in generalized factorial form and verifies immediately that each is a hypergeometric series, so that the problem becomes

$${}_2F_1[n, m; 1; x] = (1-x)^{1-m-n}\, {}_2F_1[-n+1, -m+1; 1; x].$$

This last is, however, a direct application of the known transformation formula

$${}_2F_1[a, b; c; z] = (1-z)^{c-a-b}\, {}_2F_1[c-a, c-b; c; z].$$

See, for instance, Magnus and Oberhettinger, *Special Functions*, p. 8, last line.

Also solved by L. Carlitz, H. W. Gould, Fritz Herzog, H. O. Pollok, W. F. Trench, C. C. Yalavigi, Chih-yi Wang, David Zeitlin, and the proposer.

A Property of Solutions of a Pell Equation

4836 [1959, 238]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the equation $x^2-dy^2=-1$ where d is a nonsquare integer, and suppose that $x=a$ and $y=b$ is any solution in integers. Show that there are infinitely many solutions (x, y) in which x is a multiple of a and y is a multiple of b.

I. *Solution by Georgia C. Smith, Spelman College, Atlanta, Ga.* It is easily seen that if (a, b) is a solution of the given equation, then all the number pairs (x, y) determined by $(a+b\sqrt{d})^n=x+y\sqrt{d}$, where n is odd, are also solutions. Evidently x, y are multiples of a, b, respectively.

II. *Solution by J. W. Ellis, Louisiana State University in New Orleans.* If $db^2=a^2+1$, then

$$d(4a^2+1)^2b^2=(4a^2+1)^2(a^2+1)=(4a^2+3)^2a^2+1.$$

Hence, if (a, b) is a solution of the given equation, then so also is $(4a^2+3)a$, $(4a^2+1)b$ a solution. The existence of one solution implies inductively that there are infinitely many of the type desired.

Also solved by W. J. Blundon, Jean M. Calloway, L. Carlitz, G. Di Antonio, Michael Goldberg, Peter Greiner, Fritz Herzog, Free Jamison, D. C. B. Marsh, E. J. F. Primrose, R. M. Warten, and C. C. Yalavigi.

A Further Property of Solutions of a Pell Equation

4837 [1959, 238]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the solutions (x, y) of the equation $x^2-dy^2=-1$. Show that every x is relatively prime to every y.

Solution by E. J. F. Primrose, The University, Leicester, England. It is well known that if a, b is the smallest solution of $x^2-dy^2=-1$, all (positive) solutions are given by x_r, y_r, where

$$x_r+y_r\sqrt{d}=(a+b\sqrt{d})^r, \qquad r \text{ odd}.$$

Now suppose x_r and y_s have a common factor λ. Then, if n is any odd multiple of r and s, x_n and y_n have the factor λ by Problem 4836. But this is clearly impossible since $x_n^2-dy_n^2=-1$.

Also solved by L. Carlitz, Fritz Herzog, and D. C. B. Marsh. Several readers interpreted the statement of the problem to require only that every x and its related y be relatively prime. The editor offers his apology for a statement that did not more definitely exclude this trivial interpretation.

An Integral Identity

4839 [1959, 239]. *Proposed by R. G. Bushman, University of Wichita*

Let $d(n)$ denote the number of divisors of n and $[x]$ denote the greatest integer $\leqq x$. Determine the validity of the identity

$$\int_1^x \left(\frac{1}{v}\sum_{n=1}^{[v]} d(n)\right) dv = \int_1^x \left(\frac{[v]}{v}\left[\frac{x}{v}\right]\right) dv, \qquad x \geqq 1.$$

Solution by E. N. Gilbert, Bell Telephone Laboratories, Murray Hill, N. J. The sum $\sum_{n=1}^{[v]} d(n)$ may be replaced by $\sum_{D=1}^{[v]} [v/D]$ since each positive integer D divides exactly $[v/D]$ of the integers n in $1 \leqq n \leqq [v]$. The given left-hand integral is then

$$\int_1^x \sum_{D=1}^{[v]} [v/D]dv/v = \sum_{D=1}^{[x]} \int_D^x [v/D]dv/v = \sum_{D=1}^{(x)} \int_1^{x/D} [t]dt/t$$

$$= \int_1^x \sum_{D=1}^{[x/t]} [t]dt/t = \int_1^x [x/t][t]dt/t,$$

as required.

Also solved by L. Carlitz, N. J. Fine, Fritz Herzog, Robert Weinstock, and the proposer.

Editorial Note. In some of the other solutions both given integrals were reduced to one of the forms

$$\sum_{n \leqq x} d(n) \cdot \log (x/n), \qquad \sum_{rs \leqq x} \log (x/rs),$$

where, in the latter form, the sum is taken over all ordered pairs of positive integers (r, s) such that $rs \leqq x$.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Theory of Games and Linear Programming. By S. Vajda. Wiley, New York, 1957. 106 pp. $1.75.

Each of the eleven chapters dispenses a bite-sized morsel. But in toto the reader has pleasantly digested the basic notions of game theory, linear programming, and their interconnections. The appetizer outlines the theory of games. The simplex method constitutes the *pièce de résistance*, with the inverse matrix method and the dual simplex method served as alternate entrees. Three times the seasoning and spices are liberally sprinkled, as geometrical interpretations (in two-dimensional situations) add savory flavor to the chewier algebraic development. In case the knife does not cut the meat at first, a chapter on possible complications in the simplex method reveals potential remedies for sharpening the blade. Dessert comes with duality. Then as we lean back to relax, our linear programming techniques solve our games for us. Finally as a sort of after-dinner coffee, the method of leading variables brings the repast to a close. Just 100

pages of text, and they are small—less than 120 square centimeters of material to be absorbed on a page. Unfortunately some cook misjudged the palatability of subscripts or the arrangement of them in the table decorations. Nevertheless, you should enjoy tasting this little monograph.

RICHARD A. GOOD
The University of Maryland

The Structure of Arithmetic and Algebra. By May Hickey Maria. Wiley, New York, 1958. xiv+294 pp. $5.90.

The publishers bill *The Structure of Arithmetic and Algebra* as "A clear, concise treatment of the fundamental principles of modern mathematics." The statement is incorrect in nearly all phases. The book consists of two parts which we shall review separately. The first twelve chapters contain a reasonably clear and precise—but certainly not concise—axiomatic presentation of the real numbers and careful proofs of their more important properties. The first chapter muses at length about mathematical proofs but not until the second chapter does any mathematics appear. In the next eleven chapters the real numbers are defined by seventeen axioms and some of their properties are discovered; but no concern is given to existence. Every detail of every proof is tediously referred to the appropriate definition, axiom, or pre-proved theorem in the best tradition of plane Euclidean geometry. Throughout the book the reader is faced with the unfortunate choice of memorizing one hundred and thirty-five idle, useless, and nearly meaningless abbreviations (such as TND which stands for "Theorem on Negative of a Difference") or skipping the whole thing. Since a proof is vitiated if the reader must look up every reference, the complete index of abbreviations is of use only for emergency aid. The problem of an adequate reference system in an elementary proof is eternal but it may be better solved orally by the teacher, rather than by a device in the textbook. The exercises demonstrate a further difficulty of the inordinate completeness of the text. Although the author has made a valiant effort, too often the interesting problem has been solved as a part of the text material and the exercises must be unnatural, trivial, or repetitious.

New ideas are introduced leisurely with adequate simple examples to aid the student to understand the principles involved. Too often, though, this degenerates into the philosophy that if an idea is worth stating once, it is better repeated five times.

The book is "modern" in spirit only in so far as it insists on rigor in algebra (and the demand for rigor is at least two thousand years old); it deliberately ignores the power of generalization. For example, in successive sections the set of residue classes of the integers modulo a prime number and the rationals are shown to satisfy the field postulates, but the parallelism of the systems is neither pointed out nor exploited.

The latter part of the book is loosely connected melange. The first of three

chapters concerns solutions of quadratic equations. Here again lack of generalization causes trouble. Only the real numbers have been considered so the discriminant of the equation must be arbitrarily restricted to be nonnegative.

The next chapter introduces decimal notation for the real numbers and some elementary ideas of limit, and leads the student toward the idea of countability and uncountability but never quite reaches them.

The last chapter has an over-simplified and misleading short history of number notations, a lengthy discussion of the abacus, and much detail on number bases.

While most elementary algebra books have a deplorable lack of rigor and detail of proof, and at best pay lip service to the axioms of the number system, the solution does not lie with the other extreme. It is interesting to see how the real numbers can be placed on a logical basis and derived from a few axioms. The real danger is that a student, having read such a text, would be well aware of what was involved in a careful proof and hate every tedious detail of it.

DONALD A. NORTON
University of California, Davis

Some Aspects of Analysis and Probability. Vol. 4, *Surveys in Applied Mathematics.* By I. Kaplansky, M. Hall, Jr., E. Hewitt, and R. Fortet. Wiley, New York, 1958. xi+243 pp. $9.00.

Here are four distinct survey articles bound together in one volume. They differ in content, and even more in style. Marshall Hall's article contrasts most sharply with the other three: in content, because "combinatorial analysis" is not "analysis" (as commonly understood in the trade), and in style, because the article is accessible to any mathematician, whatever his specialty. At the other extreme, Fortet writes exclusively for probabilists, showing no mercy toward amateurs. The articles by Kaplansky and Hewitt (which *are* related to each other in content) aim at a middlebrow audience.

Kaplansky begins his report on *Functional Analysis* by briefly "reviewing the progress that has been made on the problems posed by Banach." Many mathematicians will be grateful for the existence of a published account of the present state of these problems. He then gives an expert summary (in 22 pages!!) of recent work on locally convex spaces and their operators, Banach algebras, and group representations. In particular, the classification of operators is illuminated by comparison with standard equivalence relations for matrices. (The reviewer would have mentioned Grothendieck's doctoral dissertation in the paragraph about approximating completely continuous operators by operators of finite rank.)

Hewitt's *Survey of Harmonic Analysis* should be read together with the article by Mackey (Bull. Amer. Math. Soc., vol. 56, 1950, pp. 385–412). The Hewitt article gives a more or less complete catalog of theorems proved up to 1956

about function spaces defined over locally compact abelian groups. But Hewitt's point of view is uncompromisingly abstract, and he therefore has a right to ignore results that depend on special properties of (say) the reals or the integers.

In reading Fortet's *Recent Advances in Probability Theory* the reviewer found himself better qualified to appreciate the "recent advances" than the classical "probability theory" upon which these advances were based. (But on the other hand, it would be a bold mathematician who could claim expert knowledge of all four fields surveyed in this volume.) Fortet presents cogent arguments for the study of *Banach-valued* random variables, and sketches appropriate generalizations of various laws of large numbers, of the Glivenko-Cantelli theorem, and of the central limit theorem. In particular, an ordinary random variable depending on a discrete or continuous parameter can be looked at as a general random variable with values in some function space over the integers or the real line. Many current research problems deal with ordinary random variables that happen to arise as (nonlinear) functionals of Banach-valued random variables. In addition to the above abstract functional-analytic results, some of which are Fortet's and are apparently announced by him for the first time in this survey, there are two chapters dealing with estimates for the degree of approximation in central-limit-like theorems.

It is not impossible that Marshall Hall's admirable *Survey of Combinatorial Analysis* will have a permanent place among introductory expositions in this difficult, intriguing, and as yet largely uncivilized field. He has succeeded in presenting an impressive number of fundamental problems and methods under his three categories of Enumeration, Choice, and Block Designs. Many mathematical details appear in full. (The reviewer noticed, however, that Latin squares cropped up at least twice before they were finally defined on page 77. Of more moment, Pólya's name does not appear in any of the three bibliographies. Surely his Hauptsatz from Section 16 of *Kombinatorische Anzahlbestimmungen*, etc., Acta Mathematica vol. 68, deserves mention.)

In conclusion we should like to ask the publisher and authors (M. Hall again excepted) a nonmathematical question. Why publish these surveys in a book? They are intended as lively descriptions of current research, not as definitive historical monuments. The sooner they appear in print the better. But internal evidence strongly suggests that in fact there has been a time lag of more than two years. (Hewitt's article had to ignore recent work by Calderon and Herz in spectral synthesis; Kaplansky went to press before the survey in Uspehi 11 (1956) by Berezin, Gelfand, *et al.* of group representations; not to mention Malliavin, who last year proved spectral synthesis impossible for L^1 of any noncompact group.) Perhaps the best answer to our question is that existing research journals are, in fact, not very much interested in expository articles.

H. MIRKIL
Dartmouth College

Introduction to Logic and Sets. (Prelim. ed.) By Robert R. Christian. Ginn, Boston, 1958. vi+70 pp. $1.00.

This text was designed as a supplement to high school and college mathematics courses, toward enhancing facility among teachers and students in the language and ideas of sets. Although too tightly coherent for intermittent reference, it is very brief and can be covered entirely in addition to regular course work, without undue strain. The student receives more and better practice than usual at thinking and working with sets, and the book is to be recommended as quite valuable and workable at these levels. Very well chosen, developmentally ordered, and literally instructive exercise sets are a key factor to its "more leisurely, yet more systematic (treatment), than that in most texts."

(Part) I (sections) 3, 4, 5, 6 and *7 present broad portions of the propositional calculus. I.*8, *9 and *10 interestingly and informatively apply truth-functional analysis to "black box" (electronic) circuits and to switching networks. II.1 introduces "set"-notion and Venn diagrams; then II.2 presents "set-builders." II.3, 4, and 5 examine and compound singular "conditions" (denoted by one-variable open statements, and usually included in the propositional calculus) in terms of set-relationships; then II.6 embeds "conditions" in set-builders to erect the algebra of sets along traditional lines. In II.7, quantifiers (unusual ones; for instance, universal does not imply particular) develop in terms of sets, then are used to build the usual set-operators for products and sums of sets of subsets. Further quantificational material appears in II.8 and 9, and II.*10 and *11 close by sketching certain propositional inference-patterns.

The intensively set-grounded symbolism, manner, and order of exposition should delight the many teachers who are demanding all-out emphasis on sets. Comparative relationships of dependence, decidability, completeness, and consistency, however, all argue compellingly to others for the more traditional order, (1) propositional calculus, (2) quantification theory, (3) set theory. Fortunately, the choice is not exclusive, since this text's brevity leaves ample course time for alternative views.

Regrettably absent in this sort of text is the reduction of relations and functions between numbers, real-plane point-sets, and plane geometrical loci, to simple sets of ordered pairs of real numbers.

Final particulars: (1) Regarding symbolism, the more usual forms "$\{x \mid (x \in M) \wedge p(x)\}$", "$(\forall x)[(x \in M) \Rightarrow p(x)]$" and "$(\exists x)[(x \in M) \wedge p(x)]$" are as workable here as the text's versions of set-builder and quantifiers, but these expose much more crucial internal structure to view and manipulation. (2) Practice and terminology of quotation are absent and confused, respectively. (3) Of accidental but nontrivial errors, four in the text and twelve among the many exercises and answers are too few for serious complaint.

ROBERT L. STANLEY
Washington State University

Elementary Matrix Algebra. By Franz E. Hohn. Macmillan, New York, 1958. xi+305 pp. $7.50.

Without going too much into abstract algebra, the author succeeds in presenting a very excellent introduction into the subject. The treatment is, in general, suitable for a course at the junior-senior level. The chapter headings will give an adequate idea of the contents: I. Introduction to Matrix Algebra; II. Determinants; III. The Inverse of a Matrix; IV. Rank and Equivalence; V. Linear Equations and Linear Dependence; VI. Vector Spaces and Linear Transformations; VII. Unitary and Orthogonal Transformations; VIII. The Characteristic Equation of a Matrix; IX. Bilinear, Quadratic and Hermitian Forms. In addition there are three appendices on the $\sum$ and $\prod$ notations, complex numbers, and isomorphisms. Every chapter contains a wealth of exercises, ranging "from purely formal computations and extremely simple proofs to a few fairly difficult problems." Throughout the book determinants are unnecessarily stressed. In Chapter II, the "sweep-out process for evaluating determinants" is developed, in Chapter III, inversion of matrices and solution of equations by "synthetic elimination." This seems to be an unnecessary duplication of effort since both methods are just slight modifications of Gaussian elimination. The term "cancelling" (*e.g.*, p. 61) should not be used without some statement of the cancellation law in an integral domain.

The book is unusually free of errors and is written in an easily readable style.

ALBERT NEWHOUSE
University of Houston

Ordinary Differential Equations. By Wilfred Kaplan. Addison-Wesley, Reading, Mass., 1958. xv+534 pp. $9.75.

This is an introduction to the theory of ordinary differential equations by an able and prolific writer of mathematical textbooks, *e.g.*, *Advanced Calculus*, and *A First Course in Functions of a Complex Variable.* The book is happily influenced by the branch of engineering known as systems analysis or instrumentation. The book covers: basic ideas, linear equations and systems of linear equations including the application of matrices, exact differential equations, an introduction to nonlinear equations, power series solutions, numerical methods, phase plane analysis, and a final, twelfth chapter on fundamental theory, *i.e.*, existence and uniqueness theorems. Every chapter concludes with suggested references which lead the reader to many of the genuinely important books on ordinary differential equations and to a representative selection of important books on their scientific and engineering application. The reader would do well to handle and scan each of the references. This bibliographic knowledge, in addition to the basic training provided by the text, will enable the serious creative engineering mathematician to cope effectively with a significant area of technical problems. The book, in detail, is well written, modern and alive.

NATHAN GRIER PARKE, III
Carlisle, Massachusetts

Mathematics and Wave Mechanics. By R. H. Atkin. Wiley, New York, 1957. xv+348 pp. $6.00.

The author's intention has been "to select a course of study which is locally connected and yet adequate for the pursuit of modern quantum theory." The contents of the book thus consist of fifteen chapters, the first nine of which supply preparatory mathematical and physical background, *e.g.*, on integration, differentiation, multiple integrals, special functions, vectors, determinants, matrices, classical mechanics, wave motion and the electromagnetic field. The remaining six chapters deal with selected topics in quantum mechanics including a chapter on quantum chemistry and another on the theory of a quantised field.

To this reviewer it seems that the book can very well succeed in preparing the student for the pursuit of quantum mechanics, but not for attaining an understanding or mastery of it to the extent of being able to do research in it. This criticism is directed primarily to the treatment of quantum mechanics which is mostly a collection of miscellaneous, unrelated examples with little pretense of providing a thorough discussion of the physical basis or logical structure of the subject.

Granting this shortcoming, which is compelled by the inclusion of so many topics, the book should have its main value as an aid to self study for students with deficient mathematical background. Here again, the emphasis is mainly on manipulative techniques to be applied in the solution of particular problems rather than an abstract or rigorous formulation of the mathematics. The level and intent of the book are perhaps best characterized by noting that the references are preponderantly to the Cambridge Tripos examinations.

D. L. FALKOFF
Brandeis University

Integral equations. By F. Smithies. New York: Cambridge University Press, 1958. x+172 pp. $4.50.

This little book is an interesting introduction to the theory of nonsingular integral equations. The material covered corresponds roughly to the chapter on integral equations in the *Methods of Mathematical Physics* by Courant and Hilbert. The treatment of the subject matter is somewhere between the classical method, with its emphasis on piecewise continuous functions, and the modern functional analysis approach. Most of the theory is for the case of $\mathfrak{L}^2$ kernels and wide use is made of real and complex analysis; on the other hand, the author does not use the theory of linear operators on topological vector spaces. The price and the reward of this restraint are of the usual kind: the proofs are harder, the unifying concepts missing, but the convergence theorems are stronger.

The following is a list of major topics covered by the author. In Chapter I the reader is introduced to the various kinds of integral equations and an assortment of facts concerning Lesbegue integration. However, if this book were

used as a text for a graduate course, students could not be expected to learn enough about integration from Chapter I, hence courses in real and complex variables would have to be prerequisites. Chapter II defines and studies the elementary properties of resolvent kernels, characteristic value problems, the Neumann series, and certain generalizations of the Neumann series. Chapters III, V, and VII all treat various aspects of the Fredholm Theory. The approach in Chapter III is to approximate arbitrary $\mathfrak{L}^2$ kernels by kernels of finite rank. The usual theorems about characteristic values are then obtained. Chapters V and VI give explicit formulas for the solutions in the form of determinants for the case of continuous and $\mathfrak{L}^2$ kernels respectively. Chapter IV is an introduction to the study of orthonormal systems of functions. Chapters VII and VIII deal with the theory of Hermitian kernels. This is an extension of the more classical study of symmetric kernels. For the case of Hermitian kernels it is possible to describe the totality of characteristic values. Various expansion theorems are possible in this case.

Finally it should be noted that this is an unusually readable book and hence quite suitable for a graduate text.

FELIX HAAS
Wayne State University

Gödel's Proof. By E. Nagel and J. R. Newman. New York University Press, New York, 1958. x+118 pp. $1.75 paper, $2.95 cloth.

This book concerns the famous Incompleteness Theorem of Gödel, published in 1931. It is for the nonspecialist, and is based on an article in *Scientific American* in 1956. It falls into three parts: mathematical-historical background; outline of the proof; and (very briefly) discussion of its implications for mathematics. The first part is excellent, the second good, and the third open to substantial objections.

The first part, Chapters I–V, occupies, with its related appendices, most of the book. It surveys development of the axiomatic method, consistency proofs, formalization in symbolic logic, and Hilbert's search for "finitistic" proof of the absolute consistency of formalized mathematics. It is mathematical intellectual history at its best, presented with subtlety, wit and care,—suggestive and profitable for both layman and specialist.

In the second part, Chapters VI–VII, the authors face the difficulty that they must omit much detail and must thus leave room for possible misunderstanding. An intelligent selection of key steps is made and summaries of the connecting arguments are given—on the whole, a very good job. The reviewer makes two suggestions. (1) In connection with the arithmetization of metamathematics, it should be pointed out that some but not all statements of the metalanguage can be made within arithmetic. The "solution" to Richard's Paradox lies not so much in the distinction between arithmetic and metalanguage as the authors allow us to believe, but rather in the further distinction between those statements of the metalanguage that can and cannot be repre-

sented within arithmetic. A reference to work of Tarski would have been appropriate here (and at some other places). (2) In connection with the "undecidable" statement G that asserts its own unprovability, the authors state that G is true. This is ambiguous and somewhat misleading. In the case of an application of Gödel's technique to a simple axiomatization of arithmetic it is indeed the case that we can infer the truth of G by a conventional (and quite precise) argument. But Gödel's technique can also be applied to a stronger system in which all the conventional methods of mathematical argument are included. Let G be the "undecidable" arithmetical statement now produced. We cannot infer that G is true. We know that G is true if the stronger system is consistent; but at best we can only say, on empirical or aesthetic grounds, that the consistency of this stronger system is probable.

In the brief third part, Chapter VIII, the authors first assert, correctly, that the theorem seems to show the impossibility of success in Hilbert's search. They then draw the (admittedly) more controversial conclusion that the mathematical powers of the human mind transcend (in some sense) mechanical procedures. This inference is not uncommon, and is, in the reviewer's opinion, unwarranted and probably wrong. It arises from the erroneous idea, implied in the text, that an iterated application of the Gödel method for making undecidable but true inferences is in some inherent way incapable of axiomatization. In fact, such iteration out through large segments of the transfinite ordinals can be included in the axioms. Of course any such inclusion can be extended by Gödel's methods, but this is true in the same fairly trivial sense that any effective system of notation for a segment of ordinals can be extended to cover a larger segment. The limitation on a machine is one of size, not quality. The revolutionary information conveyed by Gödel is not that the human mind transcends mechanical procedures, but that mathematics itself does so. The authors also fail to make clear (see (2) above) that such iteration could break down at any point with an initial appearance of inconsistency. Recent work on ordinal notations and degrees of unsolvability sheds much light on these matters.

Apart from these criticisms, the authors merit praise for an unusually good and most welcome piece of popular exposition.

HARTLEY ROGERS, JR.
Massachusetts Institute of Technology

Lineare Algebra. By Werner Graeub. Band XCVII, Die Grundlehren der Mathematische Wissenschaften. Springer-Verlag, Berlin-Göttinger-Heidelberg, 1958. xi+219 pp. 35.70 DM (About $8.55).

Beginning on page one with axioms for a linear space, first over the real field and then over an arbitrary (commutative) field, this book develops systematically in the customary style and format of this ("Grundlehren") series the standard basic theory of linear operators in the finite dimensional case. Proofs are usually phrased in invariant form, without reference to the matrix describing a mapping, but the notation is frequently highly explicit so that A is said to be

the determinant of the matrix (α_i^k), *cf.* p. 56. Tensors are introduced as multilinear functions of vectors in a pair of dual spaces and the interplay between the tensor descriptions and abstract vector space terminology should be informative to readers acquainted with only one of these notations.

The determinant is axiomatized as a function of n vectors with specified properties and is used in a treatment of orientation. Metric spaces receive much attention both in the definite and indefinite case and affine and euclidean classification of conics and quadrics is explicit.

The elementary divisor theory concludes the book. Rational and Jordan canonical forms are obtained by adaptation of bases to invariant spaces.

A slightly unusual item is the proof (attributed to John Hilnor) by highly nonalgebraic methods that two real quadratic forms in $n>2$ variables may simultaneously be diagonalized if they do not both vanish for the same nonzero vector.

No errors were noted and the exposition is exemplary. It may appear strange to some readers that dual mappings are described by matrices (α_ν^μ) and (β_ν^μ) with $\alpha_\nu^\mu=\beta_\nu^\mu$, yet the matrices are described (p. 40) as transposes of one another under the agreement that in one case ν is the row and in the other the column index. The tensor notation thus hints at the correct conclusion that in a pair of dual spaces (best understood as a right and a left vector space over a division ring) the same matrix, and not a pair of transposed, most conveniently describes the dual mappings. (*Cf.* N. Jacobson, *Lectures in Abstract Algebra*, vol. 2, p. 58, a reference missing from the very brief bibliography.)

W. GIVENS
Wayne State University

Asymptotic Methods in Analysis. By N. G. deBruijn. Interscience, New York, 1958. xii+200 pp. $5.75.

This most entertaining and informative book is written for the nonexpert in the field and explains by examples some of the more important methods of asymptotic analysis. The mechanism of asymptotic series, the Lagrange inversion formula, the Laplace method for integrals, saddle point techniques, etc. is made plain by problems from differential equations, iterated functions, etc. The author's philosophy is that specific difficult problems, if worked out in detail, involve ideas which are general and which can be applied to general classes of problems, a view with which the reviewer concurs entirely.

MORRIS NEWMAN
National Bureau of Standards, Washington, D. C.

Algebra. By J. W. Archbold. Pitman & Sons, London, 1958. xix+440 pp. 45/-. (About $6.30).

As stated in the preface this text was written to include the algebra mentioned in the syllabuses for certain general and honors Bachelor's degrees in mathematics at the University of London.

The first three chapters contain properties of the number systems and finite series. This is followed by algebraic inequalities and then complex numbers. The next seven chapters are devoted to polynomials, polynomial equations, roots of unity, and symmetric functions. Chapters 15 and 21 are on determinants and Chapters 16 and 20 present some group theory. Chapters 17–19 and 22–24 deal with vector spaces, matrices, and quadratic forms, while the last chapter concerns discriminants and resultants.

The ordering of material could be improved by juxtaposing the two chapters on determinants and also the two on groups. Unnecessarily, vector spaces and matrices are restricted to fields of characteristic zero. Unfortunately only nonsingular linear transformations are mentioned and then only relative to quadratic forms. Most of their important properties are omitted.

In general the material is presented well. An outstanding feature is the set of exercises; some extend the theory and provoke thought while others are interesting problems that have appeared in the MONTHLY and the *Mathematical Gazette*.

This book will be useful as a reference. It is also worthy of consideration as a text in one or a sequence of undergraduate courses. For an advanced undergraduate course in modern algebra, however, a text which goes more deeply into groups, rings, fields, linear transformations, etc. and with a more abstract approach would be more suitable.

HOWARD E. CAMPBELL
Michigan State University

Mathematical Aspects of Subsonic and Transonic Gas Dynamics. By Lipman Bers. Wiley, New York, 1958. xv+164 pp. $7.75.

This extensive survey of the mathematical aspects of a specific field in gas dynamics, namely, theory of steady two-dimensional compressible flow, is a tremendous task. It includes most of the important information and results (for which the author is one of the principal contributors) from a vast source of some 400 papers published since World War II in a relatively compact book of 135 pages. The author's elucidating style and inspiring way of presenting the problems, the mathematical background and theory for their solution, the results and their extensions, and the open problems for future research make the book delightfully readable as well as valuable to applied mathematicians and aerodynamicists. The book is mainly devoted to two categories of problems: those of subsonic flows and those of transonic flows. The principal mathematical problem is, above all, the theorem of existence and uniqueness.

For subsonic flow theory, a detailed account of the mathematical background for elliptic partial differential equations is given, such as the smoothness of solutions, maximum principle, quasiconformal mapping, theory of pseudo-analytic functions, the direct method of calculus of variation and Schauder's fixed point theory. The theorem of existence and uniqueness for a flow, uniform at infinity, past an obstacle is precisely stated and the main feature and ideas

for the proof are clearly indicated and sketched. Of course, no details could be given in such a compact treatise. However, in each case, the sources are always referred to. The subsonic channel flow is treated by means of the extension of Riemann's mapping theorem. A brief account is also given for subsonic flows with free boundaries.

For the mathematical background of transonic flows, the author is mainly dealing with the uniqueness and existence of various boundary value problems of linear partial differential equations of mixed type. A special feature concerning the question of uniqueness is the presentation of yet unpublished work of Friedrichs on the new approach of the abc method as a systematic procedure for finding correctly set boundary-value problems for a wide class of linear partial differential equations. The question of "transonic controversy" (the nonexistence of transonic shockless flow past a profile) arouses a great interest and the author urges and foresees further research and progress.

Finally, the author considers transonic flows with free boundaries, in particular, the flow behind a detached shock wave due to a truncated wedge. In this particular occasion, the author did not make the important remark that the flow behind a curved shock wave is rotational. Consequently, the Chaplygin equation (3.4), as used by Frankl, is at most an approximation (by neglecting the vorticity). As a matter of fact, the exact equation involves an *a priori* unknown function (of the stream function), which is determined by the yet-to-be-found shape of the shock. [Sauer, *Gas Dynamics*, p. 137, (218); also, N.A.C.A. TN2364, p. 7, (12)]. It is indeed a new free boundary problem. A few trivial misprints are found, such as in (6.7), (8.4), (2.12), (13.9), and the lines (from the bottom) 7, p. 69; 4, p. 71; 12, p. 76; 4 and 5, p. 91.

S. S. Shu
Purdue University

NEWS AND NOTICES

Edited by Lloyd J. Montzingo, Jr., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Emeritus R. W. Brink, University of Minnesota, represented the Association at the inauguration of Dr. Harvey M. Rice as President of Macalester College.

Professor A. B. Cunningham, West Virginia University, represented the Association at the inauguration of Dr. Elvis Jacob Stahr, Jr. as President of West Virginia University on October 3, 1959.

The purpose of the handbook is to provide a collection of tested algorithms for mathematical computations of all sorts. These algorithms are to be written in ALGOL, hence will be usable on any machine for which a suitable translator is available, and even without a translator can be used as a model for programming. The descriptive language will be English. Each algorithm is to be accompanied by enough explanatory information to make it understandable, along with whatever information is available on speed, accuracy, range, or, more generally, for judging the effectiveness of the algorithm for a given type of problem. In any event, only pretested algorithms will be published.

Contributions are earnestly solicited. For the present, at least, these must necessarily be in the form of actual algorithms, along with information as to the extent and mode of testing the algorithm, estimates of accuracy, and experience in using it. As algorithms are published, information relating to published algorithms also will be welcomed. Contributions may be sent to the editor named above.

New SIAM Lecture

The Society for Industrial and Applied Mathematics has announced The John von Neumann Lecture which will be presented annually at a national meeting of the Society. The first lecture will be presented at the summer meeting in 1960. The lecturer will survey and evaluate a significant field of pure mathematics in the light of its contribution to major advances in applied mathematics. An authoritative lecture presented from this point of view is considered to be in harmony with the research and philosophy of Professor von Neumann.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 333 persons have been elected to membership by the Board of Governors on applications duly certified.

Roger W. Adair, Student, Emory University.
Allen A. Adams, Student, Central State College, Oklahoma.
Sheldon Adelberg, A.B. (New York) 79 East 36 Street, Paterson 4, New Jersey.
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Bernard S. Albert, M.B.A. (California) Senior Operations Research Analyst, North American Aviation.
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Leroy R. Amunrud, M.S. (Montana S.U.) Instr., Montana State College.
Salvatore Anastasio, B.A. (Cathedral Coll.) Part-time Instr., Iona College; Grad. Student, New York University.
Charles N. Anderson, M.Eng. (North Carolina S.C.) Instr., North Carolina State College.
John M. Anderson, Ph.D. (California) Professor, Pennsylvania State University.
Ray D. Anderson, Jr., B.S. (Portland S.C.) Grad. Student, University of Idaho.
Walter G. Andrews, Student, St. Michael's College, Vermont.
Harry W. Appelgate, B.A. (Hofstra Coll.) Programmer, Service Bureau Corp.
Loren N. Argabright, M.A. (Kansas) Lecturer, Seattle University.
Fred Armstrong, Student, St. Michael's College, Vermont.
Hack Arroe, Ph.D. (Copenhagen) Head, Physics & Math. Div., University of Denver.
Thomas S. Bagby, Jr., Student, University of Kentucky.
Malvina M. Baltrukovicz, B.A. (Hunter Coll.) Asst. Actuary, New York City Retirement System.
C. Lamar Bannister, B.S. (Jacksonville S.C.) Mathematician, Army Ballistic Missile Agency.
Joseph Barback, B.A. (Buffalo) Grad. Student, Rutgers University.
Richard C. Basinger, Student, Missouri School of Mines.
Roquez Bejarano, A.B. (California, Berkeley) Asso. Engineer, Jet Propulsion Lab.
David L. Bennett, B.Ed. (Alberta) Principal, Bashaw Schools, Alberta.
Paul W. Berg, Ph.D. (New York) Asso. Professor, Stanford University.
Mrs. Louise Berge, A.A. (California) Student, University of California, Berkeley.
Edward P. Berger, A.M. (Duke) Instr., University of Nevada.
Roy Bertoldo, B.S. (Queens Coll.) Student Actuary, New York Life Insurance Co.
Stuart M. Block, M.A. (New York) Instr., Hunter College High School.
Osei K. Bonsu, B.S. (Oregon S.C.) Grad. Student, Oregon State College.
Wilfred E. Boykin, A.B. (Emory) Grad. Student, Emory University.
Allen L. Bradfield, A.M. (Indiana) Asso. Professor, Vincennes University.
Warren Brisley, B.S. (Sydney) Lec-

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Brother Emeric, B.S. (St. John's, New York) Head of Dept., Xaverian High School, Brooklyn, New York.
Brother Theodose Brown, M.A. (Laval) Professor Academie de Quebec.
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Estella M. Smiley, M.A. (California, Los Angeles) Chairman of Dept., David Starr Jordan Junior High School, Burbank, California.

John Sniedze, 62 Park Terrace West, New York 34, New York.

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Sidney F. Star, M.S. in C.E. (C.C.N.Y.) Teacher, Thomas Edison Vocational High School, Queens, New York.

C. L. Steininger, M.A. (Toledo) Head of Dept., Rossford High School, Ohio.

John L. Stephenson, M.D. (Illinois) Biophysicist, National Heart Institute.

Arthur I. Sternhell, M.S. (New York) Staff Asst., Metropolitan Life Insurance Co.

Elmer L. Stone, B.S. in Ed. (Northern Illinois) Teacher, Warren Township High School, Gurnee, Illinois

Susan J. Strand, Student, American University.

Allen F. Strehler, Ph.D. (Wisconsin) Asst. Professor, Carnegie Institute of Technology.

C. Gordon Strong, B.A. (Alberta) Principal, Vegreville High School, Alberta.

Sylvester C. Suda, B.S. (North Dakota) Instr., Casselton Public Schools, North Dakota.

Mrs. Louise N. Sutton, M.A. (New York) Asst. Professor, Delaware State College.

Kenneth L. Swanson, B.S. (Ft. Hays, Kansas S.C.) Instr., Ashland High School, Kansas.

Noah J. Swartz, Jr., Student, Indiana Technical College; Trainee-Time Study Engineer, International Harvester.

David S. Tartakoff, Student, Phillips Exeter Academy.

Joseph P. Tassoney, M.S. (Newark Coll. of Eng.) Asst. Professor, Pennsylvania State University.

John N. Thomas, B.Ph. (Northwestern) Instr., Chicago Technical College.

Robert E. Thomas, M.S. (Kansas S.C., Pittsburg) Senior Engineer, The Martin Company.

Howard Eugene Thompson, Ph.D. (Western Reserve) Chief, Organic Section, Dugway Proving Ground.

Francis L. Thomson, A.A. (California) Student, University of California, Berkeley.

Robert R. Tinney, M.S. (Oklahoma S.U.) Instr., Gulf Coast Community College.

William R. Tomlinson, Jr., B.S. (M.I.T.) Staff Member, Operations Research Office, Johns Hopkins University.

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Charles W. Wheeler, M.B.S. (Colorado) Instr., Contra Costa Junior College.
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Stanley Winkler, Ph.D. (New York) Advisory Mathematician, International Business Machines Corp.
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John J. Wyman, Student, St. Michael's College, Vermont.
Chandrashekhar C. Yalavigi, M.S. (Gujaret) Asst. Lecturer, Government College, India.
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Frank A. Yett, Ph.D. (California, Los Angeles) Instr., Pasadena City College.
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ANNOUNCEMENT BY THE NEW SECRETARY

The separation of the offices of Secretary and Treasurer, as recently reported in this MONTHLY, vol. 66, p. 619, has now been completed. This is an announcement regarding the division of responsibilities between the Secretary and the Treasurer, insofar as this concerns the membership of the Association.

The Treasurer is responsible for payment of dues and subscriptions, ordering of publications, changes of address, and applications for membership. All correspondence regarding these matters, therefore, should continue to be sent to Professor Harry M. Gehman at the University of Buffalo, Buffalo 14, New York.

The Secretary is responsible for arrangements of national meetings of the Association, the operations of the Board of Governors, the Executive and Finance Committees and all other committees of the Association. Correspondence regarding these items, consequently, should be directed to Professor Henry L. Alder at the University of California, Davis, California.

Your new Secretary wishes to acknowledge to Professor Gehman his deep gratitude for the great help he has received during the transition of offices. During this period of transition, your new Secretary has gained an appreciation of the most effective and devoted leadership which Professor Gehman has given the Association over so many years and for which the membership is greatly in his debt. The Association should consider it fortunate indeed that Professor Gehman will continue in a position of active leadership.

HENRY L. ALDER, *Secretary*

ITINERARIES OF VISITING LECTURERS, 1959–1960

David Blackwell

Feb. 8–10	University of Kentucky, Lexington, Ky. (J. C. Eaves)
Feb. 11–12	University of the South, Sewanee, Tenn. (Gaston S. Bruton)
Feb. 15–16	Florida A. and M. University, Tallahassee, Fla. (Mrs. L. B. Clarke)
Feb. 17–19	Jackson State College, Jackson, Miss. (Benjamin H. McLemore)
Feb. 22–24	Washington University, St. Louis 5, Mo. (H. M. MacNeille)
Feb. 25–27	University of Oklahoma, Norman, Okla. (Richard V. Andree)

Mar. 7–9	Antioch College, Yellow Springs, Ohio. (Thomas C. Holyoke)
Mar. 10–12	Ohio Wesleyan University, Delaware, Ohio. (Robert L. Wilson)
Mar. 14–15	Wayne State University, Detroit 2, Mich. (Wallace Givens)
Mar. 16–17	Eastern Michigan University, Ypsilanti, Mich. (Robert S. Pate)
Mar. 18	Western Michigan University, Kalamazoo, Mich. (Charles H. Butler)
Mar. 21–22	Emmanuel Missionary College, Berrien Springs, Mich. (E. J. Specht)
Mar. 23–25	Marquette University, Milwaukee 3, Wis. (L. J. Heider)
Mar. 28–29	University of Wisconsin, 3203 North Downer Ave., Milwaukee, Wis. (M. Marden)
Mar. 31–Apr. 1	University of Illinois, Urbana, Illinois (Mahlon M. Day)
Apr. 4	Wisconsin State College, Superior, Wis. (John O. Danielson)
Apr. 5–6	University of Minnesota, Duluth, Minn. (John E. Hafstrom)
Apr. 7–8	Mankato State College, Mankato, Minn. (Warren J. Thomsen)
Apr. 11–12	Carleton College, Northfield, Minn. (Kenneth O. May)
Apr. 13–15	Iowa State Teachers College, Cedar Falls, Iowa. (H. C. Trimble)
Apr. 25–26	Fort Hays Kansas State College, Hays, Kan. (W. Toalson)
Apr. 28–29	Bethel College, North Newton, Kan. (Arnold M. Wedel)
May 2–6	Western Illinois University, Macomb, Ill. (Joseph Stipanowich) and Knox College, Galesburg, Ill. (Rothwell Stephens)
May 9–10	Beloit College, Beloit, Wis. (Ralph C. Huffer)
May 11–12	Bradley University, Peoria, Ill. (Marvin G. Moore)
May 13–14	Illinois Section of the M.A.A., Bloomington, Ill. (Donald E. Myers)
May 16–17	Millikin University, Decatur, Ill. (E. W. Ploenges)
May 18–20	Southern Illinois University, Carbondale, Ill. (W. C. McDaniel)
May 23–25	Southern Illinois University, Alton Residence Center, 2809 College Avenue, Alton, Ill. (Eric A. Sturley)

M. R. Hestenes

Feb. 1–2	Mississippi State College, State College, Miss. (Arthur Ollivier)
Feb. 4–5	University of Alabama, University, Ala. (Julian D. Mancill)
Feb. 8–9	McNeese State College, Lake Charles, La. (Stephen M. Spencer)
Feb. 11–12	Stephen F. Austin State College, Nacogdoches, Tex. (W. I. Layton)
Feb. 15–16	Central State College, Edmond, Okla. (Mrs. Dorothea Meagher)
Feb. 18–19	Wayland College, Plainview, Tex. (Dorothy McCoy)
Feb. 29–Mar. 1	St. Olaf College, Northfield, Minn. (C. S. Carlson)
Mar. 3–4	Gustavus Adolphus College, St. Peter, Minn. (Albert G. Swanson)
Mar. 7–9	Wisconsin State College, Eau Claire, Wis. (Lawrence F. Wahlstrom)
Mar. 10–11	Wisconsin State College, River Falls, Wis. (Lillian Gough)
Mar. 14–15	Macalester College, St. Paul Minn. (E. J. Camp)
Mar. 16–18	Concordia College, Moorhead, Minn. (Sigurd Mundhjeld), Moorhead State College, Moorhead Minn. (Marion V. Smith), North Dakota Agricultural College, Fargo, N. D. (A. Glenn Hill)
Mar. 28–29	Nebraska State Teachers College, Kearney, Neb. (L. M. Larsen)
Mar. 30–31, Apr. 1	University of Nebraska, Lincoln, Neb. (William G. Leavitt)
Apr. 4–5	Loras College, Dubuque, Iowa. (Louis E. Ernsdorff)
Apr. 6–8	Wartburg College, Waverly, Iowa. (John O. Chellevold)
Apr. 11–12	Simpson College, Indianola, Iowa. (Harold A. Heckart)
Apr. 13–14	Graceland College, Lamoni, Iowa. (W. S. Gould)
Apr. 25–27	University of Tulsa, Tulsa, Okla. (Ralph W. Veatch)
Apr. 29	Kansas State College, Pittsburg, Kan. (R. G. Smith)
Apr. 30	Kansas Section of the M.A.A., Pittsburg, Kan. (J. D. Haggard)
May 2–3	Kansas State University, Manhattan, Kan. (R. G. Sanger)
May 4	Washburn University, Topeka, Kan. (Paul Eberhart)

May 5–6	University of Kansas, Lawrence, Kan. (G. Baley Price)
May 9–10	Central Missouri State College, Warrensburg, Mo. (Claude H. Brown)
May 12–13	Hope College, Holland, Mich. (Jay E. Folkert)
May 16–17	Heidelberg College, Tifflin, Ohio. (William F. Steele)
May 18–20	Kent State University, Kent, Ohio. (L. E. Bush)
	S. A. Jennings
Nov. 16–18	University of Alaska, College, Alaska. (Francis D. Parker)
Nov. 19–20	Seattle Pacific College, Seattle 99, Wash. (O. Karl Krienke, Jr.)
Nov. 23–25	College of Puget Sound, Tacoma 6, Wash. (Edward G. Goman), Pacific Lutheran College, Tacoma 44, Wash. (Eugene A. Maier)
Nov. 30–Dec. 1	Occidental College, Los Angeles 41, Calif. (Charles W. Seekins)
Dec. 3–4	San Diego State College, San Diego, Calif. (C. B. Bell)
Jan. 18–19	Reed College, Portland 2, Ore. (J. B. Roberts)
Jan. 20–22	Lewis and Clark College, Portland, Ore. (Elvy L. Fredrickson), Portland State College, Portland 1, Ore. (Robert W. Rempfer)
Jan. 23–24	Oregon Council of Teachers of Mathematics, 388 Merrill Court, Eugene, Ore. (Wendell C. Hall)
Jan. 27–29	Southern Oregon College, Ashland, Ore. (Floyd L. Taylor)
Feb. 1–2	Arizona State University, Tempe, Ariz. (Lloyd L. Lowenstein)
Feb. 4–5	New Mexico Institute of Mining and Technology, Socorro, N. Mex. (Rafael Sanchez-Diaz)
Feb. 8–9	Montana State College, Bozeman, Mont. (John W. Hurst)
	Tibor Rado
Mar. 14–15	Georgetown University, Washington, D. C. and University of Maryland, College Park, Md.
Mar. 17	University of Delaware, Newark, Del.
Mar. 18–19	Villanova University, Villanova, Pa.
Mar. 21–22	Trenton State College, Trenton, N. J.
Mar. 23–26	Simmons College, Boston, Mass., Boston College, Boston, Mass., Boston University, Boston, Mass.
Mar. 28–29	University of Rochester, Rochester, N. Y.
Mar. 31–Apr. 1–2	University of Buffalo, Buffalo, N. Y. and Canisius College, Buffalo, N. Y.
Apr. 11	Connecticut College, New London, Conn.
Apr. 13–15	Wesleyan University, Middletown, Conn.
Apr. 18–19	University of New Hampshire, Durham, N. H.
Apr. 21–22	Dartmouth College, Hanover, N. H.
Apr. 25–30	University of Scranton, Scranton, Pa., Wilkes College, Wilkes Barre, Pa., Misericordia College, Dallas, Pa.
May 2–3	Hamilton College, Clinton, N. Y.
May 9–10	Washington and Lee University, Lexington, Va.
May 12–13	Virginia State College, Petersburg, Va.
May 16–17	University of Richmond, Richmond, Va.
May 19–20	University of South Carolina, Columbia, S. C.
May 22–25	Emory University, Atlanta, Ga., and Georgia Tech., Atlanta, Ga.
	Ernst Snapper
Apr. 27	University of Wisconsin, Madison, Wis. (R. H. Bing)
Apr. 28–30	Case Institute of Technology, Cleveland, Ohio. (S.W. McCuskey)
May 2–3	Muskingham College, New Concord, Ohio. (L. Coleman Knight)
May 4–5	Bowling Green State University, Bowling Green, Ohio. (F. C. Ogg)
May 7	Mount Mary College, Milwaukee, Wis., Wisconsin Section, M.A.A.

May 9–10	University of Miami, Coral Gables, Fla. (Georgia Del Franco)
May 11–13	Florida State University, Tallahassee, Fla. (Thomas L. Wade)
May 16–17	Mississippi Southern College, Hattiesburg, Miss. (B. O. Van Hook)
May 19–20	Oklahoma State University, Stillwater, Okla. (L. Wayne Johnston)

AWARD OF THE 1960 CHAUVENET PRIZE

The 1960 Chauvenet Prize will be awarded to Professor Cornelius Lanczos of the Dublin Institute for Advanced Studies for his paper entitled "Linear Systems in Self-Adjoint Form," published in the MONTHLY, Vol. 65 (1958) pp. 665–679. The prize will be presented to Professor Lanczos at the Annual Meeting of the Association to be held at the Hotel Conrad Hilton in Chicago on January 29, 1960.

This is the twelfth award of the Chauvenet Prize since its institution by the MAA in 1925. The regulations governing the award and the names of previous winners appeared in this MONTHLY, vol. 66, 1959, pp. 446–447.

CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

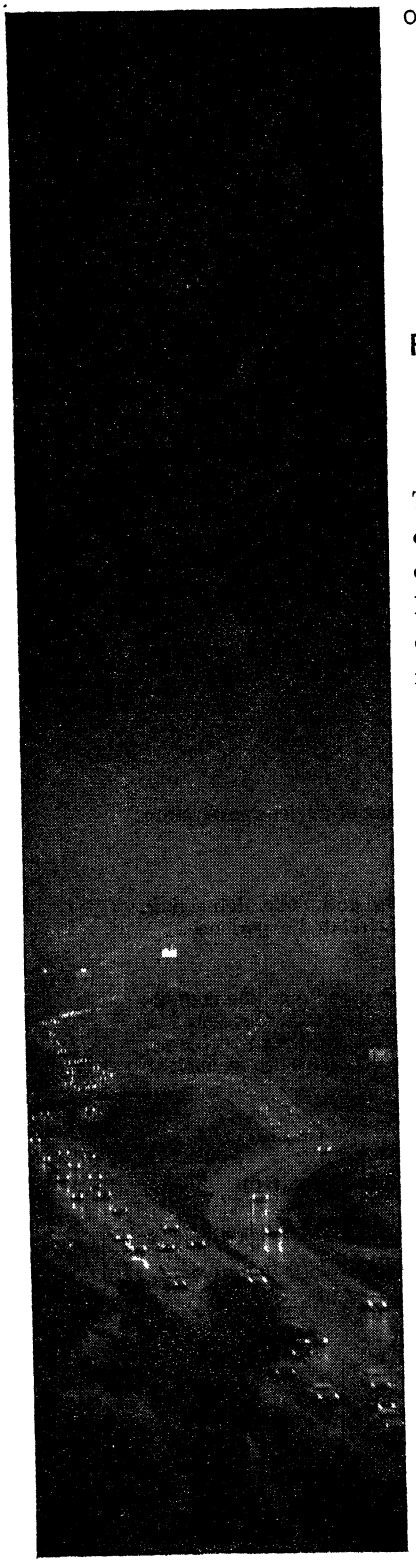
SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April 8–9, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960



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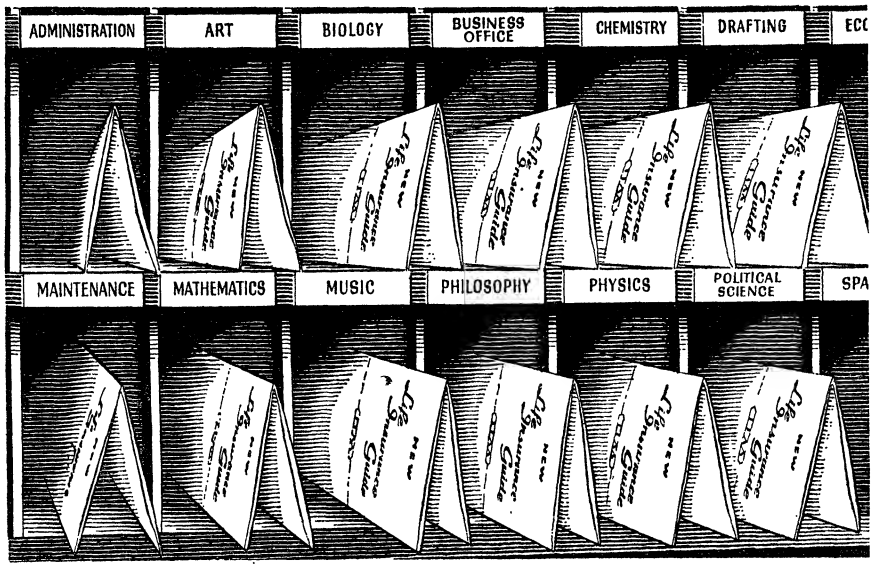
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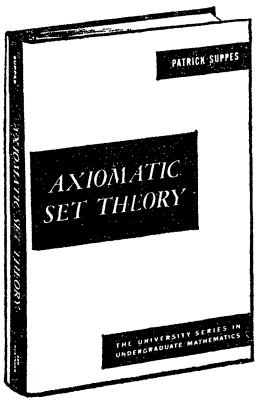
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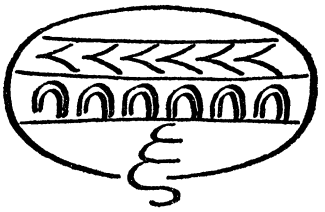
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ON THE CONCEPTS OF A PROBLEM AND PROBLEM-SOLVING*

RICHARD BELLMAN, The RAND Corporation, AND PAUL BROCK, Hughes Aircraft Company

1. Introduction. The past decade has seen startling changes in science and technology, much of it directly attributable to the use of large-scale computing devices. The power and versatility of these new devices have resolved many problems not only previously unsolved, but even previously unformulated. Yet the application of these new devices has barely begun. Now that overwhelming evidence of the effectiveness of these machines has accumulated, it is essential that serious and prolonged thought be devoted to the optimal utilization of current computers and to the future development of computers. It is clear that the field of mathematics is vitally affected by these new developments. It is worthwhile, then, to subject to critical analysis the concept of a *solution*—the primary objective of a mathematical theory—and, in addition, the even more fundamental concept of a *problem*.

In this paper, we wish in an informal fashion to discuss these matters. Our aim is to examine the role of the mathematician in providing a path from problem to solution. To do this, we shall discuss some of the different kinds of mathematical difficulties, the various ways in which computers can be used to resolve these difficulties, and a number of quite simply stated questions that give rise to these difficulties.

2. Problems. Problems may be divided roughly into three types. These are the *natural* problems arising in the world about us; the *model* problems, which we immediately recognize as an element of a class of similar problems; and, finally, the *symbolic* problems where an analytic formulation already exists. To solve a problem by mathematical techniques, we transform it step-by-step from a natural problem to a model problem and from a model problem to a symbolic problem. Obtaining the solution of the symbolic problem in analytic terms, we must interpret it suitably to obtain a solution of the model problem and then go from here to the natural problem. None of these steps are trivial and each requires experience and skill.

3. Solutions. In the course of this discussion, we introduced the concept of a *solution*, a term we shall employ on an intuitive basis. It is clear that an action, an effect which is equivalent to this solution, must bear favorable comparison with other possible actions, and that, furthermore, it must be a well-defined action with no ambiguous aspects. More important, perhaps, from our point of view, and a matter we shall refer to again, is that the solution must be commensurate with the problem. It must not contain within it problems of equal or greater complexity than the original problem. A solution of this type we shall call an *ideal solution*. In practice, we are often content with less-than-ideal solutions.

We cannot pretend that this discussion is logically sound, since it contains concepts that are undefined. Furthermore, some thought will show that such phrases as "all equivalent effects" or "equal or greater complexity" are perhaps

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even more difficult to explain than the ideas with which we started. Since we agreed upon an informal presentation, we shall accept this ambiguity as part of the price we must pay for relative simplicity. Since we know from sad experience that no amount of effort can completely remove logical ambiguities, we should not feel particularly distressed.

4. The solution process. The process or action leading from the problem reduction to the ideal solution we shall call the *solution process*. Thus, a large part of a problem process may be schematically represented as in Fig. 1 below. The connection between this diagram and a flow chart for a computer is not coincidental.

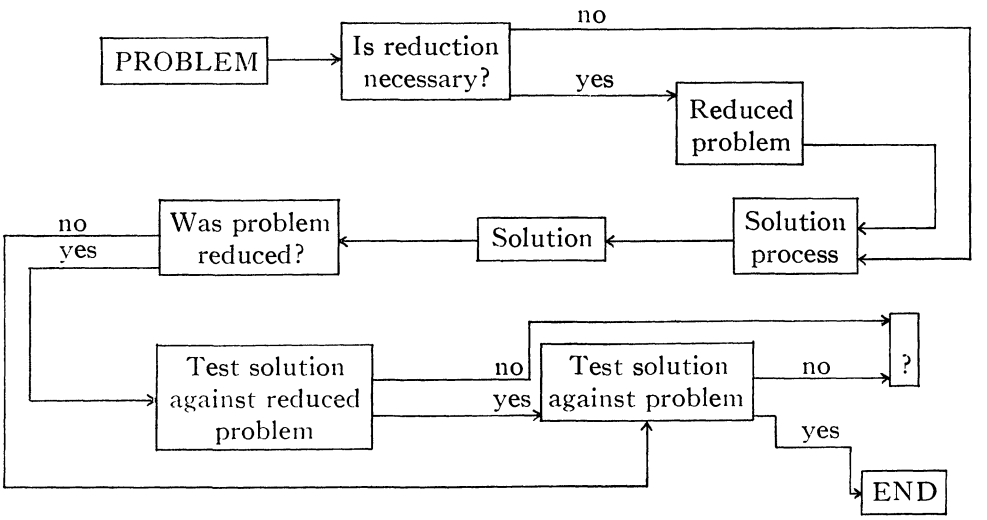


FIG. 1

Problems may not have ideal solutions because the conditions for a solution to be ideal are not satisfied, or because an appropriate solution process cannot be implemented for one of many reasons. This is the usual situation, and is not at all to be considered exceptional.

5. Applications of mathematical techniques to problem solving. Advances of mathematics may be considered to be the development of increasingly more powerful tools to apply to problems that can be transformed into symbolic form. In this connection, we may introduce the notion of a *mathematical model*, as a symbolically representable class problem. Thus, any particular system of linear algebraic equations would fall into the model problem:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, \cdots, m.$$

Since this model problem has been intensively investigated, it is only necessary to check any particular system against the several possibilities of the model to determine the number of solutions, representations for solutions, exact or approximating procedures to express solutions numerically, and so on.

If a mathematical solution to a symbolic problem exists, it must be an ideal solution in the following sense:

1. The solution must satisfy all conditions of the problem.

2. If not unique, all possible solutions must be expressible in a mathematical inductive sense. For any particular solution, mathematical equivalence must be understood.

3. Any subordinate problems posed in a solution must be solvable by well understood procedures.

It must, however, be emphasized that if a problem has been reduced to mathematical form, the existence of a mathematical solution to the reduced problem does not necessarily imply an ideal solution to the original problem.

6. Types of mathematical difficulty. We consider three classes of difficulty that may exist in a solution process. These are *analytic*, *computational*, and *conceptual*.

1. *Analytic.* At this level, we assume that the problem has been reduced to one of solving a set of equations. In most cases, however, these equations cannot be solved in terms of elementary or tabulated functions. To obtain numerical results from these equations, we must use approximate, or equivalently, iterative techniques such as Rayleigh-Ritz methods, finite difference methods, Monte Carlo methods, the Picard method of successive approximations, power series, and so on. When an analytic problem is transformed into one which permits computation, the limiting operations involved in differentiation, integration, etc., are necessarily replaced by finite operations. In certain cases, the approximating operations may not lead to a solution. But even in the cases where numbers are developed, it remains an analytical problem to determine whether these numbers are ideal solutions in the sense that has been defined. It should be noted that not all analytical problems are capable of solution in terms of existing techniques, and that even reasonable approximations may be beyond us at the moment. In spite of this, however, it is important to emphasize the fact that analytic difficulties are the *simplest* difficulties to overcome.

2. *Computational.* Although it can be argued that this is essentially an analytic difficulty, the revolutionary developments in computation force us to consider it separately. One of the paradoxical aspects of this field of study is the fact that explicit solutions may be of no use at all. A very nice example of this is furnished by the theory of linear equations. Here, Cramer's rule yields the solution as quotients of two determinants. Yet, if the degree of the linear system is large, these determinants cannot be used to grind out numerical results both because of the great number of terms arising from the straightforward evaluation of a

determinant and because of the numerical inaccuracy introduced from "round-offs" at the end of each arithmetical operation. Similarly, in the field of mathematical physics, there are large classes of problems whose solutions can be obtained in the form of infinite series or infinite integrals. In many cases however, the convergence of the expressions is so slow as to make them useless for numerical purposes. This brings to mind the remark of Heaviside: "This series diverges. Now we can compute with it!"

In a number of modern theories, we observe an interesting schizoid effect with one type of analysis used to establish properties such as existence, uniqueness, etc., and another type of analysis used to obtain numbers. Ideally, one would like an approach that can be used simultaneously for both purposes, but there is no reason to suppose that such an approach will always exist. Bigger and faster computing machines permit us to utilize algorithms that smaller machines cannot handle. Furthermore, they help conquer round-off difficulties by retaining more significant figures. Consequently, we can treat more realistic problems in finer detail. There is, however, the very real danger that a number of problems which could profitably be subjected to analysis, and so treated by simpler and more revealing techniques, will instead be routinely shunted to the computing machines. Many examples of this perverse use of these new devices can be cited.

The role of computing machines as a mathematical tool is not that of a panacea for all computational ills.

Although they extend our capabilities in this area, and will continue to do so indefinitely, they will always be of limited applicability. In fact, as will be pointed out subsequently, their prime value to the problem concept may very well be derived from a completely different direction. For this reason it is essential that the role of the computer be analyzed and discussed in great detail.

3. *Conceptual.* It is on the horns of this third dilemma that we are most frequently gored. In all parts of the physical world, and in the even more tortuous terrains of economic, industrial and military applications, the fundamental difficulty lies in formulating a clearly defined mathematical problem. No matter how difficult the equations, or complicated the analysis, we can always make some sort of approximation. What to do when there exists no such precise formulation is a difficulty of a different order of magnitude, which we shall discuss in following sections.

In order to reduce the natural problem to the resolution of equations by means of analytic algorithms, we must construct a qualitative and quantitative model of the physical process, a mathematical model. This is an art rather than a science. It is thus rather remarkable that computing machines can be used to guide the construction of models.

7. On computers. It is important to understand the value and limitations of modern large-scale digital computing equipment, and other types of automatic

computing and data-processing systems that are used in the numerical solution of problems. The digital computers that exist today have many useful attributes, principal among which are:

(a) They perform computations with extreme rapidity.
(b) They have very large memory units.
(c) They are extremely reliable.
(d) They are capable of performing binary choices.
(e) They are capable of automatic sequencing of operations.

The machines that are currently being developed and planned for the future will contain improvement over current equipment, primarily reflected by increased capabilities in items (a), (b), (c), above. If, however, one asks the question, "Will these computers eliminate any over-all computational problem that existed previously?," the answer is categorically "*No!*" Problems can always be formulated that are as intractable time-, accuracy-, and reliability-wise on any proposed computer, as the problems which we are now solving on current equipment were with the use of paper, pencil, and desk calculators. The prime advantage gained by use of the large scale equipment that exists *today* is that problems whose solutions are essential for today's technology are capable *today* of having solutions computed that are compatible with their speed and accuracy requirements. In a sense this is a revolution from an engineering, and data-handling and data-processing point of view. There is little or no gain from a theoretical point of view.

In fact, computers have given rise to a problem of practical significance that previously had only academic interest. We refer to the question of determining whether an arbitrary solution procedure is a proper algorithm. The application to computing machines of Gödel's theorem on logical indecision presents the following anomaly, see [14]. Solution procedures are often attempted where proper algorithms are unknown. Should a result be forthcoming from such a procedure, the algorithm itself is a proper one for at least a partially computable function (but need not be the correct one for the problem in question). If a properly operating computing machine must be stopped because of time considerations alone, one is faced with three possibilities:

(a) The time required for a solution was in excess of that allowed.
(b) The algorithm was proper, but the function values imposed were inadmissible.
(c) The algorithm was improper.

There is no a priori way to determine the applicable possibility.

The mathematician must utilize the computer as an engineer does for maximum efficiency. He may use it in two ways:

1. To obtain computational results which he feels are significant per se, where the computational effort is now reasonable whereas previously by hand and desk computation it was impractical.

2. To perform *mathematical* experiments which formerly were not feasible. We may wish to examine the primality of certain sequences of integers, or the behavior of the solutions of certain systems of nonlinear differential equations, or the effects of certain policies in various types of multistage decision processes.

It is generally recognized that although formal proofs are presented in deductive style, in actuality all great advances have been inductive. Discovery in the field of mathematics, as elsewhere, is a creative process.

It is worth pointing out that the experimental method was standard in the early days of number theory, and even in the early days of algebraic number theory. Much as Gauss divined the asymptotic distribution of primes from tables, and Euler the quadratic reciprocity theorem from examples, Artin formulated the Riemann hypothesis for finite fields as a result of computations based on particular primes. The inductive procedure is now capable of tremendous amplification with the aid of large-bore digital computers. Were a modern computer available to Fermat, he would not have hypothesized that the Mersenne number $2^{2^5}+1$ was a prime. He would have tested the sequence $2^{2^n}+1$ for many values of n before making a statement. On the other hand, with the aid of the SWAC at UCLA, the validity of Fermat's last theorem has been tested to unbelievable limits. It must be confessed that at the present time, very few of the computers are used for these exploratory purposes as contrasted with their use in applying known methods.

8. The principle of balance. To construct a mathematical model, we abstract certain features of the physical situation, make certain assumptions, and then pose an analytic problem whose solution gives us some information concerning the interaction of these features and the assumptions we have made. It may happen, and often does, that a very simple mathematical model gives rise to analytic problems of great intricacy, some of which may be insoluble in terms of present abilities. The simplest examples of this are in the field of number theory.

The question arises as to the physical validity of intricate solutions arising from simple models. As was indicated, it may be very difficult to derive a solution to a natural problem from the solution of a mathematical model of the problem. Concerning the third condition for a solution, however, we can prescribe an important criterion that is often overlooked: We postulate a simple

PRINCIPLE OF BALANCE: *A physically meaningful solution of a mathematical problem arising from a mathematical model of a physical process should never possess a greater degree of complexity than the mathematical model itself.*

Vague as the principle is (all useful principles of wide applicability must be so), we feel that it plays a vital role as a rule-of-thumb guide in the construction of mathematical models. Since the models determine to a great extent the ex-

penditure of research efforts and research funds, it is clear that it is worth devoting careful thought to these matters.

In this context, it must be brutally stated that a good deal of the mathematical work in the domain of applied mathematics in recent years resembles an inverted pyramid. From mathematical models of limited applicability and dubious validity have mushroomed thousands of pages of mathematical symbols, hundreds of papers and technical reports, and dozens of books. This formidable mass of equations constitutes one of the principal roadblocks to the consideration and solution of some of the most significant problems that face us. In each particular case, it must be ascertained whether a detailed analysis of a simplified model is worth as much as a crude analysis of a more realistic model. In the treatment of more realistic models by computational techniques, the computing machines can play a vital constructive role. As an accessory to the crime of turning out more reports with more graphs and more numbers based upon the apex of the inverted pyramid, it can play a quite destructive role.

Before contemplating the construction of bigger and better computers for the solution of specific problems, the foregoing comments should be duly considered.

9. A mathematical classification of problems. The mathematical problems derived from mathematical models can be separated into three categories:

1. *Deterministic problems.* These are problems where the mathematical formulation is in terms of determinate functions, *i.e.*, those having explicit values for specific values of their arguments. This does not imply that the solutions are readily accessible. For example, the problem may resolve into a nonlinear partial differential equation, the theory of which may be insufficiently developed even to guarantee the existence or uniqueness of a solution. On the other hand, the problem may be computationally impossible with existing computing equipment. The difficulties in deterministic problems are, however, always *analytic* or *computational*, never *conceptual.*

2. *Stochastic problems.* A far more important problem that arises practically is one in which all information is not known explicitly. We can consider a class of problems whose underlying structure is well undestood but where the detailed information is stochastic in character. These stochastic variables may have a known distribution function. However, in many cases, they may have unknown distribution functions, or these distributions may be approximate, based on past experiences or other extraneous information.

3. *Adaptive or learning processes.* Problems to which we assign this classification arise when the basic structure of a physical process, and hence of the associated problems, is incompletely known. As we set about determining a solution, we learn more about the process.

Although it may seem impossible at first sight to apply precise mathematical

techniques to problems which are not properly posed, this is actually not the case. The theory of games, sequential analysis, and dynamic programming all afford approaches to problems of this nature that arise from so many different applications.

10. The race against *N*!. In the remaining sections, we wish to illustrate some of the ways in which apparently innocuous problems can give rise to formidable computational and conceptual conundrums. Many of the difficult and significant problems in algebra, analysis, topology, mathematical physics, and the economic, industrial and military spheres are of combinatorial type. Let us discuss a particular problem of this nature; see [1], [2], for many further references.

Suppose that we have N men to be placed in N jobs. Assume that experience has shown that the value of placing the ith man in the jth job is given by a quantity a_{ij}, and that the value of any particular arrangement of these N men in N jobs is given by the sum of the individual values. It is clear that the first man can be placed in any of N different positions. Once this first man has been located, the second man can be placed in any of $(N-1)$ different positions. It follows, pursuing this line of argument, that the total number of possible ways in which N men can be placed in N jobs is $N!$.

If $N=10$, this number is $10!=3{,}628{,}800$, a large number, but not incredibly so. If $N=20$, a number of the same magnitude as 10, the enormity of 20! can perhaps best be understood in the following terms. Given a computing device which can evaluate the worth of any particular arrangement of 20 men in 20 jobs in a *microsecond*, it would require over 500,000 *years* to examine all of the 20! possibilities.

This then is what we mean by the race against $N!$. To treat problems of this type, highly over-simplified versions as they are of problems that arise in scheduling theory, we cannot employ direct enumerative techniques. We must develop analytic techniques and other computational algorithms. Remarkably enough, for the problem posed above, this has been done; references will be found in [1].

11. Approximate policies and mathematical experimentation. As noted above, the "assignment" problem can be resolved in a very short time by a number of elegant techniques. Slight changes, however, in the criterion for efficiency result in problems at present insoluble. It follows that it is of great interest to examine approximate policies of simple nature that can be used in situations of this type. Ideally, we would like to develop techniques that yield good approximations in those cases where we know the answer, to be used for problems without solution at present. It is here that computing machines can play an important role. With their aid, we can carry out mathematical experimentation on a scale undreamt of twenty years ago. The results of these experiments combined with analysis will yield useful approximate policies; see [3] for a discussion of the novel technique of Boldyreff. The concept of mathematical

experimentation is closely related to the study of simulation processes, [4]. These will be discussed below.

12. Detection problems, information patterns. One of the important military uses of computing machines is the SAGE system where these devices are used to keep track of enemy and friendly forces. In the case of enemy forces, it is vital to evaluate the signals of detection devices and particularly to distinguish between actual attackers and false signals arising from decoys. In order to indicate the types of problems that arise from this situation, let us discuss a simple version of the general problem.

"Suppose that we have N coins, all of the same appearance, all of equal weight except one which is heavier, and an equal arm balance. This equal arm balance may be used to weigh one batch of k coins against another, and will furnish the information that one batch is heavier, lighter, or of equal weight. The problem is to determine a weighing procedure which will minimize the maximum number of weighings required to isolate the heavy coin."

This problem, which is a favorite Sunday-supplement puzzle section question, may be easily analyzed for the following reason. If two batches of k coins are compared, two events can occur. Either the two batches balance, or one batch is heavier than the other. In the first case, we have the same problem with $N-2k$ coins, and in the second case, the same problem with k coins. We leave it to the reader to deduce the solution from this observation.

Let us now complicate the problem, by supposing that there are *two* heavy coins. Now if two batches balance, we know only that either batch contains one or no heavy coin. In place of having a simpler problem of the same type, we have a more complicated problem as a result of one observation. Furthermore, each successive test, up to a point, appears to complicate the description of the problem. The solution to this problem is unknown at present.

A problem of military import, of which detection is one part, is that of surveillance. Before we go about putting radar stations on land and sonar stations under water, we must ask ourselves what information we are looking for, how we are going to assess it, and what we are going to do about it. It seems clear that multiplying information sources and increasing sensitivity of detectors need not necessarily increase our effective defense against attack, and could conceivably weaken it, due to the effects of saturation and overload.

The problem of coin-weighing is a simple example of a search problem. Some other particular examples are:

1. Locate the maximum of a function of N variables over a given region of variability.
2. Locate the defective element in a multicomponent device. A particular aspect of this is the problem of writing handbooks for mechanics who must repair electrical or mechanical devices, see [5].

These illustrations are indicative of a fundamental problem in testing, com-

munication, and information theory in general. This is the problem of describing the state of a complex system in some efficient fashion. Both examples are representative of formidable problems in which we must simultaneously devise an information code to describe the state of the system and select a sequence of tests that can utilize this information in a most efficient fashion. A very interesting fact to emphasize is that the amount of information that is required depends heavily upon the policy that is employed. This is one of the basic principles of dynamic programming, [6]. Conversely, the policy that is employed depends equally upon the information that is available.

13. A note on computer design. Any problem involving the design of a computer or a solution by a designed computer must be in the form of a mathematical model. Of particular relevance to the problem of designing computers for specific applications is the question of obtaining "maximum information." If information is assumed to consist merely of an accumulation of data, it is safe to say that increasing the storage capacity of a device increases the amount of information we possess. Hence, in itself, it would seem to be a "good thing" to devote effort to the development of larger memories. However, we must understand how this quantity of information must be used in order to evaluate it. It is very easy to contemplate procedures which decrease in efficiency as the amount of information is increased.

A very simple example is furnished by the academic game of furnishing bibliographies. A few well-chosen papers greatly enhance the value of a paper. However, were a reader to read the fifty or so papers which are occasionally added like a tail to a kite, and then to read the papers cited in the bibliography to these, and so on, it is clear that not only would he never get to any original work, but what is worse, he would never necessarily get to read all the significant papers. It follows that one of the prime functions of an information gathering procedure is that of *rejecting* information.

14. Nonexistence of criteria of efficiency. One of the most baffling aspects of the problems that arise in military, economic, industrial, and sociological domains is the absence of any clear-cut goal. To take a very simple example, consider the problem of radar detection of enemy-manned or unmanned weapons. At first glance, the objective is very simple; we wish to recognize and destroy all enemy objects. In attempting to convert this into engineering terms, we realize very quickly that a *guarantee* of interception and destruction is at the present time beyond our abilities. We must then compromise.

This compromise takes various forms. Without paying any attention to the over-all system, we can concentrate on radar, data-processing, etc., and try to maximize the probability that an enemy object will be detected and that its position will be determined to within a desired accuracy within a prescribed time. It is clear, however, that these specifications cannot be guaranteed if a variety of enemy weapons are employed with appropriate countermeasures. The problem of countering countermeasures brings us into stochastic processes, se-

quential analysis, the theory of games, and the still more recent domain of adaptive processes.

We emphasize that the problem of component design cannot be completely separated from system performance. For surveillance, the specifications for a detection device depend to a great degree upon the type of object it is trying to detect, and the use that will be made of this information. These same difficulties plague mathematical economics.

15. Some nondeterministic examples. The problems treated in the preceding sections have been deterministic. By this we mean that each cause has a unique effect, and that this effect is known in advance. Consequently, the outcome of a sequence of decisions can be calculated before the process under study starts. Since it is obvious that very few processes of any significance will possess this property, it is imperative that we study more realistic situations. The basic problem that faces us in most situations is that of making decisions with incomplete information.

Consider, as illustrative, the problem of directing an aircraft whose aileron control is inoperable. One might ask the question, "How does an airplane fly when its aileron is flapping uncontrolled during its flight?" The equations representing the flight can be explicitly stated up to a functional representative for the displacement of the aileron but this displacement variable now has a stochastic character associated with it.

Problems of this type have a conceptual difficulty associated with them that was not present in the deterministic type problem. Mathematically, we may consider the airplane problem as a differential equation with a stochastic term in it. What is meant by the solution of such an equation? It is quite conceivable that a question of this type has no answer and yet extremely relevant information might be obtained from such a problem. Thus, one might deduce the stability of the solution of a problem of this type for certain distribution functions associated with the stochastic variable while for other types of distributions, one might expect unstable solutions. This class of problems has not been investigated extensively; see [7], [8].

16. The two-armed bandit problem. Stochastic problems divide naturally into two distinct parts. The first is the construction of a mathematical model of ignorance; the second is the application of analytic tools to treat the model. Of these, the most difficult part is that of choosing an appropriate model. It is worth dwelling upon the multiplicity of models, considering the way particular models are overworked in various areas. The impression is formed, justly or not, that the research workers in these areas feel that they are dealing with the "truth," rather than with a particular mathematical model. Let us consider the following levels of uncertainty for a known process structure:

1. The distribution functions for the stochastic effects are known.
2. The distribution functions are determined in functional form, but certain

parameters must be determined on the basis of observation and experiment.

3. The functional forms of the distribution functions are unknown.

Now suppose that we have two slot machines placed before us, each of which will cost us a dollar per play. We are told that these slot machines pay off or not on each play. We are given a limited number of plays to make. The problem is to switch back and forth in the play of the machines in such a way as to maximize the return. On the basis of all that has been said above, it should be clear that the problem as posed is meaningless, since we have not defined what we mean by the terms "return" or "maximize." In order to make it precise, we can proceed in a number of different ways. Let us follow the order of uncertainty given above.

1. The simplest model assumes that the first machine has a probability p_1 of paying a quantity q_1, and that the second machine has a probability p_2 of paying q_2. In addition, we postulate that we shall proceed so as to maximize the expected return. In this case, the solution is simple: we compare p_1q_1 and p_2q_2, and choose whichever machine has the larger product value for the total number of plays.

2. To treat this case, we assume that the p_i and q_i are fixed quantities, but that they are unknown to us. The first question that arises is that of understanding what we mean by maximizing. In order to do this, we must make precise what we mean by the term "unknown." One way of doing that is to assume that the p_i and q_i have been chosen from a given distribution. We then average the return from any particular process over this distribution. This is the usual assumption in statistics. If this assumption is unsatisfactory on the grounds that this distribution is not known, or may not exist, we can introduce the concept of a "*game against nature.*" For any given distribution, we maximize the expected return, and then choose this distribution so as to minimize this maximum expected return; see [9]. Alternatively, for those who feel that this is too pessimistic an outlook, we can merely average over some class of distributions.

3. The third category of problem can be formulated conceptually in the same terms, but the analysis is, in general, quite formidable at the present time [10], [11]. Once again, it is worth pointing out how difficult it is to describe the state of a stochastic process. A complete description of the two-armed bandit problem would require not only a specification of the amount of money obtained to date, but, in addition, the entire history of the process. This may not be a particular barrier as far as computing machine memory is concerned, but it does prevent effective use of any current analytic techniques.

In connection with the detection and tracking of aircraft to avoid air collisions, this accumulation of information could easily overwhelm the facilities of the computers. It follows that in treating these complete problems, we must find ways of rejecting information of little value. We must not think merely in terms of bigger memories and faster access times, since many of the basic problems are not in these directions at all.

17. Simulation processes. In this paper, we are concerned with general classes of mathematical problems and in the extended area of the development of mathematical models from practical considerations.

Any procedure that purports to be in a complete or partial correspondence with a practical procedure, we designate as a simulation procedure. In the most trivial sense, the flight of an aircraft can be used to deduce information about the flight of the aircraft. This would be the identity correspondence. On the other hand, the flight of an aircraft might be simulated by a system of aerodynamic and control equations. The time-dependent solutions would be equivalent to the information obtained by flying the aircraft directly. Although simulation attempts to invoke complete correspondence, it is clear that a complete duplication of a practical situation is impossible. This is because:

(a) It is impossible to consider all the variables involved. In fact, it generally is not known what all the variables are.

(b) It is impossible to represent the exact behavior of the variables of a physical situation either by a nonmathematical simulated situation, or by mathematical function. It is, however, always hoped that the simulation is close enough so that the results of the simulation approximate the true situation sufficiently closely.

Simulation processes are easier, cheaper, and sometimes safer, to construct and analyze than the actual processes. Simulation enables us to observe the effects of certain types of components and policies without worrying unduly about a specific criterion, or the determination of optimal policies. In many situations, simulated models must be used because it is unknown whether or not operational equipment is feasible. The results of the simulation determine the feasibility of constructing practical hardware.

There are many different simulators. These involve particular units of equipment, people, and mathematical representations combined in various proportions. In a full mathematical simulator we are generally faced with a deterministic problem although it is conceivable that the problem may be stochastic in nature. This is particularly true in problems involving random noise. In other types of simulations where mathematical representations are combined with equipment and people, the problem becomes completely stochastic in nature, because the effect of the equipment or of the individuals can only be approximated by stochastic functions. We do not wish to deal extensively with the problem of simulation in this paper. We only point it out as a very important origin of nondeterministic mathematical problems.

18. Adaptive problems. Let us consider the problem of optimizing the play in a chess game. At first glance, this appears to be a finite deterministic problem with a multiplicity of possibilities so extensive that it is beyond the computational facilities that exist today. However, there has never been any indication

that the game is finite in character, or that an underlying move structure exists. Aside from the win-loss requisites, the draw possibilities may be explicitly stated as:

(a) An explicit demonstration that a given board position can have no win-lose possibilities using any allowable line of play. Such a demonstration can be given in the case where two kings exist on the board, and in certain other extremely simple configurations.

(b) A demonstration that the game has entered into a cycle. By a cycle, we mean a sequence of moves leading to a situation identical to that at the beginning of the sequence, followed by the same sequence of moves.

It is entirely possible that the game can continue indefinitely without a win-loss possibility, and without a demonstrable draw situation. As a matter of fact, it has been demonstrated that noncyclic sequences of moves are possible, [12], [13]. The demonstration at any point that a win-loss condition is impossible is just an inductive extension of the over-all analysis of the game itself. Not knowing the structure of the game, we are faced with a classical imponderable question: how to optimize the play in a chess game. It is well known, however, that there are gradations in people's ability to play chess. It is highly unlikely that if the reader entered into a chess match with Bobby Fischer, he would have much of a chance of winning. Why? One is fairly confident that neither Fischer, nor Botvinnik, nor any of the other chess masters have achieved a solution to the optimal play problem that we have stated is currently imponderable. Yet, there is a strong correlation between certain individuals and success at chess.

There are many problems of this type where one can demonstrate that a solution is impossible under stated mathematical conditions, or imponderable and yet a high correlation exists between success in solving such problems and certain individual adaptive behavior. The problem thus resolves itself into establishing a substructure for analysis purposes that may produce some realistic results for problems of this type. We may consider a chess game in the small. Based on a fixed position, one may determine a probabilistic pattern that an opponent will follow. Thus, based upon the move he makes, he will put himself into a demonstratively lost situation or he will put himself in a weakened position, or he will be in a strong position. ("Weak" and "strong" positions are terms that *should* be defined, but *cannot* be in the present state of the art. Nevertheless, they are fairly obvious in meaning to competent chess players.) One might establish a probability pattern for making moves that would vary with the experience of the individuals playing the game and in such a fashion develop locally optimum procedures in a probabilistic sense. There is great interest at the present time in developing computers to play chess. As indicated, the problem is extremely difficult and progress is slow.

This type of reasoning can be applied to many other situations. Consider two opposing commanders facing each other in an essentially unknown tactical

situation. The better commander is one who can apply his forces most effectively against a best estimate of a probability distribution of possible modes of operation that are open to his counterpart. The problem is generally imponderable if a random behavior on the part of each commander is assumed in a war game analysis of such a situation.

19. Conclusion. The principal motivation for this discussion has been the increasing utilization of computer technology to solve problems of all types. We have attempted to analyze those problems that can be formulated mathematically and to show essential differences in problem concept and in problem difficulty.

The role of a computer is twofold:

1. Modern large scale computers have tended to balance practical problem-solution difficulties with the complexities of the problems arising in modern technology. There is every indication that this balance will be difficult to maintain with present computer developments.

2. Computers tremendously extend the range of mathematical testing, experimentation and inductive motivation into ever expanding areas of the mathematical unknown. They certainly tend to expand the cosmopolitan interests of mathematicians.

It must be continually remembered that a computer is not a factotum. When incorrectly used, it can be extremely damaging. Computers do not decrease the need for mathematical analysis, but rather greatly increase this need. They actually extend the use of analysis into the fields of computers and computation, the former area being almost unknown until recently, the latter never having been as intensively investigated as its importance warrants. Finally, it is up to the user of computational equipment to define his needs in terms of his problems. In any case, computers can never eliminate the need for problem-solving through human ingenuity and intelligence.

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RICCATI'S NONLINEAR DIFFERENTIAL EQUATION*

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1. Introduction. There are very few ordinary nonlinear differential equations which can be reduced to the linear form by special transformations. Among these few are the equations of Bernoulli, Jacobi, and Riccati, which have been cited in many books ([1]–[5]). The object of this paper is to extend the "conventional" transform used in solving Riccati's equation.

The so-called generalized Riccati equation is

$$dy/dx + P(x)y + Q(x)y^2 = R(x). \tag{1}$$

The transformation

$$y = z'/(Qz) \tag{2}$$

(where $z = z(x)$ and $z' = dz/dx$), is attributed to Riccati [1]. Since there is another possible transformation for the generalized Riccati equation, the transformation (2) will be called the "conventional" transformation in this paper.

The resultant linear differential equation obtained by using (2) is

$$z'' + [P - (Q'/Q)]z' - QRz = 0, \tag{3}$$

where primes stand for derivatives with respect to x. It is of value to review in a general way what Riccati did. Let

$$y = [u(x)f(x)]/g(x), \tag{4}$$

where $u(x)$, $f(x)$, and $g(x)$ are to be determined. Substitution of y and y' into (1) results in

$$u'fg + uf'g - ufg' + Pufg + Qu^2f^2 = Rg^2. \tag{5}$$

There are two places where derivatives of f and g appear. By making the sum of the third and fifth terms of the left-hand side of (5) vanish; *i.e.*, from $ufg' = Qu^2f^2$, there results

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$$f = g'/(Qu) \tag{6}$$

Therefore, (4) yields

$$y = g'/(Qg). \tag{7}$$

This is the conventional transformation.

Alternatively, by equating the second term of the left- and the right-hand side of (5); *i.e.*, from $uf'g = Rg^2$, there results $g = (uf')/R$. Therefore, (4) yields

$$y = (Rf)/f'. \tag{8}$$

When (8) is substituted into (1), the resultant linear differential equation is

$$f'' - [P + (R'/R)]f' - QRf = 0. \tag{9}$$

Thus, (8) is called the "new" transformation in this paper.

In comparing (3) and (9), it is interesting to note that:

(i) They are second-order variable-coefficient homogeneous linear differential equations.

(ii) The role played by $P(x)$ in (1) is small; this fact is also reflected in (3) and (9).

(iii) $Q=0$ in $y=g'/(gQ)$ gives trouble; yet $R=0$ in $y=(fR)/f'$ gives a trivial solution, provided $(g'/g)\neq 0$ and $(f'/f)\neq 0$, respectively.

The new transformation of (8) does not appear much different from the conventional one; however, their dissimilarities will become evident as higher-order and higher-degree nonlinear differential equations are studied in the following sections.

2. The conventional transformation. The conventional transformation of $y=g'/(Qg)$ is now applied to special forms of higher-order nonlinear differential equations. It would be of interest to find useful transformations which would reduce a typical second-order nonlinear equation such as

$$y'' + A(x)yy' + B(x)y'y^2 + C(x)y + D(x)y^2 = E(x) \tag{10}$$

to the linear form. The method used for extending the conventional transformation can be explained as follows:

Noting that $y=g'/(Qg)$ reduces the Riccati nonlinear (first-order) differential equation, the question raised is: "What type of higher-order nonlinear differential equation does this transformation linearize?" By direct differentiation of (2):

$$y'' = \frac{g'''}{Qg} - \frac{2Q'gg'' + Q''gg'}{Q^2g^2} - \frac{3Qg'g''}{Q^2g^2} + \frac{2Q'^2g'^2}{Q^3g} + \frac{2Q'g'^2}{Q^2g^2} + \frac{2g'^3}{Qg^3}. \tag{11}$$

When (11) is multiplied by g^3, it is observed that in order to eliminate nonlinear terms in g, terms such as $y'y$, and y^3 must be added with appropriate

coefficients. The resultant linear differential equations in g are obtained from these special forms of higher-order nonlinear differential equations in y by the transformation $y=g'/(Qg)$.

The second-order nonlinear differential equation

$$y'' + 3Qyy' + (W/Q)y' + Py + (Q' + W)y^2 + Q^2y^3 = R \tag{12}$$

is similarly obtained, where P, Q, R, and W are arbitrary functions of x. This process can be used to obtain linearizable equations of any order. A special case of (12) has been reported by E. L. Ince ([1], p. 318).

Reducing (12) results in the linear differential equation:

$$g''' - \frac{1}{Q}(2Q' - W)g'' + \frac{1}{Q}\left[PQ - Q'' + Q'\left(2\frac{Q'}{Q} - W\right)\right]g' - QRg = 0. \tag{13}$$

The conventional transformation $y=g'/(Qg)$ will not linearize second-degree first-order nonhomogeneous differential equations (see appendix). In fact, all higher-degree first-order nonhomogeneous differential equations are irreducible by the conventional method.

3. The new transformation. The advantages of the new transformation, *i.e.*, $y=(Rf)/f'$, are:

(i) It handles a certain special group of higher-degree first-order nonhomogeneous differential equations,

(ii) Its resultant linear differential equation does not virtually increase the order by one, while the conventional transformation does.

The first derivative of y is

$$y' = R + \frac{R'f}{f'} - \frac{Rff''}{f'^2}. \tag{14}$$

When y' is raised to the mth power, R^m appears; thus, R^m (a special form of nonhomogeneous term) is required rather than R (a general form of nonhomogeneous term) itself as in (1). The second-degree first-order nonhomogeneous differential equation and its resultant differential equation via $y=(Rf)/f'$ are respectively:

$$(y')^2 - 2Ry' - (R'/R)^2y^2 = -R^2, \tag{15}$$

$$f'' - 2(R'/R)f' = 0. \tag{16}$$

This process of deriving higher-degree first-order nonhomogeneous differential equations can be continued indefinitely. Moreover, this new application of the transformation $y=(Rf)/f'$ also solves nonlinear differential equations of orders higher than the first. For the second-order equation

$$yy'' - 2(y')^2 + 5Ry' - 3R'y = 3R^2, \tag{17}$$

the resultant linear equation is

$$f''' - 2\frac{R'}{R}f'' - \left[\frac{R''}{R} - 2\frac{R'^2}{R^2}\right]f' = 0. \tag{18}$$

This is essentially a second-order linear homogeneous differential equation. Again, this process of deriving any higher-order nonlinear equation of certain special forms continues indefinitely. Of course, as the order and degree of nonlinear differential equations increase, the two transformations, previously mentioned, lose their wide applicability, since the practical engineering and physical systems which utilize these special forms of equations are rare.

4. The extension of Riccati's equation. Two general equations which produce Riccati's equation as special forms are

$$y' + P(x)y + Q(x)y^k = R(x), \tag{19}$$

$$y' + P(x)y + Q(x)y^2 = R(x)y^k. \tag{20}$$

$k=2$ (in (19)) and $k=0$ (in (20)) provide the Riccati equation.

The transformation

$$y = \frac{a(g')^n}{Q^l g^m}, \tag{21}$$

where a, l, m, and n are constants, is applied to (19) as shown below.

$$\begin{aligned} anQ^l g^m (g')^{n-1} g'' - alQ^{l-1}Q' g^m (g')^n - amQ^l g^{m-1}(g')^{n+1} \\ + aPQ^l g^m (g')^n + a^k Q^{l(2-k)+1} g^{m(2-k)} (g')^{nk} = RQ^{2l} g^{2m}. \end{aligned} \tag{22}$$

The sum of the third and fifth terms of the left-hand side of (22) must vanish. Therefore,

$$amQ^l g^{m-1}(g')^{n+1} = a^k Q^{l(2-k)+1} g^{m(2-k)} (g')^{nk}.$$

The results

$$l = m = n = 1/(k-1), \tag{23}$$

$$a = (k-1)^{-1/(k-1)}, \tag{24}$$

$$y = \left[\frac{g'}{Qg(k-1)}\right]^{1/(k-1)}, \tag{25}$$

are obtained by equating exponents of like variables. The remaining terms reduce to

$$g'' + [P(k-1) - (Q'/Q)]g' - R[Qg(g')^{k-2}(k-1)^k]^{1/(k-1)} = 0. \tag{26}$$

Equation (26) can be made linear if $k=2$ for nonzero value of R. If $k \neq 2$ and

$k \neq 1$, the condition $R=0$ is necessary in order to have a resultant linear equation, such as

$$v'' + \lfloor P(k-1) - (Q'/Q) \rfloor g' = 0. \tag{27}$$

The transformation

$$y = \frac{bR^l g^n}{(g')^m}, \tag{28}$$

where b, l, m, and n are constants, is applied to (20). The result is

$$\begin{aligned} blR^{l-1}R'g^n(g')^m + bnR^l g^{n-1}(g')^{m+1} - mbR^l g^n(g')^{m-1}g^n \\ + bPR^l g^n(g')^m + b^2QR^{2l}g^{2n} = b^k R^{lk+1}g^{nk}(g')^{m(2-k)}. \end{aligned} \tag{29}$$

Equating the second term of the left- and right-hand sides of (29), *i.e.*, from

$$bnR^l g^{n-1}(g')^{m+1} = b^k R^{lk+1}g^{nk}(g')^{m(2-k)},$$

the results

$$= m = n = 1/(1-k), \tag{30}$$

$b=(1-k)^{1/(k-1)}$, and

$$y = \left[\frac{Rg(1-k)}{g'}\right]^{1/(1-k)}, \tag{31}$$

are derived in a manner similar to that used in obtaining (23), (24), and (25).

The remaining terms are arranged as

$$g'' - [P(1-k) + (R'/R)]g' - Q[R(1-k)^{2-k}g(g')^{-k}]^{1/(1-k)} = 0. \tag{32}$$

Equation (32) becomes a linear equation if $k=0$ for $Q \neq 0$.

Except for the special cases of $k=2$ and $R \neq 0$ in (19) and of cases $k=0$ and $Q \neq 0$ in (20), the two transformations of (25) and (31) are identical.

The Bernoulli equation is the case of $Q=0$ in (20) as shown below:

$$y' + P(x)y = R(x)y^k. \tag{33}$$

A transformation discovered by Leibnitz [1] is $Z=y^{1-k}$ where $k \neq 1$, $k \neq 0$, and $Z=Z(x)$. Equation (33) then becomes

$$Z' + P(1-k)Z = R(1-k). \tag{34}$$

The new transformation for reducing Bernoulli's equation (33) is (31), which reduces (33) to

$$g'' + [P(1-k) + (R'/R)]g' = 0. \tag{35}$$

Equation (34) is a nonhomogeneous equation, while (35) is essentially a first-order first-degree homogeneous differential equation. For easier methods of evaluating integrals, (35) has the advantage over (34).

Appendix. For simplicity, let a resultant linear equation be $g''=0$. Then

$$g' = a, \qquad g = a(x+b), \tag{A-1}$$

where a and b are constants. The result of substituting these values in (7) is $y=1/[Q(x+b)]$, and, from (A−1),

$$y' = -\frac{Q'}{Q^2(x+b)} - \frac{1}{Q(x+b)^2}. \tag{A-2}$$

Then, from (A−2),

$$(y')^2 = \frac{Q'^2}{Q^4(x+b)^2} + \frac{2Q'}{Q^3(x+b)^3} + \frac{1}{Q^2(x+b)^4}$$

and hence $(y')^2-(Q'/Q)^2y^2-2Q'y^3-Q^2y^4=0$, which is a homogeneous equation.

For the transformation $y=(Rg)/g'$, application of (A−1) yields

$$y = R(x+b), \qquad y' = R'(x+b)+R. \tag{A-3}$$

Then, from (A-3), $(y')^2-(R'/R)^2y^2-2R'y=R^2$, which is a nonhomogeneous equation.

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A THEOREM ON SELF-OVERLAPPING SEQUENCES

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1. Introduction. A finite sequence of n symbols will be said to be *self-overlapping* if the initial subsequence consisting of the first m terms is identical with the subsequence consisting of the final m terms for some m such that $0<m<n$. The sequences obtainable from a given sequence by the process of removing some initial subsequence and adding it on to the end of the sequence (*i.e.*, cycling) will be defined as *cyclically equivalent.* It will be shown that all sequences cyclically equivalent to a given sequence are self-overlapping if, and only if, the sequence is periodic in the sense that it is composed of an integral number (exceeding one) of identical subsequences.

For the sake of convenience let us refer to a subsequence consisting of the first m terms as a *prefix*, and similarly let the last m terms of the sequence be defined as a *suffix*.

Consider the sequence 011001110. The prefix consisting of the first term, a zero, is the same as the suffix consisting of the last term. Hence we can say that the sequence overlaps itself in that term, and this fact can be emphasized by writing the sequence as shown below:

011001110

011001110.

Now if the sequence is transformed by cycling the first term around to the end to obtain 110011100, another prefix, 1100, will be found to be identical to a suffix, and again the sequence is self overlapping, as shown below:

110011100

110011100.

The sequence 111001100 is cyclically equivalent to those considered above, but it is *not* self-overlapping.

THEOREM. *Every member of the set of sequences that are cyclically equivalent to a given sequence is self-overlapping if, and only if, the given sequence is made up of a chain of at least two identical subsequences.*

(In other words, if a sequence is not periodic in the sense defined above, then there exists a cyclically equivalent sequence that is not self-overlapping.)

Proof.

1. Suppose a sequence consists of a chain of two or more identical subsequences. Then the prefix comprised of the first subsequence is identical with the suffix comprised of the last subsequence and hence the sequence is self-overlapping. Since all cyclically equivalent forms of any periodic sequence are also periodic, it follows that the sufficiency portion of the theorem is true.

The remainder of the proof deals with the question of necessity. Given any sequence we shall show how it can be manipulated into a cyclically equivalent form that is self-overlapping only if it is periodic. The next eight steps consist of a partitioning process.

2. Choose an arbitrary element x, and cycle the sequence so that it starts with an x but does *not* end with an x. This can always be done unless the sequence contains only x's, in which case it is periodic.

3. Partition the sequence into E_0-elements, each beginning with a block of consecutive x's and ending just before the next x block. Let the first element of the sequence begin the first E_0-element. This uniquely defines the E_0's once step 2 has been accomplished.

4. From the set of E_0's choose those with x-blocks of *maximum* length. Select one of these with *minimum total* length and call it E_0^*. Also label as E_0^*-

elements all E_0's identical to the chosen E_0^*.

At this point let us illustrate steps 2–4, with an example. Start with the sequence,

$$abbbdcbaacbbbaacbbbdcbbacbbbc$$

Let $x=b$ and cycle the sequence to obtain

$$bbbdcbaacbbbaacbbbdcbbacbbbca.$$

The E_0's are then enclosed in parentheses to yield

$$(bbbdc)\ (baac)\ (bbbaac)\ (bbbdc)\ (bbac)\ (bbbca).$$

In accordance with step 4, E_0^* could be chosen as either $(bbbdc)$ or $(bbbca)$. If the former choice is made, then, indicating the E_0^*'s by underscoring, we obtain

$$\underline{(bbbdc)}\ (baac)\ (bbbaac)\ \underline{(bbbdc)}\ (bbac)\ (bbbca).$$

Resuming the proof:

5. If the sequence now consists entirely of E_0^*'s then the partitioning process is over. Otherwise we continue by first cycling the sequence so that it starts with an E_0^*.

6. Partition the sequence into E_1-elements by drawing partition lines just prior to the beginning of each E_0^*. Then the first E_1 begins with the first member of the sequence (an E_0^*) and contains all E_0's (if any) up to the next E_0^*, which begins the next E_1, etc. This is a unique process once steps 2–5 have been accomplished.

7. Choose one of the E_1-elements of minimum length (in terms of the original elements of the sequence) and call it, and all other E_1's identical to it, E_1^*-elements.

8. The above process can be continued as follows. Assume that the sequence has been partitioned into E_{i-1} and E_{i-1}^*-elements. If there are only E_{i-1}^*-elements then we stop the partitioning. Otherwise we cycle the sequence to start with an E_{i-1}^*. Then we form E_i-elements, each starting with an E_{i-1}^* and containing no other E_{i-1}^* (just as the E_1's were formed from the E_0^*'s and E_0's). One of the minimum length E_i-elements is then designated as E_i^* along with all other E_i's identical with it.

9. The induction process just described must terminate at some finite n, at which point the sequence will consist of a single E_k^* or of several E_k^*'s. This follows from the finiteness of the sequence and the fact that, for all i, the number of E_i^*-elements is less than the number of E_{i-1}^*-elements.

Now we shall discuss the overlap situation in terms of the partitions described above. Note that, during the partitioning process, no E_i or E_i^*-element (for any i) was ever split.

Suppose that a self-overlap is possible for the final form of the sequence.

Henceforth, the terms prefix and suffix will be reserved for the specific prefix and suffix that make the sequence self-overlapping. The subsequence common to both will be termed the overlap, with length $m<n$ (the length of the sequence).

10. Since the sequence terminates with the end of an E_i (or E_i^*) it follows that if any part of some E_i (or E_i^*)-element ($i \leqq n$) appears in the suffix, then the end of that element also appears there.

11. If an E_0^* begins in the prefix, then it must also be concluded there. The proof follows.

(a) Beginning with the first x of the E_0^*, its x's must all be matched in the suffix. Since no E_0 can have more x's than an E_0^* (see step 4 of this proof) it must follow that this block of x's is the beginning of an E_0 or E_0^* in the suffix.

(b) According to step 10 the element in the suffix must be complete there, and hence the overlap extends at least as far as the end of that element.

(c) But the E_0^* in the prefix starts concurrently with the E_0 or E_0^* in the suffix and cannot exceed it in length since, by definition of E_0^* (step 4), no E_0 with an x-block as long as the x-block of an E_0^* can be shorter than an E_0^*. Therefore there must be room in the prefix for the E_0^* to be complete.

12. Now we shall generalize the above result to show that, for all i, if an E_i^* starts in the prefix then it must be complete there. The proof is inductive. Assume that the statement is true for E_{j-1}^*. (We have shown it to be true for E_0^* in step 11.) Assume now that an E_j^*-element begins in the prefix.

(a) The E_j^* starts with an E_{j-1}^* and hence the E_{j-1}^* must be complete in the prefix (and therefore complete in the suffix as well).

(b) The E_{j-1}^* in the suffix is the start of an E_j or E_j^*. According to 10 this latter element must be complete in the suffix.

(c) But the E_j^* in the prefix starts concurrently with the E_j or E_j^* in the suffix and is no longer (by definition of the E_j^*—step 8). Hence there must be enough room in the prefix for the E_j^* to be complete there.

13. Now consider the sequence in the form of a chain of one or more identical E_j^*-elements (9). The initial E_k^* of the sequence starts in the prefix and hence, according to 12, must also end there. If this single E_k^* constituted the entire sequence, then the overlap would consist of the entire sequence, thus contradicting our assumption that $m<n$. Hence there must be at least two E_k^*'s, and the sequence is periodic. This concludes the proof.

Example. Let us continue the example presented after step 4. The underscored subsequences below are E_0^*'s, and the others are E_0's:

$$\underline{(bbbdc)}\ (baac)\ (bbbaac)\ \underline{(bbbdc)}\ (bbac)\ (bbbca).$$

The next step is to form E_1-elements, of which there are two (corresponding to the two E_0^*'s) shown in parentheses:

$$(bbbdcbaacbbbaac)\ (bbbdcbbacbbbca).$$

The second one is the shorter and it is therefore selected as E_1^* and written first

by cycling the sequence to obtain

$$\underline{(bbbdcbbacbbbca)}\ (bbbdcbaacbbbaac).$$

In the next step we obtain a single E_2-element, consisting of the entire sequence, so that the whole sequence is an E_2^* as follows:

$$\underline{bbbdcbbacbbbcabbbdcbaacbbbaac}$$

The reader may verify the fact that no overlap is possible.

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BASIC PROPERTIES OF PANDIAGONAL MAGIC SQUARES

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1. The following investigation originated from a suggestion by Chater and Chater [1] that all pandiagonal magic squares of even order are of a simple complementary pattern, and from a note by D. F. Lawden [2] showing that for such squares the determinant vanished. The question arose as to the validity of this conjecture for even squares and the possibility of a corresponding property for odd squares. If a_{rs} is the number standing in the rth row and the sth column of a square of order $2n$ the simple complementary property is that $a_{rs}+a_{r+n,s+n}=$constant. It is shown that this simple property, whilst holding for all squares of order 4, does not hold in general but can be replaced by rather more complicated properties having a complementary character. From the basic line-invariants of the type $\sum_{r=1}^{n} a_{rs}=$constant, for the square of order n, other invariants are deduced which prove to be of interest as regards the construction of magic squares and the determination of the number of independent elements in such a square. The patterns corresponding to these invariants are intriguing. Further an octagonal matrix emerges which would seem to be fundamental.

In a pandiagonal magic square the sum of the elements in any row, in any column, in any diagonal (straight or broken) add up to a constant S, the number for the square. It follows immediately that if in any pandiagonal magic square the first column is shifted to a position immediately following the last column, the new square is also pandiagonal magic. Similarly if several columns or rows are so shifted, the square remains pandiagonal magic. Thus if the square is

repeated indefinitely to either side and the resulting infinite strip is repeated indefinitely above and below, the whole plane is filled "parquet-wise." A square frame of side n placed on the plane and adjusted so as not to cut the smaller cells, will enclose a pandiagonal magic square. We may speak of these extra n^2-1 squares so formed as "variants" of the original square. Again, it follows immediately from the definition, that if any two pandiagonal magic squares of the same order are added together as matrices (*i.e.*, corresponding elements are added) the resulting square is pandiagonal magic. Hence we have,

THEOREM 1. *The matrix sum of any pandiagonal magic square and any constant multiples of its variants produces a square which is also pandiagonal magic.*

We have from here on, subtracted S/n from every element in the square so that the sum of the elements in any row, column or diagonal will be zero, and we call such modified zero-sum squares, for short, "z-squares." Since, added as matrices, the sum of any two pandiagonal squares of the same order is itself a pandiagonal square, it follows that a z-square is also a pandiagonal magic square from which those squares with constant S can be immediately obtained.

It is convenient, following Chater and Chater [1], to introduce operators R, C such that

$$Ra_{r,s} = a_{r-1,s}, \qquad Ca_{r,s} = a_{r,s+1},$$

so that R converts any row into the row immediately below that row, and by the parquet property, it converts the last row into the first one; hence $R^n=1$ and similarly $C^n=1$.

Before dealing with the general z-square of order n it is of interest to investigate those of orders 4, 5 and 6.

2. The z-square of order 4. Since the sum of the elements in any column, in any row, and in any diagonal (straight or broken) is zero, we have, operating on any element a_{rs}, the following line-invariants,

(1) column: $$(1 + R + R^2 + R^3)a_{rs} = 0,$$

(2) row: $$(1 + C + C^2 + C^3)a_{rs} = 0,$$

(3) diagonal: $$(1 + RC + R^2C^2 + R^3C^3)a_{rs} = 0,$$

(4) $$(R^3 + R^2C + RC^2 + C^3)a_{rs} = 0.$$

Subtracting (3)–(1) gives

$$R(C - 1)[1 + R(C + 1) + R^2(C^2 + C + 1)]a_{rs} = 0.$$

Hence $[1+R(C+1)+R^2(C^2+C+1)]a_{r+1,s}=Q_{r+1}$ (say), where Q_{r+1} is a quantity depending on the row $r+1$ but independent of the column s. Clearly,

$$\sum_{s=1}^{4} Q_{r+1} = [1 + R(C + 1) + R^2(C^2 + C + 1)] \sum_{s=1}^{4} a_{r+1,s} = 0$$

and therefore $Q_{r+1}=0$; hence $[1+R(C+1)+R^2(C^2+C+1)]a_{r+1,r}=0$. Since this is true for all values of r, s it is convenient to suppress the operand a_{rs} when using these operational formulae so that we may write

$$1 + R(C + 1) + R^2(C^2 + C + 1) = 0. \tag{5}$$

The elements involved in (5) form a triangle and we call this invariant the triangle-invariant. (See diagram (a).)

1	•	•	•
1	1	•	•
1	1	1	•
•	•	•	•

diagram (a)

1	1	•	•
1	1	•	•
•	•	•	•
•	•	•	•

diagram (b)

By symmetry, we may in (5) interchange the operators R, C so that

$$1 + C(R + 1) + C^2(R^2 + R + 1) = 0 \tag{6}$$

and subtracting (5)–(6) gives $(R - C)[1 + R + C + RC] = 0$, so that $[1+R+C+RC]a_{rs}=Q$, where Q is a quantity which is constant along a diagonal, upwards from left to right, and hence $\sum Q$ vanishes when taken along this diagonal, and hence $Q=0$. Hence we have established the "square-invariant" diagram (b),

$$1 + R + C + RC = 0. \tag{7}$$

From the diagrams (a), (b) it can be seen that a simpler invariant is obtained by subtracting the square-invariant from the triangle-invariant. In fact taking (5)–R(7), we obtain

$$1 + R(C + 1) + R^2(C^2 + C + 1) - R(1 + R + C + RC) = 0,$$

$$1 + R^2C^2 = 0. \tag{8}$$

This is the simple complementary property $a_{r,s}+a_{r+2,s+2}=0$ and must therefore hold generally for all z-squares of order 4.

The existence of the invariants (2), (7), (8), namely, the row, square, and complementary invariants, enables us to construct the general z-square of order 4 as economically as possible, that is, with the use of the minimum number of specified elements and therefore not involving resultant simultaneous equations from which redundant elements must be eliminated. To sketch the process briefly: Set down the elements a_{11}, a_{12}, a_{13}; then (2) determines a_{14}. Set down the element a_{21}; then (7) determines a_{22} and in turn a_{23} and then by (2) or (7), a_{24}. The first two rows being complete, the remainder of the z-square follows immediately from the complementary property (8).

If we call the independent elements a, b, c, d, the general z-square of order 4 then becomes,

a	b	c	$-a-b-c$
d	$-a-b-d$	$a-c+d$	$b+c-d$
$-c$	$a+b+c$	$-a$	$-b$
$-a+c-d$	$-b-c+d$	$-d$	$a+b+d$

It is of interest here to notice that if we put $a=1$, $b=c=d=0$, we obtain the pattern of diagram (c) in which, by extending parquet-wise to the next two rows, gives the octagonal pattern of diagram (d). By putting in turn $b=1$, $c=1$,

1	·	·	−1
·	−1	1	·
·	1	−1	·
−1	·	·	1

diagram (c)

·	1	−1	·
−1	·	·	1
1	·	·	−1
·	−1	1	·

diagram (d)

$d=1$, and the rest of the independent elements zero, we obtain patterns which are reducible to the simple octagon or to sums of octagons. Notice that the octagon as shown in diagram (d) is itself a z-square and if bordered with the appropriate number of rows and columns of zeros, still remains a z-square of any order, and hence when added to a z-square the result is still a z-square. Hence in view of (8) we have,

THEOREM 2. *All z-squares of order 4 have the simple complementary property $a_{r+2,s+2}=a_{r,s}$, and for all orders of z-squares greater than 4, there exist z-squares which do not possess this property.*

3. The z-square of order 5. Exactly as in Section 2 we have the two line-invariants,

$$1+R+R^2+R^3+R^4=0, \tag{9}$$

$$1+RC+R^2C^2+R^3C^3+R^4C^4=0. \tag{10}$$

1	·	·	·	·
1	1	·	·	·
1	1	1	·	·
1	1	1	1	·
·	·	·	·	·

diagram (e)

1	1	1	·	·
1	2	1	·	·
1	1	1	·	·
·	·	·	·	·
·	·	·	·	·

diagram (f)

·	·	·	−1	·
·	1	·	·	·
·	·	1	·	·
−1	·	·	·	·
·	·	·	·	·

diagram (g)

·	·	·	·	·
·	1	1	·	·
·	1	1	·	·
·	·	·	·	·
·	·	·	·	1

diagram (h)

Subtracting (10)–(9) and justifying the removal of the factor $R(C-1)$ we have immediately the corresponding triangle-invariant shown in diagram (e),

$$1 + R(1 + C) + R^2(1 + C + C^2) + R^3(1 + C + C^2 + C^3) = 0. \tag{11}$$

Again as in Section 2, interchanging R, C, subtracting, and justifying the removal of the factor $R-C$, we obtain the square-invariant of diagram (f),

$$(1 + R + R^2)(1 + C + C^2) + RC = 0. \tag{12}$$

The pattern of these squares is beginning to emerge and shows that the weighting increases towards the center of the square.

If we invert the triangle (e) and subtract from the square (f), (we can most easily work from the diagrams), we have

$$\begin{aligned}(1 + R + R^2)(1 + C + C^2) + RC \\ - [1 + C + C^2 + C^3 + R(1 + C + C^2) + R^2(1 + C) + R^3] = 0,\end{aligned}$$

$$- R^3 - C^3 + RC + R^2C^2 = 0, \tag{13}$$

as given in diagram (g). The negative terms of (13) may be removed by adding in a diagonal-invariant of the type (4), *viz.*, $C^3+RC^2+R^2C+R^3+R^4C^4=0$. This yields the invariant

$$RC(1 + R)(1 + C) + R^4C^4 = 0, \tag{14}$$

which is shown in diagram (h). It will be realized later that this is the counterpart for a square of order 5, of the simple complementary property already found for squares of order 4.

There is a further invariant which is important and may be derived as follows: The triangle-invariant (11) remains valid if we replace C by $1/C$ as this merely represents a reflection in a vertical line, and gives

$$1 + R(1 + C^{-1}) + R^2(1 + C^{-1} + C^{-2}) + R^3(1 + C^{-1} + C^{-2} + C^{-3}) = 0.$$

Subtracting this from (11) then gives

$$R(C - C^{-1})[1 + R(1 + C + C^{-1}) + R^2(2 + C + C^{-1} + C^2 + C^{-2})] = 0,$$

and since the line-invariant $C^{-2}+C^{-1}+1+C+C^2=0$, this becomes

$$R(C - C^{-1})[1 + R(C + 1 + C^{-1}) + R^2] = 0.$$

Hence if $[1+R(C+1+C^{-1})+R^2]a_{rs}=Q_{rs}$, then $Q_{r,s-1}=Q_{r,s+1}=Q_{r,s+3}= \cdots =Q$ (say). Now since $Q_{r,s+5}=Q_{r,s}$, it follows that these terms are equal, each being $=Q$, and further summing throughout a row, $\sum Q=0$, hence $Q=0$. So that we have an invariant

$$1 + R(C + 1 + C^{-1}) + R^2 = 0 \tag{15}$$

in which the terms form a diamond as shown in diagram (i). It must be emphasized that the division by the factor $C-C^{-1}$ is only permissible in a square of odd order and can be made to depend on the fact that $1+C^n$ has a factor $1+C$ for all odd values of n. This would seem to be the fundamental difference be-

tween squares of odd and even orders, *viz.*, the existence or nonexistence of a diamond invariant of the type (15).

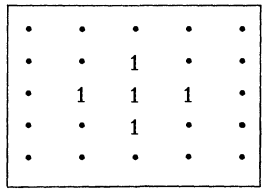


diagram (i)

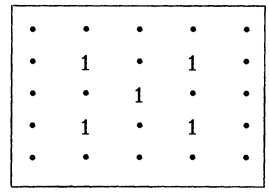


diagram (j)

We are now in a position to construct the general z-square of order 5 as economically as possible: Lay down the first four elements in each of the first two rows, namely a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{24}; the line-invariant (9) then gives a_{15} and a_{25}. Since the first two rows are now complete, repeated application of the diamond-invariant gives all the elements of the third row and further repetitions give also the fourth and fifth rows. Alternatively these can be found using the invariant (14) to give the fourth row and the fifth row then follows by using the column-invariants of the type (9). Thus it follows that the general z-square of order 5 can contain, at most, eight independent elements.

The general z-square of order 5 may be written as,

a	b	c	d	$-a-b-c-d$
e	f	g	h	$-e-f-g-h$
$g+h-a$	$-b-e-f-g$	$-c-f-g-h$	$-d+e+f$	$a+b+c+d+f+g$
$-c-d-g-h$	$a+b+c+e+f+g$	$b+c+d+f+g+h$	$-a-b-e-f$	$-b-c-f-g$
$c+d-e$	$-a-b-c-f$	$-b-c-d-g$	$a+b-h$	$b+c+e+f+g+h$

Again it is of interest to note that putting all but one, in turn, of the independent elements equal to zero, the corresponding patterns which result are each easily expressible in terms of the octagonal matrix in diagram (d).

Other symmetrical invariant patterns can easily be produced, for example, that of diagram (j) which is found by subtracting the diamond (15) from the square of diagram (f). These are of use as a check.

4. The z-square of order 6. The fundamental processes having been established we give only the main invariant patterns arising for a z-square of order 6 in the diagrams (k), $\cdots$, (p).

In the process of constructing the general z-square of order 6, three rows of five elements must be laid down, together with one element of the fourth row, so that the invariant of diagram (o) or of diagram (p) can be applied. The fifth

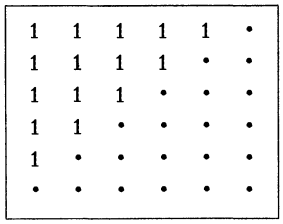


diagram (k)

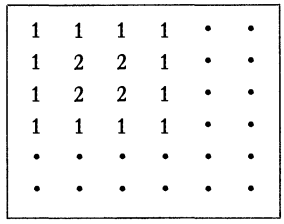


diagram (l)

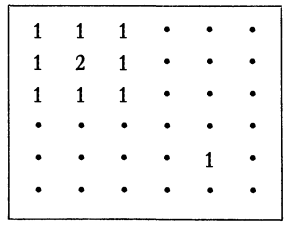


diagram (m)

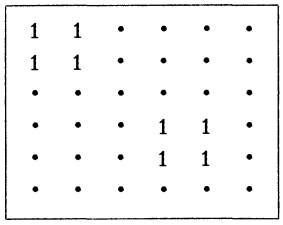


diagram (n)

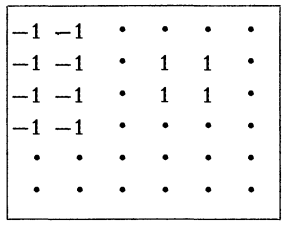


diagram (o)

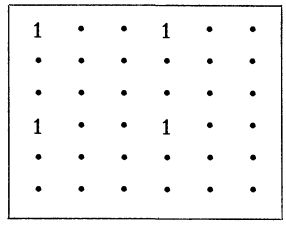


diagram (p)

row then follows by diagram (m) and the square is then completed using the column-invariant. Hence 16 independent elements are required.

Again it is found that by putting all independent elements save one, equal to zero, the various patterns obtained are each expressible in terms of the octagonal matrix. This gives rise to the conjecture that z-squares of any order can be built up entirely from numerical multiples of the octagon.

It may be mentioned that diagram (o) is derived from diagram (n) by subtracting the column-invariant from each of the first and second columns. Diagram (p) is derived from diagram (l) by multiplication by $1+R^3$ and subtracting the column-invariant from each of the first four columns. This invariant is peculiar to a z-square of order 6 and is particularly useful in its construction.

5. General considerations. z-squares of order n. It is now fairly evident that we can separate those invariants which arise generally from those which only arise in special cases, *e.g.*, the "five-spot" of diagram (j), the open-square of diagram (p) and the complementary pair (7). It follows easily from the earlier methods that for the general z-square of order n, by subtracting $\psi(RC)-\psi(R)$, where

$$\psi(R) \equiv 1 + R + \cdots + R^{n-1}, \tag{16}$$

and taking out a factor $R(C-1)$, we obtain the triangle-invariant with side $n-1$ given by

$$1 + R(1 + C) + R^2(1 + C + C^2) + \cdots + R^{n-2}(1 + C + \cdots + C^{n-2}). \tag{17}$$

Interchanging R, C, subtracting, and dividing by $R-C$ we obtain the square-invariant of side $n-2$,

$$(18)\qquad \begin{aligned}&(1+R+\cdots+R^{n-3})(1+C+\cdots+C^{n-3})\\&\quad+RC(1+R+\cdots+R^{n-5})(1+C+\cdots+C^{n-5})+\cdots,\end{aligned}$$

the last term being $(RC)^{(n-3)/2}$ or $(RC)^{(n-4)/2}(1+R)(1+C)$ according as n is odd or even, respectively. This is illustrated for the case $n=7$ in diagram (q).

1	1	1	1	1	•	•
1	2	2	2	1	•	•
1	2	3	2	1	•	•
1	2	2	2	1	•	•
1	1	1	1	1	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•

diagram (q)

1	1	1	1	•	•	•
1	2	2	1	•	•	•
1	2	2	1	•	•	•
1	1	1	1	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	1	•
•	•	•	•	•	•	•

diagram (r)

Subtracting the triangle-invariant from (18) and adding in the diagonal-invariant gives the invariant with two complementary squares of sides 1 and $n-3$ shown in diagram (r).

Alternatively, subtracting the row-invariant from each of the first $n-2$ rows of the square-invariant (q), (giving diagram (s)), and adding the column-invariant to the last two columns produces the invariant consisting of two complementary squares of sides 2 and $n-4$, as shown in diagram (t). These constitute the first two members of a sequence of "complementary squares" invariants.

•	•	•	•	•	−1	−1
•	1	1	1	•	−1	−1
•	1	2	1	•	−1	−1
•	1	1	1	•	−1	−1
•	•	•	•	•	−1	−1
•	•	•	•	•	•	•
•	•	•	•	•	•	•

diagram (s)

•	1	1	1	•	•	•
•	1	2	1	•	•	•
•	1	1	1	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	1	1
•	•	•	•	•	1	1
•	•	•	•	•	•	•

diagram (t)

•	•	1	•	•	•	•
•	1	1	1	•	•	•
1	1	2	1	1	•	•
•	1	1	1	•	•	•
•	•	1	•	•	•	•
•	•	•	•	•	•	•
•	•	•	•	•	•	•

diagram (u)

Finally, in the triangle-invariant, by replacing C by $1/C$ and subtracting, we have generally for any z-square of odd order a diamond-invariant of the type in diagram (u).

The construction of the general z-square of order n now becomes automatic using the methods already set out. First lay down arbitrarily a matrix of $n-3$ rows and $n-1$ columns. The rows are completed by the row-invariants.

For the case in which n is odd the diamond-invariant will complete the $(n-2)$th row and then an invariant of type (r) completes the $(n-1)$th row. The square is then completed by the use of column-invariants.

For the case in which n is even an additional arbitrary member is necessary in the $(n-2)$th row; this enables invariants of the type (s) to be used to complete the $(n-2)$th row. The square is then completed in the same way as in the previous case.

This gives the following theorem:

THEOREM 3. *The number of independent elements in a general z-square of order n cannot exceed* $(n-1)(n-3)$ *if n is odd, or* $(n-2)^2$, *i.e.*, $(n-1)(n-3)+1$ *if n is even.*

It is conjectured that the general z-square will actually require these number of independent constants. (The general pandiagonal magic square would of course require one extra arbitrary element.)

6. The octagonal matrix. It is convenient to represent the matrix $a_{r,s}$ by a polynomial $F(R, C)$ given by

$$F(R, C) = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{r,s} R^{r-1} C^{s-1}. \tag{19}$$

If, in this way, the octagonal matrix is represented by $\phi(R, C)$ then from diagram (d),

$$\begin{aligned} \phi(R, C) &= -R(1-R)(1-C^3) + C(1-C)(1-R^3) \\ &= (1-R)(1-C)[C(1+R+R^2) - R(1+C+C^2)] \\ &= -(1-R)(1-C)(R-C)(1-RC). \end{aligned} \tag{20}$$

The symmetry and simplicity of this representation indicates the fundamental nature of the octagonal matrix. It has been mentioned in Section 2 that wherever the octagonal matrix is placed on the "parquet" it forms, when bordered by a suitable number of rows and columns of zeros, a z-square of order n for all values of n. Combining this fact with Theorem 1 we have,

THEOREM 4. *The matrix represented by* $F(R, C)$ *such that*

$$F(R, C) = f(R, C)\phi(R, C) \tag{21}$$

is a z-square for all polynomials $f(R, C)$.

This theorem gives immediately an easy method for the construction of z-squares and hence also of pandiagonal magic squares.

It is possible to establish a converse of Theorem 4 in the following way:

If the polynomial $F(R, C)$ represents a z-square, then for each value of s, $\sum_{r=1}^{n} a_{r,s}=0$, so that $F(1, C)=0$ identically in C, and hence $F(R, C)$ is divisible by $1-R$. Let the quotient be $f(R, C)$. Now if we define $f_1(R, C)$ by the equation,

$$f_1(R, C) = f(R, C)[1 - n^{-1}\psi(R)], \tag{22}$$

where $\psi(R)$ is given by (16), we have immediately,

$$(1 - R)f_1(R, C) = (1 - R)f(R, C) - n^{-1}f(R, C)(1 - R)\psi(R) = F(R, C), \tag{23}$$

since $(1-R)\psi(R)=0$, so that $f_1(R, C)$ is a particular quotient of $F(R, C)/(1-R)$. Further, since $\psi(1)=n$, $1-n^{-1}\psi(R)$ has a factor $1-R$; in fact

$$1 - n^{-1}\psi(R) = (1 - R)\psi_1(R),$$

where

$$n\psi_1(R) = n + (n - 1)R + (n - 2)R^2 + \cdots + R^{n-2}. \tag{24}$$

Thus $F(R, C)/(1-R)=f(R, C)(1-R)\psi_1(R)=\psi_1(R)F(R, C)$, and hence by Theorem 4, there exists a particular quotient $F(R, C)/(1-R)$ which represents a z-square.

Similarly by interchanging R, C the result also holds for division by $1-C$.

Again from (19) if we put $\alpha=RC$ we have

$$F(R, C) = \sum_{r=1}^{n}\sum_{s=1}^{n} a_{r+s,s}R^{r-1}(RC)^{s-1} = F^*(\alpha, C). \tag{25}$$

In precisely the same manner $F^*(1, C)=0$ identically in C and the process shows that $F^*(\alpha, C)/(1-\alpha)$ also represents a z-square. Replacing C by $1/C$ shows that the same result holds for division by $R-C$.

Combining the results for successive divisions by $1-R$, $1-C$, $R-C$, $1-RC$, and noting the form of ϕ in (20) we have;

THEOREM 5. *If $F(R, C)$ represents a z-square then there exists a particular quotient $F(R, C)/\phi(R, C)=f(R, C)$ such that $f(R, C)$ also represents a z-square.*

This implies that the field of z-squares is, in this sense, closed under the operator ϕ.

References

1. N. Chater and W. J. Chater, A note on pan-magic squares, Math. Gaz., vol. 29, 1945, p. 92; On the determinants of pan-magic squares of even order, Math. Gaz., vol. 33, 1949, p. 94.

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UNIFORMLY CONTINUOUS SETS IN METRIC SPACES

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It is well known that every function which is continuous on a compact set is uniformly continuous on the set. The converse, however, is not true; that is, there exist sets which are not compact but which possess the property that every function which is continuous on the set is uniformly continuous on the set. N. Levine has characterized linear sets having the above property provided the image space is the set of real numbers.*

It is the purpose of this paper to further investigate sets each of which has the property that every continuous function from the set into M_1 is uniformly continuous on the set, where (M_1, d_1) is a given metric space.

DEFINITION. *Let (M, d) and (M_1, d_1) be metric spaces. Then a subset E of M is said to be U.C. (M_1) if and only if every function from E into M_1 which is continuous on E is uniformly continuous on E.*

DEFINITION. *Let (M, d) be a metric space and E a subset of M. Then E is said to U.I. (uniformly isolated) if and only if there exists a real number $\eta > 0$ such that $d(p, q) \geqq \eta$ for every $p \neq q$ in E.*

THEOREM 1. *Let (M, d) be a metric space and E a subset of M. If E is U.C. (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, then (E, d) is a complete metric space.*†

Proof. Suppose (E, d) is not complete. Then there is a Cauchy sequence $\{p_i\}_{i=1}^{\infty}$ of distinct points of E such that $\bigcup_{i=1}^{\infty} p_i$ has no limit point in E. Define $A = \bigcup_{i=1}^{\infty} p_{2i-1}$, $B = \bigcup_{i=1}^{\infty} p_{2i}$. Then A and B are disjoint closed subsets of E. Since (E, d) is normal, there exists a continuous function f from E into R_1 such that $0 \leqq f(p) \leqq 1$ for every $p \in E$, $f(p) = 0$ if $p \in A$, $f(p) = 1$ if $p \in B$.§ There is a homeomorphism h of the closed unit interval $[0, 1] \subset R_1$ onto the arc axb of M_1 such that $h(0) = a$, $h(1) = b$. It follows that the restriction $g = h/f(E)$ of h to $f(E) \subset [0, 1]$ is continuous on $F(E)$. The function $k = gf$ defined by $k(p) = g[f(p)]$ for every $p \in E$ is clearly a continuous function from E into M_1 which is not uniformly continuous on E. This gives a contradiction and the theorem is proved.

COROLLARY. *Let (M, d) be a metric space and E a subset of M. If E is U.C. (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, then E is closed.*

* N. Levine, Uniformly Continuous Linear Sets, this MONTHLY, vol. 62, 1955, pp. 579–580.

† The authors are indebted to the referee for this theorem.

§ D. W. Hall and G. L. Spencer, Elementary Topology, New York, 1955, pp. 110–121.

THEOREM 2. *Let (M, d) and (M_1, d_1) be metric spaces and E a subset of M. Then a sufficient condition for E to be U.C. (M_1) is that $E=E_1\cup E_2$, where E_1 is compact and E_2 is U.I.*

Proof. The proof is similar to that of Theorem 2 in the previously mentioned article by Levine.

DEFINITION. *Let (M, d) be a metric space and E a subset of M. Then E is U.C. $(R_1, +)$ if and only if every nonnegative real-valued function which is continuous on E is uniformly continuous on E.*

THEOREM 3. *Let (M, d) be a metric space and E a subset of M. Then a necessary and sufficient condition that E be U.C. $(R_1, +)$ is that E be U.C. (R_1).*

Proof. The sufficient condition is obvious. To prove the necessary condition, assume $E\neq\varnothing$ (the theorem is trivial if $E=\varnothing$), let E be U.C. $(R_1, +)$ and let f be any continuous real-valued function on E. Define functions f_1 and f_2 from E into R_1 as follows: for every $p\in E$, let $f_1(p)=f(p)$ if $f(p)\geqq 0$ and $f_1(p)=0$ if $f(p)<0$; and let $f_2(p)=0$ if $f(p)\geqq 0$ and $f_2(p)=f(p)$ if $f(p)<0$. Then for every $p\in E$, $f(p)=f_1(p)+f_2(p)$. Each of f_1 and f_2 is continuous on E. Moreover, each of f_1 and $-f_2$ is nonnegative and continuous on E and hence uniformly continuous on E. It follows that f_2 is uniformly continuous on E and $f=f_1+f_2$ is uniformly continuous on E, and the theorem is proved.

DEFINITION. *A metric space (M, d) is said to be a W-B space if and only if every closed and bounded subset of M is compact.*

Remark. The property of being a *W-B* space is not a topological property. (R_1, d_{R_1}), d_{R_1} being the standard metric, is a *W-B* space but (I^0, d_{R_1}) where I^0 is the open interval $0<x<1$ is not a *W-B* space although I^0 is homeomorphic to R_1.

THEOREM 4. *If E is a subset of a W-B space (M, d) and E is U.C. (R_1), then $E=E_1\cup E_2$, where E_1 is compact and E_2 is U.I.*

Proof. E is U.C. (R_1) implies that E is U.C. $(R_1, +)$. The theorem is trivial if $E=\varnothing$. Therefore assume $E\neq\varnothing$. Let p_* be a fixed point in E and consider the set $\{N(p_*, r)^- \mid r\in R_1\}$ where $N(p_*, r)^-$ is the closure of $N(p_*, r)$. There exists a point $r_*\in R_1$ such that $E-N(p_*, r_*)^-$ is *U.I.* For if not, then $E-N(p_*, r)^-$ is *U.I.* for no $r\in R_1$ implies that there exists a sequence $\{p_i\}_{i=1}^{\infty}$ of distinct points in E such that:

(i) $\lim r_i=\infty$ where $r_i=d(p_i, p_*)$; $i=1, 2, \cdots$,

(ii) $d(p_{2i-1}, p_{2i})<1/i$; $i=1, 2, \cdots$,

(iii) $r_1\leqq r_2<r_3\leqq r_4<\cdots<r_{2i-1}\leqq r_{2i}<r_{2i+1}\leqq\cdots$.

Furthermore, there is no loss in generality in assuming

(iv) $r_{2i}+2\leqq r_{2i+2}$ for all $i=1, 2, \cdots$.

It is easily verified that:

(i) $d(p_{2i}, p_{2j})\geqq 2$ if $i\neq j$,

(ii) $p_{2j-1} \notin N(p_{2i}, s_i)^-$, $i = 1, 2, \cdots$ for each $j = 1, 2, \cdots$, where $s_i = d(p_{2i-1}, p_{2i})/2$ for each $i = 1, 2, \cdots$.

(iii) $N(p_{2j}, s_j)^- \cap N(p_{2i}, s_i)^- = \varnothing$ if $i \neq j$.

A nonnegative function f from E into R_1 which is continuous on E but is not uniformly continuous on E can now be constructed contrary to E being $U.C.$ $(R_1, +)$. Define f from E into R_1 as follows: for every $p \in E$, let $f(p) = 1$ if $p \notin N(p_{2i}, s_i)^-$ for all $i = 1, 2, \cdots$ and $f(p) = d(p_{2i}, p)/s_i$ if $p \in N(p_{2i}, s_i)^-$ for some i. Then $0 \leqq f(p) \leqq 1$ for every $p \in E$. In particular, $f(p) = 0$ if $p = p_{2i}$ for some i and $f(p) = 1$ if $p = p_{2i-1}$ for some i.

It follows that $E - N(p_*, r_*)^-$ is $U.I.$ for some r_*. Define $E_2 = E - N(p_*, r_*)^-$. Since $E = (E \cap N(p_*, r_*)^-) \cup (E - N(p_*, r_*)^-)$, let $E_1 = E \cap N(p_*, r_*)^-$. E_1 is the intersection of two closed sets and hence is closed. Also, $E_1 \subset N(p_*, r_*)^-$, which implies that E_1 is bounded. It follows then that E_1 is compact and the theorem is proved.

THEOREM 5. *Let (M, d) be a W-B space and E a subset of M. If E is $U.C.$ (M_1) where (M_1, d_1) is a metric space containing a simple arc axb, then $E = E_1 \cup E_2$, where E_1 is compact and E_2 is $U.I.$*

Proof. Assume $E \neq \varnothing$ (since otherwise the theorem is trivial), and let p_* be a fixed point in E. There exists a $r_* \in R_1$ such that $E - N(p_*, f_*)^-$ is $U.I.$ For if not, then proceeding as in the proof of Theorem 4 we define the same continuous function f from E into $[0, 1] \subset R_1$. There is a homeomorphism h of the closed unit interval $[0, 1] \subset R_1$ onto the arc axb of M_1 such that $h(0) = a$, $h(1) = b$. It follows that the restriction $g = h/f(E)$ of h to $f(E) \subset [0, 1]$ is continuous on $F(E)$ in R_1. The function $k = gf$ defined by $k(p) = g[f(p)]$ for every $p \in E$ is a continuous function from E into M_1 which is not uniformly continuous on E. This is contrary to E being $U.C.$ (M_1). Hence $E - N(p_*, r_*)^-$ is $U.I.$ Define $E_2 = E - N(p_*, r_*)^-$ and $E_1 = E \cap N(p_*, r_*)^-$. E_1 is closed and bounded and hence compact; and the conclusion follows.

It is obvious that the following theorem is true.

THEOREM 6. *Let (M, d) be a metric space and E a subset of M. If (M_1, d_1) is a metric space containing a simple arc axb, then a necessary condition that E be $U.C.$ (M_1) is that $E = E_1 \cup E_2$, where E_1 is closed and bounded and E_2 is $U.I.$*

THEOREM 7. *Let (M, d) be a W-B space, E a subset of M and F a subset of E. If E is $U.C.$ (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, and if F is closed in (M, d), then F is $U.C.$ (M_1).*

Proof. This is an immediate consequence of Theorems 2 and 5.

COROLLARY. *Let (M, d) be a W-B space and $\theta = \{E\}$ a collection of subsets of M. If for every $E \in \theta$, E is $U.C.$ (M_1), where (M_1, d_1) is a metric space containing a simple arc axb, then $\bigcap_{E \in \theta} E$ is $U.C.$ (M_1).*

LEMMA. *Let (M, d) be a metric space and E a nonempty subset of M. If E is*

dense-in-itself and if $E=E_1\cup E_2$, *where* E_1 *is closed,* E_2 *is U.I., and* $E_1\cap E_2=\varnothing$, *then* $E_2=\varnothing$.

Proof. Assume $E_2\neq\varnothing$ and let $p\in E_2$. Then $p\in E\subset E'$. There exists a real $\delta_1>0$ such that $N(p,\delta_1)\cap E_1=(N(p,\delta_1)\cap E_1)-\{p\}=\varnothing$; for if not, p is a limit point of E_1 and hence $p\in E_1$ which is contrary to $E_1\cap E_2=\varnothing$. $E_2\neq\varnothing$ and is *U.I.* implies that there exists a real $\delta_2>0$ such that $d(p,q)\geqq\delta_2$ for every point $q\in E_2$ for which $p\neq q$. Let $\delta=\min(\delta_1,\delta_2)$. It follows that $N(p,\delta)\cap E=\{p\}$ and hence $(N(p,\delta)\cap E)-\{p\}=\varnothing$ which is contrary to $p\in E'$.

THEOREM 8. *Let* (M,d) *be a metric space and E a subset of M. If E is dense-in-itself, then E is U.C.* (R_1) *if and only if E is compact.*

Proof. If E is compact, then every real-valued function f which is continuous on E is uniformly continuous on E and hence E is *U.C.* (R_1).

Conversely, let E be *U.C.* (R_1) and show that E is compact. Assume $E\neq\varnothing$ since otherwise the theorem is trivial. By Theorem 6, $E=E_1\cup E_2$ where E_1 is closed and bounded and E_2 is *U.I.* Furthermore, there is no loss in generality in assuming that $E_1\cap E_2=\varnothing$. It then follows from the lemma that $E_2=\varnothing$.

It remains to show that $E=E_1$ is compact. (E,d) is a complete metric space. Therefore, E is compact if and only if E is totally bounded. If E is not totally bounded, then there exists a real number $\epsilon_1>0$ such that $E\not\subset\bigcup_{i=1}^{n}N(p_i,\epsilon_1)$ for every finite number n of points in E. Hence starting from an arbitrary point $p_1\in E$, an infinite sequence $\{p_i\}_{i=1}^{\infty}$ of distinct points in E can be constructed such that $d(p_i,p_j)\geqq\epsilon_1$ if $i\neq j$.

Now $N(p_i,\epsilon_1/4)^-\cap N(p_j,\epsilon_1/4)^-=\varnothing$ if $i\neq j$ lest $d(p_i,p_j)\leqq\epsilon_1/2<\epsilon_1$. Define a second sequence $\{p_i^*\}_{i=1}^{\infty}$ of distinct points in E as follows: $p_i^*\in N(p_i,\epsilon_1/4)$ such that (i) $d(p_i,p_i^*)<\epsilon_1/8i$ and (ii) $p_i\neq p_i^*$. It follows that $p_i^*\neq p_j^*$ if $i\neq j$, and $p_i^*\neq p_j$ for all $i,j=1,2,\cdots,n,\cdots$. Let $s_i=d(p_i,p_i^*)>0$ and consider $\{N(p_i,s_i)^-\,|\,i\in P\}$. If $i\neq j$, then $N(p_i,s_i)^-\cap N(p_j,s_j)^-=\varnothing$ and $d^{\#}[N(p_i,s_i),N(p_j,s_j)]>0$.

Define a function f from E into R_1 as follows: for every point $p\in E$, let $f(p)=1$ if $p\notin N(p_i,s_i)^-$ for all $i=1,2,\cdots$ and $f(p)=d(p_i,p)/s_i$ if $p\in N(p_i,s_i)^-$ for some i. Then $0\leqq f(p)\leqq 1$ for every $p\in E$. f is continuous on E but is not uniformly continuous on E contrary to E being *U.C.* (R_1). It follows that E is totally bounded and hence compact, and the theorem is proved.

THEOREM 9. *Let* (M,d) *be a metric space, E a subset of M, and* (M_1,d_1) *a metric space containing a simple arc axb. If E is dense-in-itself, then E is U.C.* (M_1) *if and only if E is compact.*

Proof. The sufficient condition is obvious. The proof of the necessary condition is easily patterned after the proofs of Theorems 6 and 8.

SKEW MATRICES AS SQUARE ROOTS*

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1. In the algebra of matrices it has been observed that the skew hermitian matrices exhibit a number of properties similar to those of pure imaginary numbers in the algebra of complex numbers. The present note exhibits one such property. Skew hermitian matrices are characterized as the normal square roots of negative definite, or semidefinite, hermitian matrices H. They constitute a set of generators of all like-ranked square roots of such hermitian matrices in the sense that every such square root is similar to a skew hermitian square root.

Similar results hold for the corresponding real case. Thus, with a certain additional hypothesis on H, the last two statements above remain valid if the phrases "skew hermitian," "square root" and "hermitian" are replaced respectively be "real skew," "real square root" and "real symmetric."

2. THEOREM 1. *Every skew hermitian matrix is a (normal) square root of a negative definite, or semidefinite, hermitian matrix. Conversely, every negative definite, or semidefinite, hermitian matrix possesses matrix square roots, the normal matrices among which are skew hermitian.*

Proof. Let S be skew hermitian (hence normal). Then $S^* = -S$, where S^* is the conjugate transpose of S. Hence, $S^2 = -S^*S$. Since $(S^*S)^* = S^*S$, S^2 is hermitian. Further, S^*S, being the so-called Gram matrix of S [1], is known to be positive definite or semidefinite. Hence S^2 is a negative definite, or semidefinite, hermitian matrix.

For the converse, let H be hermitian, negative definite or semidefinite. We first show that normal square roots of H do exist. Since H is hermitian $\exists$ a unitary matrix V such that V^*HV is a diagonal matrix, $\text{diag}(\lambda_1, \cdots, \lambda_n)$, and all the λ_i are real numbers. Because H is negative definite or semidefinite, the λ_i are all nonpositive. The diagonal matrix $D = (\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$ has pure imaginaries or zeros on the diagonal, hence is skew hermitian, therefore normal, and $D^2 = V^*HV$. Hence VDV^* is skew hermitian, and $(VDV^*)^2 = VD^2V^* = H$.

Now let N be any normal square root of H, *i.e.*, $NN^* = N^*N$, $N^2 = H$. Since N is normal, $\exists$ a unitary matrix $U \ni U^*NU$ is a diagonal matrix, $\text{diag}(\mu_1, \cdots, \mu_n)$. Hence $\text{diag}(\mu_1^2, \cdots, \mu_n^2) = (U^*NU)^2 = U^*N^2U = U^*HU = \text{diag}(\lambda_1, \cdots, \lambda_n)$. The λ_i, being eigenvalues of H, are all nonpositive real numbers. Hence in $U^*NU = \text{diag}(\mu_1, \cdots, \mu_n)$, each μ_i must be pure imaginary or zero, and consequently $\text{diag}(\mu_1, \cdots, \mu_n)$ is skew hermitian. Therefore $N = U\,\text{diag}(\mu_1, \cdots, \mu_n)U^*$ is skew hermitian. The corresponding real companion of Theorem 1 is

THEOREM 2. *Every real skew matrix is a (real, normal) square root of a negative definite, or semidefinite, real symmetric matrix, whose nonzero eigenvalues have even multiplicities. Conversely, every negative definite, or semidefinite, real symmetric matrix, whose nonzero eigenvalues have even multiplicities, possesses real square roots, the normal ones of which are real skew.*

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Proof. Let S be a real skew (hence normal) matrix. Since S^2 is a real hermitian matrix, we conclude exactly as before that S^2 is real, symmetric (*i.e.*, real hermitian), negative definite or semidefinite. We further observe that since the eigenvalues of a real skew matrix consist of zeros (possibly), and pairs of nonzero conjugate pure imaginary numbers, the nonzero eigenvalues of S^2, being the squares of the nonzero eigenvalues of S, consist of pairs of equal negative real numbers. The distinct nonzero eigenvalues of S^2 must therefore have even multiplicities.

For the converse, let H be an nth order real symmetric (*i.e.*, real, hermitian) negative definite, or semidefinite, matrix, whose nonzero eigenvalues have even multiplicities. As before, we first show that such an H admits real normal square roots. Since H is real, symmetric, $\exists$ a real orthogonal matrix P, such that

$$P^THP = \operatorname{diag}(-\lambda_1, -\lambda_1, -\lambda_2 - \lambda_2, \cdots, -\lambda_k, -\lambda_k, 0 \cdots 0) = D,$$

and the λ_i are all real and positive since H is real, symmetric, and negative definite or semidefinite. Then the nth order matrix F consisting of the diagonal 2×2 blocks

$$\begin{pmatrix} 0 & \sqrt{\lambda_1} \\ -\sqrt{\lambda_1} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \sqrt{\lambda_2} \\ -\sqrt{\lambda_2} & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \sqrt{\lambda_k} \\ -\sqrt{\lambda_k} & 0 \end{pmatrix},$$

and zeros otherwise, is real, skew, and satisfies $F^2 = D$. Hence (PFP^T) is real, skew, and satisfies $(PFP^T)^2 = H$.

Now let S be any normal real square root of H. Exactly as in the proof of Theorem 1 we conclude that S is skew hermitian, and since S is real, it is real skew.

3. If B is any square root of a square matrix C, and if Q is a nonsingular matrix commuting with C, then $Q^{-1}BQ$ is also a square root of C. Thus from the normal (skew hermitian) square roots of H discussed in Section 2, can be obtained other square roots of H, not necessarily normal. That not all square roots of H are so obtainable is shown by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which is a square root of the negative semidefinite hermitian matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not similar to a diagonal matrix, it is not similar to a normal square root of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We note that the ranks of the square root matrix and the argument matrix are not the same in the example. However, if S is a normal (therefore skew hermitian) matrix square root of a negative definite or semidefinite matrix H, then the rank of S must be the same as that of H, since $S^2 = -S^*S$, and from known properties of the Gram matrix, the rank of S^*S, *i.e.*, of H, is the same as that of S. Hence any square root of H which is similar to S must also have the same rank as S and H.

Further, the normal square roots of H provide a set from which can be generated by similarity transformations all those square roots of H which have the same rank as H. Specifically,

THEOREM 3. *Let H be hermitian negative definite, or semidefinite, of rank r. Then every square root of rank r of H is similar to a skew hermitian square root of H.*

The proof of Theorem 3 is facilitated by appealing to the following

LEMMA. *If A is a matrix of rank r which is similar to a diagonal matrix, then any kth root of A is similar to a diagonal matrix.*

The lemma is a direct consequence of application of a method suggested by Weitzenböck [2] of finding all kth roots of a given matrix, or, less directly, by application of the method of Franklin [3] for solving polynomial equations $p(X) = A$.

Now let B be a square root of H of the same rank as H. By the lemma there is a nonsingular matrix P such that $P^{-1}BP = \operatorname{diag}(\mu_1, \cdots, \mu_n)$ and $(P^{-1}BP)^2 = P^{-1}B^2P = P^{-1}HP = \operatorname{diag}(\mu_1^2, \cdots, \mu_n^2)$. The μ_i^2 are the eigenvalues of H, and hence are nonpositive real numbers. Hence each μ_i is zero or pure imaginary and $P^{-1}BP$ is skew. Now let U be a unitary matrix such that $U^*HU = \operatorname{diag}(\mu_1^2, \cdots, \mu_n^2)$. Then $S = U \operatorname{diag}(\mu_1, \cdots, \mu_n)U^*$ is a skew hermitian square root of H, and $B = PU^*SUP^{-1} = (UP^{-1})^{-1}S(UP^{-1})$. Thus B is similar to a skew hermitian square root of H.

One notes that $(UP^{-1})^{-1}H(UP^{-1}) = H$, and hence UP^{-1} commutes with H. Thus all like-ranked square roots of H with the same rank as H are obtained by taking similarity transforms of the skew hermitian square roots of H, using transforming matrices which commute with H.

In case H is nonsingular, then any square root of H must be nonsingular *i.e.*, must have the same rank as H. Hence,

COROLLARY 3.1. *If H is a nonsingular negative definite, or semidefinite, hermitian matrix, then every square root of H is similar to a skew hermitian square root of H.*

In case H, hermitian negative definite, or semidefinite, has distinct eigen-

values, it is easily seen from the Weitzenböck analysis that the diagonalized form U^*HU of H, with U unitary, has only diagonal, hence skew hermitian, square roots, whence all the square roots of H are given by UDU^*, D skew hermitian. Hence,

COROLLARY 3.2. *If H is hermitian negative definite, or semidefinite, with distinct eigenvalues, then every square root of H is skew hermitian.*

The real analogue of Theorem 3 is:

THEOREM 4. *Let H be a real symmetric negative definite, or semidefinite, matrix of rank r, whose nonzero eigenvalues have even multiplicities. Then every real square root of rank r of H is similar, over the real field, to a real skew square root of H.*

Proof. Let R be a real square root of H of the same rank as H. By the lemma, as in the proof of Theorem 3, R is similar to a diagonal matrix $D=P^{-1}RP$, whose nonzero diagonal entries are conjugate pure imaginaries and which can be taken to be a direct sum of blocks of the form

$$C = \begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix},$$

λ real, and a zero matrix; then $P^{-1}HP=D^2$ is composed of corresponding diagonal blocks

$$\begin{pmatrix} -\lambda^2 & 0 \\ 0 & -\lambda^2 \end{pmatrix}$$

and a zero block. Now the unitary matrix

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

transforms C into

$$U^*CU = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}.$$

Hence a unitary matrix V which is a direct sum of such U and an identity block, transforms D into V^*DV, a direct sum of blocks of the form

$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$$

and a zero matrix, and leaves $P^{-1}HP$ unaltered, $V^*P^{-1}HPV=P^{-1}HP$. Thus $K=V^*DV$ is a real skew square root of

$$P^{-1}HP = \operatorname{diag}(-\lambda_1^2, -\lambda_1^2, \cdots, -\lambda_k^2, -\lambda_k^2, 0 \cdots 0).$$

Let W be a real orthogonal matrix such that $W^THW=P^{-1}HP$. Then $WKW^T=S$

is a real skew square root of H, and $R = PDP^{-1} = PVKV^*P^{-1} = (PVW^T)(WKW^T)(WV^*P^{-1}) = (WV^{-1}P^{-1})^{-1}S(WV^{-1}P^{-1})$. Hence R is similar to the real skew square root S of H, but the transforming matrix $WV^{-1}P^{-1}$ may be complex. However, since the real matrices R and S, viewed as matrices over the complex field, have the same invariant factors, they have, as matrices over the real field, likewise all invariant factors in common and hence are similar over that field. Hence $\exists$ a real matrix Q such that $R = Q^{-1}SQ$.

As in Section 3, there follows

COROLLARY 4.1. *If H is a nonsingular negative definite, or semidefinite, real symmetric matrix, then every real square root of H is similar over the real field to a real, skew square root of H.*

Since a real symmetric negative definite, or semidefinite, matrix which admits real skew square roots must have even multiplicities on its nonzero eigenvalues, a real analogue of Corollary 3.2 would have to read something like: "If H is a real symmetric negative definite, or semidefinite, matrix, whose eigenvalues all have multiplicity two, then every square root of H is real, skew." Such an analogue is not valid as the following example shows. Let

$$H = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix},$$

α real and $\neq 0$. Then any real matrix M similar to

$$S = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

is a real square root of H, and not all such M are real, skew.

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2. R. Weitzenböck, Über die Matrixgleichung $X^2 = A$, Nederl. Akad. Wetensch. Proc., vol. 35, 1932, pp. 157–161.
3. P. Franklin, Algebraic matrix equations, J. Math. Phys., vol. 10, 1932, pp. 135–143.

CORRECTIONS

H. P. Mulholland and C. A. B. Smith, An inequality arising in genetical theory, this MONTHLY, vol. 66, 1959, pp. 673–683. The right-hand side of (1), page 674 should be $(v^Tuv)^n$.

H. L. Turrittin, Linear differential or difference equations with constant coefficients, this MONTHLY, vol. 66, 1959, pp. 869–875. Equations (6) should be $\overline{X} = PY_f(t)\overline{K}_1$ and $\overline{X}(t) = PY_f(t)\overline{K}_2(t)$.

MATHEMATICAL NOTES

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A GENERALIZATION OF CEVA'S THEOREM

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Ceva's theorem in elementary geometry deals with a triangle $A_1A_2A_3$ and a point P in general position in its plane. The lines A_1P, A_2P, A_3P, intersect the sides A_2A_3, A_3A_1, A_1A_2, respectively, in points I_1, I_2, I_3. The theorem states that

$$\frac{A_2I_1}{A_3I_1}\cdot\frac{A_3I_2}{A_1I_2}\cdot\frac{A_1I_3}{A_2I_3}=(-1)^3.$$

We propose to generalize this result by considering a plane polygon $A_1(x_1, y_1)$, $\cdots$, $A_n(x_n, y_n)$, and $\frac{1}{2}d(d+3)-1$ points (P) in general position in its plane.

A preliminary example, with $n=5$, $d=2$, will help to clarify what follows. Each vertex A_r of a pentagon $A_1A_2A_3A_4A_5$ determines with four fixed points $P_1P_2P_3P_4$ a unique conic Q_r. Name the six points in which A_1A_2 intersects Q_3, Q_4, and Q_5, P_{12}^i $(i=1, \cdots, 6)$; name the six points in which A_2A_3 intersects Q_4, Q_5, and Q_1, P_{23}^i; define similarly P_{34}^i, P_{45}^i, P_{51}^i. What is to be proved is that the product

$$(A_1P_{12}^1\cdots A_1P_{12}^6)(A_2P_{23}^1\cdots A_2P_{23}^6)\cdots(A_5P_{51}^1\cdots A_5P_{51}^6)$$

is equal to $(-1)^5$ times the similar product taken in the opposite direction, *i.e.*, to

$$(-1)^5(A_1P_{51}^1\cdots A_1P_{51}^6)(A_5P_{45}^1\cdots A_5P_{45}^6)\cdots(A_2P_{12}^1\cdots A_2P_{12}^6).$$

The theorem can now be stated in general terms.

THEOREM. *Let $Q_r(x, y)=0$ be the unique curve of degree d determined by the points (P) and the vertex* A_r of a plane polygon $A_1, \cdots, A_n$. Let $A_{r(t,t+1)}^s$ denote the product of all the signed lengths of the segments joining A_s to the d points of intersection of $Q_r=0$ with the line A_tA_{t+1}. Then*

$$\prod_{r=1}^{n}\prod_{s=r+1}^{r+n-2}\frac{A_{r(s,s+1)}^s}{A_{r(s,s+1)}^{s+1}}=(-1)^n.$$

Proof. The proof that follows uses a result of Newton's which we recall.† Let the two points $B(b_1, \cdots, b_m)$, $C(c_1, \cdots, c_m)$ determine a line in euclidean

* We shall identify A_h and A_k provided $h\equiv k \pmod n$.

† See G. Salmon, Higher Plane Curves, 3rd ed., Dublin 1879, p. 108.

m-space which intersects $H(x_1, \cdots, x_m)=0$, a hypersurface of order g. Then

$$P_b/P_c = H(b_1, \cdots, b_m)/H(c_1, \cdots, c_m), \tag{1}$$

where P_b is the product of the directed distances from B to the g points of intersection and P_c, the corresponding product for C. When this last formula is applied to A_s, A_{s+1}, and the intersection of the line passing through them with the curve $Q_r=0$, we obtain

$$\frac{A^{s}_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}} = \frac{Q_r(x_s, y_s)}{Q_r(x_{s+1}, y_{s+1})}$$

and hence

$$\prod_{s=r+1}^{r+n-2} \frac{A^{s}_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}} = \frac{Q_r(x_{r+1}, y_{r+1})}{Q_r(x_{r+n-1}, y_{r+n-1})} = \frac{Q_r(x_{r+1}, y_{r+1})}{Q_r(x_{r-1}, y_{r-1})} = J_r.$$

To show that $\prod_1^n J_r=(-1)^n$, write Q_r as $\sum_{i+j\leq d} {}_ra_{ij}x^iy^j$. Since all the Q_r together form a linear pencil, the points $\bar{x}_r$ in $\frac{1}{2}d(d+3)$-space, whose homogeneous coordinates are ${}_ra_{ij}$, lie on a straight line. Furthermore, $\bar{x}_r$ is the unique point of intersection of this line (L) with the hyperplane

$$W_r = \sum_{i+j\leq d} (x_r)^i(y_r)^jX_{ij} = 0.$$

It is also clear that $W_p(\bar{x}_r)=Q_r(x_p, y_p)$ for any r, p. Applying (1) to the points $\bar{x}_{r+1}$, $\bar{x}_{r-1}$ and the intersection $\bar{x}_r$ of the line (L) through them with the hyperplane $W_r=0$, we obtain

$$\prod_1^n \frac{\bar{x}_{r-1}\bar{x}_r}{\bar{x}_{r+1}\bar{x}_r} = \prod_1^n \frac{W_r(\bar{x}_{r-1})}{W_r(\bar{x}_{r+1})} = \prod_1^n \frac{Q_{r-1}(x_r, y_r)}{Q_{r+1}(x_r, y_r)} = \prod_1^n J_r.$$

Hence $\prod_1^n J_r$ turns out to be the product of the n distances $\bar{x}_r\bar{x}_{r+1}$ taken in one direction divided by the same product taken in the opposite direction, and so, must equal $(-1)^n$.

The same method may readily be extended to prove the theorem for $A_1, \cdots, A_n$ in three or more dimensions so long as the number of points (P) (which would now be in general position in the space determined by the A's) is such that the points (P) will determine with each A_r a unique hypersurface.

We can also deduce two geometrical interpretations of the cross ratio (c) of any linear pencil of four d-ic hypersurfaces. First it is clear that $c=(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4)$, where the $\bar{x}_i$ are related to the four hypersurfaces Q_i as above. Secondly, choose A_1 arbitrarily on Q_1 and A_2 on Q_2. Then

$$\frac{A^{1}_{3(1,2)}}{A^{2}_{3(1,2)}} \div \frac{A^{1}_{4(1,2)}}{A^{2}_{4(1,2)}} = \frac{Q_3(A_1)}{Q_3(A_2)} \div \frac{Q_4(A_1)}{Q_4(A_2)} = \frac{\bar{x}_1\bar{x}_3}{\bar{x}_1\bar{x}_4} \div \frac{\bar{x}_2\bar{x}_3}{\bar{x}_2\bar{x}_4} = (\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4) = c.$$

FUNCTIONS OF FINITE BAIRE TYPE

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The function space technique has been used, over the past 40 years, to obtain simple proofs of a great variety of facts regarding real and complex functions. In this note, the method is applied to the functions of finite Baire type to show that (a) there are sequences of functions of finite Baire type which converge uniformly to functions not of finite Baire type, but that (b) there are functions of Baire type ω_0 which are not the limits of any uniformly convergent sequences of finite Baire type.

The definition given in Hausdorff's book, [1], of functions of Baire type η, η a limiting ordinal, (different from the definition used here) indicates that he may have been aware of these facts. Nevertheless, the proof given here has a certain simplicity and appeal.

All functions considered are bounded real functions on the closed interval $[0, 1]$. Let B_0 be the set of continuous functions and, for every $\xi<\omega_1$, let B_ξ consist of all bounded functions that are limits of sequences of functions belonging to $\bigcup_{\eta<\xi} B_\eta$. The functions in B_ξ are the functions of Baire type ξ. Let C_ξ be the functions of Baire type ξ which are not of lower Baire type. The following are well known:

(a) For every $\xi<\omega_1$, C_ξ is nonempty.

(b) For every $\xi<\omega_1$, the limit of every uniformly convergent sequence of functions in B_ξ is itself in B_ξ.

(c) If $a_0=0<a_1<\cdots<a_n=1$, and the function f_i, defined on $[a_{i-1}, a_i)$, is of finite Baire type, $i=1, \cdots, n$, then the function, f, defined on $[0, 1]$ as $f=f_i$ on $[a_{i-1}, a_i)$, $f(1)=0$, is also of finite Baire type.

The proofs of (a), (b), (c) may be found in [1] and (b), (c) in [2]. The notations B_ξ, C_ξ will also denote the above classes of functions defined on subintervals of $[0, 1]$. The domain will always be clear from the context.

THEOREM 1. *There is a uniformly convergent sequence in $B=\bigcup_{n<\omega_0} B_n$ whose limit belongs to C_{ω_0}.*

Proof. B is a vector space. For every $f\in B$, let

$$\|f\| = \sup_{t\in[0,1]} |f(t)| .$$

Then B is a normed vector space. Now (b) says that, for every n, B_n is a Banach space. In particular, B_n is a closed proper subspace of B and so is nowhere dense in B. Hence $B=\bigcup_{n<\omega_0} B_n$ is of the first category and so is not complete. Thus, there are sequences in B which converge uniformly to functions of Baire type ω_0.

The following lemma is needed for the proof of Theorem 2.

LEMMA 1. *If E is a normed vector space and F is a closed vector subspace of E, there is an $x\in E$ such that $\|x\|=1$ and $\|x-y\|>\frac{1}{2}$ for every $y\in F$.*

For the proof, see [3].

THEOREM 2. *There is an $f\in C_{\omega_0}$ such that f is not the limit of any uniformly convergent sequence in B.*

Proof. Let $a_0=0<a_1<\cdots<a_n<\cdots$, $\lim_{n\to\infty} a_n=1$. For every n, let f_n be a function defined on $[a_{n-1}, a_n]$, f_n of type C_n such that $\|f_n\|=1$ and $\|f_n-f\|>\frac{1}{2}$ for every $f\in B_{n-1}$. For every n, define g_n such that $g_n=f_1$ on $[a_0, a_1)$, $g_n=f_2$ on $[a_1, a_2)$, $\cdots$, $g_n=f_n$ on $[a_{n-1}, a_n)$, and $g_n=0$ on $[a_n, 1]$. By (c), $g_n\in B$. The sequence g_n converges to a function $g\in B_{\omega_0}$. But $\|g-f\|>\frac{1}{2}$ for every $f\in B$. Hence g is not the limit of any uniformly convergent sequence in B.

References

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ON THE EQUIVALENCE OF TWO SURFACES

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One of the fundamental classification theorems for surfaces states that any surface is topologically equivalent to a surface with at most two cross-caps.* This result rests on the fact that the sphere with three cross-caps is equivalent to the sphere with one cross-cap and a handle, or what is an equivalent statement, that *a Klein bottle with a boundary matched with the boundary of a cross-cap is topologically equivalent to a torus with a boundary matched with the boundary of a cross-cap.* The latter result is also rather interesting in view of the fact that the Klein bottle and torus themselves are *not* equivalent surfaces. The standard proofs of the italicized statement involve so-called "cutting and pasting" arguments, and the role of the cross-cap in the equivalence is obscured. Below is given a short informal proof which avoids cutting and pasting and which clarifies the "catalytic" role of the cross-cap.

We begin with the surface K, a Klein bottle with a boundary matched with the boundary of a cross-cap (Fig. 1). We shall show that K can be continuously

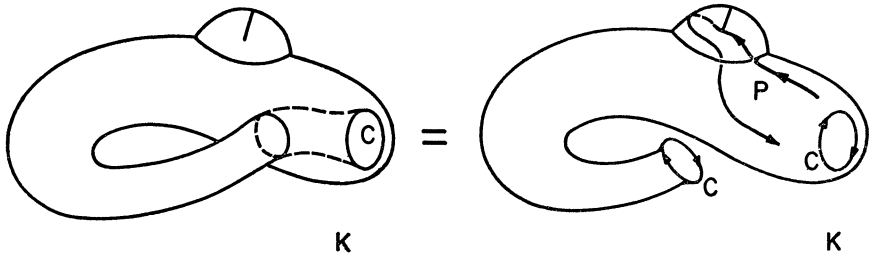


FIG. 1

* Due to A. W. Tucker; cf. S. Lefschetz, Introduction to Topology, Princeton, 1949, p. 85.

deformed into a torus with a boundary matched with the boundary of a cross-cap.

Let C be the simple closed curve on K at which the neck joins the mouth of the Klein bottle, and give C an orientation as shown. Consider a path P on K from the junction C up to the cross-cap, through the self-intersection, around the back of the cross-cap, and down again to the front of K. If C is moved continuously along this orientation-reversing path, then having traversed P, C will have the reverse of its initial orientation. Hence in the resulting surface T, (Fig. 2) the neck of the bottle will meet the surface at C without having to pass through T. Since topological equivalence is preserved in deforming C continu-

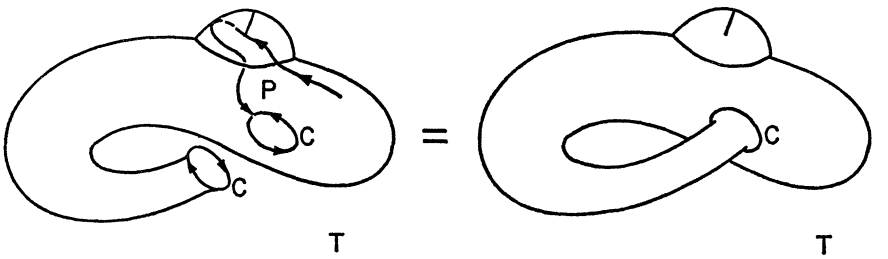


FIG. 2

ously along P, the surfaces K and T are equivalent. And as T is just a torus with a boundary matched with the boundary of a cross-cap, the proof is complete.

Notice that the important property of the cross-cap in the equivalence is its nonorientability. In particular, the italicized statement holds if the cross-cap is replaced by any nonorientable surface with a boundary.

ON SOME OF THE CLASSICAL ORTHOGONAL POLYNOMIALS

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C. K. Chatterjee [2] gave an interesting result for Bessel polynomials [4], namely,

$$y_n y_{n+1} = 1 + \sum_{k=0}^{n} (2k+1)x y_k^2. \tag{1}$$

This same result was also independently derived by W. A. Al-Salam [1]. Chatterjee further used the differentiation relation for Bessel polynomials, $x^2 y_n' = (nx-1)y_n + y_{n-1}$ to obtain another interesting analogue of Rodrigues' formula,

$$(x^3(d/dx))^k(y_n e^{-1/x}x^{-n}) = x^{-(n-k)}e^{-1/x}y_{n-k}. \tag{2}$$

He also derived a corresponding result with the help of the relation $x^2 y_{n-1}' = y_n - (nx+1)y_{n-1}$.

In the present note, we present similar sets of results for all the classical orthogonal polynomials (except those of Gegenbauer and Jacobi, where the results become cumbersome). Along with them we give a new set of differentiation formulas. The notations are as in [3].

We follow the following plan:

(a) will contain the analogue of Al-Salam's and Chatterjee's result (1) and a determinantal representation.*

(b) will contain the analogue of Chatterjee's result (2).

(c) will stand for the particular cases of the following lemma of the author:

LEMMA. *If U is a function of x of class C^n, then*

$$(d^n/dx^n)(\exp U) = \exp U((d/dx) + U')\cdot 1, \tag{3}$$

where $U' = \partial U/\partial x$ and $((d/dx)+U')\cdot 1$ is to be taken in the iterative sense.

1. Legendre polynomials.

(a) This has a recurrence relation

$$(n + 1)P_{n+1} - (2n + 1)xP_n + nP_{n-1} = 0. \tag{4}$$

Following Chatterjee or Al-Salam, we get

$$(n + 1)P_{n+1}P_n = \sum_{k=0}^{n} (-1)^{k-1}(2k + 1)xP_k^2.$$

Now (4) shows that $P_n(x)$ may be written as a determinant $P_n(x) = ((-1)^n/n!)$ det (a_{ij}), where

$$\begin{aligned} a_{ii} &= -(2n - 2i + 1)x, && 1 \leqq i \leqq n; \\ a_{i,i+1} &= a_{i+1,i} = n - i, && 1 \leqq i \leqq n - 1; \\ a_{ij} &= 0, && \text{otherwise.} \end{aligned} \tag{5}$$

(b) $P_n(x)$ satisfies the recurrence relation $(1-x^2)P_n' = nP_{n-1} - nxP_n$. This may be written

$$[(1 - x^2)^{3/2}(d/dx)][(1 - x^2)^{-n/2}P_n] = n(1 - x^2)^{-(n-1)/2}P_{n-1}(x).$$

Iterating this expression K times $(1 \leqq K \leqq n)$ with the operator $(1-x^2)^{3/2}(d/dx)$ and writing it as $(1-x^2)^{3/2}(d/dx)^K$, we get

$$\begin{aligned} &[(1 - x^2)^{3/2}(d/dx)]^K[(1 - x^2)^{-n/2}P_n] \\ &\qquad = n(n - 1)\cdots(n - K + 1)(1 - x^2)^{-(n-k)/2}P_{n-k}. \end{aligned} \tag{6}$$

(c) Rodrigues' formula for Legendre polynomials is $P_n(x) = (1/(2^n n!))$ $(d/dx)^n(x^2-1)^n$. Taking $U = n \log (x^2-1)$ in (3), we get

$$P_n(x) = (1/(2^n n!))(x^2 - 1)^n[(d/dx) + (2nx)/(x^2 - 1)]^n \cdot 1. \tag{7}$$

* The latter was suggested to me by Professor P. L. Bhatnagar.

We merely quote the results for the rest of the polynomials

2. Tchebycheff polynomials.

(a) Since $T_n-2xT_{n-1}+T_{n-2}=0$, we get

$$T_nT_{n+1} = 2x\sum_{k=0}^{n}(-1)^{n-k}T_k^2. \tag{8}$$

Also, $T_n(x)=(-1)^n \det(b_{ij})$, where

$$\begin{aligned} b_{11} &= -x; \quad & b_{ii} &= -2x, & 2 &\leqq i \leqq n; \\ b_{i,i+1} &= b_{i+1,i} = 1, & & & 1 &\leqq i \leqq n-1; \\ b_{ij} &= 0, & & & & \text{otherwise.} \end{aligned} \tag{9}$$

(b) Since $(1-x^2)T_n' = nT_{n-1}-nxT_n$, we get

$$\begin{aligned} &[(1-x^2)^{3/2}(d/dx)]^K[(1-x^2)^{-n/2}T_n] \\ &\qquad = n(n-1)\cdots(n-K+1)(1-x^2)^{-(n-K)/2}T_{n-k}, \end{aligned} \tag{10}$$

$$T_n(x) = (2^n n!)/(2n)!(x^2-1)^n[(d/dx)+(2n-1)x)/(x^2-1)]^n\cdot 1. \tag{11}$$

3. Hermite polynomials.

(a) Here we get

$$H_nH_{n+1} = 2x\sum_{k=0}^{n}(-1)^k 2^k[n!/(n-k)!]H_{n-k}^2. \tag{12}$$

4. Laguerre polynomials.

(a) We have

$$(n+1)L_nL_{n+1} = \sum_{k=0}^{n}(-1)^{n-k}(2k+1-x)L_k^2 \tag{13}$$

and $L_n(x)=((-1)^n/n!)\det(c_{ij})$, where

$$\begin{aligned} c_{ii} &= x-2n+2i-1, & & & 1 &\leqq i \leqq n; \\ c_{i,i+1} &= c_{i+1,i} = n-i & & & 1 &\leqq i \leqq n-1; \\ c_{ij} &= 0, & & & & \text{otherwise.} \end{aligned} \tag{14}$$

5. Bessel polynomials.

(a) See Chatterjee or Al-Salam. We have $y_n(x)=\det(d_{ij})$, where

$$\begin{aligned} d_{11} &= 1+x, & d_{ii} &= (2i-1)x, & 2 &\leqq i \leqq n; \\ d_{i,i+1} &= -1, & d_{i+1,i} &= 1, & 1 &\leqq i \leqq n-1; \\ d_{ij} &= 0, & & & & \text{otherwise.} \end{aligned} \tag{15}$$

(b) See Chatterjee [2].

(c) We get

$$y_n(x) = (x^{2n}/2^n)[(d/dx) + (2(1 + nx)/x^2)]^n \cdot 1. \tag{16}$$

It may be of interest to point out here that in the Sections 1, 2, 4, the formulas under (b) when $K=n$, give $n!$

I wish to express my thankfulness to Professors R. S. Krishnan and P. L. Bhatnagar for their encouragement and interest in this work. Thanks are also due to the referee for his valuable suggestions.

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ON THE NUMBER OF ZEROS OF A CUBIC RECURRENCE

S. J. Scott, The University of North Carolina and The North American Air Defense Command

The cubic recurrence $(T)=(T_0, T_1, \cdots)$ is defined by the real numbers T_0, T_1, T_2 and the relation

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \qquad (n = 0, 1, \cdots)$$

where P, Q, and R are real numbers. When $T_k=0$, k is said to be a zero of (T). If any term of (T) is zero, it is no restriction to assume $T_0=0$.

M. F. Smiley [1] recently proved that if $(T)\neq(0, 0, 0, \cdots)$ and if the zeros of the companion of (T),

$$f(z) = z^3 - Pz^2 + Qz - R,$$

are nonvanishing real numbers with distinct absolute values, then (T) has no more than 3 zeros. This note extends these results as follows:

THEOREM. *A cubic recurrence (T), distinct from (T'): $T_n' = Uu^n[1-(-1)^n]$, $n=0, 1, \cdots$, can have no more than three zeros when the zeros of its companion are nonvanishing real numbers.*

In view of Smiley's results it suffices to prove the theorem when the zeros of the companion of (T), u, v, and w, are such that

(1) $u=-v$; $|u|\neq|w|$,

(2) $u=v\neq w$, or

(3) $u=v=w$.

Case 1. $u=-v$; $|u|\neq|w|$. The term T_n may be expressed as $T_n = Uu^n + V(-u)^n + Ww^n$. Thus $T_0=T_j=0$ if and only if $U+V+W=0$ and $Uu^j + V(-u)^j + Ww^j=0$. Since $W=0$ yields $U=-V$ and $T_n = Uu^n[1-(-1)^n]$, i.e. $(T)=(T')$, we may assume $W\neq0$. If j is even we have $u^j=w^j$, $|u|=|w|$, contrary to $|u|\neq|w|$. Hence j is odd. If (T) has 3 zeros $0<j<k$ then j and k are

odd and a suitable translate of (T) has zeros 0 and $k-j$ with $k-j$ even. We have just shown that this is impossible. Thus, in this case, (T) has at most two zeros.

Case 2. $u=v\neq w$. We have $T_n=(U+nV)u^n+Ww^n$, as well as $V\neq 0$, since $V=0$ and $T_0=T_j=0$ gives $(T)=(T')$. If $W=0$, the expression for T_n reduces to $T_n=nVn^n$ when $T_0=0$ and, in this case, (T) has exactly one zero. Now, $T_0=0$ if and only if $U+W=0$; hence, for $W\neq 0$, $T_n=0$ if and only if

$$1+nVU^{-1}=z^n, \tag{1}$$

where $z=w/u$. If z is positive, then (1) has at most one solution $n>0$. If z is negative, then (1) has at most two positive solutions. Consequently, (T) has at most three zeros when $u=v\neq w$.

Case 3. $u=v=w$. We have $T_n=(U+nV+n^2W)u^n$. If $T_0=0$ then $U=0$ and $T_n=0$, for $n>0$, if and only if $V+nW=0$. This is possible for at most one value of n. Consequently, (T) has at most two zeros in this final case.

Remark. Keeping the assumption that u, v, w are real, it is possible to determine those (T) with more than three zeros even if one or more of u, v, w is zero. Besides the (T') recurrences, one obtains those (T) such that $T_k=0$ for $k\geqq 3$.

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CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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PLOTTING BOOLEAN FUNCTIONS

WALTER E. STUERMANN, University of Tulsa

By deforming the conventional truth table of logic so as to apply it to Boolean functions and displaying it graphically, one secures certain advantages, such as greater visual comprehension of the nature of the functions and a larger scope of applicability for the truth table.

Consider the following Boolean function of four variables and six products in disjunctive normal form, where $+$ denotes the Boolean sum, the immediate juxtaposition of elements designates the Boolean product, and $\bar{x}$ is the complement of x:

$$F = xyz + \bar{x}\bar{y}z + xy\bar{z} + \bar{x}y\bar{z} + \bar{x}yz\bar{w} + \bar{x}\bar{y}\bar{z}\bar{w}.$$

Now construct a reference framework as shown in Figure 1. The first or top horizontal line is assigned to the variable x. This line is divided into two equal

A REMARK ON LINEAR DIFFERENCE EQUATIONS

DAVID ZEITLIN, Remington Rand Univac, St. Paul, Minnesota

It is well known that the order of the linear difference equation,

$$\sum_{j=0}^{n} p_j(k)\Delta^j y(k) = R(k), \qquad p_n(k) \equiv 1, \tag{1}$$

is not necessarily n; it may be less. If we express the differences in terms of the functional values, $y(k)$, $y(k+1)$, $\cdots$, $y(k+n)$, the order of (1) is then defined to be the difference between the largest and smallest arguments involved. We present now a simple test for the order of (1) which can be applied without resorting to formal decompositions of the differences to functional values. Since

$$\Delta^j y(k) = \sum_{i=0}^{j} (-1)^i \binom{j}{i} y(k + j - i),$$

we have

$$\sum_{j=0}^{n} p_j(k)\Delta^j y(k) = \sum_{j=0}^{n} p_j(k) \sum_{i=0}^{j} (-1)^i \binom{j}{i} y(k + j - i) = \sum_{r=0}^{n} B_r(k) y(k + r),$$

where $B_r(k) = \sum_{j=r}^{n} (-1)^{j-r} \binom{j}{r} p_j(k)$, $0 \leqq r \leqq n$. Thus, (1) is of order n if $B_0(k) \not\equiv 0$. If $B_0(k) \equiv 0$, then (1) is of order $n - r_0$ if $B_{r_0}(k) \not\equiv 0$ and $B_r(k) \equiv 0$, $0 \leqq r \leqq r_0 - 1$.

Example. Suppose $p_j(k) = \binom{n}{j}$, $j = 0, 1, \cdots, n$. Then

$$B_r(k) = \binom{n}{r} \sum_{j=r}^{n} (-1)^{j-r} \binom{n-r}{j-r} = \binom{n}{r} \sum_{s=0}^{n-r} (-1)^s \binom{n-r}{s}.$$

If $r = n$, $B_n(k) = 1$; if $0 \leqq r \leqq n-1$, $B_r(k) \equiv 0$, since $B_r(k)$ is then the binomial expansion of $(1-1)^{n-r}$. Thus (1) is of order zero and reduces to $y(k+n) = R(k)$. In terms of symbolic operators, we have the formula,

$$y(k + n) \equiv E^n y(k) \equiv (1 + \Delta)^n y(k) = \sum_{j=0}^{n} \binom{n}{j} \Delta^j y(k),$$

which is readily established by mathematical induction.

A NOTE ON DIAGONABLE MATRIX POLYNOMIALS

D. W. ROBINSON, Brigham Young University

This note provides a necessary and sufficient condition for a matrix polynomial to be diagonable. Although the author is not aware of any mention of the result in the literature, it is doubtful that it has not previously been noticed. However, the result does provide an interesting classroom exercise in the use of principal idempotents of a matrix.

The author is indebted to the referee for several helpful suggestions.

A PROBABILITY RELATED TO THE EUCLIDEAN ALGORITHM

H. S. THURSTON, University of Alabama

If α and $\beta \neq 0$ are integers of the quadratic field $\mathrm{Ra}(\sqrt{m})$ where m is a square-free, negative, rational integer, what is the probability that integers μ and ν of the field can be found such that $\alpha = \beta\mu + \nu$ where $N(\nu) < N(\beta)$? This question arose in a course in the theory of algebraic numbers and its solution was found to involve an interesting bit of integral calculus.

In the complex plane, the points representing the integers of $\mathrm{Ra}(\sqrt{m})$ are located at the vertices of *cells*, parallelograms whose bases are of length one unit and whose sides are inclined at an angle ϕ. If $m \equiv 2$ or 3 (mod 4), the height of the cell is $\sqrt{(-m)}$ and $\phi = \pi/2$. If $m \equiv 1$ (mod 4) the height is $\frac{1}{2}\sqrt{(-m)}$ and $\phi = \mathrm{Arctan}\ \sqrt{(-m)}$. The existence of μ and ν requires that the point $z = \alpha/\beta$ shall be within at least one of the unit circles with center at a vertex of a cell. Thus the required probability is that percentage of the total area of a cell which is covered by such circles.

As is well known, if m is -1, -2, -3, -7 or -11, the probability is unity. For other negative values of m, let us consider the cell $ABCD$ with vertices $A(0, 0)$ and $B(1, 0)$, and lying in the first quadrant (Figs. 1, 2). The circles $r = 1$ and $r = 2\cos\theta$ cut AD and BC, respectively, at E and F, and intersect at G on

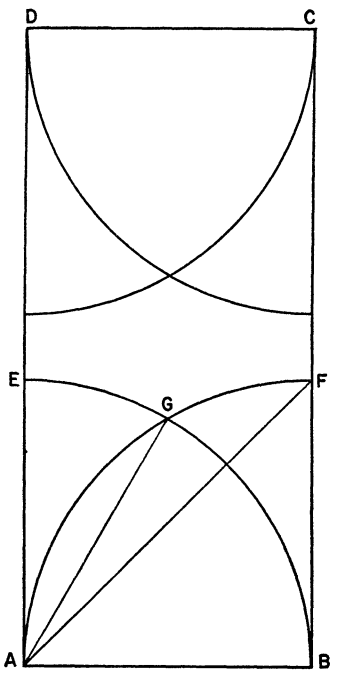


FIG. 1: $m = -5$.

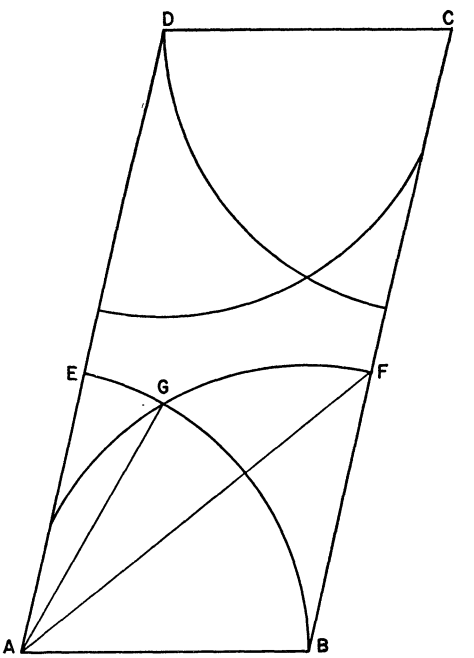


FIG. 2: $m = -19$.

the line $\theta=\pi/3$. The area inside the cell and covered by these circles is made up of the area A_1 of the triangle ABF, the area A_2 of the region FAG bounded by the lines $\theta=\phi/2$, $\theta=\pi/3$, and the circle $r=2\cos\theta$, and the area A_3 of the sector GAE. These areas are given as follows:

$$A_1 = \tfrac{1}{2}\sin\phi,\quad A_2 = 2\int_{\phi/2}^{\pi/3}\cos^2\theta\,d\theta,\quad \text{and}\quad A_3 = \frac{1}{2}\int_{\pi/3}^{\phi}d\theta.$$

If $m\equiv 2$ or 3 (mod 4), $\phi=\pi/2$, the area is obviously independent of m, and is readily shown to be $(2\pi+3\sqrt{3})/12$. If $m\equiv 1$ (mod 4) the areas are

$$A_1 = \tfrac{1}{2}\sin\phi,\quad A_2 = \pi/3 - \phi/2 + \sqrt{3}/4 - \tfrac{1}{2}\sin\phi,\quad A_3 = \phi/2 - \pi/6,$$

where $\phi=\text{Arctan}\ \sqrt{(-m)}$, and the total area is again $(2\pi+3\sqrt{3})/12$. Since the circles with centers at C and D cover an area equal to this, and since the areas of the cells in the two cases are, respectively, $\sqrt{(-m)}$ and $\frac{1}{2}\sqrt{(-m)}$, the probability of the existence of μ and ν is

$$p = (2\pi + 3\sqrt{3})/6\sqrt{(-m)} \text{ if } m \equiv 2 \text{ or } 3 \pmod 4 \quad \text{and} \quad m \leqq -5$$

and

$$p = (2\pi + 3\sqrt{3})/3\sqrt{(-m)} \text{ if } m \equiv 1 \pmod 4 \quad \text{and} \quad m \leqq -15.$$

It will be instructive for the students to draw figures for $m=-3$, -7, -11, and -15. These will show the gradual evolution, when $m\equiv 1$ (mod 4), from the case in which every point of a cell is covered by at least three of the circles to that in which a small region is not covered by any circle. Of particular interest are the cases $m=-7$ and $m=-15$. In the former case the unit circles with centers at A and C are tangent at the mid-point of AC, but every point of the cell is within at least one of the circles. For $m=-15$, the circles with centers at B and D are tangent at the midpoint of BD, and two small regions on opposite sides of the center of the cell are not covered by any of the circles. In this case the probability of the existence of μ and ν is approximately .988.

A REMARK ON THE RANK OF A MATRIX

ITIRO MURASE, University of Tokyo

Consider an $m\times n$ matrix $A=\|a_{ij}\|$. It consists of m row vectors of $V_n(F)$. We define the row space of A as the vector space spanned by the row vectors $(a_{i1}, \cdots, a_{in})$, $1\leqq i\leqq m$, and the row rank of A is defined as the dimension of the row space of A. Similarly we define the column space of A and the column rank of A. Then we have

THEOREM. *The column rank of any matrix is equal to its row rank.*

This common dimension of the row space and the column space is called the rank of the matrix. To prove this theorem and to find the rank of a given matrix A is a starting point of the theory of matrices. I think that the most appropriate

MATHEMATICAL EDUCATION NOTES

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THE SYRACUSE UNIVERSITY "MADISON PROJECT"

ROBERT B. DAVIS, Syracuse University

This is an extremely brief report on the Syracuse University "Madison Project." Longer accounts appear in the M.I.T. *Technology Review* and elsewhere (see references).

The project is concerned with mathematics teaching, particularly in grades 3–9 (but also, to a lesser extent, with the process of learning mathematics on the college and graduate levels). It operates mainly in New York State and in Connecticut. (The name, incidentally, derives from the name of the first school in which the project operated, Madison Junior High School in Syracuse, New York.)

The main objective of the project is a behavioral study of how people learn mathematics. A two-year program in algebra, suitable for presentation to children beginning as early as grade 3, is one concrete result of the project's studies. A workbook to accompany this algebra course is another concrete by-product of project research.

The workbook. The workbook provides for a two-year sequence in algebra, with the following features:

(i) An unusual effort is made to present basic mathematical concepts clearly, and to keep the logic as nearly 100% honest as possible. This latter requirement implies that most of the logic is inductive, by generalization from experience, but a small amount of very simple deductive logic is included (*e.g.*, one of the rare pieces of deductive logic—if a single substitution into an open sentence results in a false statement, the open sentence was not an *identity*, because for identities *every* substitution produces a true statement).

(ii) The language itself is kept casual and unobtrusive. Mathematical concepts are developed from experience with mathematical situations, not by exposition.

(iii) The student learns mainly from his experiences in analyzing a sequence of problems. For example, by plotting linear functions by trial and error, he discovers how to identify and make use of the slope coefficient. We know from experience that even low-I.Q. children can readily discover these things for themselves, if we merely give them an appropriate sequence of problems. *The course includes extremely little exposition*—the workbook itself is mainly a sequence of questions, such as: (1) Graph $y=2x+1$; (2) Graph $y=3x+1$; (3) Graph $y=5x+4$; etc.

(iv) This course has been taught with notable success to gifted fourth and

fifth graders, to low-I.Q. 7th graders, and to many other groups, beginning as early as grade 3, and including a full range of I.Q.'s from low-I.Q. classes to gifted classes.

(v) The workbook deals only with algebra. It can be used as occasional enrichment material: a fourth-grade teacher, for example, might introduce some one or two topics from the workbook. In this way, both teachers and school systems can introduce more algebra into the grades by a gradual process. This flexibility is a big advantage over a closely integrated program which might have to be adopted as a whole or else not at all.

(vi) The program lays great stress on identifying the basic ideas (and especially the *hard* ideas) of mathematics and introducing them *early*, so that they may be progressively refined in many subsequent years of study. The project advocates a first appearance of the concept of limit, for example, as early as grade 4, in simple form. Specifically, we can seek numbers which, when multiplied by themselves, result in a product of 2. The germ of the entire concept of limit is contained in this problem, and can be brought out in subsequent study during the next six or eight years.

(vii) Since mathematics, we believe, consists in large measure of mastery of the art of creative problem solving, students practice this at every step of their study. This is a corollary of our nonexpositional approach of teaching by asking a sequence of questions.

Why algebra in the grades? We have many reasons for wishing to begin the study of algebra as early as grade 3 or 4. To list just a few:

(i) We believe a great deal of algebra is *easier* than a great deal of arithmetic.

(ii) We believe that many algebraic concepts and tools lie at the heart of arithmetic, and can be used both to give deeper insight into arithmetic and to simplify the study of arithmetic.

(iii) We believe that continual practice in creative problem solving is essential to the true nature of mathematics, and that algebra provides more (and easier) material for the practice of this art than arithmetic does.

Other aspects of the project. As mentioned above, the workbook is merely a concrete by-product of project research. We believe that problems in the teaching of mathematics should not be regarded as national crises to be overcome by gigantic crash programs. Rather, we insist that a wiser approach is to regard mathematics education as a uniquely fruitful field for studying the behavior of the human mind; curricula, courses, tests, and so on should appear as "applied" bonuses that stem from a program of basic research.

The experience of four years has convinced us that in working with the schools it is necessary to operate on a regional basis, with collaboration between teachers and university mathematicians. A program brought in from outside cannot and will not be installed successfully within a local school. If you want a better program in a school, you have to cultivate it and grow it right there on the spot.

The use of films. Many observers have been impressed by the project's success in getting fourth graders *to discover for themselves* the coefficient rules for quadratic equations, and so on. We feel this success is closely related to the conduct of our classes. This creates two problems: First, how can we show teachers just what we mean when we speak of "discovery method," "making mathematics an intellectual challenge rather than dreary routine," and so on? Second, how can we study project classes in order to determine why they *do* work so well? We hope that films (with sound) may provide an answer to both questions.

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United States Registry of Junior and Senior High School Science and Mathematics Teaching Personnel

The United States Registry of Junior and Senior High School Science and Mathematics Teaching Personnel has been made possible through grants by the National Science Foundation and the Future Scientists of America Foundation of NSTA to the National Council of Teachers of Mathematics and the National Science Teachers Association (NSTA). At present, the registry consists of more than 120,000 individuals and their school addresses on coded cards punched to indicate fields of teaching responsibility.

Information regarding the conditions under which this name list can be obtained for the distribution of professional type materials can be obtained for mathematics by writing to Myrl H. Ahrendt, Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street N.W., Washington 6, D. C.

The numbers of teachers in the registry are as follows: (Note—since many teachers have more than one subject responsibility, totals do not agree with the sum of the group items.)

Mathematics		76,600	Science		68,400
Grades 7–8	32,000		General Science	33,600	
Grades 9–12	48,600		Biology	21,800	
Dept. Heads	14,500		Chemistry	14,700	
Supervisors (State, County & Local)	308		Physics	12,500	
			Dept. Heads	12,700	
			Supervisors (State, County & Local)	339	

Advanced Mathematics by TV

The University of Notre Dame is expanding its experiment with closed circuit television as a medium for instruction in advanced mathematics.

Approximately 180 high school and college teachers, the majority of them nuns, enrolled for a 1959 summer course in "Higher Algebra and Analytic Geometry" via TV. Arnold Ross, head of Notre Dame's mathematics department conducted the course from the studios of WNDU-TV, the university-operated station. Except for a rotating studio audience of about 20, the students viewed his lectures on monitors set up in six classrooms of the Engineering Building two blocks away.

This experiment grew out of an experimental two-week series of lectures on "Boolean Algebra" offered by Ross via closed circuit TV in the summer of 1958. Although Ross was the only person appearing on the TV screen, he stressed that the course was not a one-man project. Supplementing his daily one-hour lectures were seminars conducted three times a week by other faculty members of the mathematics department. The seminar leaders worked closely with Dr. Ross in evaluating the students' progress.

—From a report which appeared in several Indiana newspapers and the Providence (R. I.) Journal.

NASDTEC Study of Certification Requirements for Teachers of Secondary-School Mathematics and Science

The American Association for the Advancement of Science has received a grant from the Carnegie Corporation of New York for a study of certification requirements for teachers of secondary-school science and mathematics. The grant is based on a proposal addressed to the Carnegie Corporation by the National Association of State Directors of Teacher Education and Certification (NASDTEC). Since this organization is unincorporated, the grant has been made to AAAS which will hold the funds and administer the grant in cooperation with NASDTEC. J. R. Mayor, Director of Education, AAAS, will serve as director of the certification study, and William Viall, formerly chief certification officer in New York state and immediate past president of NASDTEC, has been appointed associate director. E. G. Begle is a member of the advisory board for the study. The offices will be in the AAAS building, 1515 Massachusetts Avenue, N.W., Washington.

The study, planned as an eighteen months' enterprise, has the following goals:

1. To develop procedures whereby representatives of logically interested organizations and state certification and accreditation officials may work together effectively in the development of teacher preparation programs, and to prepare suggested guides for program approval by state certification officers.
2. To reach common agreements in regard to qualitative factors in the preparation and certification of teachers of science and mathematics.
3. To identify areas of instruction in science and mathematics for which preparation programs should be developed and requirements established.
4. To provide continuity in teacher-education programs in science and mathematics beyond the baccalaureate degree in order to facilitate the fulfillment of requests to teach these subjects in any state.

The study is planned to involve state action under which in each state there would be appointed a state committee under the Director of Teacher Education and Certification. The committee would include scientists as well as teachers and professional educators. Four regional groups will be designated to study materials available on teacher certification in science and mathematics, including those prepared by scientific groups and to draft what they conceive to be an ideal program. The result of the regional studies

will be referred to the states and eventually a national conference of certification officers and other interested persons will be held.

Among the purposes of the national conference would be the making of plans for the development of reciprocity agreements among the states. In the proposal it is pointed out that in quite a number of instances several states in a region have reciprocity agreements which affect candidates having certificates in the elementary field because agreements have been reached in general as to what constitutes a sound program for the preparation of elementary teachers. No such agreement exists for secondary level teachers nor seems possible without action of the type proposed in this document.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1401. *Proposed by Emil Grosswald, Institute for Advanced Study*

Prove that for every integer $k \geqq 1$,

$$\frac{d^{k-1}}{du^{k-1}}\left(\frac{1}{e^u - 1}\right) = (-1)^{k-1} \sum_{r=1}^{k} \frac{\gamma_k(r)}{(e^u - 1)^r},$$

where the constants $\gamma_k(r)$ are all positive and satisfy

$$\gamma_k(r) = \sum_{\nu=0}^{r-1} (-1)^\nu \binom{r-1}{\nu} (r - \nu)^{k-1}.$$

In particular, show that $\gamma_k(k) = (k-1)!$.

E 1402. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let r denote the radius of the inscribed sphere of a tetrahedron T and let r_i $(i = 1, 2, 3, 4)$ denote the radii of the exspheres of T which touch one face of T and the other three faces of T produced. Show that $\sum_{i=1}^{4} r_i \geqq 8r$, with equality if and only if T is isosceles.

E 1403. *Proposed by T. J. Rivlin, IBM Corp., Yorktown Heights, N. Y.*

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, with the sequence of coefficients a_3, a_5, a_7, $\cdots$, a_m $(m = n,\ n$ odd; $m = n-1,\ n$ even) either all nonnegative or all nonpositive. Show that:

(a) there exists a number y_n such that $[p(1)-p(-1)]/2=p'(y_n)$,
(b) if n is odd and >1, $|y_n| \leqq (1/n)^{1/(n-1)}$,
(c) if n is even and >2, $|y_n| \leqq [1/(n-1)]^{1/(n-2)}$,
(d) the bounds on $|y_n|$ are the best possible,
(e) when $n=1$, 2 we may take $y_n=0$.

E 1404. *Proposed by H. N. Ward, Harvard University*

(a) In Euclidean 2-dimensional space R^2, let L be the lattice of points with integral coordinates, and T any nondegenerate triangle (including its boundary) whose vertices belong to L. Let T' be any translation of $-T(=\{-x|x\in T\})$ which has members of L as vertices; that is, $T'=p-T$, where $p\in L$. Show that if $T\cap T'\neq\phi$, then $T\cap T'$ contains a member of L.

(b) Show that an analogous statement in Euclidean n-dimensional space R^n, $n>2$, where simplexes are used for triangles, fails.

E 1405. *Proposed by L. L. Garner, University of North Carolina*

Find (a) an expression for the general term, (b) the sum of the first n terms of the series $1+22+333+\cdots$.

SOLUTIONS

The 10^{1000} Digit in the Sequence of Natural Numbers

E 1371 [1959, 512]. *Proposed by G. C. Thompson, Security Mutual Life Ins. Co., Binghamton, N. Y.*

On p. 457 of Vol. I of Dickson's *History of the Theory of Numbers* appears the statement: "E. Barbier asked what is the 10^{1000}th digit written if the series of natural numbers be written down." What is this digit?

Solution by E. S. Eby, U. S. Navy Underwater Sound Laboratory, Fort Trumbull, New London, Conn. The total number of digits used to write all integers of less than $n+1$ digits is $S_n=[(9n-1)10^n+1]/9$; this can be verified by mathematical induction. Now $S_{997}<10^{1000}<S_{998}$. Hence the 10^{1000}th digit occurs in an integer of 998 digits, and, in fact, is the rth digit of $10^{997}+q$, where $10^{1000}-S_{997}=998q+r$, $0\leqq r<998$. Evaluation of q and r show the 10^{1000}th digit to be the 31st digit of the integer

$$100\ 31173\ 45802\ 71654\ 41995\ 10131\ 37385\cdots$$

so that the answer to E. Barbier's question is 3.

Also solved by Louis and Frances Bauer, John Brown, N. J. Fine, A. R. Hyde, John Jordan, Sidney Kravitz, O. W. Laing, W. L. McDaniel, D. C. B. Marsh, G. M. Roe, C. M. Sandwick, Sr., D. R. Sondergeld, and the proposer.

Not all the solvers found the answer 3.

Existence of a Derivative

E 1372 [1959, 512]. *Proposed by J. R. Munkres, Princeton University*

Let $f(x)$ be continuous on the closed interval $[0, 1]$ and have a derivative on the open interval $(0, 1)$. Suppose $|xf'(x)-f(x)+f(0)| < x^2M$ for all x on $(0, 1)$, where M is fixed. Does $f'(0)$ exist?

Solution by N. J. Fine, Institute for Advanced Study. The right-hand derivative exists. We may suppose that $f(0)=0$. Let $h(x)=f(x)/x$ for $0<x<1$. It is sufficient to show that $\lim_{x\to 0} h(x)$ exists. Now h is differentiable in $0<x<1$, and

$$f'(x) = [xh(x)]' = xh'(x) + h(x),$$

whence $xf'(x)-f(x)=x^2h'(x)$. By hypothesis, therefore,

$$|x^2h'(x)| < Mx^2, \text{ or } |h'(x)| < M \qquad (0 < x < 1).$$

If $\{x_n\}$ is any sequence in $(0, 1)$ tending to 0, then

$$h(x_m) - h(x_n) = h'(\xi)(x_m - x_n),$$

where ξ is between x_m and x_n, so $|h(x_m)-h(x_n)| < M|x_m-x_n| < \epsilon$, provided that $m, n > N(\epsilon)$. Thus $\{h(x_n)\}$ is a Cauchy sequence, and hence convergent. Since $\{x_n\}$ was arbitrary, $\lim_{x\to 0} h(x)$ exists.

Also solved by J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), E. S. Eby, H. B. Emerson, George Glauberman, L. C. Graue, Alfred Gray, J. Hooley, R. H. Hou, A. R. Hyde, D. C. B. Marsh, Q. G. Mohammad, D. L. Muench, D. S. Passman, W. A. Peterson, John Rainwater, W. A. Riley, J. T. Rosenbaum, E. C. Schlesinger, Jack Silver, J. V. Whittaker, and the proposer. Late solution by Sidney Glusman.

Editorial Note. The hypothesized continuity of $f(x)$ at $x=0$ is unnecessary.

A Power Series

E 1373 [1959, 512]. *Proposed by W. A. Veech, Dartmouth College*

Find the radius of convergence and the function $f(z)$ represented by

$$f(z) = \sum_{n=0}^{\infty} s_n z^n \quad \text{if} \quad s_n = \sum_{k=0}^{n} k\binom{n}{k}.$$

Solution by E. S. Eby, U. S. Navy Underwater Sound Laboratory, Fort Trumbull, New London, Conn. Since

$$s_n = \sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n-1} n\binom{n-1}{k} = n2^{n-1},$$

we have

$$f(z) = \sum_{n=0}^{\infty} s_n z^n = \sum_{n=1}^{\infty} n2^{n-1}z^n = z\sum_{n=0}^{\infty} (n+1)(2z)^n = z(1+2z)^{-2},$$

with radius of convergence $1/2$.

Also solved by Norman Bauman, A. P. Boblétt, Julian Braun, D. A. Breault, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), A. E. Danese, H. B. Emerson, N. J. Fine, Michael Goldberg, L. D. Goldstone, George Glauberman, Cornelius Groenewoud, J. R. Hanne, J. C. Hickman, J. Hooley, R. H. Hou, A. R. Hyde, John Jordan, Morton Kupperman, T. V. Laxminavasimhan and K. R. Rajagopalan (jointly), W. L. McDonald, D. C. B. Marsh, Q. G. Mohammad, C. S. Ogilvy, F. R. Olson, D. S. Passman, John Rainwater, E. M. Scheuer, Jack Silver, Benjamin Sims, Robert Spira, Eric Sturley, J. R. Swenson, W. F. Trench, Chih-yi Wang, David Zeitlin, and the proposer.

Another Square-Covering Problem

E 1374 [1959, 513]. *Proposed by Yonder Page, Kitchawan, N. Y.*

What is the minimum d such that the unit square may be covered with five sets each having diameter d?

Solution by J. L. Selfridge, University of California at Los Angeles, and R. A. Willoughby, IBM Corp., Yorktown Heights, N. Y. Use a square of side 32 with corners at $(\pm 16, \pm 16)$. Draw appropriate polygons with corners at $(\pm 8, -16)$, $(\pm 8, -3)$, $(\pm 16, 3)$, $(0, 3)$, $(0, 16)$ to show that $d^2 \leqq 425$. Suppose d smaller. There are four corner sets. None covers 1/4 the perimeter. The fifth set touches some side, say the bottom. The fifth set covers $(0, 3)$. The lower corner sets cover $(\pm 8, -16)$ respectively. The upper corner sets cover $(\pm 16, 3)$. No set covers $(0, 16)$. Thus for the unit square $d = 5\sqrt{17}/32$. Note that the fifth set will not fit into a circle of diameter d.

Also solved by Norman Bauman, Julian Braun, J. W. Ellis, Michael Goldberg, P. C. Hammer, and O. Tannenbaum.

Editorial Note. See E 1311 [1958, 775].

Construction of a Triangle Given A, m_a, t_a

E 1375 [1959, 513]. *Proposed by L. D. Goldstone, N. Y. State Public Works Lab., Albany, N. Y.*

Construct a triangle given A, m_a, t_a, that is, given a vertex angle and the median and angle bisector issued from this vertex.

I. *Solution by Victor Thébault, Tennie, Sarthe, France.* Let BAC be the sought triangle whose angle A, angle bisector $AL = t_a$, and median $AM = m_a$ are given. Let B' be the reflection of B in A. Then CB' is parallel to AM and equal to $2m_a$, since M bisects BC. Through L draw the parallel to BA to cut CB' in L'. Since, on the one hand, AL is the bisector of angle A, and on the other hand, BB' and LL' are parallel, we have

$$AB'/AC = AB/AC = LB/LC = L'B'/L'C.$$

Comparison of the first and last of these ratios shows that AL' bisects angle CAB'. Triangle LAL' is then right-angled at A, and since we know a side $(AL = t_a)$ and an angle (angle $ALL' = A/2$) of the triangle, we can construct the triangle and obtain $AL' = t_a' = t_a \tan A/2$. In triangle $B'AC$ we then know a side $(CB' = 2m_a)$, the opposite angle $(= \pi - A)$, and the corresponding angle

bisector ($t_a' = AL'$). The construction of this triangle is classic: on $B'C = 2m_a$, describe the arc containing angle $\pi - A$; then angle bisector AL' passes through the midpoint I of the supplementary arc and

$$IA - IL' = t_a', \qquad (IA)(IL') = (IB')^2 = m_a^2/\sin^2 A/2,$$

whence (knowing their difference and their product) IA and IL' can be constructed. Triangle $B'AC$ being constructed, it suffices to find the symmetric of B in A in order to construct triangle BAC.

Note. This problem was proposed by Desmons (*Journal de Mathématiques Elémentaires et Spéciale de Longchamps*, 1881, p. 92, question 301) and solved with the use of calculus by Debray (*ibid*, p. 398). The preceding geometric solution, inspired by that of L. Bickart (*Journal de Vuibert*, 31e année, p. 61, question 6345), constitutes a geometric solution of the problem of finding the points of intersection of a conic and a circle centered at one of the vertices of the conic. An angle $xAy = A$ being given, as well as a point L on its bisector, it suffices to observe that Ax, Ay intercept on an arbitrary line through L a segment whose midpoint M describes a hyperbola h having vertices A and L and whose asymptotes are parallel to Ax, Ay. Problem E 1375 is that of finding the intersections of h and the circle of center A and radius m_a.

II. *Solution by Roscoe Woods, State University of Iowa.* Let ABC be the required triangle. Let the bisector (internal) of angle A intersect BC in U and let A' be the midpoint of BC. Let M denote the foot of the perpendicular from A' to AU. If $2\theta = A$, it is readily seen that $AM = [(b+c)\cos\theta]/2$. If segment AM can be constructed then point A' can be found, since $AA' = m_a$ is given. Consequently the line through U (known since $AU = t_a$) and A' determine the vertices B and C, since the angle A is given. From the figure it is obvious that in order to have a solution, m_a must be greater than or equal to t_a.

Since $4m_a^2 = 2b^2 + 2c^2 - a^2$ and $a^2 = b^2 + c^2 - 2bc\cos A$ we find, by use of the relation $2\sin^2\theta = 1 - \cos A$, that $4m_a^2 = (b+c)^2 - 4bc\sin^2\theta$. A short calculation shows that $t_a(b+c) = 2bc\cos\theta$. If bc is eliminated from these two expressions, a quadratic equation in $b+c$ is obtained. From the positive root of this equation an expression for AM is readily found, namely

$$AM = \tfrac{1}{2}(b+c)\cos\theta = \tfrac{1}{2}t_a\sin^2\theta + \sqrt{\{[\tfrac{1}{2}t_a\sin^2\theta]^2 + m_a^2\cos^2\theta\}}.$$

Segment AM may now be constructed. From the right angle RST, with angle $R = \theta$ and hypotenuse $RT = t_a$, the projection of ST upon RT has length $t_a\sin^2\theta$. Also, from a right triangle with hypotenuse m_a and one acute angle θ, $m_a\cos\theta$ may be found. The details of the construction of AM may be omitted since they are obvious. There are two solutions if m_a is greater than AM, one solution when $AM = m_a(= t_a)$, and no solution when m_a is less than AM.

Also solved by Josef Langr, D. C. B. Marsh, Beckham Martin, and the proposer. N. A. Court furnished some remarks.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

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PROBLEMS FOR SOLUTION

4877 [1959, 921] Corrected. *Proposed by O. P. Aggarwal and Irwin Guttman, University of Alberta.*

Show that $\int_0^a \exp(-\frac{1}{2}t^2)\,dt=(\frac{1}{2}\pi-aI)^{1/2}$, where

$$I=\int_{a^2}^{2a^2}\frac{\exp(-\frac{1}{2}\omega)d\omega}{\omega\sqrt{(\omega-a^2)}}.$$

4887. *Proposed by Z. A. Melzak, McGill University*

What is the shortest length of infinitely strong, infinitely thin and perfectly inextensible string with which a smooth sphere of radius 1 can be tied up so that the resulting parcel can be picked up and carried by the string without danger of slipping apart?

4888. *Proposed by Emil Grosswald, University of Pennsylvania*

Determine the values of α, for which the following series converge:

$$S_1(\alpha)=\sum_{m=0}^{\infty}\frac{(m+1)^{m-1}}{m!}\alpha^m,\qquad S_2(\alpha)=\sum_{m=0}^{\infty}\frac{(m+1)^{m-1}}{m!}\alpha^{m+1}.$$

Show also that for those values of α, $S_2=\log S_1$.

4889. *Proposed by M. S. Klamkin, A VCO Research, Wilmington, Mass.*

If k points are distributed at random at the vertices of a regular n-gon, determine the probability that the center of gravity of the k masses lies in a circle of radius r about the center of the n-gon. What does the probability function reduce to when $n\to\infty$?

4890. *Proposed by H. S. Shapiro, New York University*

Let $f(t)$ be positive, integrable and strictly decreasing in $(0,\infty)$, and suppose that the Fourier cosine transform of f is strictly decreasing in $(0,\infty)$. Then,

$$\int_0^{\infty} f(t)e^{izt}dt$$

is regular and schlicht in the upper half-plane.

4891. *Proposed by I. S. Gâl, Cornell University*

Let X be a regular topological space and let A be a dense subset of X such that every net with values in A has a non-void adherence in X. Show that X is compact. Is regularity necessity?

4892. *Proposed by Andrew Kraus, Boulder, Colorado*

Show that if p is an odd prime and $k \geqq 1$, then

$$(p^{k-1}+1)\cdots(p^k-1)+p^{p^{k-1}-k} \equiv 0 \pmod{p^{p^{k-1}}}.$$

Note that for $k=1$ this is Wilson's theorem.

SOLUTIONS

Linear Independence

4797 [1958, 452; 1959, 730]. *Proposed by D. J. Newman, Brown University.*

Prove that all expressions like $7\sqrt{19}/4-3\sqrt{7}+8\sqrt{6}/5$ are irrational. More specifically, prove that the square roots of the square-free integers are linearly independent over the rationals.

III. *Solution by Harley Flanders, University of California, Berkeley.* We use the rudiments of field theory to prove the theorem:

$\mathbf{T}_n$: *Let $a_1, \cdots, a_n$ be positive integers, coprime in pairs, no one of which is a perfect square. Then the 2^n algebraic integers*

$$(a_1^{e_1}\cdots a_n^{e_n})^{1/2}, \qquad e_j = 0, 1,$$

are linearly independent over the field Q of rationals.

$\mathbf{T}_1$ is true and we attack $\mathbf{T}_n$ where $n \geqq 2$; and we have the induction hypothesis for smaller n. We define fields $K_0=Q$, $K_n=K_{n-1}(\sqrt{a_n})$. Then $[K_n: K_{n-1}] \leqq 2$, hence $[K_n: Q] \leqq 2^n$. Also 1, $\sqrt{a_n}$ span K_n linearly over K_{n-1}, hence the 2^n numbers above span K_n over Q. To prove they are linearly independent, hence a basis, we must prove $[K_n: Q]=2^n$. But by induction $[K_{n-1}: Q]=2^{n-1}$, so we must merely prove $[K_n: K_{n-1}]=2$. If not, then $K_n=K_{n-1}$, $\sqrt{a_n} \in K_{n-1}$. We have

$$\sqrt{a_n} = b + c\sqrt{a_{n-1}}, \qquad b, c \in K_{n-2}.$$

Thus $a_n=b^2+c^2a_{n-1}+2bc\sqrt{a_{n-1}}$. Since $\sqrt{a_{n-1}} \notin K_{n-2}$, we have $bc=0$.

Case 1. $c=0$. Then $\sqrt{a_n} \in K_{n-2}$ which contradicts $\mathbf{T}_{n-1}$ applied to $a_1, \cdots, a_{n-2}, a_n$.

Case 2. $b=0$. Then

$$\sqrt{a_n} = c\sqrt{a_{n-1}}, \qquad \sqrt{(a_{n-1}a_n)} = a_{n-1}c \in K_{n-2}.$$

This again contradicts $\mathbf{T}_{n-1}$ for the numbers $a_1, \cdots, a_{n-2}, a_{n-1}a_n$.

This completes the proof. The only real fact from field theory which we have used is the relation for linear dimension $[K: L][L: F]=[K: F]$ where $K \supset L \supset F$.

Also solved by Melvin Hausner, and Eugene Levine.

Radius of Convergence

4840 [1959, 317]. *Proposed by P. T. Bateman, University of Illinois*

Suppose $a_0>0$, $a_1 \geqq 0$, $a_2 \geqq 0$, $\cdots$, and put $A_n=a_0+a_1+\cdots+a_n$. Show that if as $n \to \infty$, $A_n \to \infty$ and $a_n/A_n \to 0$, then $\sum a_n x^n$ has radius of convergence unity.

Solution by Yi Chang, Fenn College, Cleveland, Ohio. Let r and R be the radii of convergence of $\sum a_n x^n$ and of $\sum A_n x^n$, respectively. Since $\lim A_n=\infty$, $\sum a_n x^n$ diverges for $x=1$, whence $r \leqq 1$.

Now $\lim a_n/A_n=0$ $(a_n \geqq 0)$ implies $r \geqq R$. It also implies $1=\lim A_n/A_n$ $=\lim (A_{n-1}+a_n)/A_n=\lim A_{n-1}/A_n$, whence $R=1$. Therefore $r=1$, as required.

Also solved by R. C. Bollinger, N. J. Fine, Leopold Flatto, L. R. Ford, Jr., Paul Gordon, Fritz Herzog, Joseph Lehner, A. E. Livingston, Lee Lorch, M. D. Mavinkure, A. A. Mullin, John Rainwater, James Reill, Peter Renz, J. E. Vinson, R. J. Whitley, Albert Wilansky, David Zeitlin, and the proposer. E. E. Kohlbecker finds the essential part of the result in Hardy, *Divergent Series*, p. 65.

A Set Everywhere Dense in the Plane

4843 [1959, 317]. *Proposed by Leopold Flatto, Brooklyn Polytechnic Institute*

Let S be a set everywhere dense in the plane. Does it follow that S is dense on some line segment in the plane?

Solution by Leo Moser, University of Alberta. No. In fact we construct a set S, dense in the plane, which contains no three collinear points. Let $Z_1, Z_2, \cdots$ be an enumeration of the algebraic points in the complex plane. We now let S be the points $S_1, S_2, \cdots$ defined as follows: $S_1=Z_1$, $S_2=Z_2$, while for $n>2$, $S_n=Z_n+2^{-n}e^{2\pi i/f(n)}$ where $f(n)$ is the smallest positive integer for which S_n is not collinear with any two of $S_1, S_2, \cdots, S_{n-1}$. Since we have infinitely many concyclic points to pick from and only a finite number of lines to avoid, $f(n)$ will certainly exist. The set S will have no three collinear points but since the Z's are dense and $|S_n-Z_n|=2^{-n}$ the set S will be dense.

Also solved by Donald Coleman, N. J. Fine, Fred Galvin, Fritz Herzog, Arthur Rosenthal, and the proposer.

Editorial Note. Coleman notes that Sierpinski has proved (*Fund. Math.*, vol. 1, 1920, pp. 112–115) that there exists a (non-measurable) subset of the plane which meets every closed set of positive measure, but which intersects any line in at most two points.

Rosenthal has shown (*S.-B. Bayer. Akad. Wiss.*, 1922, pp. 221–240—in particular p. 227) the existence of sets S everywhere dense in the plane which on every straight line have exactly 2 (or more generally, n) points.

The Ring of Polynomials in the Partial Differential Operators

4844 [1959, 318]. *Proposed by D. S. Kahn, Princeton University*

Consider the ring Q of polynomials in the partial differential operators $\partial/\partial x_1, \cdots, \partial/\partial x_n$ over the field of rational functions of $x_1, \cdots, x_n$. (1) Determine all (two-sided) ideals of Q. (2) Prove that Q is a primitive ring.

Solution by the proposer. In the case of two variables, we prove that the only ideals are (O) and Q.

A term $a_{mn}D_x^m D_y^n$ is said to have degree $m+n$. A polynomial is said to have deg p if p is the degree of the term of highest degree with non-zero coefficients. Let $\mathfrak{A} \neq 0$ be an ideal in Q and let A have least degree among non-zero elements of $\mathfrak{A}$. If deg $A = 0$, we are done. Assume deg $A = p > 0$. Then

$$(xy)^{-1}(A)(xy) = \sum_{i=0}^{p} a_{i,p-i} D_x^i D_y^{p-i} + (\text{terms of deg} < p) \in \mathfrak{A}.$$

Therefore $\overline{A} = A - (xy)^{-1}(A)(xy) \in \mathfrak{A}$ and $\overline{A} \neq 0$ with deg $< p$. This contradicts the minimality of deg A, whence $\mathfrak{A} = Q$.

Finally it is known that any simple ring with identity is primitive.

The same method of proof holds for n variables.

A Maximum Related to Orthogonal Matrices

4845 [1959, 426]. *Proposed by Fulton Koehler, University of Minnesota*

Let α be a real square matrix $[a_{ij}]$ and let $D(\alpha) = \sum_{i \geqq j} a_{ij}$. Find the maximum of $D(\alpha)$ for the group of orthogonal matrices of given order n; and show that, as $n \to \infty$, the maximum is asymptotic to $(n/\pi) \log n$.

Solution by the proposer. For given n, the quantity $D(\alpha)$ is a continuous function of the n^2 elements a_{ij}, and the manifold of points (a_{ij}) corresponding to orthogonal matrices is compact; hence, $D(\alpha)$ has a maximum. Let $D(\alpha)$ assume the maximum for $\alpha = \gamma = [c_{ij}]$.

Let $\tilde{\gamma}(\theta)$ be the orthogonal matrix whose kth column is composed of the elements $c_{ik} \cos\theta + c_{im} \sin\theta$ $(i = 1, \cdots, n)$, whose mth column $(k \neq m)$ is composed of the elements $c_{im} \cos\theta - c_{ik} \sin\theta$ $(i = 1, \cdots, n)$, and whose other columns are identical with those of γ. Then

$$D(\tilde{\gamma}) - D(\gamma) = \sum_{i=k}^{n} \{c_{ik}(\cos\theta - 1) + c_{im} \sin\theta\} + \sum_{i=m}^{n} \{c_{im}(\cos\theta - 1) - c_{ik} \sin\theta\} \leqq 0$$

for all real θ. For infinitesimal θ this can be written

$$\left\{ \sum_{i=k}^{n} c_{im} - \sum_{i=m}^{n} c_{ik} \right\} \theta + O(\theta^2) \leqq 0$$

which is possible only if

$$\sum_{i=k}^{n} c_{im} - \sum_{i=m}^{n} c_{ik} \qquad (m, k = 1, \cdots, n). \tag{1}$$

Let $s_{km} = \sum_{i=k}^{n} c_{im}$, $\sigma = [s_{km}]$. Then σ is symmetric, and

$$D(\gamma) = s_{11} + \cdots + s_{nn} = \lambda_1 + \cdots + \lambda_n, \tag{2}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic values of σ. From the orthogonality of γ and the symmetry of σ we have

$$\sigma^2 = \sigma\sigma^* = \begin{bmatrix} n & n-1 & n-2 & \cdots & 1 \\ n-1 & n-1 & n-2 & \cdots & 1 \\ n-2 & n-2 & n-2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}, \tag{3}$$

and it can seen by inspection that

$$(\sigma^2)^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & & -1 & 2 \end{bmatrix}. \tag{4}$$

Let $2-\lambda = 2\cos\theta$. Then

$$\text{Det}\,[(\sigma^2)^{-1} - \lambda E] = (2\cos\theta - 1)\frac{\sin n\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta} = \frac{\sin(n+1)\theta - \sin n\theta}{\sin\theta}$$

from the well-known formula

$$\begin{vmatrix} 2\cos\theta & -1 & 0 & \cdots & 0 \\ -1 & 2\cos\theta & -1 & \cdots & 0 \\ 0 & -1 & 2\cos\theta & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\theta \end{vmatrix} = \frac{\sin(m+1)\theta}{\sin\theta},$$

where m is the order of the determinant on the left. Hence the characteristic values of $(\sigma^2)^{-1}$ are $2(1-\cos r\pi)$, where $r = (2k-1)/(2n+1)$ and $k = 1, \cdots, n$. The characteristic values of σ are therefore $\pm(1-\cos r\pi)^{-1/2}/\sqrt{2}$, and

$$D(\gamma) \leqq (1/\sqrt{2}) \sum_{k=1}^{n} (1 - \cos r\pi)^{-1/2}. \tag{5}$$

We now show that equality must hold in (5). Let a matrix A of order n be defined by

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then $\sigma = A\gamma$. Let U be the orthogonal matrix such that $U\sigma^2U^*$ is the diagonal matrix whose kth diagonal element is $\frac{1}{2}(1-\cos r\pi)^{-1}$. Let δ be the diagonal matrix of order n whose kth diagonal element is $(1-\cos r\pi)^{-1/2}/\sqrt{2}$, and let $\gamma_1 = A^{-1}U^*\delta U$. Then $\gamma_1\gamma_1^* = A^{-1}U^*\delta^2U(A^{-1})^* = A^{-1}\sigma^2(A^{-1})^* = A^{-1}(A\gamma)(\gamma^*A^*)(A^{-1})^* = \gamma\gamma^* = E$, hence γ_1 is orthogonal. Also $D(\gamma_1)$ is the sum of the diagonal elements of $A\gamma_1 = U^*\delta U$ which is equal to

$$\sum_{k=1}^{n} (1/\sqrt{2})(1 - \cos r\pi)^{-1/2}.$$

Hence

$$D(\gamma) \geqq D(\gamma_1) = \sum_{k=1}^{n} (1/\sqrt{2})(1 - \cos r\pi)^{-1/2}. \tag{6}$$

For given n, the required maximum is therefore

$$\sum_{k=1}^{n} \frac{1}{\sqrt{2}} (1 - \cos r\pi)^{-1/2} = \sum_{k=1}^{n} (2 \sin \tfrac{1}{2}r\pi)^{-1}$$

$$= \sum_{k=1}^{n} 1/r\pi + \sum_{k=1}^{n} \left\{(2 \sin \tfrac{1}{2}r\pi)^{-1} - (r\pi)^{-1}\right\} = (n/\pi) \log n + O(n),$$

since $\sum_{k=1}^{n} 1/(2k-1) = \frac{1}{2} \log n + O(1)$ and $(2 \sin \frac{1}{2}\theta)^{-1} - \theta^{-1}$ is bounded for $0 < \theta < \pi$.

Hermitian Matrices

4846 [1959, 427]. *Proposed by Olga Taussky, California Institute of Technology*

Show that a matrix which is similar to a real diagonal matrix is the product of two hermitian matrices one of which is positive definite. (The converse is known.)

Solution by W. V. Parker, Alabama Polytechnic Institute. If matrix A is similar to the real diagonal matrix D, then there exists a non-singular $n \times n$ matrix P such that $PAP^{-1} = D$. We may write $P = UH$, where U is a unitary matrix and

H is positive definite hermitian. Then $UHAH^{-1}U^* = D$ and $HAH^{-1} = H_1$ where H_1 is hermitian. It follows that $A = H^{-1}H_1H^{-1}H^2 = (H^{-1})^2HH_1H$, where $H^{-1}H_1H^{-1}$ and HH_1H are hermitian, and H^2 and $(H^{-1})^2$ are positive definite hermitian.

Also solved by A. C. Aitken, Wallace Givens, P. C. Greiner, Arthur F. Kaupe, Hans Schneider, H. Schwerdtfeger, and the proposer.

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EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Principles and Applications of Random Noise Theory. By Julius S. Bendat. Wiley, New York, 1958. xxi+431 pp. $11.00.

While in this book mathematical methods are cleverly introduced as needed in the development of the physical theory, the mathematical portions are definitely limited to the mathematics necessary for immediate application. Included, however, as part of a chapter is an excellent but abbreviated course on probability. The relationship between the correlation functions of the statistician and power spectral density functions of the engineer is clearly delineated. The autocorrelation function $Ae^{-k|\tau|}\cos c\tau$ where A, k, and c are nonnegative constants is shown to be of considerable theoretical as well as experimental importance. The description, in some detail, of investigations of methods of controlling and evaluating this fundamental autocorrelation function is perhaps the primary concern of the book.

The author spells out his objectives in the preface and follows the table of contents with single paragraph summaries of each of the ten chapters. Following each chapter is a list of references, and finally following the last chapter is a quite complete bibliography of the generally accessible literature.

The author is to be congratulated for conceiving and carrying out so well the task of supplying such a well-connected and authoritative account of a virgin field, a field in which his own contributions have been most extensive.

HERBERT A. MEYER
University of Florida

Programming the IBM 650 Magnetic Drum Computer and Data-Processing Machine. By Richard V. Andree. Holt, New York, 1958. vi+114 pp. $2.95. (paper bound).

This book consists of a set of notes with many simple examples and exercises devoted to programming the IBM 650. The first four chapters are self-contained (71 pp.) giving the reader simple programming in machine language only, flow

charting, console operation, looping, stepping, testing, branching, scaling, examples of loss of accuracy, and use of subroutines. The remaining part of the book gives very brief discussions of the symbolic optimum assembly program, the Bell interpretive system, and the For Transit compiler with reference to standard manuals for more complete treatment.

Although very little use is made of symbolic programming, the book can be useful at any school with an IBM 650. Even if symbolic programming is introduced early by the instructor, the examples and exercises are easy to put into the symbolic form.

Also a minimum of the details of loading and general procedure is presented, thus making the text adaptable as introductory material for any IBM 650 computing center.

The author's friendly and informal style makes the text pleasant and easy to read.

DAVID B. DEKKER
University of Washington

Editorial Note. Since Professor Andree is the editor of this section, the editor-in-chief has assumed responsibility for editing this review.

The Theory of Groups. By Marshall Hall, Jr. Macmillan, New York, 1959. xiii+434 pp. $8.75.

This is a book which I wish I could put in the hands of every graduate student who has shown an interest in the elements of group theory. The first ten chapters would give him an excellent foundation in group theory, and there would still remain ten chapters for his delight. I will speak of the latter, some of which are unique.

Many mathematicians are vaguely aware that there is a famous problem for groups, called the Burnside problem. But what proportion, even if we restrict attention to algebraists alone, can state the problem, let alone distinguish it from the several forms of the restricted Burnside problem? This book supplies most of the tools currently used for the Burnside problem, proves many of the known theorems in the subject (with a few omissions where the only known proofs would severely try the patience of the reader) and brings the history of the problem as close to the present moment as is humanly possible. No other book goes half so far in this direction.

Who can direct a student to a book containing the theory of representations of a group of finite order n over a field of characteristic prime to n? This theory is contained *in principle* in H. Weyl's *The Classical Groups* (see footnote, p. 100) and has been given in many a graduate course, but Hall seems to be the first to publish a detailed account within the covers of a book. The advantages are numerous.

And, finally, there is no substitute in English for the eighty pages which Hall devotes to the connections between groups and projective planes. Car-

michael's *Theory of Groups of Finite Order*, which does something along this line, appeared before the existence of most of the present material.

To this reviewer, the book seems like a lively and attractive grandson of Burnside's *Theory of Groups of Finite Order*, a grandson, let it be added, with a marked personality of his own.

R. H. BRUCK
University of Wisconsin

Sampling Opinions. An Analysis of Survey Procedures. By Frederick F. Stephan and Philip J. McCarthy. Wiley, New York, 1958. 451 pp. $12.00.

From a pragmatic point of view, statistics is the art of making inferences about populations based upon samples from these populations. This book studies the problems of applying basic techniques of statistics to investigating human populations. The emphasis is on socio-economic phenomena, but there is much of interest for all scientific investigators. The book is written in a nonmathematical fashion, but is unlike most statistical cookbooks in two important respects: a) It reflects a sophisticated training in modern theoretical statistics on the part of the authors, and b) it should be of considerable interest to an audience with such training. To elaborate on the second point, I question that much of the book will make sense to one not versed in statistics, despite its nonmathematical appearance.

MEYER DWASS
Northwestern University

Studies in Linear and Non-Linear Programming. Edited by Kenneth Arrow, Leonid Hurwicz, and Hirofumi Uzawa. Stanford University Press, Stanford, California, 1958. 229 pp. $7.50.

This book is the second in the Stanford Mathematical Studies in the Social Sciences and is a well-integrated collection of research papers that deal with the fields of linear and nonlinear programming. In addition to the contributions of the editors there are papers by Hollis B. Chenery, Selmer M. Johnson, Samuel Karlin, Thomas Marschak, and Robert M. Solow. Although the original problems studied were in the area of transportation, assignment, and storage, the theory as it is developed here may be applied to general problems involving the optimal or feasible allocation of limited resources.

In addition to an introductory first chapter, that also summarizes the main results, there are three parts. Part I presents the basic mathematical structure in the form of fundamental existence and duality theorems for linear, convex, and concave programming. In Part II, the gradient of a function is defined and applied to solve constrained maximization problems by means of successive approximations. Part III is devoted to applying and extending the techniques to problems in price speculation, machine assignment, and interindustry analysis.

Because it is primarily devoted to the exposition of general results, there

should be at least three types of users of this excellent book. First, it should appeal to those who are interested in research in mathematical programming and allocation theory. It should be of special use to those in mathematics, mathematical economics, and operations research who are concerned with solving industrial and economic policy problems. Finally, with the further addition of examples and problems, this book would prove to be an excellent basis for an advanced course in mathematical programming to be given to graduate students in the mathematical or management sciences. The only modification this reviewer would make would be to include material on programming with uncertainty.

BURTON V. DEAN
Case Institute of Technology

Introduction to Mathematical Analysis with Applications to Problems of Economics. By Paul H. Daus and William M. Whyburn. Addison-Wesley, Reading, Mass., 1958. 244 pp. $6.50.

This book covers such material as elementary notions of analytic geometry, differentiation and integration, partial differentiation, maxima and minima including constraint conditions and the method of Lagrange multipliers, curve fitting (straight line, parabola, exponential and power, laws by method of least squares), correlation. Spread throughout these mathematical topics are the applications to economics, such as supply and demand curves, market equilibrium, effect of taxation, elasticity, marginal revenue, profit under monopoly, production functions, indifference maps, utility index, relative preference, etc. The material is well written and the large number of interesting problems are more than adequate for a one-semester course. Especially recommended for students of business and economics.

CLETUS OAKLEY
Haverford College

Methods in Numerical Analysis. By Kaj L. Nielsen. Macmillan, New York, 1957. xiii+382 pp. $7.25.

Since this text appeared, the reviewer has used it in two long sessions and in two summer sessions for a beginning course for technical students pursuing studies in numerical analysis. The reviewer has had opportunity to examine the first printing (1956) and the second printing (1957). A number of harmless and/or annoying misprints in the first printing were corrected in the second printing.

Many teachers prefer to use a textbook with errors on the theory that a careful students will learn much more by discovering an error and reconstructing the material in correct fashion. Perhaps the reviewer is such a teacher, for there are numerous errors in Nielsen's book, and yet he has continued to use Nielsen's book.

The reviewer finds no basis in fact for the statement on page 42: "If the nth

differences of a function tabulated at equally spaced intervals are constant, the function is a polynomial of degree n."

The answer list is poorly prepared. For example, Exercise 7a, page 168 uses five significant digit tabular data, and the answers are six significant digit numbers.

In many places the author writes that "to a first order approximation $\Delta N = \sum_{i=1}^{n} \Delta x_i(\partial f/\partial x_i)$" or some similar statement. This reviewer instructs all his students to replace the equality sign above by some symbol to indicate approximate equality, or order of magnitude of equality, or the use of a phrase like leading error terms.

Nielsen's style of writing appeals to many technical students who are not majoring in mathematics. If the class is taught by a mathematician who will make sure that the pupils discover and weed out the errors in the book, this text can be used with profit.

ROBERT E. GREENWOOD
The University of Texas

Linear Programming: Methods and Applications. By Saul I. Gass. McGraw-Hill, New York, 1958. xii+233 pp. $6.75.

This is the first textbook on linear programming which gives a comprehensive treatment of the subject and is suitable for use in a college mathematics course. The covering is remarkably complete, including an adequate survey of matrices, vectors, convex sets and linear inequalities, a detailed description of the theoretical and computational aspects of methods of solution, and applications. It is the author's intent that the text be appropriate for a one-semester course at the senior or first-year-graduate level. Such a course would have great value for students of many disciplines, and for this purpose the book warrants unreserved recommendation.

After an introductory chapter in which the general linear programming problem is stated and several examples are given, there follows a compact discussion of essential mathematical background (matrices, convex sets, etc.). Chapter 3 treats properties of solutions to linear programming problems, especially extreme point solutions. Chapter 4 describes the simplex method and Chapter 5 deals with duality in linear programming. Recent and/or special methods are covered in the next four chapters (the revised simplex method, degeneracy procedures, parametric linear programming, additional computational techniques.)

The remaining three chapters are concerned with applications. No attempt has been made to encompass all of these, but a good variety of representative examples is covered. The transportation problem and its more popular variants are studied in detail. Unfortunately, the max-flow, min-cut method of Ford and Fulkerson receives only passing mention. One might also wish that at least one example of a nonlinear problem solvable by piecewise linear approximation to

the objective form had been included. The final chapter treats the equivalence of the zero-sum two-person game and the linear programming problem.

Each chapter is followed by a short set of exercises suitable for pencil-and-paper solution, and the book has been written so that recourse to a computer is not required. The exercises achieve their purpose, but this reviewer would have appreciated more "motivated" problems. A comprehensive bibliography is appended, containing 102 general references and 91 items arranged by area of application.

L. E. WARD, JR.
University of Oregon

Elementary Algebra. By D. S. Russell, Allyn and Bacon, Boston, 1959. ix+297 pp. $4.50.

The author introduces this text as "designed . . . for the college student who has had no previous training in algebra as well as for . . . an intensive review of the basic fundamentals." The presentation is, however, as well suited for high school as it is for college. The author states that "the theory is presented in a simple style." This is surely true, although at times one might wish for a less intuitive approach, particularly in a text for college use. In some places there is an unfortunate lack of precision of statement. For examples: "a^3 means that the number represented by a is multiplied together three times"; (How does one multiply a number together?), and "a *real* number is any number that can be associated with a point on a line." The associative, commutative and distributive laws could have been well introduced where they are first used in formulas, rather than later. Numerical multiplication by use of binomial factors is treated in a short section. This is an interesting inclusion often omitted in other texts. In the section on quadratics the discriminant is not mentioned as such, although some exercises provide opportunity for its use. Problems are numerous and generally well chosen although more on rates and distances, mixtures, and fractions might have been included in the "stated problems." "Review tests" included seem to have worthwhile possibilities as teaching aids. This is an adequate text but has little to recommend it over some others on the market.

JAMES L. SIMPSON
Montana State College

Calculus. By R. E. Johnson and F. L. Kiokemeister. Allyn and Bacon, Boston, 1959. xii+634 pp. $7.75.

This text is designed for students who have had algebra, trigonometry, plane geometry, and analytic geometry. It contains applications of the calculus with an excellent discussion of its rigorous basis. Illustrative examples are used to motivate new concepts in advance of their theoretical discussion.

The definite integral is carefully handled with a discussion of upper and lower integrals, and the integral as the limit of a sequence of Riemann sums.

There is a chapter on the basic properties of continuous and differentiable

functions. This chapter contains an excellent discussion of approximation by Taylor polynomials which should be of real help to the students in handling Taylor series and its remainder later.

Convergence of infinite series is handled with more than usual care, and proofs are given for termwise differentiation and integration of power series. Solid analytic geometry is treated both with and without vectors. One whole chapter and part of another are devoted to vectors.

Second order linear differential equations with constant coefficients are stressed in the final chapter. Material in this chapter and on vectors will help the engineering student whose mathematics lags behind his need for it in other courses.

The text contains all the traditional calculus material and should be readily adaptable to a variety of courses.

Finally there are collections of facts and formulas from trigonometry and analytics, a table of integrals, various numerical tables, and answers to odd numbered problems.

The book set up is attractive, and the material well presented. The reviewer recommends the text as a worthy continuation of the high standard set by the book *Calculus with Analytic Geometry* by the same authors.

W. N. HUFF
University of Oklahoma

Linear Operators, Part I: General Theory, by N. Dunford and J. T. Schwartz. Interscience, New York, 1958. xiv+858 pp. $23.00.

The present volume is the first of a two-volume work which is designed to give a self-contained and comprehensive survey of linear analysis. The first volume is primarily devoted to a study of Banach spaces and the linear transformations between such spaces. In addition to a wealth of material concerning the theory of linear operators, one finds many special features worthy of note. One such is an extensive treatment of the theory of integration in a form which applies to both vector-valued and extended real-valued measures. Another is a detailed comparison of the properties of 28 special spaces. Complete proofs are presented of all propositions except those within the exercise lists. The latter, incidentally, are distributed throughout the volume and contain a multitude of the applications of the theory to classical analysis. Each chapter includes notes of a historical nature and notes concerning the differing directions in which the theory has developed. The volume closes with a list of references 96 pages in length and covering the material of both the current and prospective volumes.

This volume should be an invaluable reference work, both from the point of view of the worker in the field of linear analysis and from that of a teacher who is presenting some aspects of this theory. The mathematical world will surely find this volume a most useful one for a considerable time to come.

PAUL CIVIN
University of Oregon

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of California, Riverside: Dr. R. L. McKinney has been promoted to Assistant Professor; Dr. Auguste Forge, Duke University, has been appointed Visiting Assistant Professor; Dr. R. E. Haymond, University of Oregon, has been appointed Instructor; Miss Dorothy I. Koehler, Cornwall College, Montego Bay, Jamaica, has been appointed Associate.

University of Florida: Assistant Professor Diran Sarafyan is on leave of absence to study in Rome, Italy, on a National Science Foundation Faculty Fellowship; Dr. L. R. Luckenbach has been appointed Int. Assistant Professor; Mrs. Joyce C. Cundiff, Alabama Polytechnic Institute, Mr. R. V. Gentry, Martin Company, Mr. J. W. Kenelly, and Mr. J. W. Meux have been appointed Instructors; Mr. R. E. Sellers, Radio Corporation of America, has been appointed Int. Instructor.

Eastern Michigan University: Assistant Professor Madeline Early has been promoted to Associate Professor; Mr. B. L. Goosey, Pennsylvania State University, has been appointed Assistant Professor.

Iowa State Teachers College: Associate Professor E. Glenadine Gibb has been promoted to Professor; Assistant Professor Augusta Schurrer has been promoted to Associate Professor; Mrs. Ina Mae Silvey has been promoted to Assistant Professor; Miss Ruth L. Erckmann, Central High School, Sioux City, Iowa, has been appointed Assistant Professor; Mr. R. L. Yount, Illinois Wesleyan University, and Mr. J. E. Bruha have been appointed Instructors.

Lafayette College: Dr. John Bender, Rutgers University, and Dr. J. S. Klein, Wilson College, have been appointed Associate Professors; Mr. John Robinson, Operations Research, Inc., Adelphi, Maryland, has been appointed Assistant Professor.

University of Louisville: Professor Guy Stevenson has retired as Head of the Department of Mathematics, but will continue as Professor of Mathematics; Associate Professor W. H. Spragens, Jr. has been promoted to Professor and Head of the Department of Mathematics.

University of Massachusetts: Miss Lorraine D. Lavallee, University of Michigan, Miss Carmela M. Landolfi, Ohio State University, Mr. R. M. Kennedy, and Mr. Y. S. Farsakh have been appointed Instructors.

Pennsylvania State University: Assistant Professor C. C. Faith is at Heidelberg University, Germany, on a NATO fellowship; Associate Professor R. G. Ayoub has a grant awarded by the American Mathematical Society and supported by the U. S. Air Force Office of Scientific Research, to study at Harvard University and Oxford University, England; Dr. R. P. Kanwal, Mathematics Research Center, University of Wisconsin, has been appointed Associate Professor; Dr. W. L. Harkness, Michigan State University, and Dr. D. H. Husemoller, University of Rochester, have been appointed Assistant Professors; Professor Beatrice L. Hagen has retired with the title Professor Emeritus.

Rutgers, The State University: Professor E. R. Ott, Chairman of the program in applied and mathematical statistics, has been named Director of the newly established Statistics Center; Dr. M. B. Wilk is Director of Research at the Center; Drs. R. S. Pinkham, M. E. Wescott, and H. F. Dodge, will serve on the staff of the Center.

Texas A. & M. College: Assistant Professors M. E. Tittle and S. A. Sims have been

promoted to Associate Professors; Mr. H. D. Perry has been promoted to Assistant Professor; Professor W. L. Porter has retired with the title Professor Emeritus; Mr. C. M. Pearcy has been appointed Assistant Professor; Mr. James Harris has been appointed Instructor; Mr. Peter Terwey, on leave from Lamar State College of Technology, has been appointed Visiting Assistant Professor.

University of Western Ontario: Assistant Professor William Wehlau has been promoted to Associate Professor; Dr. G. E. Cross, Victoria College, British Columbia, Canada, and Dr. J. Sanders, Glasgow University, Scotland, have been appointed Assistant Professors; Mr. R. M. Ellis, University of California, Los Angeles, has been appointed Instructor.

Wayne State University: Professor Sze-Tsen Hu was conferred the distinguished degree of Doctor of Science by the University of Manchester, England; Associate Professor O. G. Owens has been promoted to Professor; Assistant Professor Bess E. Allen has been promoted to Associate Professor; Dr. Franz Schnitzer has been promoted to Assistant Professor; Mr. Perry Scheinok, Indiana University, has been appointed Instructor.

Mr. J. F. Blackburn, I.B.M. Eastern Regional Office of the Data Processing Division, has joined the International Business Machines Research Center staff as Manager of Research Sales and Contracts at the Mohansic Laboratory in Yorktown, New York.

Professor A. T. Brauer, University of North Carolina, has been promoted to a Kenan Professorship.

Mr. Sol Broder, Brooklyn College, has accepted a position as Supervisor of Engineering Applications with Republic Aviation Corporation, Farmingdale, Long Island, New York.

Brother Joseph Heisler, C.S.C., University of Notre Dame, has been appointed Mathematics Teacher at the Archbishop Curley High School, Miami, Florida.

Mr. Frederick Brundage, University of Kansas, has accepted a position as Computer Analyst with the Military Products Division of the International Business Machines Corporation, Owego, New York.

Dr. E. W. Cheney, Convair-Astronautics, San Diego, California, has accepted a position as Member of the Technical Staff with Space Technology Laboratories, Inc., Los Angeles, California.

Mr. S. H. Coleman has been appointed Junior Instructor at the University of Virginia.

Mr. Demetrios Counes has accepted a position as Mathematician with the Lehigh Design Company, Newark, New Jersey.

Professor A. H. Diamond, Stevens Institute of Technology, has been appointed Professor at Webb Institute, Glen Cove, Long Island, New York.

Mr. Chester Domoracki has accepted a position as Methods Planner with the Western Electric Company.

Mrs. Jeanne C. Gardner, University of Pittsburgh, has been appointed Instructor at the University of Arkansas.

Mrs. Aiko M. Hormann, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has accepted a position as Senior Analyst with the Systems Development Corporation, Santa Monica, California.

Mr. R. R. Hornby, University of Nebraska, has accepted a position as Associate Research Engineer with the Boeing Airplane Company, Wichita, Kansas.

Dr. J. E. Householder, Sr., University of Colorado, has been appointed Assistant Professor at Humboldt State College.

Mr. C. M. Hughey has accepted a position as Planning Engineer with the Western Electric Company, Inc., Winston-Salem, North Carolina.

Mr. R. L. Jacobsen, Stanford University, has been appointed Assistant Instructor at the State University of Iowa.

Mr. A. J. Kainen, Georgia Institute of Technology, has been appointed Assistant Professor at the University of Tampa.

Dr. Ferdinand Kertes has retired as Head of the Department of Mathematics at Perth Amboy High School, New Jersey.

Dr. R. K. Meany has been appointed Assistant Professor at Texas Christian University.

Mr. L. F. Nichols, Picatinny Arsenal, Dover, New Jersey, has been appointed Instructor at Stevens Institute of Technology.

Dr. Gordon Raisbeck has taken a one year leave of absence from the Bell Telephone Laboratories to accept a temporary appointment to the staff of the Advanced Research Projects Division of the Institute of Defense Analyses, Washington, D. C.

Mr. K. A. Retzer, Saunemin High School, Saunemin, Illinois, has been appointed Assistant Professor at Illinois State Normal University.

Mr. Winston Riley, III, International Business Machines Corporation, Cleveland, Ohio, has joined the Operations Research Branch of the Corporation for Economic and Industrial Research, Arlington, Virginia.

Mr. J. F. H. Schluep, Cato Meridian Central School, Cato, New York, has been appointed Associate Professor at the State University of New York, College of Education at Oswego.

Mr. M. C. Schwartz, Philco Corporation, has been appointed Lecturer at Harpur College of the State University of New York.

Mr. E. R. Willard, University of Rochester, has been appointed Instructor at Hamilton College.

Professor Emeritus Joseph Bowden, Adelphi College, died in June, 1959. He was a charter member of the Association.

Professor Emeritus Arthur Rosenthal, Purdue University, died September 8, 1959. He was a member of the Association for eighteen years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE OCTOBER MEETING OF THE INDIANA SECTION

The annual fall meeting of the Indiana Section of the Mathematical Association of America was held on October 30, 1959, at Butler University, Indianapolis, Indiana, in conjunction with the meeting of the Indiana Academy of Science. Professor G. N. Wollan, Chairman of the Mathematics Section of the Academy presided at the morning session and Professor K. H. Carlson, Chairman of the Indiana Section of the Association presided at the afternoon session.

Professor Carlson was elected Chairman of the Mathematics Section of the Academy for 1960.

The following program was presented:

Morning Session—10 A.M. to 1 P.M.

1. *Computers and the collegiate mathematics program,* a symposium, by Professor P. T. Mielke, Wabash College, Professor Seymour Parter, Indiana University, Dr. K. L. Nielsen, Allison Division of General Motors, and Dr. John Lawrence, Chicago Office of I.B.M.

2. *Some advantages of Riemann-Stieltjes integration*, by Professor G. L. Krabbe, Purdue University.

3. *Uncertainty and entropy*, by Professor J. H. Abbott, Purdue University.

Afternoon session—3:30 P.M. to 5:30 P.M.

4. *Generalized functions—a survey*, by Professor Michael Golomb, Purdue University.

5. *A new introduction to the ideas and methods of trigonometry*, by Professor R. J. Thomas, DePauw University.

6. *A new look at least squares*, by Professor Richard Dowds, Butler University.

7. *A definition of polynomials on topological groups, and some examples*, by Professor F. W. Carroll, Purdue University.

C. F. BRUMFIEL, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University, Oklahoma, on October 23, 1959. Professor E. P. Richardson, Chairman of the Section presided. There were 114 persons in attendance, including 64 members of the Association.

The following officers were elected for a one-year term: Chairman, Professor Katherine Mires, Northwestern State College; Vice-Chairman, Professor R. G. Laatsch, University of Tulsa; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association and is devoted to expository papers of particular interest to high school teachers. Research papers are presented in the spring meeting. The following papers were presented:

1. *Mann's theorem on the Landau-Schnirlmann alpha plus beta hypothesis*, by Professor Charles Nicol, University of Oklahoma.

2. *Probability and inductive inference*, by Dr. F. A. Graybill, Oklahoma State University of Agriculture and Applied Science.

3. *Recent developments in geometry*, by Professor T. K. Pan, University of Oklahoma.

4. *Metric postulates for plane geometry*, by Professor Gene Levy, University of Oklahoma.

5. *The relation between abstract algebra and the high school mathematics curriculum*, by Professor J. E. Hoffman, Oklahoma State University of Agriculture and Applied Science.

6. *Vectors in high school geometry*, by Professor Arthur Bernhart, University of Oklahoma.

7. *Experience with the UICSM project*, by Professor Eunice Lewis, University of Oklahoma.

8. The *luncheon speaker* was Professor Jack Forbes of Ball State Teachers' College, Muncie, Indiana. Professor Forbes spoke to the combined membership of the Oklahoma Section of the Mathematical Association of America and the Oklahoma Council of Teachers of Mathematics.

9. Another combined session of these two groups was a *panel discussion* led by three of the teachers who participated as members of the writing group for secondary school text materials which met at Boulder, Colorado, during the summer of 1959.

Committee reports included the report of the high-school lecture committee, which is serving as a clearing house for invitations from high school mathematics groups desiring college speakers. Finances will not permit sponsoring a regular traveling lecturer, but the

committee has served as a clearing house for information concerning available speakers.

The high school contest committee reports that 57 schools participated in the contest during the spring of 1959, and that Booker T. Washington High School of Idabel, Oklahoma, had the highest team score in Region 8, which includes Wyoming, Utah, Colorado, Oklahoma, Texas, New Mexico, and Arizona. This school placed four students on the Region 8 Honor Roll.

The touring lectureship committee reported that a proposal had been submitted jointly with the Oklahoma and Arkansas Academies of Science to sponsor visiting lectures in mathematics in Oklahoma and Arkansas. Later, the proposal was withdrawn on the grounds that it was too late in the year to secure lecturers of the quality desired who could devote full time to the project. The committee reports they plan to submit another proposal and hope it will receive earlier action.

R. V. ANDREE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April 30, 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK, City College, New York, April 2, 1960.

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA, St. Olaf College, Northfield, May 7, 1960.

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, University of Oklahoma, Norman, April 15–16, 1960.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960

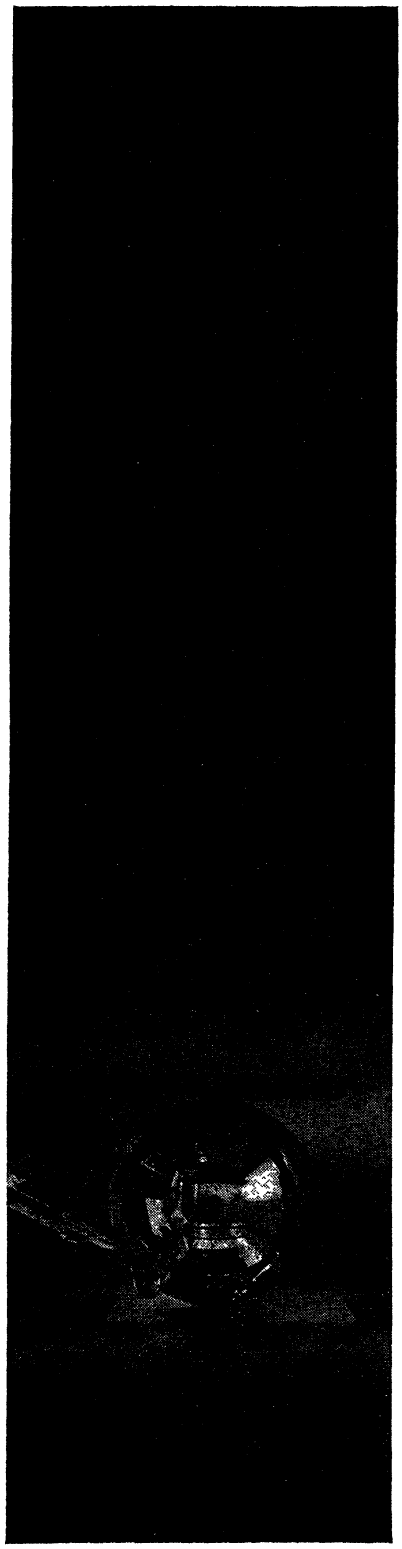
SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April 8–9, 1960.

TEXAS, San Antonio College, April 8–9, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.



One of a series

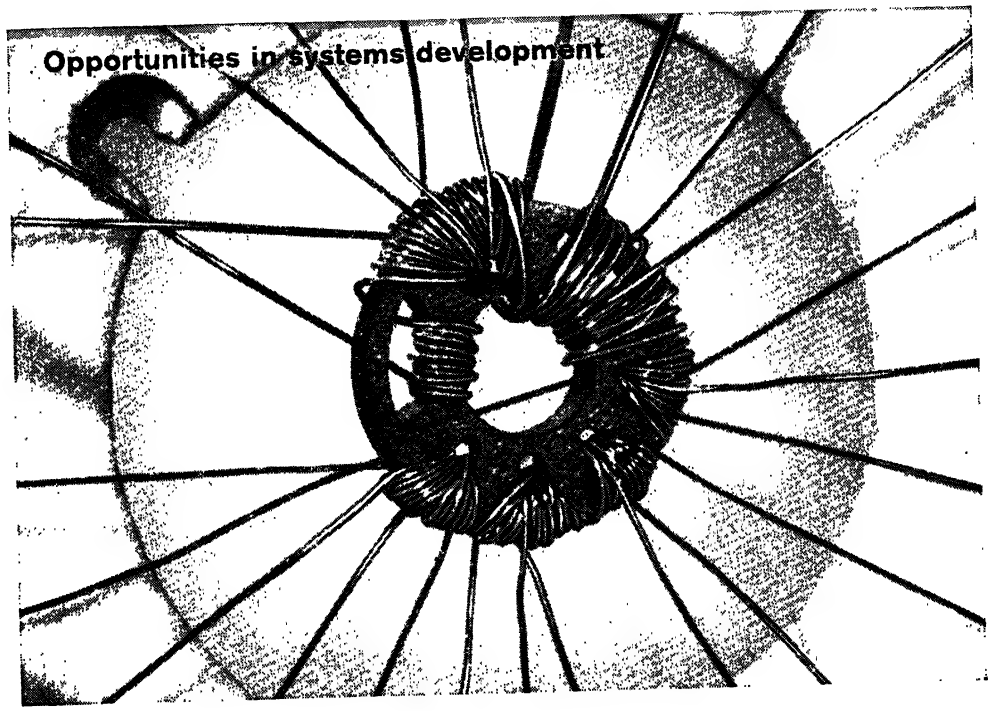
On the riddle of rolling friction

It doesn't take much to roll a hard ball across a hard, smooth, level surface —actually only about 0.00001 times the normal force acting vertically on the ball. But by careful measurement of this tiny rolling force, scientists at the General Motors Research Laboratories are helping to unravel the riddle of rolling friction.

An important relationship recently uncovered in this fundamental study: the rolling force is proportional to the volume of material that is stressed above a certain level. As a result, a GM Research group have not only confirmed the hypothesis of *how* a rolling ball loses energy (Answer: elastic hysteresis) but also have learned *where* this lost energy is dissipated (Answer: in the interior of the material, not on the surface).

The purpose of friction research at the GM Research Laboratories is to learn more about the elastic and inelastic behavior of materials. This knowledge—of academic interest now—will eventually give GM engineers greater control of energy lost through friction. This is but one more example of how General Motors lives up to its promise of "More and better things for more people."

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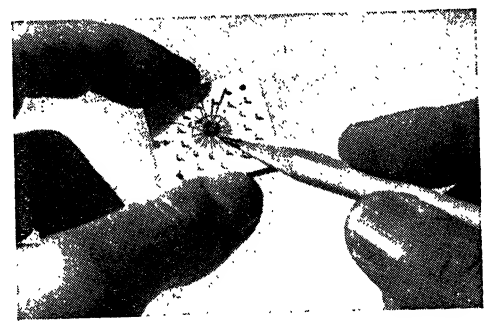
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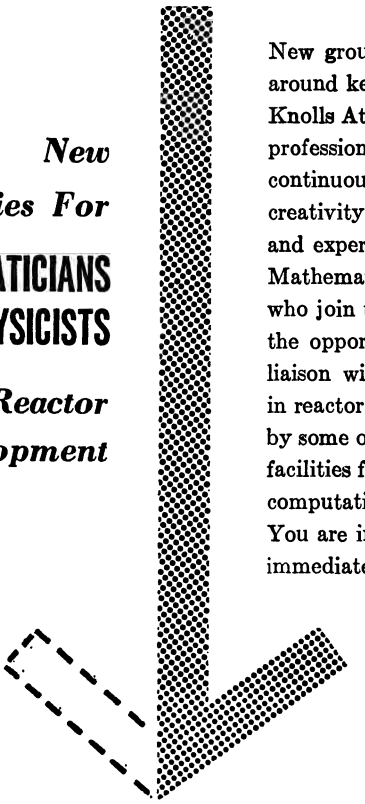
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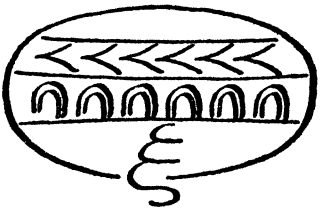
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THE CALCULATION OF THE COMPLETE ELLIPTIC INTEGRAL OF THE THIRD KIND

MORGAN WARD, California Institute of Technology

1. Introduction. The difficulty of computing an elliptic integral of the third kind is well known; even a computation of the complete integral

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 + \nu \sin^2 \phi)\sqrt{(1 - k^2 \sin \phi)}} \tag{1.1}$$

for numerical values of its parameters ν and k is a time-consuming operation. Apparently it is only very recently that any systematic tabulation has been made. I develop here a very efficient way to compute this integral when the parameters ν and k are real. The procedure is well adapted to a digital computer, and is so easily programmed that no tables should be necessary in the future.

When ν is zero, the procedure reduces to Gauss' method for computing the complete elliptic integral of the first kind by the use of Landen's transformation. However no knowledge of elliptic functions or integrals is required for the developments which follow. Any reader who has had a first course in function theory can easily understand the proofs.

2. The computation procedure. The process of computation is iterative, but takes slightly different forms according as the parameter ν is positive or negative. We let $\nu = \epsilon\mu^2$, $\epsilon = \pm 1$.

If $\epsilon = +1$, we speak of the positive case; if $\epsilon = -1$, of the negative case. In either case, we denote the integral (1.1) by $L(\mu, k)$. If $\mu = 0$, the integral reduces to the complete integral of the first kind. We denote it then by $L(k)$:

$$L(k) = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}}. \tag{2.1}$$

The parameters μ and k are assumed to be real, and

$$0 \leqq k < 1, \qquad 0 \leqq \mu \text{ if } \epsilon = 1; \qquad 0 \leqq \mu < 1 \text{ if } \epsilon = -1. \tag{2.2}$$

Write for brevity

$$k' = \sqrt{(1 - k^2)}, \qquad \mu' = \sqrt{(1 + \epsilon\mu^2)}, \tag{2.3}$$

and let $k^* = (1-k')/(1+k')$ and $\mu^* = \mu k/\{(1+\mu')(1+k')\}$. We shall prove in Section 4 of this paper that

$$L(\mu, k) = \frac{2}{1 + k'} \mu'^{-1}(2L(\mu^*, k^*) - L(k^*)). \tag{2.4}$$

Now $k^* = k^2/(1+k')^2 < k^2$ so that if we iterate (2.4), the successive values of μ and k obtained tend rapidly to zero. Since $L(0, 0) = L(0) = \frac{1}{2}\pi$, the following procedure suggests itself for obtaining an approximation $A(\mu, k)$ to $L(\mu, k)$.

First step. Compute $k_1, \cdots, k_r$ *and* $\mu_1, \cdots, \mu_r$ *by the formulas*

$$\mu_1 = \mu,\ k_1 = k; \qquad k_{n+1} = \frac{1 - k_n'}{1 + k_n'}, \qquad \mu_{n+1} = \frac{\mu_n k_n}{(1 + \mu_n')(1 + k_n')}. \tag{2.5}$$

The algorithm is continued until k_{r+1} and μ_{r+1} are sufficiently small. More specifically for $n = 1, 2, \cdots$, let

$$K_n = \prod_{s=1}^{n} 2/(1 + k_s'), \qquad M_n = \prod_{s=1}^{n} \mu_s'^{-1}, \tag{2.6}$$

and let

$$E_n = \begin{cases} 2^n K_n M_n k_{n+1}^2 & \text{in the positive case;} \\ 2^n K_n M_n(\mu_{n+1}^2 + \tfrac{1}{2}k_{n+1}^2) & \text{in the negative case.} \end{cases} \tag{2.7}$$

We require that E_r be negligible.

Second step. Compute the approximations $A(\mu_{r+1}, k_{r+1}), A(\mu_r, k_r), A(\mu_{r-1}, k_{r-1}), \cdots A(\mu_1, k_1) = A(\mu, k)$ *and* $A(k_{r+1}), A(k_r), A(k_{r-1}), \cdots A(k_2)$ *by the formulas*

$$\begin{aligned} A(\mu_{r+1}, k_{r+1}) &= A(k_{r+1}) = \tfrac{1}{2}\pi, \\ A(\mu_n, k_n) &= \frac{2}{1 + k_n'}\,\frac{1}{\mu_n'}\,(2A(\mu_{n+1}, k_{n+1}) - A(k_{n+1})), \\ A(k_n) &= \frac{2}{1 + k_n'}\,A(k_{n+1}), \qquad n = r, r-1, \cdots, 1. \end{aligned} \tag{2.8}$$

Third step. Take $A(\mu, k)$ *as an approximation to* $L(\mu, k)$; *the error in doing so is negligible, for*

$$0 < L(\mu, k) - A(\mu, k) < E_r. \tag{2.9}$$

It will be observed that the first two steps of the computation are iterative processes very well adapted to a digital computer or even to an ordinary desk calculator.

3. The rapidity of convergence. The convergence of the process is extremely rapid. It is evident from (2.2) and (2.5) that if μ and k are positive, then $0 < k_{n+1} < k_n^2$ and $0 < \mu_{n+1} < \frac{1}{2}(\mu_n^2 + k_n^2)$. We shall show later that *if* $0 < k_t, \mu_t < \frac{2}{3}$ *and* $n \geqq 1$, *then*

$$0 < k_{n+t} < (\tfrac{1}{3}k_t)^{2^n}, \qquad 0 < \mu_{n+t} < \mu_t(\tfrac{1}{3}k_t)^{2^{n-1}}. \tag{3.1}$$

We shall also show that if $0 < k_n, \mu_n < .1$, then

$$2/(1 + k_n') = 1 + \theta_n(\tfrac{1}{4}k_n^2), \qquad 1/\mu_n' = 1 - \epsilon\phi_n(\tfrac{1}{2}\mu_n^2), \tag{3.2}$$

with $1 < \theta_n < 1.006$, $1 < \phi_n < 1.008$.

It follows from (3.2) that the infinite products $K=\prod_1^\infty \{2/(1+k_n')\}$, $M=\prod_1^\infty (1/\mu_n')$, converge and, from (3.1), that the partial products M_r and K_r in the error term E_r change very little as r increases when μ_r and k_r are small. It is also evident from the inequalities (3.1) that E_r may be made arbitrarily small by choosing r large enough; that is, the procedure converges in the usual sense. The following numerical data for the negative case illustrate the rapidity of convergence.

The data is taken from a sytematic computation of E_r to twenty places for μ and k ranging independently over the values .1, .2, $\cdots$, .8, .9. The computation was programmed for the California Institute Datatron by Mr. George W. Logemann, and the complete results will be given elsewhere. Evidently the most unfavorable case is when $\mu=k=.9$. Mr. Logemann found that $E_3=.000174$, $\cdots$, $E_4=.00000\ 00000\ 5197$, $\cdots$, $E_5<10^{-18}$. We conclude then that four or five iterations will suffice for most ordinary purposes.

4. Derivation of the iterative formula. We here derive formula (2.4) on which the computation procedure rests.

The functions $(1+\epsilon\mu^2\sin^2\phi)^{-1}$ and $(1-k^2\sin^2\phi)^{-1/2}$ may evidently be expanded in Fourier series which converge uniformly to them in the closed interval $[-\pi, \pi]$. Let these expansions be

$$\frac{1}{1+\epsilon\mu^2\sin^2\phi}=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos n\phi, \tag{4.1}$$

$$\frac{1}{\sqrt{(1-k^2\sin^2\phi)}}=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}b_n\cos n\phi. \tag{4.2}$$

Then

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos n\phi d\phi}{\sqrt{(1-k^2\sin^2\phi)}}. \tag{4.3}$$

It will be shown at the close of this section that $a_{2n+1}=0$ and that

$$a_{2n}=2\mu'^{-1}\epsilon^n\cdot\{\mu/(1+\mu')\}^{2n}. \tag{4.4}$$

Thus by (4.1)

$$\frac{1}{(1+\epsilon\mu^2\sin^2\phi)\sqrt{(1-k^2\sin^2\phi)}}$$
$$=\mu'^{-1}\left(\frac{1}{\sqrt{(1-k^2\sin^2\phi)}}+2\sum_1^\infty\epsilon^n\left(\frac{\mu}{1+\mu'}\right)^{2n}\frac{\cos 2n\phi}{\sqrt{(1-k^2\sin^2\phi)}}\right).$$

The series on the right is uniformly convergent with respect to ϕ. Hence integrating term-wise over $[-\pi, \pi]$ we obtain from (2.1) and (4.3) the formula

$$L(\mu, k) = \tfrac{1}{4}\pi\mu'^{-1}\left(b_0 + 2\sum_1^\infty \epsilon^n\{\mu/(1+\mu')\}^{2n}b_{2n}\right). \tag{4.5}$$

We next obtain another expression for the coefficients b_{2n} in this series. Let $w=\phi+i\zeta$ be a complex variable. Then there exists a positive number δ depending on k alone such that $\Delta(w)=(1-k^2\sin^2 w)^{-1/2}$ is regular in the strip $|\operatorname{Im} w|<\delta$. Since $\Delta(w)$ is also periodic with period 2π in the whole plane, it is representable everywhere in the strip by the Fourier series

$$\Delta(w) = \tfrac{1}{2}b_0 + \sum_1^\infty b_n \cos nw, \tag{4.6}$$

with the b_n given by formula (4.3). From now on, we confine ourselves to the rectangle $|\operatorname{Re} w| \leqq \pi$, $|\operatorname{Im} w| < \delta$.

Let T stand for the mapping of the w-plane induced by letting $z=e^{iw}$. Then T transforms the rectangle above into a circular ring R in the z plane: $R=\{z \mid 0<r<|z|<R\}$, where $e^{-\delta}=r<1<R=e^{\delta}$. Furthermore if

$$F(z) = T\Delta(w), \tag{4.7}$$

then $F(z)$ is regular in R. But $T\cos nw=\frac{1}{2}(z^n+z^{-n})$. Hence if we let $b_{-n}=b_n$, we have by (4.6) and (4.7) the Laurent expansion of $F(z)$ in R: $F(z)=\sum_{-\infty}^{\infty}\frac{1}{2}b_nz^n$.

Consequently by Laurent's theorem,

$$b_n = \frac{1}{\pi i}\int_\Gamma z^{n-1}F(z)dz, \qquad n = 0, 1, \cdots. \tag{4.8}$$

Here Γ is the unit circle described once counterclockwise. Now let

$$k = \sin\tfrac{1}{2}\beta, \qquad 0 \leqq \beta < \pi. \tag{4.9}$$

Then we find from (4.7) after a little algebra that

$$F(z) = \frac{2\csc\frac{1}{2}\beta z}{\sqrt{\{(z^2+\tan^2\frac{1}{4}\beta)(z^2+\cot^2\frac{1}{4}\beta)\}}}. \tag{4.10}$$

Hence the only singularities of $F(z)$ within Γ are branchpoints of order two at $z=\pm i\tan\frac{1}{4}\beta$. Join these points by a cut along the axis of imaginaries. Then the contour Γ in (4.8) may be shrunk down to a path consisting of two small circles about the branch points and the two sides of the cut. If the radius of each of these circles is ρ, their contribution to the line integral is of order $\sqrt{\rho}$. If therefore we set $z=i\tan\frac{1}{4}\beta\sin\phi$ along the cut and pass to the limit by letting ρ tend to zero, we find that

$$b_n = \frac{4}{\pi}i^n\csc\tfrac{1}{2}\beta\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\sin^n\phi\tan^{n+1}\frac{1}{4}\beta d\phi}{\sqrt{(1-\tan^4\frac{1}{4}\beta\sin^2\phi)}}.$$

Hence b_n is zero when n is odd, and when n is even,

$$b_{2n} = \frac{8}{\pi}\frac{(-k^*)^n}{1+k'}\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\phi\,d\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}}. \tag{4.11}$$

Here we have written k^* for $\tan^2\frac{1}{4}\beta$ and replaced the multiplier $\csc\frac{1}{2}\beta\tan\frac{1}{4}\beta$ by $(1+\cos\frac{1}{2}\beta)^{-1}$ which equals $(1+k')^{-1}$ by (4.9) and (2.3). We easily find from the same formulas that

$$k^* = (1-k')/(1+k'). \tag{4.12}$$

(4.11) is the new expression for b_{2n} which we were seeking. Introduce it into the series (4.5) for $L(\mu, k)$, and let

$$\mu^* = \frac{\mu}{1+\mu'}\sqrt{k^*} = \frac{\mu}{1+\mu'}\frac{k}{1+k'}. \tag{4.13}$$

Then (4.5) becomes

$$L(\mu,k) = \frac{2\mu'^{-1}}{1+k'}\left(\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}} + 2\sum_1^\infty(-\epsilon)^n\mu^{*2n}\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\phi}{\sqrt{(1-k^{*2}\sin^2\phi)}}\right). \tag{4.14}$$

Now the series $(1-k^*\sin^2\phi)^{-1/2}+2\sum_1^\infty(-\epsilon\mu^{*2})^n\sin^{2n}\phi(1-k^*\sin^2\phi)^{-1/2}$ is uniformly convergent in ϕ, and is essentially a geometric progression whose sum may be written

$$\frac{2}{(1+\epsilon\mu^{*2}\sin^2\phi)\sqrt{(1-k^{*2}\sin^2\phi)}} - \frac{1}{\sqrt{(1-k^{*2}\sin^2\phi)}}.$$

Hence on integrating term-wise from 0 to $\frac{1}{2}\pi$ and comparing it with (4.14) and the integral of its sum with (1.1) and (2.1), we obtain the iteration formula (2.4), with the values of k^* and μ^* given in Section 2.

It remains to verify the values given for the Fourier coefficients a_n in the series (3.1) for $(1+\epsilon\mu^2\sin^2\phi)^{-1}$. Let $\mu=\tan\frac{1}{2}\alpha$, $0\leqq\alpha<\pi$, in the positive case; $\mu=\sin\frac{1}{2}\alpha$, $0\leqq\alpha<\pi$, in the negative case. Then we find that

$$T(1+\epsilon\mu^2\sin^2\phi)^{-1} = \begin{cases}4\cot^2\frac{1}{2}\alpha z^2[(z^2-\tan^2\frac{1}{4}\alpha)(z^2-\cot^2\frac{1}{4}\alpha)]^{-1} & \text{if } \epsilon=+1,\\ 4\csc^2\frac{1}{2}\alpha z^2[(z^2+\tan^2\frac{1}{4}\alpha)(z^2+\cot^2\frac{1}{4}\alpha)]^{-1} & \text{if } \epsilon=-1.\end{cases}$$

This result may be written

$$T(1+\epsilon\mu^2\sin^2\phi)^{-1} = \frac{4\mu^{-2}z^2}{(z^2-\epsilon\tan^2\frac{1}{4}\alpha)(z^2-\epsilon\cot^2\frac{1}{4}\alpha)}.$$

Hence by analogy with (4.8)

$$a_n = \frac{4\mu^{-2}}{\pi i}\int_\Gamma\frac{z^{n+1}dz}{(z^2-\epsilon\tan^2\frac{1}{4}\alpha)(z^2-\epsilon\cot^2\frac{1}{4}\alpha)}.$$

Thus by the residue theorem a_n equals $8\mu^{-2}$ times the sum of the residues of the integrand at its two poles $\pm \sqrt{\epsilon} \tan \frac{1}{4}\alpha$ within Γ. On evaluating these residues, we find that $a_{2n+1}=0$ and that a_{2n} has the value stated in (4.4), so that the proof is complete.

5. Proofs of the basic inequalities. The inequalities (3.1) are proved as follows. We have $k_{t+1}=(1-k_t')/(1+k_t')=k_t^2/(1+k_t')^2$. Hence $k_{t+1}<\frac{1}{3}k_t^2$ if and only if $1+k_t'>\sqrt{3}$. This works out to $k_t^2<\sqrt{(3+2\sqrt{3})}=.681$. Hence: *If* $k_t<\frac{2}{3}$, *then* $k_{t+1}<\frac{1}{3}k_t^2$. This statement is the case $n=1$ of the inequality (3.1) for k_{t+n}. The general inequality follows by an easy induction. The inequality for μ_{t+n} is proved similarly.

To prove that (3.2) is valid, assume that $0<k_n<.1$. Then

$$\frac{2}{1+k_n'}=\frac{2(1-k_n')}{1-k_n'^2}=\frac{2}{k_n^2}\sqrt{(1-k_n^2)}).$$

Now on expanding the function $1-\sqrt{(1-x)}$, $0<x<1$ by Taylor's theorem with remainder and then substituting k_n^2 for x, we obtain the formulas

$$\frac{2}{1+k_n'}=\theta_n(\tfrac{1}{4}k_n),\quad \theta_n=1+\tfrac{1}{2}k_n^2+\frac{5}{16}\frac{k_n^4}{(1-\theta k_n^2)^{7/2}},\qquad 0<\theta<1.$$

Hence

$$1<\theta_n<1+\tfrac{1}{2}(.01)+\frac{5}{16}\frac{.0001}{(.99)^{7/2}}<1.006.$$

The formula for μ_n' in (3.2) is proved in a similar fashion.

We preface the proof of the inequality (2.9) by two lemmas.

LEMMA 5.1. *If* $0<\mu,\ k<.1$, *then*

(5.1) $\qquad 0<L(\mu,k)-\frac{1}{2}\pi<\frac{4}{5}k^2$ *if* ϵ *is positive,*

(5.2) $\qquad 0<L(\mu,k)-\frac{1}{2}\pi<\frac{4}{5}(\mu^2+\frac{1}{2}k^2)$ *if* ϵ *is negative.*

Proof. By Taylor's theorem with remainder, where $0<\theta<1$,

$$(1-k^2\sin^2\phi)^{-1/2}=1+\tfrac{1}{2}k^2\sin^2\phi+\frac{3}{8}k^4\frac{\sin^4\phi}{(1-\theta k^2\sin^2\phi)^{5/2}}$$
$$>1+\tfrac{1}{2}k^2\left(1+\frac{3}{4}\frac{k^2}{(1-k^2)^{5/2}}\right)\sin^2\phi.$$

Since $k<.1$,

$$1+\frac{3}{4}\frac{k^2}{(1-k^2)^{5/2}}<1+\frac{3}{4}\frac{.01}{(.99)^{5/2}}=1.0077.$$

Hence

$$\frac{1}{(1 - k^2 \sin^2 \phi)^{1/2}} < 1 + \alpha(\tfrac{1}{2}k^2) \sin^2 \phi, \qquad \alpha = 1.0077. \tag{5.3}$$

Similarly

$$\frac{1}{1 - \mu^2 \sin^2 \phi} < 1 + \beta\mu^2 \sin^2 \phi, \qquad \beta = 1.01031.$$

Therefore

$$0 < \frac{1}{(1 - \mu^2 \sin^2 \phi)\sqrt{(1 - k^2 \sin^2 \phi)}} - 1$$
$$< \tfrac{1}{2}\alpha k^2 \sin^2 \phi + \beta\mu^2 \sin \phi + \alpha\beta(\tfrac{1}{2}\mu^2 k^2) \sin^4 \phi.$$

On integrating these inequalities from 0 to $\frac{1}{2}\pi$ we obtain when ϵ is negative the inequality

$$0 < L(\mu, k) - \tfrac{1}{2}\pi < \tfrac{1}{4}\pi\alpha(1 + \tfrac{3}{8}\beta\mu^2)(\tfrac{1}{2}k^2) + \tfrac{1}{4}\pi(1 + \tfrac{3}{16}k^2\alpha)\mu^2.$$

The coefficients of $\frac{1}{2}k^2$ and μ^2 turn out to be .79446 and .794996 which are both less than $\frac{4}{5}$. This completes the proof of the inequality (5.2).

If ϵ is positive, it follows from (5.3) that

$$L(\mu, k) - \tfrac{1}{2}\pi < L(k) - \tfrac{1}{2}\pi - (\tfrac{1}{2}\alpha k^2)(\tfrac{1}{4}\pi).$$

Since $\frac{1}{4}\alpha\pi < \frac{4}{5}$, (5.1) follows, completing the proof of the lemma.

LEMMA 5.2. *If*

$$A(\mu_n, k_n) = \frac{2\mu_n'^{-1}}{1 + k_n'} (2a(\mu_{n+1}, k_{n+1}) - A(k_{n+1})),$$

$$A(k_n) = \frac{2}{1 + k_n'} A(k_{n+1}),$$

for $n = 1, 2, \cdots$, *then*

$$L(\mu, k) - A(\mu, k) = 2^n M_n K_n (L(\mu_{n+1}, k_{n+1}) - A(\mu_{n+1}, k_{n+1}))$$
$$- \sum_{s=1}^{n} 2^{s-1} M_s K_s (L(k_{s+1}) - A(k_{s+1}), \tag{5.2}$$

where K_n *and* M_n *are given for all* n *by* (2.6).

Proof. We know by the iteration formula that for all n,

$$L(\mu_n, k_n) = \frac{2}{1 + k_n'} \mu_n'^{-1} (2L(\mu_{n+1}, k_{n+1}) - L(k_{n+1}),$$

$$L(k_n) = \frac{2}{1+k_n'} L(k_{n+1}).$$

Hence since $\mu = \mu_1$ and $k = k_1$,

$$\begin{aligned} L(\mu, k) - A(\mu, k) &= \frac{2}{1-k_n'} \mu_1'^{-1}(2L(\mu_2, k_2) - L(k_2) - \frac{2}{1+k_1'} \mu_1'^{-1}(2A(\mu_2, k_2) - A(k_2)) \\ &= 2K_1M_1(L(\mu_2 k_2) - A(\mu_2, k_2) - K_1M_1(L(k_2) - A(k_2)). \end{aligned}$$

Hence (5.2) is true when $n=1$. But is easy to show that if it is true for n, it is true for $n+1$. Hence it is true for all n by mathematical induction. Now

$$L(k_{s+1}) = \frac{2}{1+k_{s+1}'} \frac{2}{1+k_{s+2}'} \cdots \frac{2}{1+k_n'} L(k_{n+1}),$$

$$A(k_{s+1}) = \frac{2}{1+k_{s+1}'} \frac{2}{1+k_{s+2}'} \cdots \frac{2}{1+k_n'} A(k_{n+1}).$$

Hence (5.2) may be written

$$\begin{aligned} L(\mu, k) - A(\mu, k) &= 2^n K_n M_n (L(\mu_{n+1} k_{n+1}) - A(\mu_{n+1}, k_{n+1}) \\ &\quad - K_n \Big(\sum_{s=1}^{n} 2^{s-1} M_s \Big) (L(k_{n+1}) - A(k_{n+1})). \end{aligned} \tag{5.3}$$

Now suppose that $A(k_{n+1}, \mu_{n+1}) = A(k_{n+1}) = \frac{1}{2}\pi$. Then if ϵ is positive, $L(\mu_{n+1}, k_{n+1}) < L(k_{n+1})$. Hence when ϵ is positive, we obtain from (5.3) the inequalities.

$$\begin{aligned} L(\mu, k) - A(\mu, k) &< 2^n K_n M_n (L(k_{n+1}) - \tfrac{1}{2}\pi) \\ &> K_n \Big(2^n M_n - \sum_{s=1}^{n} 2^{s-1} M_s \Big) (L(K_{n+1}, \mu_{n+1}) - \tfrac{1}{2}\pi). \end{aligned}$$

Now $M_1 < M_2 < \cdots < M_n$. Hence $\sum_{s=1}^{n} 2^{s-1} M_s < 2^n M_n$. Hence $L(\mu, k) - A(\mu, k)$ is positive from the second inequality. And by the first inequality and Lemma 5.1, it is less than $2^n K_n (\frac{2}{5} k_{n+1}^2)$.

If ϵ is negative, $L(\mu_{n+1}, k_{n+1}) > L(k_{n+1})$. Hence we obtain from (5.3) the inequalities

$$L(\mu, k) - A(\mu, k) > K_n \Big(2^n M_n - \sum_{s=1}^{n} 2^{s-1} M_s \Big) (L(k_{n+1}) - \tfrac{1}{2}\pi) > 0,$$

$$\begin{aligned} L(\mu, k) - A(\mu, k) &< 2^n K_n M_n (L(\mu_{n+1}, k_{n+1}) - \tfrac{1}{2}\pi) \\ &< 2^n K_n M_n \tfrac{4}{5} (\mu_{n+1}^2 + \tfrac{1}{2} k_{n+s}^2), \end{aligned}$$

by Lemma 5.1, and this completes the proof of the inequality (2.9).

In conclusion, it is worth noting that the method may evidently be extended

to compute a complete integral of the third kind of the form

$$\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1+\nu\sin^2\phi)\sqrt{(1+k^2\sin^2\phi)}}$$

for ν and k real and ν greater than -1.

ANALYTICAL EXPRESSIONS AND ELEMENTARY FUNCTIONS

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The title is very nearly meaningless. When Professor Ritt discusses integration in finite terms an elementary function is one (a not too elementary) thing. In common classroom usage it is another—roughly, any differentiable function frequently used by Euler. When an author presents us with an analytic expression for a function, the expression usually has little or nothing to do with analytic functions in any strict sense. The adjective has become a synonym for "nonverbal". As examples we give*

$$\operatorname{sgm}\sin^2\frac{(n-1)!+1}{n}\pi \quad \text{and} \quad \operatorname{sgm}\sin^2\frac{(n-1)!^2}{n}\pi,$$

characteristic functions for the sets of nonprime and prime integers. The spirit of the formula constructing game is clear. Its rules are not.

Consider $[x]$,† the characteristic of x regarded as a logarithm, the greatest integer not greater than x. We find

$$[x] = x - \tfrac{1}{2} + (1/\pi)\tan^{-1}\cot\pi x \qquad x \neq 0, \pm 1, \pm 2, \cdots.$$

How do we fill in the gaps at integral x? Using limits is too much like hunting quail with an elephant gun. It's difficult to find functions which are not expressible as limits of limits.

The admission of one simple additional function somehow seems more sporting. Logical candidates are null $x=\lim_{n\to\infty}(1+|x|)^{-n}$, the characteristic function of the set (0), or $\operatorname{sgm} x=\lim_{n\to\infty} x^{1/(2n+1)}=x/(|x|+\text{null } x)$. Thus, for $T(x)=(2/\pi)\sin^{-1}\sin\frac{1}{2}\pi x \operatorname{sgm}\cos\frac{1}{2}\pi x$, we have $[x]=x-\max(T(x), T(x+1))$. Even this procedure may be circumvented if we adopt the convention that an expression meaningless for $x=a$, but suitably defined in a neighborhood, defines $f(a)=\frac{1}{2}\{f(a-)+f(a+)\}$. Then $\operatorname{sgm} x=x/|x|$ and $T(x)=(2/\pi)\tan^{-1}\tan\frac{1}{2}\pi x$.

It is interesting to see how far one may go with modest assumptions. If we agree $\cos x$, $\cos^{-1} x$, x, and real constants are "common" functions of x and

* Cf. Louis Brand, Advanced Calculus, New York, 1955, p. 84.

† Cf. Arthur Porges, An analytical expression for $[X]$, this MONTHLY, vol. 66, 1959, pp. 706–707.

postulate that sums, differences, products, quotients, and common functions of common functions are common, we obtain the following:

1. $\sin x = \cos(\frac{1}{2}\pi - x)$ is common.
2. The trigonometric functions are common.
3. $\sin^{-1} x = \frac{1}{2}\pi - \cos^{-1} x$ is common.
4. $\sqrt{x} = \cos(\frac{1}{2}\cos^{-1}(2x-1))$, $0 \leqq x \leqq 1$, is common.
5. $\sqrt{x} = \sqrt{(x/x+1)}/\sqrt{(1/x+1)}$, $0 \leqq x < \infty$, is common.
6. For positive x, integral p, and q a power of 2, $x^{p/q}$ is common.
7. $\tan^{-1} x = \sin^{-1}\{x/\sqrt{(1+x^2)}\}$ is common.
8. The inverse trigonometric functions are common.
9. $|x| = \tan \cos^{-1} \cos \tan^{-1} x = \sqrt{x^2}$ is common.
10. For common $f(x)$ and $g(x)$, $2 \max(f(x), g(x)) = f(x) + g(x) + |f(x) - g(x)|$ is common. Maxima and minima of sets of functions are common.
11. The Heaviside function (not defined for $x=0$), $h(x) = (x + |x|)/(2|x|)$, is common.
12. Characteristic functions of intervals, such as $g(x) = \min(h(x-a), h(-x+b))$, are common.
13. Piecewise common functions are common. For example, if $f(x) = f_i(x)$ on the interval $a_{i-1} < x < a_i$ which has $g_i(x)$ as characteristic function, then $f(x) = \sum_{i=1}^{n} f_i(x) g_i(x)$.

Although this last function is not defined at points a_i, removable discontinuities may be removed with a little more effort.* First, a reflection type procedure allows us to extend the domain of any common function to the entire x-axis. Thus $f(x)$ defined on $a \leqq x \leqq \infty$ coincides with $s(x) = f(|x-a| + a)$ on that interval and the latter is defined for all x. Similarly, we may extend $f(x)$ defined on $a \leqq x \leqq b$ to the function $s(x) = f(\phi(x))$, where

$$\phi(x) = \frac{a+b}{2} + \frac{b-a}{\pi} \sin^{-1} \sin\left(\frac{\pi}{b-a}\left(x - \frac{a+b}{2}\right)\right).$$

Now we affix to the graph of $f(x)$ steeply slanting half-lines. Choose m large, let $r(x) = f(a) + m(x-a)$, $t(x) = f(b) - m(x-b)$, and define $f^*(x) = \min(r(x), s(x), t(x))$. If $f(x)$ is common on $a \leqq x \leqq b$, and $g(x)$ is common on $b \leqq x \leqq c$, and $f(b) = g(b)$, then $\max(f^*(x), g^*(x))$ is common on $a \leqq x \leqq c$, is defined for $x = b$, and coincides with $f(x)$ or $g(x)$ to the left or right of $x = b$.

If it is desired that piecewise common functions be defined at the points where jump discontinuities occur, it is necessary to accept, say, sgm x as common. Then null $x = 1 + \min(\text{sgm } x, \text{sgm } (-x))$ is common. The Heaviside function $H(x) = \frac{1}{2}(1 + \text{sgm } x - \text{null } x)$ is common, as well as characteristic functions of open intervals, $G(x) = \min(H(x-a), H(-x+b))$. The piecewise common functions are common since they may be written as $\sum f_i(x) G_i(x) + \sum k_i \text{ null } (x - a_i)$. By means of the latter term, these are given arbitrary values k_i at the juncture points a_i.

* An exception occurs when $f'(a_i+)$ and $f'(a_i-)$ are both ∞ or both $-\infty$.

SOME PROPERTIES AND APPLICATIONS OF DOUBLY STOCHASTIC MATRICES*

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1. Introduction. Since the appearance of a paper by H. Weyl [30] in 1949 there has been a considerable amount of effort, on the part of many authors, devoted to questions concerning the relationships among eigenvalues, singular values, and quadratic forms associated with a linear transformation. Almost all of these results can be formulated in terms of a single general extreme value problem involving doubly stochastic matrices. In what follows, we will indicate how this has been done and we will also discuss the current state of a pair of conjectures of van der Waerden and Kakutani on doubly stochastic matrices.

An n-square matrix $S=(s_{ij})$ is called *doubly stochastic* (d.s.) if

$$(1.1) \qquad s_{ij} \geqq 0, \qquad \sum_{j=1}^{n} s_{ij} = 1, \qquad \sum_{i=1}^{n} s_{ij} = 1, \qquad 1 \leqq i, j \leqq n.$$

The totality of n-square d.s. matrices will be denoted by Ω_n. The conditions (1.1) immediately imply that Ω_n is convex and it is readily checked that if P is a permutation matrix then $P \in \Omega_n$. As we shall see, Ω_n is actually a polyhedron with the permutation matrices as vertices. (Birkhoff's theorem).

2. The Birkhoff theorem. The result of Birkhoff [3] goes as follows:

Any doubly stochastic n-square matrix S is in the convex hull of m permutation matrices for $m \leqq (n-1)^2+1$.

Several proofs and generalizations of this result are known [4], [22], [23]. They all depend on showing that there is a permutation σ of $1, \cdots, n$ such that $s_{i\sigma(i)}>0$ for $i=1, \cdots, n$. For if this can be shown let $t=\min_i s_{i\sigma(i)}>0$ and let P be a permutation matrix with $P_{i\sigma(i)}=1$, $i=1, \cdots, n$. Then if $t<1$ (if $t=1$, $S=P$ follows from (1.1)) the matrix $(1-t)^{-1}(S-tP)$ has at least one more zero entry than S. The proof is completed then by an induction on the number of positive entries of S. Mirsky showed directly that a σ with the desired properties exists and the result also follows immediately from the Frobenius-König theorem as in [4]. The condition (1.1) shows that Ω_n is of dimension $(n-1)^2$ and thus in general $m \leqq (n-1)^2+1$.

The way in which we shall use the result is as follows: suppose f is a convex (concave) function on Ω_n to the reals; then f will assume its maximum (minimum) value at a vertex of Ω_n; *i.e.*, $f(S) \leqq f(P)$ ($f(S) \geqq f(P)$) for any $S \in \Omega_n$ and P an appropriate permutation matrix.

3. Quadratic forms and eigenvalues. Let A be any normal transformation on unitary n-space to itself and let $u_1, \cdots, u_n$ be an orthonormal (o.n.) set of

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eigenvectors of A corresponding respectively to the eigenvalues $\lambda_1, \cdots, \lambda_n$. If $x_1, \cdots, x_n$ is any other o.n. set then $S=(|(x_i, u_j)|^2)$, $i, j=1, \cdots, n$, is in Ω_n. Moreover, if $z=((Ax_1, x_2), \cdots, (Ax_n, x_n))$ and $\lambda=(\lambda_1, \cdots, \lambda_n)$ then

$$z = S\lambda. \tag{3.1}$$

The relation (3.1) can be exploited immediately to yield an interesting extremal result due to K. Fan [5]:

If A is positive semidefinite Hermitian (p.s.d.) *with eigenvalues* $0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n$ *then*

$$\prod_{j=1}^{k} (Ax_j, x_j) \geqq \prod_{j=1}^{k} \lambda_j, \qquad 1 \leqq k \leqq n, \tag{3.2}$$

for any o.n. *set* $x_1, \cdots, x_k$.

To see this simply set $g(z)=g(z_1, \cdots, z_n)=(\prod_{j=1}^{k} z_j)^{1/k}$ and observe that $g(z)$ is a concave function for nonnegative variables z_j. Also $f(S)=g(S\lambda)$ is concave in S for $S \in \Omega_n$. It follows that

$$\left(\prod_{j=1}^{k} (Ax_j, x_j) \right)^{1/k} = f(S) \geqq f(P) \geqq \left(\prod_{j=1}^{k} \lambda_j \right)^{1/k},$$

where P is a permutation matrix in Ω_n. From (3.2) we have

$$\prod_{j=1}^{n} a_{jj} \geqq \det A \quad (\textit{Hadamard determinant theorem})$$

by setting $k=n$ and $x_i=\epsilon_i$, the unit vector with 1 in position i. Also, if B is a second p.s.d. matrix then

$$\begin{aligned} (\det (A+B))^{1/n} &= \left(\prod_{j=1}^{n} [(Ax_j, x_j) + (Bx_j, x_j)] \right)^{1/n} \\ &\geqq \left(\prod_{j=1}^{n} (Ax_j, x_j) \right)^{1/n} + \left(\prod_{j=1}^{n} (Bx_j, x_j) \right)^{1/n} \\ &\geqq (\det A)^{1/n} + (\det B)^{1/n} \quad (\textit{Minkowski determinant inequality}), \end{aligned}$$

where $x_1, \cdots, x_n$ is an o.n. set of eigenvectors of $A+B$.

The result (3.2) has been extended by several authors in two directions: first to more complicated functions of the quadratic forms and second to a wider class of matrices than the p.s.d. ones [24], [12], [14], [15], [16], [17], [6], [7], [8].

4. Sets associated with Ω_n and the result of H. Weyl. Let $b_1 \geqq \cdots \geqq b_n$ be an ordered set of real numbers and define $\Omega_n(b)$ to be the set of all real points $t=(t_1, \cdots, t_n)$ satisfying

$$(4.1) \qquad \sum_{j=1}^{k} t_{i_j} \leqq \sum_{j=1}^{k} b_j, \qquad \sum_{j=1}^{n} t_j = \sum_{j=1}^{n} b_j,$$

for all index sets $1 \leqq i_1 < \cdots < i_k \leqq n$, $1 \leqq k \leqq n-1$. It is simple to verify that $\Omega_n(b)$ is a compact convex set. However a much stronger result of Hardy, Littlewood and Pólya [9] is the following:

$\Omega_n(b)$ is the set of all vectors Sb for $S \in \Omega_n$.

Several proofs of this are available in [9], [24], [23], [2], [10]. A reasonably succinct argument can be made in terms of support functions and by induction on n. For, denote by $L_n(b)$ the convex hull of the points Pb for P a permutation matrix. The support function of $L_n(b)$ is then $f(u) = \max_P (u, Pb)$ and thus $L_n(b)$ is the intersection of the half spaces $(u, t) \leqq f(u)$. By setting $u_0 = \epsilon_{i_1} + \cdots + \epsilon_{i_k}$ (again ϵ_{i_j} is the unit vector with 1 in position i_j and 0 elsewhere), where $i_1, \cdots, i_k$ is an index set in (4.1), we can check that if $t \in L_n(b)$ then

$$\sum_{j=1}^{k} t_{i_j} = (u_0, t) \leqq f(u_0) = \max_P (u_0, Pb) = \sum_{j=1}^{k} b_j,$$

with equality for $k = n$. Hence $L_n(b) \subseteq \Omega_n(b)$. Now suppose $t \in \Omega_n(b)$ and set $g(t) = (u, t)$ for a fixed u. Now $g(t)$ is a linear function of t defined on the set of t given by (4.1). Therefore it assumes its maximum value for $t = \bar{t}$ on one of the bounding hyperplanes in (4.1), say

$$\sum_{j=1}^{k} t_{i_j} = \sum_{j=1}^{k} b_j, \qquad 1 \leqq k \leqq n-1.$$

Let $\bar{t}^1 = (\bar{t}_{i_1}, \cdots, \bar{t}_{i_k})$, $\bar{t}^2 = (\bar{t}_{i_{k+1}}, \cdots, \bar{t}_{i_n})$, $b^1 = (b_1, \cdots, b_k)$, $b^2 = (b_{k+1}, \cdots, b_n)$ where $i_{k+1} < \cdots < i_n$ is the set of integers in $1, \cdots, n$ complementary to $i_1, \cdots, i_k$. We check that $\bar{t}^1 \in L_k(b^1)$ and $\bar{t}^2 \in L_{n-k}(b^2)$. For suppose $\alpha_1, \cdots, \alpha_r$ is a subset of the integers $i_1, \cdots, i_k$. Then $\sum_{j=1}^{r} \bar{t}_{\alpha_j} \leqq \sum_{j=1}^{r} b_j$ and also if $\beta_1, \cdots, \beta_\rho$ is a subset of $i_{k+1}, \cdots, i_n$ then

$$\sum_{j=1}^{\rho} \bar{t}_{\beta_j} + \sum_{j=1}^{k} \bar{t}_{i_j} \leqq \sum_{j=1}^{\rho+k} b_j.$$

Hence

$$\sum_{j=1}^{\rho} \bar{t}_{\beta_j} \leqq \sum_{j=1}^{\rho+k} b_j - \sum_{j=1}^{k} \bar{t}_{i_j} = \sum_{j=1}^{\rho+k} b_j - \sum_{j=1}^{k} b_j = \sum_{j=k+1}^{\rho+k} b_j.$$

Since $k < n$ and $n-k < n$ we may use induction to conclude that $\bar{t}^1 = S_1 b^1$ and $\bar{t}^2 \in S_2 b^2$ for $S_1 \in \Omega_k$, $S_2 \in \Omega_{n-k}$. Then since $i_1, \cdots, i_n$ is simply a permutation of $1, \cdots, n$ we have that $\bar{t} = Q(S_1 b^1 \dotplus S_2 b^2) = Q(S_1 \dotplus S_2) b \in L_n(b)$ for Q an appropriate permutation matrix ($\dotplus$ indicates direct sum). But then $(u, t) = g(t) \leqq g(\bar{t}) = (u, \bar{t}) \leqq f(u)$; so $(u, t) \leqq f(u)$ for all u and $t \in L_n(b)$.

A. Horn obtained the considerably stronger result that

If $t \in \Omega_n(b)$ then $t = Sb$ where S is an orthostochastic (o.s.) *matrix.*

An o.s. matrix S is one for which there exists an orthogonal matrix U such that $s_{ij} = u_{ij}^2$. The following example due to A. J. Hoffman shows that not every d.s. matrix is o.s.:

$$S = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Horn uses his result to obtain the interesting fact that the *set of all main diagonals of n-square rotation matrices is the convex hull of the points* $(\pm 1, \cdots, \pm 1)$ *of which an even number of coordinates are* -1. Also in this connection some information concerning the structure of points lying on the support planes of $\Omega_n(b)$ is known [13].

Now let A be an arbitrary nonsingular n-square matrix with eigenvalues λ_j ordered according to absolute value: $|\lambda_1| \geqq \cdots \geqq |\lambda_n|$. Let $\alpha_1 \geqq \cdots \geqq \alpha_n$ be the positive square roots of the eigenvalues of AA^* (the α_j are called the *singular values* of A). In 1949 H. Weyl [30] proved that

$$\prod_{j=1}^{k} |\lambda_j| \leqq \prod_{j=1}^{k} \alpha_j, \qquad 1 \leqq k \leqq n \tag{4.2}$$

with equality for $k = n$. Weyl established (4.2) by comparing the dominant eigenvalue and singular value of the kth compound of A. By taking logarithms on both sides of (4.2) and letting $\log |\lambda| = (\log |\lambda_1|, \cdots, \log |\lambda_n|)$, $\log \alpha = (\log \alpha_1, \cdots, \log \alpha_n)$ we have from the Hardy, Littlewood, Pólya result that

$$\log |\lambda| = S \log \alpha \tag{4.3}$$

for $S \in \Omega_n$. As an application of (4.3) we have a special case of a result of Fan [7] proved also by Pólya [25] and originally by Weyl [30] under additional hypotheses:

$$\sum_{j=1}^{k} |\lambda_j|^s \leqq \sum_{j=1}^{k} \alpha_j^s, \qquad 1 \leqq k \leqq n,\ s > 0. \tag{4.4}$$

The result in (4.4) follows easily by defining the convex function $g(t) = \sum_{j=1}^{k} e^{st_j}$ and noting that

$$\sum_{j=1}^{k} |\lambda_j|^s = g(\log |\lambda|) = g(S \log \alpha) \leqq g(P \log \alpha) \leqq \sum_{j=1}^{k} \alpha_j^s,$$

for P an appropriate permutation matrix. (This is Pólya's proof.) An exhaustive treatment of the inequalities connecting eigenvalues and singular values is found in [7], [8], [11], [1].

5. Other functions on Ω_n and the permanent problem. In 1926 B. L. van der Waerden [29] proposed the following problem: let $p(S)$ denote the *permanent* of S,

$$p(S) = \sum_{\sigma} \prod_{i=1}^{n} s_{i\sigma(i)},$$

where the sum is over all permutations σ of $1, \cdots, n$; question: is the minimum value of this function on Ω_n assumed uniquely for $S=J_n$, the d.s. matrix all of whose entries are $1/n$? For $n=2$ the problem is trivial and for $n=3$ the answer is "yes." In addition to this it is known that if the minimum occurs in the relative interior of Ω_n (*i.e.*, d.s. matrices with positive entries) then it occurs uniquely at J_n [20]. P. Erdös observed that two consequences of the van der Waerden conjecture are: (a) for any $S\in\Omega_n$ there exists a permutation σ such that $\prod_{i=1}^{n} s_{i\sigma(i)} \geqq (1/n)^n$; (b) there exists a permutation σ such that $\sum_{i=1}^{n} s_{i\sigma(i)} \geqq 1$ and $s_{i\sigma(i)}>0$. Conjecture (a) seems about as hard as the original problem but (b) appears as a corollary of a considerably stronger result [21]:

If $S\in\Omega_n$ and has k eigenvalues of absolute value 1 (1 *itself is always an eigenvalue*) *then there exists a permutation σ such that $s_{i\sigma(i)}>0$, $i=1, \cdots, n$ and $\sum_{i=1}^{n} s_{i\sigma(i)} \geqq k$. The case of equality can be removed unless S can be brought to the form*

$$J_{n_1} \dotplus \cdots \dotplus J_{n_k}, \qquad \sum_{j=1}^{k} n_j = n \tag{5.1}$$

by permutation of rows and columns.

Let the p-minor of s_{ij} be the permanent of the $(n-1)$-square matrix obtained by deleting the ith row and jth column of S. The following question has arisen in connection with the conjecture of van der Waerden; is it possible for some $S\in\Omega_n$ that

(i) the p-minors of all positive entries of S are equal with common value $p(S)$,

(ii) the p-minors of all zero entries of S are equal with common value $p(S)+\beta$ for $\beta>0$?

If the answer to this is "no" for general n (which is the case for $n=3$ and 4) then the permanent problem will be solved. The separation of p-minors into two equal sets can occur of course, but apparently not for $\beta>0$:

$$S = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

6. Kakutani's conjecture. A partial ordering can be introduced in Ω_n as fol-

lows: $A<B$ if there exists $C\in\Omega_n$ for which $A=CB$. A partial ordering of real n-tuples can also be defined by: $a<b$ if and only if for every convex function ϕ

$$\sum_{i=1}^{n}\phi(a_i)\leqq\sum_{i=1}^{n}\phi(b_i).$$

A result in [9] states that $a<b$ *if and only if* $a\in\Omega_n(b)$. Thus if $a<b$, $a=Sb$ for some $S\in\Omega_n$. Now Kakutani conjectured that *if* $Aa<Ba$ *for all real n-tuples* a, *then* $A<B$. The result holds for $n=3$ but A. Horn constructed A and B for $n=4$ for which it fails [27]:

$$A=\begin{pmatrix}\frac{1}{4}&0&\frac{3}{8}&\frac{3}{8}\\ \frac{1}{4}&0&\frac{3}{8}&\frac{3}{8}\\ \frac{1}{4}&\frac{1}{2}&\frac{5}{32}&\frac{3}{32}\\ \frac{1}{4}&\frac{1}{2}&\frac{3}{32}&\frac{5}{32}\end{pmatrix},\qquad B=\begin{pmatrix}\frac{1}{4}&0&\frac{1}{4}&\frac{1}{2}\\ \frac{1}{4}&0&\frac{1}{2}&\frac{1}{4}\\ \frac{1}{4}&\frac{3}{4}&0&0\\ \frac{1}{4}&\frac{1}{4}&\frac{1}{4}&\frac{1}{4}\end{pmatrix}.$$

However in case B is nonsingular an argument of S. Sherman [28] will establish the result. Recently S. Schreiber [26] gave a simpler argument for B nonsingular as well as geometric interpretation of this partial ordering in Ω_n.

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CONVERGENCE IN ORDERED INTEGRAL DOMAINS

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1. Introduction. All the terms used in defining convergence of a sequence of real numbers have meaning in any ordered integral domain. Accordingly, we say: a sequence of elements $\{a_n\}$ of an ordered integral domain D is convergent if there exists an element a with the property that for every element $\epsilon>0$ there is an integer N such that $|a_n-a|<\epsilon$ for $n>N$. (The double use of the inequality sign, to denote inequalities in D and in the domain of ordinary integers, will cause no confusion.) We write, as usual, $a_n\to a$.

If D is archimedean ordered (*i.e.*, whenever $a>0$, $b>0$ there exists an integer n such that $na>b$) it is order-equivalent to a subdomain of the real numbers, and all the usual questions concerning convergence in D have the usual answers.

The following example illustrates what may happen if the ordering of D is not archimedean.

Let D consist of all real numbers expressible in the form $\sum_{i=0}^{k} \alpha_i \pi^i$, $k \geqq 0$, the α_i rational. Call this element positive (or negative) if α_k (the last non zero coefficient) is positive (or negative) in the usual sense. In this domain: (a) if $a_n \to \pi$, $b_n \to 0$ and $b_n \neq 0$ for every n, then $\{a_n b_n\}$ does not converge, (b) if $a_n = \pi + 1/n$, $b_n = \pi - 1/n$, then $a_n b_n \to \pi^2$, but $\{a_n b_{n+1}\}$ does not converge.

2. Preliminaries. If the ordered integral domain D has a smallest positive element ϵ and $a_n \to a$, then $|a_n - a| < \epsilon$ for $n > N$ implies $a_n = a$ for all such n; thus, $\{a_n\}$ can converge if and only if all a_n are equal for n sufficiently large. (We shall call such a sequence trivial.) Since this situation is without interest, we suppose henceforth that D has no smallest positive element.

The following results are easily established:

(1) A convergent sequence has a unique limit.

(2) A convergent sequence has the Cauchy property (*i.e.*, if $a_n \to a$ then for every $\epsilon > 0$ there is an N such that $|a_n - a_m| < \epsilon$ when $n > N$, $m > N$.

(3) If $a_n \to a$, $b_n \to b$ then $a_n \pm b_n \to a \pm b$.

(4) Every subsequence of a convergent sequence converges to the same limit as the sequence.

For example, to establish (1), suppose $a_n \to a$, $a_n \to b$. Let $\epsilon > 0$. Let δ be chosen so that $0 < \delta < \epsilon$. Let $|a_n - a| < \delta$ for $n > N$ and $|a_n - b| < \epsilon - \delta$ for $n > N'$. Then, for $n > \max(N, N')$, $|a - b| \leqq |a - a_n| + |a_n - b| < \epsilon$. Hence $a - b = 0$.

3. Bounded elements. We call an element a of D bounded if there exists an element $\alpha > 0$ such that $\alpha|a| \leqq 1$. If a and b are bounded, let $\alpha|a| \leqq 1$, $\beta|b| \leqq 1$ where $\alpha > 0$, $\beta > 0$. If $\gamma = \min(\alpha, \beta)$, then $\gamma|a \pm b| \leqq \gamma|a| + \gamma|b| \leqq \alpha|a| + \beta|b| \leqq 2$. Let $0 < \delta < 1$ and $\epsilon = \min(\delta, 1 - \delta)$. Since $2\delta + 2(1 - \delta) = 2$, therefore either $2\delta \leqq 1$ or $2(1 - \delta) \leqq 1$. Hence $2\epsilon \leqq 1$. Therefore $\epsilon\gamma|a \pm b| \leqq 2\epsilon \leqq 1$. Thus, $a \pm b$ are bounded. Also, since $\alpha\beta|ab| \leqq 1$, ab is bounded.

Since 1 is a bounded element and the sum, difference and product of two bounded elements are bounded, the bounded elements of D form a subdomain. Since this subdomain contains 1 it contains every integral element; hence every integral element of D is bounded.

If $|a| \leqq b$ and b is bounded, clearly a is also bounded. Hence every element of an archimedean ordered domain is bounded, since for every a there is an integral element n such that $|a| \leqq n$.

If a is bounded and $\alpha|a| \leqq 1$ where $\alpha > 0$, then for every $\epsilon > 0$ we have $(\alpha\epsilon)|a| \leqq \epsilon$. Thus, there exists a $\delta > 0$ such that $\delta|a| \leqq \epsilon$. Conversely, if for a given element a there exists for every $\epsilon > 0$ a $\delta > 0$ such that $\delta|a| \leqq \epsilon$ then (taking $\epsilon = 1$) a is bounded. It is this property of bounded elements that is used so frequently in proofs of convergence theorems in the real domain.

4. Product sequences. The main difference between convergence in D and

convergence in the real domain occurs in connection with the product sequence $\{a_nb_n\}$ of $\{a_n\}$ and $\{b_n\}$.

Let $a_n \to a$, $b_n \to b$, so that $a_n = a+\alpha_n$, $b_n = b+\beta_n$ where $\alpha_n \to 0$, $\beta_n \to 0$. Then $a_nb_n = ab+(a\beta_n+b\alpha_n)+\alpha_n\beta_n$. Since $|\alpha_n| < 1$ for n sufficiently great and $\beta_n \to 0$, therefore $\alpha_n\beta_n \to 0$.

Case 1. *If* $\{a_n\}$, $\{b_n\}$ *are both trivial, then obviously* $a_nb_n \to ab$.

Case 2. *If* $\{a_n\}$ *is trivial and* $\{b_n\}$ *is not, then* $\{a_nb_n\}$ *converges if and only if* a *is bounded, in which case* $a_nb_n \to ab$.

Proof. Since $\{a_n\}$ is trivial, $a_n = a$ for all sufficiently large n. Suppose $\{a_nb_n\}$ converges. Then for m and n sufficiently large $|a_nb_n - a_mb_m| = |a|\,|b_n - b_m| < 1$. Since $\{b_n\}$ is nontrivial, there exist such m and n for which $b_n - b_m \neq 0$. Thus, a is bounded.

Conversely, suppose a is bounded. Let $\epsilon > 0$. Let δ be a positive element so chosen that $\delta|a| \leqq \epsilon$. Let N be chosen so that $|\beta_n| < \delta$ for $n > N$, which can be done since $\beta_n \to 0$. Then $|a\beta_n| \leqq \delta|a| \leqq \epsilon$ for $n > N$. Hence $a\beta_n \to 0$. Since $\alpha_n = 0$ for n sufficiently great, therefore $b\alpha_n \to 0$. Thus, $a_nb_n \to ab$.

Case 3. *If neither* $\{a_n\}$ *nor* $\{b_n\}$ *is trivial and* a *is bounded, then* $\{a_nb_n\}$ *converges if and only if* b *is bounded, in which case* $a_nb_n \to ab$.

Proof. Since a is bounded, Case 2 implies $a\beta_n \to 0$. If b is bounded then, by Case 2, $b\alpha_n \to 0$. Hence $a_nb_n \to ab$. Conversely, if $a_nb_n \to c$ then $b\alpha_n = a_nb_n - ab - a\beta_n - \alpha_n\beta_n \to c - ab$. Hence, by Case 2, b is bounded.

Case 4. *If neither* $\{a_n\}$ *nor* $\{b_n\}$ *is trivial and neither* a *nor* b *is bounded, then* $\{a_nb_n\}$ *converges if and only if there exist elements* $\alpha \neq 0$, $\beta \neq 0$ *such that* $a\alpha+b\beta$ *is bounded and* $\alpha\alpha_n = \beta\beta_n$ *for all* n *sufficiently great, in which case* $a_nb_n \to ab$.

Proof. (a) Suppose $a_nb_n \to c$. Then, letting $\gamma_n = a\beta_n + b\alpha_n$, we have $\gamma_n = a_nb_n - ab - \alpha_n\beta_n \to c - ab$. Let N be so chosen that for $n > N$, $m > N$

$$|\gamma_n - \gamma_m| = |a(\beta_n - \beta_m) + b(\alpha_n - \alpha_m)| < 1, \quad |\alpha_n - \alpha_m| < 1, \quad |\beta_n - \beta_m| < 1.$$

Let i, j be integers so chosen that $i > N$, $j > N$, $|\beta_i - \beta_j| \neq 0$. Let $\alpha = \beta_i - \beta_j$, $\beta = \alpha_i - \alpha_j$. Then $0 < |\alpha| < 1$, $|\beta| < 1$. Also $|a\alpha+b\beta| = |\gamma_i - \gamma_j| < 1$, so that, since a is not bounded, $\beta = 0$. Let $\gamma = a\alpha+b\beta$. By Cases 1 and 2, we have $\beta\gamma_n \to \beta(c-ab)$ and $\gamma\alpha_n \to 0$, since β and γ are bounded. Also, $\beta\gamma_n = a\beta_n\beta + b\alpha_n\beta = a(\beta_n\beta - \alpha_n\alpha) + \gamma\alpha_n$, so that $a(\beta_n\beta - \alpha_n\alpha) = \beta\gamma_n - \gamma\alpha_n \to \beta(c-ab)$. But $\beta_n\beta - \alpha_n\alpha \to 0$ since $\beta_n\beta \to 0$ and $\alpha_n\alpha \to 0$. Since $\{a(\beta_n\beta - \alpha_n\alpha)\}$ is convergent and a is not bounded, Case 2 implies that $\{\beta_n\beta - \alpha_n\alpha\}$ is a trivial sequence. Thus, $\alpha_n\alpha = \beta_n\beta$ for all n sufficiently great. And, since $\beta(c-ab) = 0$, we have $c = ab$.

(b) Conversely, suppose there exist elements α and β with the properties stated above. Then $\alpha(a\beta_n + b\alpha_n) = (\alpha a + \beta b)\beta_n$ for n sufficiently great, since $\alpha\alpha_n = \beta\beta_n$. By Case 2, since $\beta_n \to 0$ and $\alpha a + \beta b$ is bounded, $(\alpha a + \beta b)\beta_n \to 0$. Therefore also $\alpha(a\beta_n + b\alpha_n) \to 0$. Hence $a\beta_n + b\alpha_n \to 0$ (for otherwise for some $\epsilon > 0$ we

would have $|a\beta_n+b\alpha_n| \geqq \epsilon$ for infinitely many values of n; for these values of n we would have $|\alpha(a\beta_n+b\alpha_n)| \geqq |\alpha|\epsilon>0$, which is impossible). Thus, $a_nb_n \to ab$.

Remark. If $a_n \to a$ and a is not bounded, for a given b there is "at most one" sequence $\{b_n\}$ converging to b such that $\{a_nb_n\}$ is convergent. That is, if $b_n \to b$, $b_n' \to b$, and $\{a_nb_n\}$, $\{a_nb_n'\}$ both converge, then $b_n=b_n'$ for all n sufficiently great. For, if both these product sequences converge, then $a_nb_n-a_nb_n' = a_n(b_n-b_n') \to 0$. But $b_n-b_n' \to 0$ and 0 is a bounded element, so that if $\{b_n-b_n'\}$ were not a trivial sequence then $\{a_n(b_n-b_n')\}$ could not converge (by Case 2 if $\{a_n\}$ is trivial and by Case 3 if it is not). Hence $\{b_n-b_n'\}$ must be a trivial sequence.

5. Quotient sequences. Let $a_n \to a$, $b_n \to b$ and suppose $c_n=a_n/b_n$ exists for every n. If $c_n \to c$ then, since $a_n=b_nc_n \to a$, we have $a=bc$. Thus a/b exists and $c=a/b$.

Assuming a/b exists, we state without proof the following conditions for the convergence of $\{c_n\}$:

Case 1. $\{c_n\}$ *is trivial (in which case it is convergent) if and only if* $ab_n=ba_n$ *for all* n *sufficiently great.*

Case 2. *If* $\{c_n\}$ *is nontrivial and* a/b *is bounded, then* $\{c_n\}$ *is convergent if and only if* b *is bounded.*

Case 3. *If* $\{c_n\}$ *is nontrivial and* a/b *is not bounded, then* $\{c_n\}$ *converges if and only if either* (a) $\{b_n\}$ *is trivial or* (b) $\{b_n\}$ *is nontrivial,* b *is not bounded, and there exist elements* $\alpha \neq 0$, $\beta \neq 0$ *such that* β *is bounded and* $\alpha(a_n-a) = (\alpha(a/b)+\beta b_n)(b_n-b)$ *for all* n *sufficiently great.*

Remark. If $\{a_n\}$ and $\{b_n\}$ are nontrivial and converge to limits a and b which are not bounded, it is impossible for both the product and quotient sequences to converge. In fact, if $\{a_nb_n\}$ converges, it is impossible for both a/b and a_n/b_n (for all n sufficiently large) to exist.

For, suppose $\{a_nb_n\}$ converges and a/b, a_n/b_n exist for $n>N$. Then (referring to Case 4 of the preceding section) $\gamma=a\alpha+b\beta$ is bounded and divisible by b. Let $\gamma=bd$. Since γ is bounded, there exists an element $x>0$ such that $(x|d|)|b| = x|\gamma| \leqq 1$. Since b is not bounded, this requires $x|d|=0$. Hence $d=0$, so that $\gamma=0$. Thus $a\alpha=-b\beta$.

Since $\alpha\alpha_n=\beta\beta_n$ for all n sufficiently large, we have, for these values of n, $\alpha(a_n-a)=\beta(b_n-b)$, $\alpha a_n-\beta b_n=\alpha a-\beta b=-2\beta b$; hence $2\beta b$ is divisible by b_n when n is large enough. Since $2\beta\beta_n=2\beta(b_n-b)=2\beta b_n-2\beta b$, therefore $2\beta\beta_n$ is also divisible by b_n.

For n sufficiently large, $|\beta_n|<1$, so that β_n is bounded. Reference to the proof of Case 4 in the preceding section shows $0<|\beta|<1$, so that β is bounded. Thus, $2\beta\beta_n$ is bounded and nonzero so that, since b_n is one of its factors, b_n is also bounded. Since $|b| \leqq |b_n|+|b-b_n| \leqq |b_n|+1$ for n sufficiently large, b is bounded, which is a contradiction of the hypothesis.

6. Quotient fields. Let F be the quotient field of D ordered so as to preserve the ordering of D. Let $a_n \to a$ nontrivially in D. Suppose D contains an element b which is not bounded. Since $\{a_n\}$ is a nontrivial sequence, therefore for infinitely many values of n, $|a_{n+1}-a_n|>0$. Since b is not bounded, for these values of n we have $|a_{n+1}-a_n|\,|b|>1$. Thus, in F, $|a_{n+1}-a_n|>1/|b|$. Hence $\{a_n\}$ cannot converge in F since it does not have the Cauchy property in F.

Suppose, on the other hand, that every element of D is bounded. If $\epsilon>0$ in F, let $\epsilon=\alpha/\beta$ where α and β are positive elements of D. In D, $a_n-a\to 0$, so that $\beta(a_n-a)\to 0$. Hence there exists an N such that $|\beta(a_n-a)|<\alpha$ for $n>N$. Thus, for $n>N$, $|a_n-a|<\alpha/\beta=\epsilon$. Therefore $a_n\to a$ in F.

7. Sequence $\{a^n\}$. There are integral domains in which $\{a^n\}$ does not converge for any $a\neq 0, 1$. Suppose, however, that it does converge, say to b. If the sequence is trivial then $a^n=a^m$, $m>n$, for sufficiently great n, so that $a^{m-n}=1$. Hence $a=0$, 1, or -1. Clearly for $a=0$, 1 we have convergence and for $a=-1$ we do not.

Suppose now that $a^n\to b$ and $\{a^n\}$ is not trivial. The sequence $\{aa^n\}$ may be regarded as a subsequence of $\{a^n\}$, and thus convergent to b, and as the product sequence of $\{a^n\}$ and $\{a\}$ and thus, since it does converge, convergent to ab. Hence $ab=b$, so that $b=0$. Also, since $|a^n|<1$ for n sufficiently large, $|a|<1$.

To consider the converse problem, let D' denote the subdomain consisting of all the bounded elements of D, ordered according to the ordering of D. If $0<a<1$ then a is in D'; if $a^n\to 0$ in D then $a^n\to 0$ in D', and conversely.

We now show: if and only if D' is archimedean ordered does $0<a<1$ imply $a^n\to 0$.

(a) We show first: if D' contains an element $b>0$ such $b^n\to 0$ and $Nb\geqq 1$ for some integer N, then D' is archimedean ordered. Let ϵ, δ be positive elements of D'. Since ϵ is a bounded element of D, there is an element $\alpha>0$ in D such that $0<\alpha\epsilon\leqq 1$. Since $b^n\to 0$ there exists a K such that $b^n<\alpha\delta$ for $n>K$. Since $Nb\geqq 1$, we have $1\leqq N^nb^n<N^n\alpha\delta$ for $n>K$. Hence, $\epsilon<N^n\alpha\delta\epsilon\leqq N^n\delta$. Thus $N^n\delta>\epsilon$, and the desired result is established since N^n is an integer.

Now, suppose D contains an element a between 0 and 1 such that $\{a^n\}$ and $\{(1-a)^n\}$ both converge. Since either $2a\geqq 1$ or $2(1-a)\geqq 1$, it follows from the preceding, with $N=2$, that D' is archimedean ordered.

(b) Conversely, suppose D' is archimedean ordered. Let $0<a<1$. Let k be a positive integer so chosen that $k(1-a)>a$. (This can be done since $1-a$ is in D'). By induction on n it follows that $na^n\leqq ka$ for all positive integral n. (For, if $na^n\leqq ka$, then $(n+1)a^{n+1}=(na^n+a^n)a\leqq(ka+a)a\leqq ka$). Let ϵ be a positive element of D' and let integer N be so chosen that $N\epsilon>a$. Then, for $n>kN$, $kNa^n<na^n\leqq ka<kN\epsilon$, from which $a^n<\epsilon$. Thus, $a^n\to 0$.

8. Sequence $\{a/n\}$. Suppose a is a positive element of D divisible by every positive integral element n. If $a/n\to b$, then $a/(n(n+1))=(a/n)-(a/(n+1))\to 0$. But $\{a/(n(n+1))\}$ is a subsequence of $\{a/n\}$ and, therefore, converges to b.

Hence $b=0$. Therefore, also, a is a bounded element, since $a/n<1$, or $a<n$, for n sufficiently great. If, in addition, $Na\geqq 1$ for some integer N, then D', the subring of bounded elements, must be archimedean. For, if δ is a positive element of D' then $N\delta a/n\to 0$, since $N\delta$ is a bounded element. Therefore, if ϵ is any positive element of D', we have $\delta/n\leqq N\delta a/n<\epsilon$ for n sufficiently great, so that $\delta<n\epsilon$.

Conversely, if D' is archimedean ordered and a is bounded then $a/n\to 0$. For, if ϵ is a positive element of D' and K is an integer so chosen that $K\epsilon>a$, then $a/n<\epsilon$ for $n>K$.

9. Limitability. In D let $a_n\to a$ and suppose $S_n=(a_1+\cdots+a_n)/n$ exists for $n=1, 2, \cdots$. In the real domain $\{S_n\}$ also converges to a, which shows that a convergent sequence is limitable by Cesàro or Hölder means of the first order to the same number as the limit of the sequence. For a general integral domain, even a field, this may not be true. For instance, in a non-archimedean ordered field if $a_1=1$, $a_n=0$ for $n>1$, then $a_n\to 0$ but $\{S_n\}=\{1/n\}$ does not converge.

However, we now show, if $a_n\to a$ in D and $S_n\to b$, then $b=a$. Let $a_n=a+\alpha_n$. Then $\alpha_n\to 0$. Also $S_n=a+T_n$, where $T_n=(\alpha_1+\cdots+\alpha_n)/n$. Since $a_n\to a$ and $S_n\to b$, $T_n\to b-a$. Hence $T_n=b-a+\beta_n$, where $\beta_n\to 0$. We have

$$n(\beta_n-\beta_{n-1})=n(T_n-T_{n-1})=n\left(\frac{\alpha_1+\cdots+\alpha_n}{n}-\frac{\alpha_1+\cdots+\alpha_{n-1}}{n-1}\right)$$

$$=\alpha_n+(\alpha_1+\cdots+\alpha_{n-1})\left(1-\frac{n}{n-1}\right)=\alpha_n-T_{n-1}.$$

Hence $n(\beta_n-\beta_{n-1})\to a-b$.

Suppose $a-b>0$. (This involves no loss of generality since if $a-b<0$ we may let $\gamma_n=-\beta_n$ and we have $\gamma_n\to 0$, $n(\gamma_n-\gamma_{n-1})\to b-a>0$). Choose α and γ so that $0<\gamma<\alpha<a-b$. Let $|n(\beta_n-\beta_{n-1})-(a-b)|<(a-b)-\alpha$ for $n>N'$. Then $\alpha-(a-b)<n(\beta_n-\beta_{n-1})-(a-b)<(a-b)-\alpha$. Hence $n(\beta_n-\beta_{n-1})>\alpha$ for $n>N'$.

Let $|\beta_n|<\gamma$ for $n>N''$. Let $N>\max(N', N'')$ and let N''' be so chosen that

$$\frac{1}{N+1}+\frac{1}{N+2}+\cdots+\frac{1}{n}>1 \qquad \text{for } n>N'''.$$

(Here we are dealing with ordinary rational numbers). By induction on n, making use of $n(\beta_n-\beta_{n-1})>\alpha$, we have, for $n>N$,

$$n(n-1)\cdots(N+1)(\beta_n-\beta_N)$$

$$>\alpha\left[\frac{n(n-1)\cdots(N+1)}{N+1}+\frac{n(n-1)\cdots(N+1)}{N+2}+\cdots\right.$$

$$\left.+\frac{n(n-1)\cdots(N+1)}{n}\right].$$

From this it follows that, for $n>\max(N, N''')$,

$$n(n-1)\cdots(N+1)(\beta_n-\beta_N) > \alpha n(n-1)\cdots(N+1),$$

so that $\beta_n-\beta_N>\alpha$ or $\beta_n>\alpha+\beta_N$.

Since $n(\beta_n-\beta_{n-1})>\alpha>0$, for $n>N'$, $\beta_{n-1}<\beta_n$. Since $\beta_n\to 0$, this requires that $\beta_n<0$ for $n>N'$. Thus, $-\beta_N=|\beta_N|<\gamma$, so that $\beta_N+\alpha=\gamma+\beta_N+(\alpha-\gamma)>\alpha-\gamma>0$. Hence, for $n>\max(N, N''')$ we have $\beta_n>\alpha+\beta_N>\alpha-\gamma>0$, contradicting the fact that $\beta_n\to 0$. Thus $a-b>0$ is impossible and our assertion that $a=b$ is proved.

We now show:

(a) If D is archimedean ordered and $a_n\to a$, then $S_n\to a$ provided the S_n all exist. In the notation above, let $\epsilon>0$ and suppose $|\alpha_n|<\epsilon$ for $n>M$. Let K be an integer, so chosen that $|\alpha_1+\cdots+\alpha_M|<K\epsilon$. For $n>\max(M, K)$ we have

$$\begin{aligned} |nT_n| &= |\alpha_1+\cdots+\alpha_M+\alpha_{M+1}+\cdots+\alpha_n| \\ &\leqq |\alpha_1+\cdots+\alpha_M| + |\alpha_{M+1}|+\cdots+|\alpha_n| \\ &< K\epsilon+(n-M)\epsilon < 2n\epsilon, \end{aligned}$$

so that $|T_n|<2\epsilon$. Let $\delta>0$ and let η be so chosen that $0<\eta<\delta$. Then either $2\eta\leqq\delta$ or $2(\delta-\eta)\leqq\delta$. Hence, with the proper choice of ϵ, we have $|T_n|<2\epsilon\leqq\delta$ for n sufficiently large, so that $T_n\to 0$ and $S_n\to a$.

(b) Suppose D contains a positive element a divisible by every positive integral element and such that $Ma\geqq 1$ for some integer M. Then, if every convergent sequence (or merely every trivial sequence) is limitable by first order Cesàro means, D is archimedean ordered.

For, if b is any element of D then, letting $a_1=ab$, $a_n=0$ for $n>1$, we have $a_n\to 0$, $S_n=ab/n\to 0$. Thus, $(a/n)|b|<1$ for some n, so that b is bounded. Thus D', the subring of bounded elements, is D itself. Since (taking $b=1$) $\{a/n\}$ is convergent and $Ma\geqq 1$, by the preceding section, D is archimedean ordered.

10. Domains with valuations. Let I be an integral domain with a valuation function $\phi(x)$ whose values lie in the ordered integral domain D. That is, (1) for each x in I, $\phi(x)$ is a unique element in D; (2) $\phi(0)=0$; (3) $\phi(x)>0$ if $x\neq 0$, (4) $\phi(xy)=\phi(x)\phi(y)$; (5) $\phi(x+y)\leqq\phi(x)+\phi(y)$.

If $\{x_n\}$ is a sequence of elements in I, we say it is convergent (relative to the valuation function) if there exists an element x in I such that $\phi(x_n-x)\to 0$ in D. We call an element x in I bounded if $\phi(x)$ is a bounded element in D. All of the above results for convergence in D can easily be restated to apply to convergence in I relative to $\phi(x)$.

A GENERALIZED INDEPENDENCE CONDITION AND ERROR CORRECTION CODES*

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The notion of independence of a set of vectors is fundamental to the theory of vector spaces. In this note we define a generalization of this concept for vector spaces over finite fields and present an application to the theory of error correction codes. Before proceeding to this generalization we shall digress to present a few facts concerning error correction codes.

Let M be a finite metric space and C a subset of M such that elements in C are separated from one another by at least d_0 units. Such a subset will be called a code. In order to motivate this terminology consider a communication system in which the transmitting station selects its messages from C. If the transmission channel linking the transmitting and receiving stations is noiseless there will be no difficulty in recovering the transmitted message. If noise is present let us assume that the received message is an element of M which is "close" to the transmitted message. If, in fact, the distance a transmitted message moves may be bounded by $\frac{1}{2}d_0-\epsilon$ (for some $\epsilon>0$) there is an element of C, namely the transmitted message, which is closest to the received message. Decoding, or the recovery of the transmitted message, is accomplished by finding that (unique) element of C closest to the received message. We say that information has been encoded by restricting the class of transmittable messages to C and thus removing or lessening the chance of ambiguity arising due to noise.

Now suppose that M consists of all 2^n sequences of n symbols, each symbol assuming one of two states which we denote by zero and one. If the elements of M are identified with the vertices of a unit-edge cube in E^n we define the distance between vertices to be the minimum number of edges that must be traversed in going from one vertex to the other. An error is said to have occurred in the ith symbol if the transmitted and received ith symbols do not agree. Errors are introduced by the channel performing a chance experiment on each of the n symbols; the ith symbol is received as transmitted with probability p and altered with probability $1-p$. It is assumed that these chance experiments are independent and have the same probability p associated with them. If $d_0=2e+1$ and if no more than e alterations occur in the transmission of a message then there is a unique element of C closest to the received message and this is the most probable transmitted message. In this case C is an e-order error correction code. A central problem is the determination of the largest code C with a fixed n and e. Such error correction codes have been studied by Hamming [1], Slepian [2] and Reed [3].

DEFINITION 1. *Let V be a finite-dimensional vector space over a finite field F. A set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is k-independent if every subset of k of these vectors*

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is independent. The set is k-dependent if some subset of k of these vectors is dependent.

DEFINITION 2. *With V and F as in Definition* 1, *a set of vectors* $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ *is k-full if the set is k-independent and not the proper subset of any k-independent set. A set is k-maximal if, among all k-full sets, it has a maximum number of elements.*

Let $C^1 = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and denote by C^m the set of all elements of the form

$$\sum_{i=1}^{m} a_i \mathbf{v}_{j_i},\ a_i \neq 0,\ a_i \in F,\ i = 1, \cdots, m,\ 1 \leqq j_1 \cdots < j_m \leqq n. \tag{1}$$

Let $C^0 = \{\mathbf{0}\}$, the null vector in V.

A k-full set plays the role of a basis in V; to show this we prove the

LEMMA. *Let* C^1 *be k-independent. Then* C^1 *is k-full if and only if* $\cup_{i=0}^{k-1} C^i = V$.

Proof. If C^1 is k-full and $\cup_{i=1}^{k-1} C^i \subset V$, choose $\mathbf{v}$ in $V - \cup_{i=0}^{k-1} C^i$ and let $D^1 = C^1 \cup \{\mathbf{v}\}$. Now

$$\mathbf{v} + \sum_{i=1}^{m} a_i \mathbf{v}_{j_i} \neq \mathbf{0} \tag{2}$$

for any choice of the a_i and $m \leqq k-1$, since equality would imply $\mathbf{v} \in \cup_{i=0}^{k-1} C^i$. But (2) implies that D^1 is k-independent, a contradiction of the hypothesis. Conversely, if $\cup_{i=0}^{k-1} C^i = V$, then given $\mathbf{v} \in V$ we have a (not necessarily unique) representation

$$\mathbf{v} = \sum_{i=1}^{m} a_i \mathbf{v}_{j_i} \tag{3}$$

for some choice of the a_i and $m \leqq k-1$. Equation (3) shows that $\mathbf{v}$ may not be added to C^1 without $C^1 \cup \{\mathbf{v}\}$ becoming k-dependent. Thus C^1 is k-full and the lemma is proved.

Let J_2 be the field $\{0, 1\}$ and $\Omega^n = J_2 \times \cdots \times J_2$ (n copies). Ω^n is an n-dimensional vector space over J_2 with addition defined in the following manner: If

$$\mathbf{v} = (a_1 \cdots a_n) = (a_i), \qquad \mathbf{w} = (b_1 \cdots b_n) = (b_i), \tag{4}$$

then

$$\mathbf{v} + \mathbf{w} = (c_i), \qquad c_i = a_i \oplus b_i, \tag{5}$$

where $\oplus$ denotes addition modulo 2.

Following Hamming [1], we define a metric d by

$$d(\mathbf{v}, \mathbf{w}) = \sum_{i=1}^{n} (a_i \oplus b_i). \tag{6}$$

DEFINITION 3. *A subset C of Ω^n is a k-code if* $\mathbf{v}, \mathbf{w} \in C$, $\mathbf{v} \neq \mathbf{w}$, *imply* $d(\mathbf{v}, \mathbf{w}) \geqq k+1$.

If it is further assumed that C is a subspace of Ω^n (of dimension m, $1 \leqq m \leqq n$), then Slepian [2] has shown that the elements of C are of the form (up to a permutation of the coordinates)

$$(\alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_r), \qquad m + r = n, \tag{7}$$

$$\beta_j = \sum_{i=1}^{m} \gamma_{ij}\alpha_i \pmod 2, \qquad \gamma_{ij} \in J_2, j = 1, \cdots, r. \tag{8}$$

In this case C is said to be a *systematic k-code.* For such a code let

$$\mathbf{v}_i = (\gamma_{i1} \cdots \gamma_{ir}), \qquad i = 1, \cdots, m; \tag{9}$$

$$\mathbf{v}_{m+i} = (\underbrace{0 \cdots 0}_{i-1\ 0\text{'s}}\ 1\ \underbrace{0 \cdots 0}_{r-i\ 0\text{'s}}), \qquad i = 1, \cdots, r; \tag{10}$$

$$C^1 = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}. \tag{11}$$

The following equivalence is to be proved:

THEOREM. *A necessary and sufficient condition that a subspace C (defined by* (7) *and* (8)*) be a systematic k-code is that the set C^1 of* (11) *be k-indpendent.*

Proof. Let C be a systematic k-code and suppose that for some p indices $1 \leqq j_1 < \cdots < j_p \leqq n$ we have

$$\mathbf{v}_{j_1} + \cdots + \mathbf{v}_{j_p} = \mathbf{0}. \tag{12}$$

Let R be the mapping

$$R\colon \Omega^m \to \Omega^r, \qquad R((\alpha_1 \cdots \alpha_m)) = (\beta_1 \cdots \beta_r), \tag{13}$$

where the β_j are given by (8). Let f be the mapping

$$\begin{gathered} f\colon \Omega^n \to \Omega^r, \\ f((u_1 \cdots u_n)) = R((u_1 \cdots u_m)) + (u_{m+1} \cdots u_n). \end{gathered} \tag{14}$$

f is a linear transformation of Ω^n onto Ω^r with kernel C. Next, let

$$\mathbf{x}_i = (\underbrace{0 \cdots 0}_{i-1\ 0\text{'s}}\ 1\ \underbrace{0 \cdots 0}_{n-i\ 0\text{'s}}), \qquad i = 1, \cdots, n. \tag{15}$$

Equations (7)–(8) and (13)–(14) show that $\mathbf{v}_i = f(\mathbf{x}_i) \in C^1$, $i = 1, \cdots, n$. Let $\mathbf{x}$ be an arbitrary element in C and

$$\mathbf{y} = \mathbf{x} + (\mathbf{x}_{j_1} + \cdots + \mathbf{x}_{j_p}). \tag{16}$$

Since $\mathbf{x}$ is in C, $f(\mathbf{x}) = \mathbf{0}$ and thus $f(\mathbf{y}) = \mathbf{v}_{j_1} + \cdots + \mathbf{v}_{j_p} = \mathbf{0}$ by (12). Thus $\mathbf{y}$ is in C and $d(\mathbf{x}, \mathbf{y}) = p$. Since C is a k-code, $p \geqq k+1$ and we conclude that C^1 is k-independent.

Now assume that C^1 is k-independent and choose $\mathbf{x}$ and $\mathbf{y}$ to be distinct elements in C. Then for some p and choice of indices,

$$\mathbf{x} = \mathbf{y} + (\mathbf{x}_{j_1} + \cdots + \mathbf{x}_{j_p}). \tag{17}$$

Since $f(\mathbf{x}) = f(\mathbf{y}) = \mathbf{0}$, we have $\mathbf{v}_{j_1} + \cdots + \mathbf{v}_{j_p} = 0$. But $p \geqq k+1$ since C^1 is k-independent and $d(\mathbf{x}, \mathbf{y}) = p$, from which we conclude that C is a k-code.

By this theorem, the construction of (binary) error correction codes is equivalent to the generation of k-independent sets. At present there are procedures for generating such full sets, but no method is known for the systematic construction of k-maximal sets.

The author wishes to thank Professor J. Wolfowitz for his helpful suggestions.

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SOME PACKING PROBLEMS

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How many queens may be packed on an ordinary chess board without any pair challenging each other? How many rooks, kings, bishops or horses?

A little experimenting soon leads to the suspicion that the maxima are (with a fairly obvious notation) $Q=8$, $R=8$, $K=16$, $B=14$, $H=32$. These scores may be achieved in various ways, for example, as shown in Figure 1.

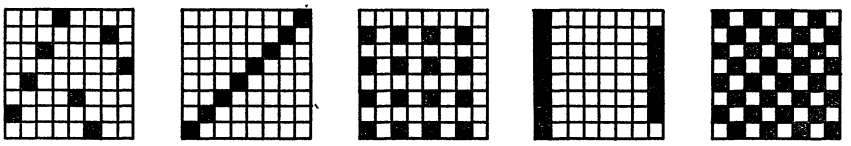


FIG. 1

The demonstration that these packings are best possible is quite easy and involves only the most elementary methods.

In the case of rooks or queens, more than eight would certainly require that two or more share a row and hence challenge. For bishops it is enough to count the parallel diagonals (fifteen) and observe that both extremes (one square long) cannot be used simultaneously. The maximum of 32 for horses would follow at once if a maximum of 8 could be established for a 4×4 chessboard. But if more than 8 could be packed on this smaller board then at least one of the 2×2 corners would contain 3 or 4 horses. That this is impossible is easily discovered by an examination of the few possible cases. The result for kings follows in precisely the same way.

Much more interesting is the generalization to an $N\times N$ board. The same arguments used for $N=8$ suffice in the general case when rooks or bishops are concerned. With an obvious extension of our notation and $N>1$,

$$R_N = N, \qquad B_N = 2N - 2.$$

The case of queens is more elusive. Since only one queen may occur in each row or column, it is convenient to describe a solution as a permutation of the integers $1, \cdots, N$. For each column from left to right we merely indicate the row occupied. The above solution for $N=8$ thus becomes

$$2, 4, 6, 8, 3, 1, 7, 5.$$

Now clearly $Q_N \leqq N$. That equality may actually be achieved for every $N>3$ is proved by the following very unattractive separation into cases:

N	SOLUTION
$4+6j$	$2, 4, \cdots, n, 1, 3, \cdots, n-1.$
$5+6j$	$2, 4, \cdots, n-1, 1, 3, \cdots, n.$
$6+6j$	$2, 4, \cdots, n, 1, 3, \cdots, n-1.$
$7+6j$	$2, 4, \cdots, n-1, 1, 3, \cdots, n.$
$8+12j$	$2, 4, \cdots, n, 3, 1, 7, 5, \cdots, n-1, n-3.$
$9+12j$	$n, 4, 6, \cdots, n-1, 3, 1, 7, 5, \cdots, n-2, n-4, 2.$
$14+12j$	$n-1, 2, 4, \cdots, n, 3, 1, 7, 5, \cdots, n-3, n-5.$
$15+12j$	$n, n-2, 2, 4, \cdots, n-1, 3, 1, \cdots, n-4, n-6.$

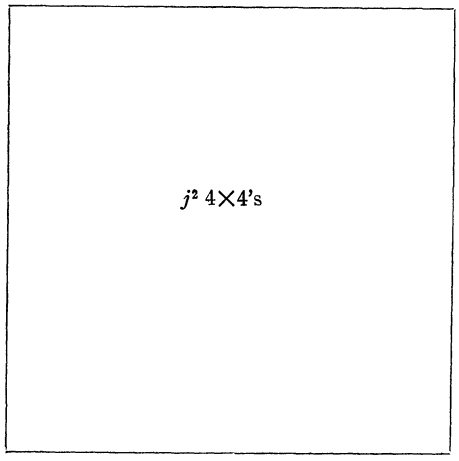
We conclude that $Q_N = N$.

For the horse problem one soon suspects that the diagonal pattern shown earlier is best possible. The implied result would be $H_N = \frac{1}{2}N^2$, N even; $H_N = \frac{1}{2}(N^2+1)$, N odd. To show that these values cannot be exceeded a few preliminary results are helpful. For the small rectangular boards listed below a brief examination of cases suffices to establish the maxima shown.

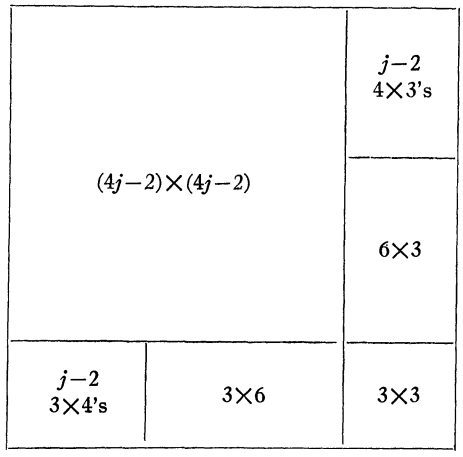
BOARD	2×4	3×4	4×4	3×3	3×6	5×5	6×6
MAXIMUM	4	6	8	5	9	13	18

The labor involved can be reduced to a very comfortable size by partitioning the larger of these boards. For example the 6×6 may be partitioned into four 2×4's around a central 2×2. That only two of these central squares may be occupied follows quickly by trial.

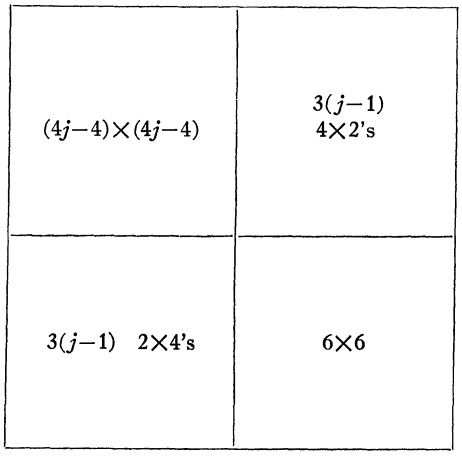
The general $N\times N$ board may now be partitioned as follows:



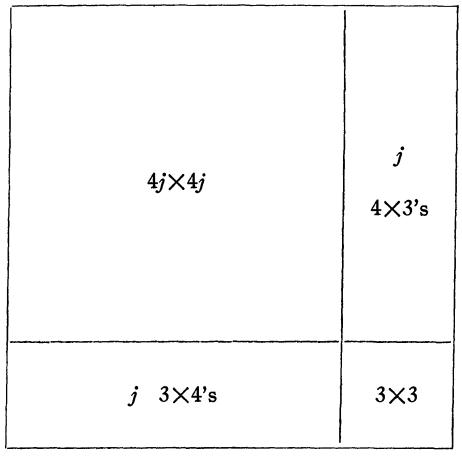
$N=4j$:
$H_N \leqq 8j^2 = \frac{1}{2}N^2$.



$N=4j+1$:
$H_n \leqq \frac{1}{2}(4j-2)^2 + 5 + 2\cdot 9 + 12(j-2) = \frac{1}{2}(N^2+1)$.



$N=4j+2$:
$H_N \leqq 8(j-1)^2+18+24(j-1) = \frac{1}{2}N^2$.



$N=4j+3$:
$H_N \leqq 8j^2+5+12j = \frac{1}{2}(N^2+1)$.

Finally, the kings. Here again one suspects that the pattern shown earlier is "optimal." The implied result is

$$K_N = \tfrac{1}{4}N^2, \ N \text{ even}; \qquad K_N = \tfrac{1}{4}(N+1)^2, \ N \text{ odd}.$$

That these values cannot be exceeded may be shown just as in the case of horses. However, the following argument, which avoids the split into cases, is interesting.

Suppose some optimal pattern is placed on the board. We will show that this

pattern can be changed to the one above by simply moving various pieces one square. First, if square A is unoccupied (Fig. 2), then exactly one of the squares B, C, D, must be occupied. (If none were occupied we could add another king at A, violating the assumption that our original pattern is optimal.) So, by mov-

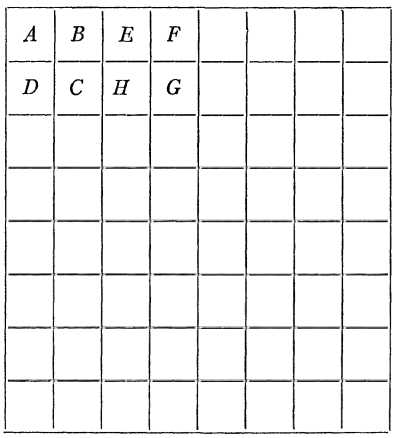


FIG. 2

ing this piece, A will be occupied and B, C, D, free. Of course, if A were occupied in the original pattern, then B, C, D, would be free and no move would be necessary. Now the same argument may be applied to E, F, G, H, leaving E occupied and G, H, F, free, and so on around the board. The result is the transformation of the original pattern, assumed optimal, to our regular pattern, which therefore must be optimal also.

It is interesting to observe that all these maximum problems are special cases of the following:

Determine the maximum of $\sum_{i=1}^{N} x_i$, *where* x_i *is either* 0 *or* 1, *subject to the constraints* $x_i \sum_{j=1}^{N} a_{ij}x_j = 0$, $i = 1, \cdots, N$.

The a_{ij} are the elements of a symmetric matrix A and are also 0 or 1; in particular, $a_{ii} = 0$. A may be termed the "challenge" matrix, each of its rows indicating with 1's the positions challenged from a particular square. For kings on a 3×3 board, the challenge matrix could be as shown in Figure 3.

We may further observe that if an optimal solution is

$$x_i = \begin{cases} 1 & \text{if } i = i_1, \cdots, i_m = \{S\}, \\ 0 & \text{otherwise,} \end{cases}$$

the constraints reduce to $\sum_{j \in \{S\}} a_{ij} = 0$, $i \in \{S\}$ and since $a_{ij} \geqq 0$, it follows that $a_{ij} = 0$ when $i \in \{S\}, j \in \{S\}$. That is, the matrix A has a zero principal minor of order M. One now sees that the problem is equivalent to finding a principal

0	1	0	1	1	0	0	0	0
1	0	1	1	1	1	0	0	0
0	1	0	0	1	1	0	0	0
1	1	0	0	1	0	1	1	0
1	1	1	1	0	1	1	1	1
0	1	1	0	1	0	0	1	1
0	0	0	1	1	0	0	1	0
0	0	0	1	1	1	1	0	1
0	0	0	0	1	1	0	1	0

FIG. 3

minor of A having maximum order and consisting only of zeros. In the example above, this maximum order is, of course, four, and the appropriate elements have been underscored (Fig. 3). In general, no useful scheme for discovering this minor exists.

A CLASS OF GROUPS ASSOCIATED WITH LATIN SQUARES*

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Let (a_{ij}) be the $n\times n$ Latin square whose rows are the successive cyclic permutations of the integers 0, 1, $\cdots$, $n-1$, so that $a_{ij}\equiv i+j \pmod n$. Let

$$(x_i): x_0, x_1, \cdots, x_{n-1},$$

be an arbitrary permutation of the integers 0, 1, $\cdots$, $n-1$. If the rows (columns) are numbered 0, 1, $\cdots$, $n-1$ from top to bottom (left to right), the element x_i will be found in column i and row r_i, $i=0, 1, \cdots, n-1$, where $r_i\equiv x_i-i \pmod n$. The permutation (x_i) will be called a 1-*permutation* if $r_i\neq r_j$ for $i\neq j$, that is, if the integers $r_0, r_1, \cdots, r_{n-1}$, also form a permutation of the integers 0, 1, $\cdots$, $n-1$. We seek, for a given n, the number $N(n)$ of 1-permutations. $N(n)$ is the number of ways n distinct integers can be selected from the Latin square (a_{ij}) so that no two integers are in the same row or in the same column.

If n is even, $N(n)=0$. To see this, let (x_i) be an arbitrary permutation and let r_i be defined as above. Then

$$\sum_{i=0}^{n-1} r_i \equiv \sum_{i=0}^{n-1} (x_i - i) \equiv \sum_{i=0}^{n-1} x_i - \sum_{i=0}^{n-1} i \equiv 0 \pmod n.$$

* Presented to the International Congress of Mathematicians 1958, Edinburgh, Scotland on August 16, 1958.

Also $\sum_{i=0}^{n-1} i = \frac{1}{2}n(n-1)$ and is therefore congruent to 0 (mod n) if and only if n is odd. Hence the r's, which are integers in the range 0, 1, $\cdots$, $n-1$, cannot all be distinct if n is even.

The function $N(n)$ is unknown at present for n odd but the following values have been calculated: $N(1)=1$, $N(3)=3$, $N(5)=15$, $N(7)=133$, $N(9)=2025$, $N(11)=37{,}851$. In an effort to shed some light on the values of $N(n)$ for larger values of n, we investigate a certain class of groups.

We first determine, for each positive integer n, an abstract group $\mathfrak{G}$ of order $6n^2\phi(n)$, where $\phi(n)$ is the familiar Euler ϕ-function, and then relate $\mathfrak{G}$ to the problem at hand. $\mathfrak{G}$ is defined as the product of four of its subgroups, $\mathfrak{G}_{12}$, $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{K}$, as follows. $\mathfrak{G}_{12}$ is the group of order 12 generated by two elements C and R with the generating relations

$$C^2 = R^2 = (CR)^6 = I, \tag{1}$$

where I is the identity of the group.

If we put

$$B = CR, \quad D = CB^2, \quad E = CB^3, \quad F = CB^4, \quad G = CB^5,$$

it is easy to verify that $\mathfrak{G}_{12}$ is indeed of order 12 and has the multiplication table

I	C	R	D	E	F	G	B	B^2	B^3	B^4	B^5
C	I	B	B^2	B^3	B^4	B^5	R	D	E	F	G
R	B^5	I	B	B^2	B^3	B^4	D	E	F	G	C
D	B^4	B^5	I	B	B^2	B^3	E	F	G	C	R
E	B^3	B^4	B^5	I	B	B^2	F	G	C	R	D
F	B^2	B^3	B^4	B^5	I	B	G	C	R	D	E
G	B	B^2	B^3	B^4	B^5	I	C	R	D	E	F
B	G	C	R	D	E	F	B^2	B^3	B^4	B^5	I
B^2	F	G	C	R	D	E	B^3	B^4	B^5	I	B
B^3	E	F	G	C	R	D	B^4	B^5	I	B	B^2
B^4	D	E	F	G	C	R	B^5	I	B	B^2	B^3
B^5	R	D	E	F	G	C	I	B	B^2	B^3	B^4

We note that the group $\mathfrak{G}_{12}$ is independent of the value of n.

Let $m_1, m_2, \cdots, m_{\phi(n)}$ be the positive integers not greater than and prime to n. They form a group $\mathfrak{M}^*$ under multiplication, modulo n, of order $\phi(n)$. We define a group $\mathfrak{M}$, with elements

$$M_{m_1}, M_{m_2}, \cdots, M_{m_{\phi(n)}},$$

to be an abstract group isomorphic to $\mathfrak{M}^*$.

Finally, $\mathfrak{A}$ and $\mathfrak{K}$ are defined to be cyclic groups, each of order n, with generators A and K, respectively. M_{n-1} is an element of $\mathfrak{M}$ if $n>1$ since then $n-1$ is a positive integer prime to n. Let m be an arbitrary element of $\mathfrak{M}^*$ and m', its inverse (so that $mm' \equiv 1 \pmod{n}$); then $M_m M_{m'} = I$ in $\mathfrak{M}$. We now stipulate, in addition to (1), the following generating relations.

$$(CR)^3 = M_{n-1}; \tag{2}$$

$$M_m C = CM_m, \qquad M_m R = RM_m; \tag{3}$$

$$AK = KA; \tag{4}$$

$$AC = CA^{n-1}, \qquad AR = RK^{n-1}, \qquad AM_m = M_m A^{m'}, \tag{5}$$

$$KC = CA^{n-1}K, \qquad KR = RA^{n-1}, \qquad KM_m = M_m K^{m'}. \tag{6}$$

If W, X, Y, Z, are arbitrary elements of $\mathfrak{S}_{12}$, $\mathfrak{M}$, $\mathfrak{A}$, $\mathfrak{K}$, respectively, it follows, by use of the relations (3), (4), (5), (6), that the product of two elements of the form $WXYZ$ is again an element of this form. There are $12n^2\phi(n)$ products of this form, but because of the relation (2), these fall into half as many distinct pairs of equal elements. We thus have

THEOREM 1. *The elements C, R, M_m, A, K, defined as above, and subject to the generating relations* (1) *through* (6), *generate a group $\mathfrak{S}$ of order $6n^2\phi(n)$.*

To relate the group $\mathfrak{S}$ to 1-permutations and the evaluation of $N(n)$, we will find it convenient to introduce a new symbol for a permutation (x_i) of the integers $0, 1, \cdots, n-1$. Let (y_i) also be a permutation of the integers $0, 1, \cdots, n-1$; we write the symbol $(x_i | y_i)$ to designate the permutation which has the integer x_i in the y_ith place (place numbers are read from left to right), $i=0, 1, \cdots, n-1$. Thus, the permutation (x_i) can now be written as $(x_i | i)$; the permutation obtained by reversing the order of the elements of (x_i) can be written as $(x_i | n-1-i)$ or $(x_{n-1-i} | i)$. If $(x_i | y_i)$ is a permutation, the permutation obtained by moving each element two spaces to the left (the 0th element will go into the $(n-2)$nd place, the first element into the $(n-1)$st place) is $(x_i | y_i - 2)$. It is understood, of course, that any integer which is an element of a permutation or the place number of an element or a subscript on either can be replaced by any other integer congruent to it (mod n).

THEOREM 2. *If (x_i) and (y_i) are permutations of the integers* $0, 1, \cdots, n-1$, *the permutation $(x_i | y_i)$ is a 1-permutation if and only if $(x_i - y_i)$, $i=0, 1, \cdots, n-1$, is a permutation of the integers* $0, 1, \cdots, n-1$.

The proof of the theorem follows immediately from the observation that, for a given i, the element x_i of the permutation $(x_i|y_i)$ is found in row x_i-y_i and column y_i of the Latin square (a_{ij}).

As an immediate consequence of this theorem, we have

THEOREM 3. *If the permutation $(x_i|y_i)$ is a 1-permutation, so are the permutations $(y_i-x_i|y_i)$, $(y_i|x_i)$, $(mx_i|my_i)$, $(x_i+a|y_i)$, $(x_i|y_i+a)$, where m is an integer relatively prime to n and a is any integer.*

The 1-permutations $(y_i-x_i|y_i)$ and $(y_i|x_i)$ are called the *complement* and the *reverse*, respectively, of the 1-permutation $(x_i|y_i)$; the 1-permutations $(mx_i|my_i)$, $(x_i+a|y_i)$, $(x_i|y_i+a)$ are said to be *multiplicatively*, *additively*, and *cyclically equivalent*, respectively, to the 1-permutation $(x_i|y_i)$.

We now consider a number of operators that transform 1-permutations into 1-permutations. In particular, we define operators $\overline{I}$, $\overline{C}$, $\overline{R}$, $\overline{M}_m$, $\overline{A}$, $\overline{K}$, where n is an odd positive integer and m is an integer not greater than and prime to n, by the statements

$$(7)\quad \begin{aligned} &\overline{I}(x_i|y_i) = (x_i|y_i), \quad \overline{C}(x_i|y_i) = (y_i - x_i|y_i), \quad \overline{R}(x_i|y_i) = (y_i|x_i), \\ &\overline{M}_m(x_i|y_i) = (mx_i|my_i), \quad \overline{A}(x_i|y_i) = (x_i + 1|y_i), \quad \overline{K}(x_i|y_i) = (x_i|y_i - 1). \end{aligned}$$

Products of the operators are defined as usual; for example,

$$\begin{aligned} \overline{C}^2(x_i|y_i) &= \overline{C}[\overline{C}(x_i|y_i)] = \overline{C}(y_i - x_i|y_i) = (x_i|y_i) = \overline{I}(x_i|y_i), \\ \overline{C}\overline{R}(x_i|y_i) &= \overline{C}[\overline{R}(x_i|y_i)] = \overline{C}(y_i|x_i) = (x_i - y_i|x_i). \end{aligned}$$

It is readily seen, by direct computation and straightforward use of the definitions, that the operators defined above satisfy relations identical to the relations (1) through (6). Hence, corresponding to any element $WXYZ$, defined as above, of the group $\mathfrak{G}$ there is an operator $\overline{W}\overline{X}\overline{Y}\overline{Z}$ that transforms a 1-permutation into a 1-permutation. Furthermore, if $W_1X_1Y_1Z_1 \cdot W_2X_2Y_2Z_2 = W_3X_3Y_3Z_3$, then $\overline{W}_1\overline{X}_1\overline{Y}_1\overline{Z}_1 \cdot \overline{W}_2\overline{X}_2\overline{Y}_2\overline{Z}_2 = \overline{W}_3\overline{X}_3\overline{Y}_3\overline{Z}_3$.

We observe, however, that if $T_1 = W_1X_1Y_1Z_1$ and $T_2 = W_2X_2Y_2Z_2$ are distinct elements of $\mathfrak{G}$, the corresponding operators $\overline{T}_1 = \overline{W}_1\overline{X}_1\overline{Y}_1\overline{Z}_1$ and $\overline{T}_2 = \overline{W}_2\overline{X}_2\overline{Y}_2\overline{Z}_2$ may be identical for a particular 1-permutation; that is, the two operators, although they correspond to distinct elements of $\mathfrak{G}$, may both transform a given 1-permutation into the same 1-permutation. It is to be emphasized that whether the operators $\overline{T}_1$ and $\overline{T}_2$ are or are not identical depends on the 1-permutation upon which they act. Thus, 0, 2, 1 and 1, 0, 2 are two of the three 1-permutations for $n=3$; we have

$$\overline{C}(0, 2, 1) = \overline{I}(0, 2, 1) = (0, 2, 1),$$

where $\overline{C}(0, 2, 1)$ is an abbreviation for $\overline{C}(0, 2, 1|0, 1, 2)$, etc. On the other hand,

$$\overline{C}(1, 0, 2) = (2, 1, 0) \neq \overline{I}(1, 0, 2) = (1, 0, 2).$$

Let p be a specific 1-permutation for some fixed n and let $\mathfrak{K}(p)$ be the totality of elements of $\mathfrak{G}$ whose corresponding operators leave p unaltered. Then $\mathfrak{K}(p)$ is a subgroup of $\mathfrak{G}$ of order $h(p)$, say. The subgroup $\mathfrak{K}(p)$ determines a partitioning of $\mathfrak{G}$ into $t=6n^2\phi(n)/h(p)$ cosets, each of the form $g\mathfrak{K}(p)$. The $h(p)$ operators corresponding to the $h(p)$ elements of $\mathfrak{G}$ in one coset will all transform p into the same 1-permutation q and this set of $h(p)$ operators will contain all the operators that transform p into q. It follows that the operators will transform p into t distinct 1-permutations, say, $p=p_1, \cdots, p_t$. Let $\overline{Q}_1, \cdots, \overline{Q}_t$ be a set of operators such that $\overline{Q}_i$ transforms $p=p_1$ into p_i, $i=1, \cdots, t$. The $\overline{Q}$'s form a group $\overline{\mathfrak{Q}}(p)$ of order t; it follows that $\mathfrak{K}(p)$ is an invariant subgroup of $\mathfrak{G}$ and $\overline{\mathfrak{Q}}(p)$ is (simply) isomorphic to the factor group $\mathfrak{G}/\mathfrak{K}(p)$.

The 1-permutations $p_1, \cdots, p_t$ form a transitive system; the various operators merely permute these p's among themselves and no operator, corresponding to an element of $\mathfrak{G}$, will transform a 1-permutation not in this set into a 1-permutation of this set.

We can continue in the same manner and separate the $N(n)$ 1-permutations into a certain number of disjoint transitive systems such that the operators corresponding to the elements of $\mathfrak{G}$ merely permute the 1-permutations of a transitive system among themselves.

These ideas can be pushed a bit further. Let p and q be distinct 1-permutations (for the same n). As above, they determine invariant subgroups $\mathfrak{K}(p)$ and $\mathfrak{K}(q)$ of $\mathfrak{G}$. If p and q belong to the same transitive system, $\mathfrak{K}(p)$ and $\mathfrak{K}(q)$ will be of the same order; if p and q belong to different transitive systems, $\mathfrak{K}(p)$ and $\mathfrak{K}(q)$ may be of the same or of different orders. In either case, $\mathfrak{K}(p, q) = \mathfrak{K}(p) \cap \mathfrak{K}(q)$ will be an invariant subgroup of $\mathfrak{G}$ such that the operators corresponding to its elements will leave both p and q individually unaltered. More generally, if $p, q, r, \cdots, u$ is any set of 1-permutations, then

$$\mathfrak{K}(p, q, \cdots, u) = \mathfrak{K}(p) \cap \mathfrak{K}(q) \cap \cdots \cap \mathfrak{K}(u)$$

will be an invariant subgroup of $\mathfrak{G}$ whose associated operators will leave $p, q, \cdots, u$ individually unaltered. In particular, if $p, q, \cdots, v$ is the set of all $N(n)$ 1-permutations, then $\mathfrak{K}=\mathfrak{K}(p, q, \cdots, v)=\mathfrak{K}(p)\cap\mathfrak{K}(q)\cap \cdots \cap\mathfrak{K}(v)$ will be an invariant subgroup of $\mathfrak{G}$ whose associated operators will leave each of the $N(n)$ 1-permutations individually unaltered.

If $n=3$, $N(3)=3$; let p, q, r be the three distinct 1-permutations. The group $\mathfrak{G}$ is of order 108; the subgroups $\mathfrak{K}(p), \mathfrak{K}(q), \mathfrak{K}(r)$ are distinct but each is of order 36 so that p, q and r form a single transitive system; the subgroup $\mathfrak{K}$ is of order 18. This means that in addition to the identity operator, there are 5 operators that actually alter at least one of the three 1-permutations.

If $n=5$, $N(5)=15$, and $\mathfrak{G}$ is of order 600. The 15 1-permutations form a single transitive system; if p is any one of the 15 1-permutations, $\mathfrak{K}(p)$ is a subgroup of order 40. The subgroup $\mathfrak{K}$ reduces to the identity of $\mathfrak{G}$.

It appears that for $n>3$, $\mathfrak{K}$ always reduces to the identity of $\mathfrak{G}$.

As a final example, we take $n=11$, then $\mathfrak{G}$ is of order 7260 and $N(11)=37{,}851$. These 1-permutations are divided into 11 disjoint transitive sets. We list a representative 1-permutation from each set and give the total number of distinct transforms of the representative 1-permutation.

1-permutations											Number of distinct transforms
0	2	4	6	8	10	1	3	5	7	9	33
0	3	6	9	1	4	7	10	2	5	8	66
0	2	4	8	3	9	1	10	5	7	6	726
0	2	5	8	3	1	10	9	6	4	7	726
0	2	4	6	9	1	10	5	3	8	7	3630
0	2	4	7	9	1	5	10	3	6	8	3630
0	2	5	8	10	9	3	6	4	7	1	3630
0	2	5	9	6	10	4	3	1	8	7	3630
0	2	4	7	1	10	9	6	3	5	8	7260
0	2	4	7	1	10	9	5	3	8	6	7260
0	2	4	7	1	10	9	3	6	8	5	7260
											37851

The group $\mathfrak{G}$ provides some additional information about the 1-permutations and $N(n)$. Each operator corresponding to an element of the subgroup $\mathfrak{G}_{12}\mathfrak{M}$ of $\mathfrak{G}$ transforms a 1-permutation beginning with the integer 0 into a 1-permutation also beginning with the integer 0 (although an operator with this property is not necessarily an operator corresponding to an element of $\mathfrak{G}_{12}\mathfrak{M}$).

The operator $\overline{C}$ transforms a 1-permutation beginning with the integers 0, x into a 1-permutation beginning with the integers 0, $1-x$. Hence the number of 1-permutations beginning with 0, x is equal to the number beginning with 0, $1-x$.

Similarly, if $n=2k-1$, then $(k, n)=1$ and $2k\equiv 1 \pmod{n}$. The operator $\overline{M}_k\overline{R}$ transforms a 1-permutation beginning with 0, 2 into a 1-permutation beginning with 0, k. Hence the numbers of 1-permutations beginning with 0, 2; 0, k; 0, $n-1$ are equal.

The operator $\overline{A}$ transforms a 1-permutation beginning with the integer x into a 1-permutation beginning with the integer $x+1$; hence the number of 1-permutations beginning with one integer is the same as the number of 1-permutations beginning with any other integer (in the range 0, 1, $\cdots$, $n-1$) It follows that $N(n)\equiv 0 \pmod{n}$.

FOUR INTERSECTING SPHERES

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Summary. Four intersecting spheres (A), (B), (C), (D), taken three at a time determine four pairs of points (e).

To study the configuration formed by those points we consider: (a) The sphere (M) orthogonal to the given spheres; (b) the tetrahedron $(T)=ABCD$ formed by the centers of those spheres; (c) the polar reciprocal tetrahedron $(T')=A'B'C'D'$ of (T) with respect to (M); (d) the "adjoint spheres" (A'), (B'), (C'), (D'), with centers at the vertices of (T') and orthogonal to (M).

The eight points (e) give rise to $2^4=16$ "tetrahedrons of intersection" of the given spheres. They may be grouped into eight "pairs of complementary tetrahedrons of intersection" (abbreviated to c.t.i.), each pair involving all the eight points.

The tetrahedrons of each pair are perspective, the center of perspectivity being the center M of the sphere (M). Their eight planes of perspectivity coincide with the eight planes of similitude of the four adjoint spheres.

The radical plane of the circumspheres of a pair of c.t.i. coincides with the plane of perspectivity of the two tetrahedrons considered. The two spheres are inverse with respect to (M), are cut isogonally by each of the given spheres, etc.

1. Preliminaries. a. Given four spheres (A), (B), (C), (D), whose centers form a tetrahedron $(T)=ABCD$, the "central tetrahedron" of the given spheres, let

$$(e) \qquad P,\ P';\ Q,\ Q';\ R,\ R';\ S,\ S'$$

be, respectively, the pairs of points of intersection of the triads of spheres

$$(g) \qquad (B),\ (C),\ (D);\ (C),\ (D),\ (A);\ (D),\ (A),\ (B);\ (A),\ (B),\ (C).$$

The points of intersection (e) located on each sphere are listed in

TABLE 1

(A)			$Q,\ Q';$		$R,\ R';$		$S,\ S'$
(B)	$P,\ P';$				$R,\ R';$		$S,\ S'$
(C)	$P,\ P';$		$Q,\ Q';$				$S,\ S'$
(D)	$P,\ P';$		$Q,\ Q';$		$R,\ R'$		

b.i. Two points of the same pair of (e) will be referred to as "complementary points of intersection" of the four given spheres.

ii. A line joining two noncomplementary points of intersection will be said to be a "line of intersection" of the given spheres. Their two complementary points determine the "complementary line of intersection." Two pairs of complementary points of intersection, say, $P,\ P';\ Q,\ Q'$ determine two pairs of

complementary lines of intersection: PQ, $P'Q'$; PQ', $P'Q$.

iii. Three points of intersection, no two of them complementary, determine a "plane of intersection." The three complementary points determine the "complementary plane of intersection" of the given spheres.

iv. Four points of intersection, no two of them complementary, form a "tetrahedron of intersection" of the four given spheres. The four remaining points of intersection form the "complementary tetrahedron of intersection"; this locution is abbreviated to c.t.i. in what follows, to save space.

2. Perspective tetrahedrons. a. The lines PP', QQ', RR', SS', are the radical axes of the triads of spheres (g), respectively (Sec. 1a), hence these four lines meet in the orthogonal center M of the four given spheres, which point is also the center of the orthogonal sphere (M) of those spheres.

If we take one point, and only one, in each of the four pairs of points (e) (Sec. 1a) we obtain $2^4=16$ tetrahedrons of intersection which may be grouped into eight pairs of c.t.i. (Sec. 1b iv). Two such tetrahedrons involve all the eight points (e).

b. THEOREM. *A pair of c.t.i. are perspective.*

Indeed, pairs of corresponding vertices of two such tetrahedrons are alined on the orthogonal center M of the given spheres (Sec. 2a). The eight pairs of c.t.i. have thus the same center of perspectivity. Their eight planes of perspectivity will be discussed later.

3. The radical tetrahedron. a. The polar planes $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ of the orthocenter M (Sec. 2a) for the spheres (A), (B), (C), (D), respectively, form a tetrahedron $(T')=A'B'C'D'$ which, for want of a better term, may be referred to as the "radical tetrahedron" of the four given spheres. The following properties of the radical tetrahedron (T') are mentioned here, for future reference.

b. The sphere (M) being orthogonal to the four spheres (A), (B), (C), (D), the faces of (T') are, respectively, the radical planes of the four pairs of spheres (M), (A); (M), (B); (M), (C); (M), (D).

c. The radical plane $B'C'D'$ of the two orthogonal spheres (M), (A) (Sec. 3b) is also the polar plane of the center A of (A) with respect to (M). Similarly for the other faces of (T'). Thus: The central tetrahedron $(T)=ABCD$ (Sec. 1a) and the radical tetrahedron (T') of the four given spheres are polar reciprocal with respect to the orthogonal sphere (M) of the given spheres.

d. The vertex A' of (T') is the pole of the face BCD of (T) for (M) (Sec. 3c), hence the perpendicular from M upon the plane BCD passes through A'. Now that perpendicular is the radical axis PP' of the three spheres (B), (C), (D) (Sec. 2a), hence: The vertices of the radical tetrahedron of four given spheres lie

on the four radical axes of those spheres taken three at a time.

We have thus the tetrads of collinear points:

(j) $\qquad MMP'A',\ MQQ'B',\ MRR'C',\ MSS'D'.$

4. Harmonic relations. a. Consider any two of the four concurrent lines (j) (Sec. 3d), say, $MQQ'B'$, $MRR'C'$. Their plane cuts the sphere (D) along a small circle (d) (Fig. 1) passing through the points Q, Q', R, R', since these four points lie both in the plane considered and on the sphere (D) (Table 1).

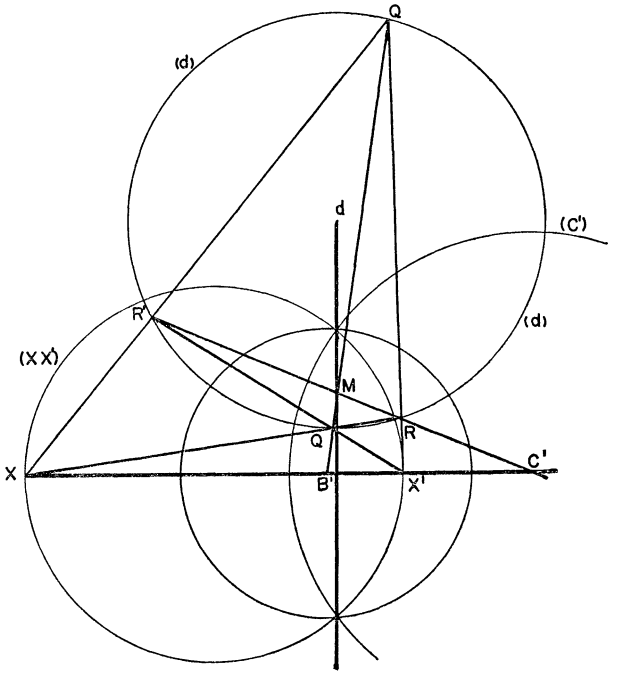


FIG. 1

The diagonal triangle of the complete quadrangle $QQ'RR'$ inscribed in (d) has for its vertices the points M, $X=(QR,\ Q'R')$, $X'=(QR',\ Q'R)$; the diagonal triangle MXX' being polar to (d), the line XX' is the polar of M for that circle.

On the other hand, the polar plane $A'B'C'$ of M for the sphere (D) (Sec. 3a) cuts the plane $(MQQ'B',\ MRR'C')$ of the circle (d) along the line $B'C'$, and this line is the polar of M for (d) ([1], p. 141, art. 426), hence the line $B'C'$ coincides with XX', and we have ((j), Sec. 3d): $B'=(MQQ',\ XX')$, $C'=(MRR'C',\ XX')$.

Thus the points B', C' are harmonically separated from the point M by two pairs of points Q, Q'; R, R', respectively.

Considering the pair of lines $MPP'A'$, $MSS'D'$ of (j) we obtain similar results for the points A', D'. We have thus the four groups of *harmonic* pairs of points:

(*h*) MA', PP'; MB', QQ'; MC', RR'; MD', SS',

located on four concurrent lines.

b. The points X, X' (Sec. 4a) are determined by the two pairs of complementary lines of intersection QR, $Q'R'$; QR', $Q'R$ derived from the two pairs of complementary points of intersection Q, Q'; R, R'. Moreover, X, X' lie on the edge $B'C'$ of (T') joining the vertices B', C' of (T') (Sec. 4a) situated on the lines MQQ', MRR'.

Similarly for any other pair of the four lines (*j*) (Sec. 3d). Hence: i. *A pair of complementary lines of intersection of the four given spheres meet on an edge of the radical tetrahedron of the four spheres*; ii. *The other pair of complementary lines of intersection determined by the same two pairs of complementary points of intersection* (Sec. 16ii) *meet on the same edge of the radical tetrahedron as the first pair of lines*. iii. *The two points of intersection thus obtained on the edge of the radical tetrahedron are harmonically separated by the two vertices of that tetrahedron located on that edge.*

Indeed, in the complete quadrangle $QQ'RR'$ (Sec. 4a) the two vertices X, X' of the diagonal triangle MXX' are harmonically separated by the sides $MQQ'B'$, $MRR'C'$.

5. The adjoint set of spheres. a. The four spheres (A'), (B'), (C'), (D'), having for centers the vertices of the radical tetrahedron $(T')=A'B'C'D'$ (Sec. 3) and orthogonal to the sphere (M) (Sec. 2a) are said to be the "adjoint spheres" of the given spheres (A), (B), (C), (D).

b. The point A' is common to the radical planes of (M) with the three spheres (B), (C), (D), respectively (Secs. 3b, 3d), hence the sphere (A') which is orthogonal to the sphere (M), by construction, is also orthogonal to each of the spheres (B), (C), (D). Similarly for the other vertices of (T'). Thus: *The adjoint spheres* (A'), (B'), (C'), (D') *are respectively orthogonal to the corresponding triads of spheres* (*g*) (Sec. 1a).

c. The center A' of the sphere (A') orthogonal to the spheres (B), (C), (D) lies on their radical axis which is the line PP' (Sec. 1a). We have thus another proof of a property noted already (Sec. 3d).

6. Centers of similitude. a. The two spheres (B'), (C') are orthogonal to the sphere (D) (Sec. 5b), and their centers lie in the plane (MQQ', MRR') of (d) (Sec. 5c), hence this plane cuts the two spheres (B'), (C') along two great circles (b'), (c') orthogonal to the circle (d) ([1], p. 176, ex. 19a). (Fig. 1).

The two diagonal points X, X' (Sec. 4a) are conjugate with respect to the circle (d) (Sec. 4a), hence the circle (XX') having XX' for diameter is orthogonal to (d) ([2], p. 181, art. 387).

Furthermore, the points X, X' are harmonically separated by the centers B', C' of the circles (b'), (c') (Sec. 4biii), therefore (XX') is the circle of simili-

tude of the circles (b'), (c') ([2], p. 214, art. 481; [3]). Hence the points X, X' are the two centers of similitude of the two great circles (b'), (c') [5], and therefore also of the two spheres (B'), (C'). Consequently: *Given four intersecting spheres, the two pairs of complementary lines of intersection determined by two pairs of complementary points of intersection* (Sec. 1aii) *meet in two points on an edge of the radical tetrahedron of the given spheres. The two points thus obtained are the two centers of similitude of the two adjoint spheres having their centers on the edge considered.*

7. Axes of similitude. a. Consider any pair of complementary planes of intersection (Sec. 1biii), say, PQR, $P'Q'R'$.

The pair of complementary lines of intersection QR, $Q'R'$ intersect in a point X on the edge $B'C'$ of (T') (Sec. 6), and X is a center of similitude of the spheres (B'), (C'). Similarly the points $Y=(RP, R'P')$, $Z=(PQ, P'Q')$ lie, respectively, on the edges $C'A'$, $A'B'$, of (T'), and Y, Z are centers of similitude of the pairs of spheres (C'), (A'); (A'), (B'), respectively.

On the other hand, the point $X=(QR, Q'R')$ lies both in the plane PQR and in the plane $P'Q'R'$, for the lines QR, $Q'R'$ lie in the planes PQR, $P'Q'R'$, respectively; similarly for the points Y and Z. Hence the three points X, Y, Z lie on the straight line common to the three planes PQR, $P'Q'R'$, $A'B'C'$, and the line XYZ is an axis of similitude of the three spheres (A'), (B'), (C'). Consequently: *A pair of complementary planes of intersection of four intersecting spheres meet in a face of the radical tetrahedron of the given spheres, and this common line is an axis of similitude of three adjoint spheres.*

b. Consider the plane $XYZ-M$ determined by the axis XYZ (Sec. 7a) and the orthogonal center M. The harmonic conjugate of this plane with respect to the pair of complementary planes of intersection PQR, $P'Q'R'$ passing through XYZ meets the lines MPP', MQQ', MRR', in the harmonic conjugates of M with respect to the pairs of points P, P'; Q, Q'; R, R', that is, in the points A', B', C', respectively ((*h*), Sec. 4a). Thus the axis of similitude common to the planes PQR, $P'Q'R'$ lies in the face $A'B'C'$ of the tetrahedron (T').

Similarly for any other pair of complementary planes of intersection. Thus: *Two complementary planes of intersection are coaxal with that face of the radical tetrahedron* (T') *which contains the vertices of* (T') *lying on the lines joining the three pairs of complementary points of intersection* ((*e*), Sec. 1) *which determine the pair of planes considered.*

8. Planes of similitude. a. Consider now any two c.t.i., (Sec. 1 biv) say, $PQRS$ and $P'Q'R'S'$. Their faces are four pairs of complementary planes of intersection, and therefore meet in four lines which are four axes of similitude of the adjoint spheres (Sec. 7a).

On the other hand, the two tetrahedrons considered are perspective (Sec. 2a), hence the four axes of similitude lie in the plane of perspectivity of the two tetrahedrons. Thus ([1], pp. 157 ff.): *The plane of perspectivity of two c.t.i. of*

four intersecting spheres is a plane of similitude of the adjoint spheres of the four given spheres.

b. The four axes of similitude determined by the four pairs of corresponding faces of two c.t.i., considered lie in four different faces of the radical tetrahecron (T'). The face of (T') containing the line of intersection of a given pair of corresponding faces of any two c.t.i. is readily determined. For instance, if the two c.t.i. $PQ'RS'$, $P'QR'S$ be considered, their pair of corresponding faces PRS', $P'R'S$ intersect in the face $A'C'D'$ of (T'), by Section 7b.

The preceding proposition (Sec. 8a) may be stated in other words: i. *The eight planes of perspectivity of the eight pairs of c.t.i. of four given intersecting spheres coincide with the eight planes of similitude of the four adjoint spheres of the given spheres.* ii. *The eight planes of perspectivity may be grouped into two tetrahedrons which are mutually harmonic and which form a desmic system with the radical tetrahedron of the four given spheres* ([1], p. 240, art. 736).

d. It may also be noticed that: The radical tetrahedron (T') is perspective to each tetrahedron of intersection, the center of perspectivity being the orthogonal center of the given spheres ((*j*), Sec. 3d and (*h*), Sec. 4a), the planes of perspectivity coinciding with the planes of similitude of the adjoint spheres (Secs. 8a, 8c).

9. Spheres of intersection. a. The circumsphere of a tetrahedron of intersection is, by definition, a "sphere of intersection" of the four given spheres. The circumspheres of two c.t.i are two "complementary spheres of intersection."

b. Consider two complementary spheres of intersection, say, $(O)=PQRS$, $(O')=P'Q'R'S'$. The two points P, P' are collinear with the center M of the sphere (M) and lie on the sphere (B) orthogonal to (M), hence P, P' are inverse for (M). Similarly for the other pairs of points (*e*) (Sec. 1a). Hence the spheres (O), (O') are inverse with respect to (M) as sphere of inversion. Consequently ([1], pp. 218, 219; arts. 672, 673): *Two complementary spheres of intersection of four given spheres are coaxal with the orthogonal sphere of the latter, and have that sphere as one of their spheres of antisimilitude.*

c. The two points P, P' of the spheres (O), (O') being inverse for the sphere (M), are antihomologous on the spheres (O), (O'). Similarly for the other pairs of points (*e*). Hence any two corresponding faces of the two perspective tetrahedrons $PQRS$, $P'Q'R'S'$ (Sec. 2), say, PQR, $P'Q'R'$ are antihomologous planes for the spheres (O), (O') and therefore intersect in the radical plane of the two spheres. Thus the radical plane of the spheres (O), (O') coincides with the plane of perspectivity of the tetrahedrons $PQRS$, $P'Q'R'S'$.

The same argument holds for any other pair of c.t.i. and their respective circumspheres. Hence: *The eight radical planes of the eight pairs of complementary spheres of intersection of four given intersecting spheres coincide with the eight planes of similitude of the group of adjoint spheres of the given spheres* (Sec. 8c).

10. Isogonal spheres. a. The spheres (O), (O') being coaxal with the sphere (M) (Sec. 9b), their centers O, O' are collinear with M, and the line of centers MOO' is perpendicular to the radical plane of (O), (O'). Similarly for any other pair of complementary spheres of intersection of the four given spheres. Thus (Sec. 9c): *Given four intersecting spheres, the perpendicular dropped from their orthogonal center upon the plane of perspectivity of a pair of their c.t.i. passes through the circumcenters of those two tetrahedrons.*

b. The given sphere (A) cuts the two spheres $(O)=PQRS$, $(O')=P'Q'R'S'$ along the circles QRS, $Q'R'S'$, respectively (see Sec. 1, Table 1, line 1), and (A) is orthogonal to the sphere of antisimilitude (M) of (O), (O') (Sec. 9b); hence (A) cuts the two spheres (O), (O') isogonally ([1], p. 223, art. 685). The same sphere (A) cuts isogonally any other pair of complementary spheres of intersection of the four given spheres, for analogous reasons.

The same argument applied to the sphere (A) is valid for any other of the four given spheres (Sec. 1). Consequently: *Given four intersecting spheres, each of them cuts isogonally any complementary pair of their spheres of intersection.*

11. Spheres of antisimilitude. a. The two adjoint spheres (A'), (B') are coaxal with their two spheres of antisimilitude ([1], p. 185), and the sphere (M) being orthogonal to the former two spheres (Sec. 5a), is also orthogonal to the latter two ([1], p. 179, art. 566). Thus the orthogonal sphere (M) of the four spheres (A'), (B'), (C'), (D'), (Sec. 5a) is orthogonal to their six pairs of spheres of antisimilitude.

A plane of similitude, say, σ of the four adjoint spheres contains six centers of similitude, and the six corresponding spheres of antisimilitude form a coaxal net (N) ([1], p. 196, art. 616), the conjugate pencil of which includes the sphere (M) and has for its axis the perpendicular k from M upon σ. Now the plane σ is the radical plane of a pair of complementary spheres of intersection, say, (I), (I') (Sec. 9c), and if (U) is a sphere of antisimilitude having its center in σ, this sphere (U) is orthogonal to (M) and therefore also to the spheres (I), (I') coaxal with (M) (Sec. 9b). Consequently: *Given four intersecting spheres, any pair of complementary spheres of intersection is orthogonal to six spheres of antisimilitude of the adjoint group of spheres. The six centers of those six spheres lie in the plane of similitude which coincides with the radical plane of the two spheres of intersection.*

The reader may consider, in this connection, the spheres of similitude of the adjoint spheres considered.

b. Each pair of complementary spheres of intersection being orthogonal to six spheres of antisimilitude (Sec. 10a), the eight pairs of such spheres are orthogonal to $8\times 6=48$ spheres of antisimilitude. Now the four adjoint spheres have only twelve spheres of antisimilitude, hence each of them is orthogonal to four pairs of complementary spheres of intersection. Thus: *A sphere of antisimiltitude of the adjoint spheres of four given intersecting spheres is orthogonal to*

eight spheres of intersection of the given spheres.

12. A reciprocity. We considered two sets of spheres (A), (B), (C), (D); (A'), (B'), (C'), (D'), having a common orthogonal sphere (M), the tetrahedrons $(T)=ABCD$, $(T')=A'B'C'D'$ formed by their centers being polar reciprocal with respect to (M).

We started with the first set of spheres (A), $\cdots$, the sphere (M), and the tetrahedron (T); from this set of elements we derived the spheres (A'), $\cdots$, and (T'). The two tetrahedrons (T), (T') being polar reciprocal for (M), if we started with the second set of spheres (A'), $\cdots$, and the sphere (M), we would, by the same construction, obtain the first set (A), $\cdots$; the two tetrahedrons would interchange roles: (T') would be the central tetrahedron and (T) the radical tetrahedron.

13. Special cases. In the preceding considerations it has been tacitly assumed that no two of the four given spheres (A), $\cdots$ are orthogonal. If those four spheres are all mutually orthogonal, the central tetrahedron (T) is orthocentric and coincides with its polar reciprocal tetrahedron (T') for the sphere (M). This case has been considered elsewhere [4].

A number of intermediate cases may be considered, like (the symbol $\perp$ stands for "orthogonal"): i. $(A)\perp(B)$, $(C)\perp(D)$; ii. $(A)\perp(B)$, $(A)\perp(C)$, $(A)\perp(D)$; iii. $(A)\perp(C)$, $(A)\perp(D)$, $(B)\perp(C)$, $(B)\perp(D)$.

For lack of space, if for no other reason, these considerations are left for the entertainment of the interested reader.

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MATHEMATICAL NOTES

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VOLUMES AND AREAS OF CROSS-SECTIONS

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Consider two bodies B_1, B_2 in ordinary space containing the point z and with positive volumes $V(B_i)$. For any unit vector w let $A(B_i, w)$ be the area of the intersection of B_i with the plane normal to w and through z.

Investigations on area lead to the problem, see [2]:

If B_1, B_2 are convex and have z as center, does $A(B_1, w) > A(B_2, w)$ for all w imply $V(B_1) > V(B_2)$?

This very simple sounding problem is still open. One would expect an affirmative answer, even if convexity is replaced by starshapedness with respect to z, or when the condition that z be the center of the B_i is dropped. Here we give examples which show that both assumptions are essential:

(*) *There are bodies of revolution B_1, B_2 starshaped with respect to their common center z with the property*

$$A(B_1, w) > A(B_2, w) \quad \text{for all } w, \quad \text{but } V(B_1) < V(B_2). \tag{1}$$

There are convex bodies of revolution B_i containing z and satisfying (1).

For the construction introduce geographic coordinates u, θ, $-\pi/2 \leqq u \leqq \pi/2$, $0 \leqq \theta < 2\pi$ on the unit sphere S^2. Let $f(u, \theta)$ be a rotation invariant, even, continuous, function on S^2, *i.e.*,

$$f(u, \theta) = f(u), \qquad f(u) = f(-u). \tag{2}$$

Putting $g(s, v) = f[\text{arc sin } (\sin v \cdot \sin s)]$, the value of the integral of f with respect to the arc length over a great circle forming the angle v, $0 \leqq v \leqq \pi$, with the equator $u = 0$ is, because of (2),

$$I_{\pi - v} = I_v = \int_0^{2\pi} g(s, v)ds = 2\int_0^{\pi} g(s, v)ds$$

$$= 2\int_0^{\pi/2} g(s, v)ds + 2\int_{\pi/2}^{\pi} g(\pi - s, v)ds = 4\int_0^{\pi/2} g(s, v)ds, \qquad 0 \leqq v \leqq \pi/2.$$

In this v-interval $\sin v \cdot \sin s$ increases for each s, $0 < s \leqq \pi/2$, hence

$$I_v \geqq I_{\pi/2}, \text{ for } 0 \leqq v \leqq \pi, \text{ if } f(u) \text{ decreases in } [0, \pi/2]. \tag{3}$$

In spherical coordinates r, u, θ with z as the point $r = 0$ we define bodies B_h as bounded by $r = F(u, h) = e^{2|u|} + h(\cos 2u + 2^{-1})e^{-2|u|}$, $|u| \leqq \pi/2$. Then $F(u, h)$ satisfies (2) as function of u, hence B_h is star shaped with respect to z, has z as center and is a surface of revolution.

$$V(B_h) = 3^{-1}\int_0^{2\pi} d\theta \int_{-\pi/2}^{\pi/2} F^3(u, h) \cos u du = 4\pi 3^{-1}\int_0^{\pi/2} F^3(u, h) \cos u du,$$

$$V'(B_0) = \left.\frac{dV(B_h)}{dh}\right|_{h=0} = 4\pi\int_0^{\pi/2} F^2(u, 0)(\cos 2u + 2^{-1})e^{-2u}du$$

$$= 4\pi\int_0^{\pi/2} e^{2u}(\cos 2u + 2^{-1})du = \pi(-e^{\pi} - 1) + \pi(e^{\pi} - 1) = -2\pi.$$

Thus $V(B_h) < V(B_0)$ for small $h > 0$.

We want to show that $A(B_h, w) > A(B_0, w)$ for small h and all w. Instead of w we may use the previous angle v, because B_h is a body of revolution, so that $A(B_h, v)$ is the area of the intersection of B_h with a plane intersecting the plane $u = 0$ at the angle v. Then as above

$$A(B_h, \pi - v) = A(B_h, v) = 2\int_0^{\pi/2} F^2[\arcsin(\sin v \cdot \sin s), h]ds, \quad 0 \leqq v \leqq \pi/2,$$

$$A'(B_0, v) = \frac{dA(B_h, v)}{dh}\bigg|_{h=0} = 4\int_0^{\pi/2} [\cos 2 \arcsin(\sin v \cdot \sin s) + 2^{-1}]ds.$$

Since $f(u) = \cos 2u + 2^{-1}$ decreases for $0 \leqq u \leqq \pi/2$ we conclude from (3) that $A'(B_0, v) \geqq A'(B_0, \pi/2) = \pi$ for $0 \leqq v \leqq \pi$. It follows that $A(B_h, v) > A(B_0, v)$ for small h and all v, which proves the first part of (*).

The second part is a consequence. For B_h is symmetric with respect to $u = 0$. The body B_h^* bounded by $r = F(u, h)$ for $0 < u \leqq \pi/2$ and the disk $0 \leqq r \leqq F(0, h)$, $u = 0$, is convex for small h. This is obvious for B_0 because $r = F(u, 0) = e^{2u}$, $0 \leqq u \leqq \pi/2$, is part of a spiral, and continuity shows the convexity of B_h for small $h > 0$.

On the other hand

$$v(B_h^*) = V(B_h)/2, \qquad A(B_h^*, v) = A(B_h, v)/2 \text{ for } v \neq 0, \pi,$$

$$A(B_h^*, v) = A(B_h, v) \qquad \text{for } v = 0, \pi.$$

This proves the second part of (*). In spite of the discontinuity of $A(B_h^*, v)$ at $v = 0$ the relations (1) remain valid if z is replaced by a point on the normal to the disk at its center, inside B_h^* and close to that center.

For any two bodies which are star shaped with respect to z and have z as center, $A(B_1, w) = A(B_2, w)$ for all w, implies $B_1 = B_2$; hence all the more $V(B_1) = V(B_2)$. This fact, remarkable in view of our example, follows from a theorem of Funk [3]; see also [2].

If B_1 is an ellipsoid with center z and B_2 is any body containing z, then $A(B_1, w) < A(B_2, w)$ does imply $V(B_1) < V(B_2)$;

See [1]. The discussion there assumes that B_2 is convex, but the result of [1] on which this discussion is based, extends to very general sets B_2.

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ON THE MAXIMUM MODULUS OF POLYNOMIALS*

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1. Introduction. Let $p(z)=a_0+a_1z+\cdots+a_nz^n$ be a polynomial of degree n in the complex variable z. Let $M(p, r)=[\max |p(z)|, |z|=r]$. Let P denote the set of p such that $M(p, 1)=1$. Every p can be normalized by dividing it by $M(p, 1)$ so that $p/M(p, 1)\in P$. Let Q be the set of $p\in P$ such that $p(z)\neq 0$ for $|z|<1$. The following results concerning the size of $M(p, r)$ for $p\in P$ and $p\in Q$ are known.

THEOREM A. (*S. Bernstein, cf.* [1]). *If $p\in P$ then $M(p, R)\leqq R^n$ for $R\geqq 1$, with equality only for $p(z)=\lambda z^n$, $|\lambda|=1$.*

THEOREM B. (*Ankeny and Rivlin* [2]). *If $p\in Q$ then $M(p, R)\leqq(1+R^n)/2$ for $R\geqq 1$ with equality only for $p(z)=(\lambda+\mu z^n)/2$, $|\lambda|=|\mu|=1$.*

THEOREM C. (*Zarantonello-Varga* [3]). *If $p\in P$ then $M(p, r)\geqq r^n$ for $r\leqq 1$ with equality only for $p(z)=\lambda z^n$, $|\lambda|=1$.*

A much simpler proof of Theorem C than is given in [3] is implicit in [1].

It is our purpose here to present a result that stands in relation to Theorem C as Theorem B does in relation to Theorem A. That is, we give an improved bound from below for $M(p, r)$ in the case $p\in Q$. This is done in Section 2 in Theorem 1.

In Section 3 we obtain as a consequence of Theorem 1 a fairly sharp bound (from both above and below) on the maximum modulus of polynomials having all their zeros on the unit circle. Other consequences of Theorem 1 are also discussed.

2. THEOREM 1. *If $p\in Q$ then $M(p, r)\geqq((1+r)/2)^n$ for $r\leqq 1$, with equality only for $p(z)=((\lambda+\mu z)/2)^n$, $|\lambda|=|\mu|=1$.*

Proof. Suppose $p\in Q$. Let $z_j=R_je^{i\phi_j}$, $j=1, \cdots, n$, be the zeros of p. Since $p\in Q$, $R_j\geqq 1$, $j=1, \cdots, n$. Suppose that $|p(e^{i\alpha})|=M(p, 1)=1$; then for $r<1$ (when $r=1$ there is nothing to prove),

$$M(p, r) = M(p, r)/M(p, 1) \geqq |p(re^{i\alpha})| / |p(e^{i\alpha})|$$

and

$$\frac{|p(re^{i\alpha})|}{|p(e^{i\alpha})|} = \prod_{j=1}^{n} \frac{|re^{i\alpha}-R_je^{i\phi_j}|}{|e^{i\alpha}-R_je^{i\phi_j}|}. \tag{1}$$

We claim now that

$$\left|\frac{re^{i\alpha}-R_je^{i\phi_j}}{e^{i\alpha}-R_je^{i\phi_j}}\right| \geqq \frac{r+R_j}{1+R_j}, \qquad j=1, \cdots, n, \tag{2}$$

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with equality holding if, and only if, $\phi_j=\alpha-\pi$. This may be established by minimizing the function

$$h(\phi) = \frac{\left| re^{i\alpha} - R_j e^{i\phi} \right|^2}{\left| e^{i\alpha} - R_j e^{i\phi} \right|^2} - \frac{(r + R_j)^2}{(1 + R_j)^2}$$

for $0 \leqq \phi < 2\pi$. An easy computation gives

$$h(\phi) = \frac{2R_j(1 - r)(R_j^2 - r)(1 + \cos(\alpha - \phi))}{\left| e^{i\alpha} - R_j e^{i\phi} \right|^2 (1 + R_j)^2},$$

and since $r<1$ and $R_j \geqq 1$ we have $h(\phi) \geqq 0$ and $h(\phi)=0$, if, and only if, $\cos(\alpha-\phi) = -1$, or $\alpha-\phi=\pi$, or $\phi=\alpha-\pi$. Thus (2) is established.

Furthermore, since $R_j \geqq 1$ we have

$$(r + R_j)/(1 + R_j) \geqq \tfrac{1}{2}(1 + r), \qquad j = 1, \cdots, n \tag{3}$$

with equality if, and only if, $R_j=1$. To establish (3) we need only note that since $r<1$ the function $(r+x)/(1+x)$ is strictly monotone increasing for $x>0$.

Now, using inequalities (2) and (3) in (1) we obtain

$$M(p, r) \geqq ((1 + r)/2)^n \tag{4}$$

and equality holds in (4) if, and only if, $R_j=1$, $\phi_j=\alpha-\pi$, $j=1, \cdots, n$. Thus we must have $p(z)=\gamma(z+e^{i\alpha})^n$. But $M(p, 1)=1$; hence $\left|\gamma(2e^{i\alpha})\right|^n=1$ and $\gamma=\beta 2^{-n}$ where $\left|\beta\right|=1$. The proof of Theorem 1 is complete.

3. By combining Theorem 1 with Theorem B we can obtain fairly sharp bounds from above and below for a polynomial all of whose zeros have absolute value one. More precisely, we have

THEOREM 2. *If* $p \in P$, *and* $p(z)=0$ *implies* $\left|z\right|=1$, *then*

$$\left\{\tfrac{1}{2}(1 + \rho)\right\}^n \leqq M(p, \rho) \leqq \tfrac{1}{2}(1 + \rho^n), \qquad \rho \geqq 0.$$

Proof. In view of Theorems 1 and B we need only to show (i) $M(p, \rho) \leqq (1+\rho^n)/2$, $\rho<1$ and (ii) $M(p, \rho) \geqq ((1+\rho)/2)^n$, $\rho>1$.

To this end put $q(z)=z^n p(1/z)$. Then $q \in Q$ and so:

(i) by Theorem B, if $R=1/\rho>1$

$$\tfrac{1}{2}(1 + R^n) \geqq M(q, R) = R^n M(p, \rho);$$

hence $M(p, \rho) \leqq \frac{1}{2}(1+\rho^n)$.

(ii) by Theorem 1, if $r=1/\rho<1$, $((1+r)/2)^n \leqq M(q, r)=r^n M(p, \rho)$; hence

$$M(p, \rho) \geqq \left\{\tfrac{1}{2}(1 + \rho)\right\}^n.$$

Another consequence of Theorem 1 is that for $p \in Q$ we have, taking $r=0$,

$$\left| p(0) \right| = \left| a_0 \right| \geqq 2^{-n}.$$

Indeed, suppose $p(0) \neq 0$, and let a be the absolute value of a zero of smallest

absolute value of p (we are not assuming $p \in Q$ here). Then putting $q(z) = p(az)$ we see that $q/M(q, 1) \in Q$ and so $|q(0)| \geqq M(q, 1)2^{-n}$ or

$$|p(0)| = |a_0| \geqq 2^{-n}M(p, a). \tag{5}$$

Equation (5) may be expressed as

THEOREM 3. *If $M(p, \rho) > |a_0| 2^n$ for some $\rho > 0$ then the polynomial $p(z)$ has at least one zero in $|z| < \rho$.*

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A NOTE ON WALSH-FOURIER SERIES

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Let $F(t)$ be a function such that $|e^{-st}F(t)| < Ae^{-(s-a)t}$ where a and A are constants. Then $\int_0^\infty e^{-st}F(t)dt$ exists for $R(s) > a$ and defines a new function $f(s)$ called the Laplace transform of the determining function $F(t)$. If one uses the Walsh function as the object function the following infinite product expansion may be proved.

$$\prod_{k=0}^{\infty}\left\{\frac{2}{1 + x^{1/2^k}}\right\} = \frac{2 \log x}{1 - x^2}, \qquad 0 < x < 1. \tag{1}$$

Although this formula may be proved directly, I think the following derivation is enlightening.

Since the Walsh functions may not be familiar to many, I shall define them by using the Rademacher functions:

$$\phi_0(x) = 1 \quad (0 \leqq x < 1/2), \quad \phi_0(x + 1) = \phi_0(x),$$
$$\phi_0(x) = -1 \quad (1/2 \leqq x < 1); \qquad \phi_n(x) = \phi_0(2^n x), \quad n = 1, 2, \cdots.$$

These Rademacher functions may now be used to define the Walsh functions as follows:

$$\psi_0(x) = 1; \qquad \psi_n(x) = \phi_{n_1}(x) \cdots \phi_{n_r}(x)$$

for $n = 2^{n_1} + \cdots + 2^{n_r}$, where the n_i are uniquely determined by $n_{i+1} < n_i$. Using the above definition of the Walsh function, one may prove many interesting properties such as

$$\psi_{2n+1}(x) = \psi_{2n}(x)\psi_1(x), \tag{A}$$

$$\psi_{2n}(x) = \psi_n(2x), \tag{B}$$

$$(C) \qquad \psi_{2n}(x + 1/2) = \psi_{2n}(x).$$

Walsh has proved that the sequence $\{\psi_n(x)\}$ forms a complete orthonormal set, and every periodic function $g(x) \in L(0, 1)$ has a Walsh-Fourier series

$$g(x) \sim C_0 + C_1\psi_1(x) + \cdots + C_n\psi_n(x) + \cdots,$$

where

$$C_n = \int_0^1 \psi_n(x)g(x)dx; \qquad n = 0, 1, \cdots.$$

Consider the Laplace transform of $\psi_n(t)$ which certainly has period 1; then for $R(s) > 0$

$$f_n(s) = \int_0^\infty e^{-st}\psi_n(t)dt = \sum_{k=0}^\infty \int_k^{k+1} e^{-st}\psi_n(t)dt = C_n(s) \sum_{k=0}^\infty e^{-ks} = C_n(s)\{1/(1 - e^{-s})\}$$

where $C_n(s)$ is the nth Walsh-Fourier coefficient of the function which is equal to e^{-st} for $0 \leqq t < 1$ and which has period 1. Since the function is of bounded variation and continuous on the right one may write $e^{-st} = \sum_{n=0}^\infty C_n(s)\psi_n(t)$ by a theorem due to J. L. Walsh stated in [1]. The Walsh-Fourier coefficients for e^{-st} give rise to the following interesting recurrence formulas:

$$(a) \qquad C_{2n}(s) = 1/2(1 + e^{-s/2})C_n(s/2),$$

$$(b) \qquad C_{2n+1}(s) = 1/2(1 - e^{-s/2})C_n(s/2),$$

$$(c) \qquad C_{2n+1}(s) = \{(1 - e^{-s/2})/(1 + e^{-s/2})\}C_{2n}(s),$$

$$(d) \qquad C_n(s) = (1/s)(1 - e^{-s})P_n(s),$$

where

$$P_n(s) = \prod_{j=1}^r \frac{1 - \exp(-s/2^{n_j+1})}{1 + \exp(-s/2^{n_j+1})}; \quad P_0(s) \equiv 1,$$

$$n = 2^{n_1} + \cdots + 2^{n_r}; \qquad n_1 > \cdots > n_r \geqq 0.$$

Recurrence formulas (a) and (b) are worthy of proof, because they make use of the properties of the Walsh function. Applying property (B); one may derive equation (a) as follows:

$$C_{2n}(s) = \int_0^1 e^{-st}\psi_{2n}(t)dt = \int_0^1 e^{-st}\psi_n(2t)dt = 1/2 \int_0^2 e^{-(s/2)t}\psi_n(t)dt$$

$$= 1/2 \int_0^1 e^{-(s/2)t}\psi_n(t)dt + 1/2 \int_1^2 e^{-(s/2)t}\psi_n(t)dt.$$

Substituting $t = u + 1$ into the second integral and remembering that $\psi_n(t)$ has period one, we obtain

$$C_{2n}(s) = 1/2C_n(s/2) + 1/2e^{-(s/2)}\int_0^1 e^{-(s/2)u}\psi_n(u)du = 1/2(1 + e^{-(s/2)})C_n(s/2).$$

Applying properties (A), (B) and $\psi_1(t)=\phi_0(t)$; equation (b) may be derived as follows:

$$C_{2n+1}(s) = \int_0^1 e^{-st}\psi_{2n+1}(t)dt = \int_0^1 e^{-st}\psi_{2n}(t)\psi_1(t)dt$$
$$= \int_0^{1/2} e^{-st}\psi_n(2t)dt - \int_{1/2}^1 e^{-st}\psi_{2n}(t)dt.$$

substituting $t=(1/2)u$ into the first integral and $t=u+1/2$ into the second integral and applying property (C), one obtains

$$C_{2n+1}(s) = 1/2\int_0^1 e^{-(s/2)u}\psi_n(u)du - e^{-(s/2)}\int_0^{1/2} e^{-su}\psi_{2n}(u)du$$
$$= 1/2C_n(s/2) - 1/2e^{-(s/2)}\int_0^1 e^{-(s/2)u}\psi_n(u)du = 1/2(1 - e^{-(s/2)})C_n(s/2).$$

These recurrence formulas may now be used to derive* a general product expansion which reduces to equation (1). Let $f(s, t)=e^{-st}$ $(0\leqq t<1)$, $f(s, t+1)=f(s, t)$. Then for all t,

$$f(s, t) = \sum_{n=0}^{\infty} C_n(s)\psi_n(t) = \sum_{n=0}^{\infty} C_{2n}(s)\psi_{2n}(t) + \sum_{n=0}^{\infty} C_{2n+1}(s)\psi_{2n+1}(t)$$
$$= 1/2(1 + e^{-s/2})\sum_{n=0}^{\infty} C_n(s/2)\psi_n(2t) + \tfrac{1}{2}(1 - e^{-s/2})\psi_1(t)\sum_{n=0}^{\infty} C_n(s/2)\psi_n(2t)$$
$$= 1/2(1 + e^{-s/2})\{1 + \lambda(s)\psi_1(t)\}f(s/2, 2t).$$

If we define $g(s, t)=\{s/(1-e^{-s})\}f(s, t)$, this functional equation reduces to

$$g(s, t) = [1 + \lambda(s)\psi_1(t)]g(s/2, 2t), \tag{2}$$

where $\lambda(s)=\tanh(s/4)$. By iterating (2), we get

$$g(s, t) = \prod_{k=0}^{n-1}[1 + \lambda(s/2^k)\psi_{2^k}(t)]\cdot g(s/2^n, 2^n t).$$

It is easy to see that $g(s/2^n, 2^n t)\to 1$ uniformly in t, so

$$g(s, t) = \prod_{k=0}^{\infty}(1 + \lambda(s/2^k)\psi_{2^k}(t)).$$

Putting $s=-2\log x$, we get

* The referee suggested this proof for (1).

$$\frac{-2x^{2t}\log x}{1-x^2} = \prod_{k=0}^{\infty}\left\{\frac{2}{1+x^{\psi_{2^k}(t)/2^k}}\right\} \qquad (0 \leqq t < 1).$$

Substituting $t=0$, we obtain equation (1).

An alternative proof of the product expansion (1) may also be obtained as follows:

$$e^{-st} = \sum_{n=0}^{\infty} C_n(s)\psi_n(t) \qquad (0 \leqq t < 1, s \neq 0)$$

$$= \sum_{n=0}^{\infty} \{(1 - e^{-s})/s\} P_n(s)\psi_n(t)$$

$$= (1/s)(1 - e^{-s}) \sum_{n=0}^{\infty} \{\tanh(s/2^{n_1+2}) \cdots \tanh(s/2^{n_r+2})\}\psi_n(t).$$

Putting $t=0$, one obtains

$$1 = (1/s)(1 - e^{-s}) \sum_{n=0}^{\infty} P_n(s)$$

which reduces to

$$s/(1 - e^{-s}) = 1 + \tanh(s/2^2) + \tanh(s/2^3) + \tanh(s/2^3)\cdot\tanh(s/2^2) + \tanh(s/2^4) + \cdots.$$

Hence

$$s/(1 - e^{-s}) = \prod_{k=0}^{\infty}(1 + \tanh(s/2^{k+2})), \qquad R(s) > 0. \tag{3}$$

Using the transformation $s = -2\log x$ for $0<x<1$, one may easily show that

$$s/(1 - e^{-s}) = -2\log[x/(1 - x^2)] \quad \text{and} \quad 1 + \tanh(s/2^{k+2}) = 2/(1 + x^{1/2^k}).$$

Substituting into (3), we obtain

$$-2\log[x/(1 - x^2)] = \prod_{k=0}^{\infty}\{2/(1 + x^{1/2^k})\} \qquad 0 < x < 1.$$

Note that

$$\lim_{x\to 1-}\{-2\log[x/(1 - x^2)]\} = 1.$$

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PARALLELOGRAMS INSCRIBED IN CONVEX CURVES

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There is a wide class of theorems concerning convex curves of the form: Within any convex curve of a certain type can be found a convex curve of another type.

For example each convex curve F, surrounding a closed set K of area A and minimum width d, contains a triangle of area $2\sqrt{3}A/4\pi$, a radially symmetric convex set of area $2A/3$ ([1] p. 116), a circle of radius $d/3$ ([1] p. 112), a curve of constant width $d/(3-\sqrt{3})$ ([2] p. 141). Each convex curve F of constant width contains a radially symmetric convex curve of area $(.84)A$ ([2] p. 131). Some of these "best possible" results have been extended to higher dimensions, some have not. In the latter class belongs*

THEOREM 1. *If F is a convex curve surrounding a closed set K of area A then there is a parallelogram inscribed† in F of area $A/2$. Moreover there is no parallelogram of area strictly larger than $A/2$ inscribed in F if and only if F is a triangle.*

Proof. Assume first that F is neither a triangle nor a quadrilateral. Choose four support lines of F enclosing a polygon of the smallest possible area. Since F is not a triangle it is easily seen that these four lines are distinct. In addition the midpoint of each of the four sides of this quadrilateral of minimal area must lie in F. Were this not the case, a simple continuity argument could be used to construct a quadrilateral of smaller area containing K. Thus the parallelogram with the four midpoints of a minimal quadrilateral as vertices lies in K. Thus, since K is not a quadrilateral, K contains a parallelogram with area strictly larger than $A/2$.

Next assume that F is a triangle. Since a parallelogram has four vertices two lie on one side of F. A straight forward examination of such parallelograms shows that the largest has area $A/2$.

Finally let $ACBD$ be a convex quadrangle other than a trapezoid such that AD, $BC=P$ and AC, $BD=Q$ are the exterior diagonal points and C is inside the diagonal triangle. The vertex C is then between A and Q and also between B and P. We now make D the origin of a system of oblique Cartesian coordinates and assign coordinates as follows: $A(a, 0)$, $B(0, b)$, $C(x_0, y_0)$, where all nonzero values are assumed positive. The points P and Q are also located on the positive semiaxes. The condition of convexity is expressed as

$$x_0/a + y_0/b > 1. \tag{1}$$

For the proof it seems necessary to distinguish between two cases.

Case 1. We assume that $2x_0>a$, $2y_0>b$. The points $(2x_0, 0)$, $(0, 2y_0)$, and D are the vertices of a triangle which contains the given quadrangle. Now the

* This problem arose in conversation with Donald C. Benson.

† The theorem remains valid if "inscribed in F" is replaced by "contained in K."

parallelogram with sides $x=0$, $x=x_0$, $y=0$, $y=y_0$ has area equal to one half that of the triangle and hence more than half that of the quadrangle.

Case 2. Assuming that $2x_0 \leqq a$, we find from (1) that $2y_0 > b$. We draw the parallels to the axes through the midpoint of AQ. These together with the axes determine a parallelogram whose area is half the area of triangle DAQ. But the original quadrangle is inscribed in the latter. Again the desired conclusion follows.

The trapezoid can be dealt with by making P an ideal point. Otherwise the proofs remain unchanged. This concludes the proof.

Macbeath has shown ([3] pp. 59–60) that if K is a convex set of volume V in n-dimensional space then K contains a parallelotope of volume V/n^n and is contained in one of volume $n!\,V$. From his argument of ([3] p. 58) and Theorem 1 of this paper one deduces

THEOREM 2. *If K and K' are convex sets in the plane with nonzero areas A and A' respectively then there is an affine image of K', $\sigma K'$, contained in K, such that area $\sigma K'$ is greater than $A/4$, and also an affine image $\tau K'$, containing K such that area $\tau K'$ is less than $4A$.*

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CIRCULANTS AND THEIR GROUPS*

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Let the elements in the first row of the symbol for a determinant of order n be $x_0, x_1, \cdots, x_{n-1}$. If we permute the elements cyclically, beginning with x_{n-1} in the second row, x_{n-2} in the third row etc., the result is a circulant [1], C_n, of order n. That is, $C_n = |x_{r-c}|$ $r, c = 0, 1, \cdots, n-1$. It is known [2] that every rational function belongs to some group. The purpose of this note is to derive the group to which C_n belongs.

Let the permutation $P = (x_0 x_1 \cdots x_{n-1})$ and apply P^α to the factored form of

$$C_n = \prod_{j=0}^{n-1} \sum_{k=0}^{n-1} \epsilon^{jk} x_k, \qquad \epsilon = \exp(2\pi i/n).$$

The factor $r_j = \sum_{k=0}^{n-1} \epsilon^{jk} x_k$ becomes

$$\sum_{k=0}^{n-1} \epsilon^{jk} x_{k+\alpha} = \epsilon^{-j\alpha} \sum_{k=0}^{n-1} \epsilon^{j(k+\alpha)} x_{k+\alpha} = \epsilon^{-j\alpha} r_j.$$

* Presented to the Southeastern Section of the Association, March 15, 1958.

than n permutations which leave C_n unaltered by changing each r_j into a constant multiple of itself. This contradicts the lemma.

THEOREM. *The order of G is $n\phi(n)$ if n is odd and $(n/2)\phi(n)$ if n is even.*

COROLLARY 1. *A circulant of prime order p belongs to the metacyclic* [3] *group of degree p.*

COROLLARY 2. *An isomorphism is established by the correspondence $\{H\}U$ and V, where U is a permutation on the x's and V permutes the r's in the corresponding inverse order.*

COROLLARY 3. *A circulant of order n is a symmetric function if and only if $n=3$.*

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SOME INFINITE SERIES FOR $\zeta(n+1)$

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Briggs, Chowla, Kempner, and Mientka [1] have shown that

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}\left\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\right\}, \tag{1}$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n}\left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right\}. \tag{2}$$

By the same methods it can be shown that

$$\zeta(n+1) = \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n^2}, \qquad \text{for } n \geqq 1, \tag{3}$$

where

$$s_{n-1}(k_n) = \sum_{k_{n-1}=n-1}^{k_n-1} \frac{s_{n-2}(k_{n-1})}{k_n}, \qquad s_0(k_1) = 1. \tag{4}$$

If $n=2$, (3) reduces to (1). Also, using the same notations,

$$\sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n^3} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}}, \qquad \text{for } n \geqq 1, \tag{5}$$

which reduces to (2) for $n=1$. Finally it can be shown that

$$\zeta(n+1) = \sum_{m=1}^{\infty} \frac{1}{m^n} \sum_{k=1}^{m} \frac{1}{k} - \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^n}, \qquad \text{for } n \geqq 2. \tag{6}$$

Since, for $|x|<1$,

$$\log\left(\frac{1}{1-x}\right)=\sum_{k=1}^{\infty}\frac{x^k}{k},$$

it can be shown by induction that

$$\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n=\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)}{k_n}x^{k_n}. \tag{7}$$

To do this consider

$$\left(\frac{1}{1-x}\right)\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n=\left\{\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)x^{k_n}}{k_n}\right\}\left\{\sum_{k=0}^{\infty}x^k\right\}$$

$$=\sum_{m=n}^{\infty}\left(\sum_{k_n=n}^{m}\frac{s_{n-1}(k_n)}{k_n}\right)x^{m+1},$$

so that

$$\frac{1}{(n+1)!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^{n+1}=\sum_{m=n}^{\infty}\frac{1}{m+1}\left(\sum_{k_n=n}^{m}\frac{s_{n-1}(k_n)}{k_n}\right)x^{m+1}$$

$$=\sum_{k_{n+1}=n+1}^{\infty}\frac{s_n(k_{n+1})}{k_{n+1}}x^{k_{n+1}}$$

Consider [3]

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}}{e^t-1}\,dt,$$

and let $e^{-t}=1-x$, $s=n+1$. Thus

$$\zeta(n+1)=\int_0^1\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n\frac{dx}{x}$$

$$=\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)}{k_n}\int_0^1 x^{k_n-1}dx=\sum_{k_n=n}^{\infty}\frac{s_{n-1}(k_n)}{k_n^2},$$

which is (3). Since

$$\frac{n!}{k^{n+1}}=\int_0^{\infty}t^ne^{-kt}dt,$$

we have by summing on k, setting $e^{-t}=1-x$, and using (7) and (3)

$$\sum_{k=1}^{m}\frac{1}{k^{n+1}}=\int_0^1\frac{1}{n!}\left\{\log\left(\frac{1}{1-x}\right)\right\}^n[1-(1-x)^m]\frac{dx}{x}$$

$$= \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n} \left\{ \int_0^1 x^{k_n-1}dx - \int_0^1 x^{k_n-1}(1-x)^m dx \right\}$$

$$= \zeta(n+1) - \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n} \cdot \frac{\Gamma(m+1)\Gamma(k_n)}{\Gamma(m+1+k_n)} .$$

Hence

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}} = \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)(k_n-1)!}{k_n} \sum_{m=1}^{\infty} \frac{(m-1)!}{(k_n+m)!} .$$

Let $f(m)=(m-1)!/k_n(m+k_n-1)!$; so that $f(1)=1/(k_n)!k_n$, $f(m)-f(m+1)=(m-1)!/(m+k)!$ and $\sum_{m=1}^{\infty}(m-1)!/(k_n+m)!=f(1)$; and thus (5) is obtained:

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}} = \sum_{k_n=n}^{\infty} \frac{s_{n-1}(k_n)}{k_n^3} .$$

Finally, let $R_{n+1}(m+1)=\sum_{k=m+1}^{\infty} 1/k^{n+1}$, $s_1(1)=0$ and consider

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \frac{1}{k^{n+1}} = \sum_{m=1}^{\infty} \left(\sum_{j=1}^{m} \frac{1}{j} - \sum_{j=1}^{m-1} \frac{1}{j} \right) R_{n+1}(m+1)$$

$$= \sum_{m=1}^{\infty} [s_1(m+1) - s_1(m)] R_{n+1}(m+1)$$

$$= \sum_{m=1}^{\infty} s_1(m)[R_{n+1}(m) - R_{n+1}(m+1)]$$

$$= \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \frac{1}{m} \right) \frac{1}{m^{n+1}}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^{n+1}} \sum_{k=1}^{m} \frac{1}{k} - \zeta(n+2)$$

which is equivalent to (6).

Formula (6) can also be written in the form

$$\zeta(n+1) = \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{k^{n-1}+m^{n-1}}{(mk)^n} - \zeta(n) \right), \qquad \text{for } n \geqq 2.$$

It does not appear obvious that there is a direct connection between these formulas and those of Brun [2] (setting $s=n+1$),

$$\zeta(n+1) = \frac{n+1}{n} + \sum_{r=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{2^{rn}}{(2^r+m)^{n+1}},$$

$$\zeta(n+1) = \frac{n+1}{n} + \sum_{m=3}^{\infty} \frac{(-1)^{m-1}}{m^{n+1}} \sum_{k=0}^{[\sqrt{(m-1)}]} 2^{kn}.$$

References

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A VECTOR SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

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In the following note we adopt the summation convention for repeated indices; the indices r and j will always have the range $1, \cdots, n$. Let $e_1, \cdots, e_n$ be an orthonormal base system of an n-dimensional vector space.* We define the vector product of the $n-1$ vectors $V_i = v_i^r e_r$, $i = 1, \cdots, n-1$, to be the vector obtained by expanding the determinant,

$$V = \begin{vmatrix} e_1 & e_2 & \cdots & e_n \\ v_1^1 & v_1^2 & \cdots & v_1^n \\ \cdot & \cdot & \cdots & \cdot \\ v_{n-1}^1 & v_{n-1}^2 & \cdots & v_{n-1}^n \end{vmatrix} \tag{1}$$

in terms of elements in the first row. For any vector U, $U \cdot V = 0$ if and only if the vectors $U, V_1, \cdots, V_{n-1}$ are linearly dependent. Now consider the n points $(b_j^1, \cdots, b_j^n)$ which determine a hyperplane

$$A_r x^r = M, \qquad M \neq 0 \text{ if } \det b_j^r \neq 0. \tag{2}$$

If P and Q are any two points whose coordinates, c^r and d^r, satisfy (2), we say that P and Q lie in the plane, and define the vector $[P, Q]$ by $[P, Q] = (d^r - c^r)e_r$. We also define the coefficient vector of (2) by $C = A_r e^r$ ($e^r = e_r$ since the system is orthonormal). The vector $[P, Q]$ is orthogonal to the coefficient vector, C, for $[P, Q] \cdot C = M - M = 0$.

THEOREM 1. *The points $(b_j^1, \cdots, b_j^n)$ whose $\det b_j^r \neq 0$ determine at least one set of $n-1$ linearly independent vectors in the hyperplane determined by the points.*

Proof. (constructive)

$$0 \neq \det b_j^r = \begin{vmatrix} b_1^1 & b_1^2 & \cdots & b_1^n \\ b_2^1 - b_1^1 & b_2^2 - b_1^2 & \cdots & b_2^n - b_1^n \\ \cdot & \cdot & \cdots & \cdot \\ b_n^1 - b_1^1 & b_n^2 - b_1^2 & \cdots & b_n^n - b_1^n \end{vmatrix}. \tag{3}$$

* See, for example, Louis Brand, Vector Analysis, New York, 1957, Ch. 9.

Therefore the vectors $(b_s^r - b_1^r)e_r$, $s = 2, \cdots, n$, which lie in the plane, are linearly independent.

THEOREM 2. *The vector product, V, of $n-1$ vectors, V_i, in the hyperplane $A_r x^r = M$, is orthogonal to any vector, U, in the plane. If the $n-1$ vectors, V_i, are linearly independent, then U is a linear combination of these vectors.*

Proof. Let $V_i = v_i^r e_r$, $i = 1, \cdots, n-1$, and $U = u^r e_r$. Since $A_r v_i^r = 0$, $A_r u^r = 0$, and not every $A_r = 0$,

$$U \cdot V = \begin{vmatrix} u^1 & u^2 & \cdots & u^n \\ \overset{1}{v_1} & \overset{2}{v_1} & \cdots & \overset{n}{v_1} \\ \cdot & \cdot & \cdots & \cdot \\ \overset{1}{v_{n-1}} & \overset{2}{v_{n-1}} & \cdots & \overset{n}{v_{n-1}} \end{vmatrix} = 0.$$

Thus U, $V_1, \cdots, V_{n-1}$ are linearly dependent. Hence if the V_i are linearly independent, then U is a linear combination of the V_i.

THEOREM 3. *The coefficient vector, C, of the hyperplane, $A_r x^r = M$, determined by n points whose* det $b_j^r \neq 0$ *is a scalar product of the vector product of $n-1$ linearly independent vectors in the plane.*

Proof. Denote the vectors $b_j^r e_r$ by b_j. Then we have for $s = 2, \cdots, n$, $C \cdot (b_s - b_1) = 0$, and $V \cdot (b_s - b_1) = 0$ by Theorem 2. Further $C \cdot b_1 = M \neq 0$, and let $V \cdot b_1 = K$. Since the vectors, V_i, which define V are linearly independent by Theorem 2, the second to n-th rows of the nonzero det (3) are linear combinations of the second to n-th rows of det (1); hence $K \neq 0$. Let $b_1 = X_1$ and $b_s - b_1 = X_s$ for $s = 2, \cdots, n$. Then combining the above results we have $[(1/K)V - (1/M)C] \cdot X_i = 0$, $i = 1, \cdots, n$, where the X_i are n linearly independent vectors by the proof of Theorem 1. Hence $(1/K)V - (1/M)C = 0$, or $C = (M/K)V$.

Now consider the n simultaneous linear equations in n variables, x_r in which a_j^r are constants,

$$a_j^r x_r = M, \qquad M \neq 0, \qquad \det a_j^r \neq 0. \tag{4}$$

In (4) we may interpret the $(a_j^1, \cdots, a_j^n)$ as n points and the x_r as the components of the coefficient vector of the hyperplane determined by the points. By the method of Theorem 1 we construct $n-1$ linearly independent vectors whose vector product is a scalar product of the coefficient vector. Thus if the vector product is $V = v^r e_r$, we have $Nv^r = x_r$. To find N we substitute in one of the equations (4). If det $a_j^r = 0$, then the vectors $(a_s^r - a_1^r)e_r$, $s = 2, \cdots, n$, are linearly dependent, and Theorem 3 does not apply; the constructed vector product $V = 0$.

Example. Solve the system:

$$\begin{array}{rrrrr} 2x+ & y+ & z- & w= & 3 \\ x- & y- & z+ & 2w= & 2 \\ x+ & 2y+ & z- & 3w= & 1 \\ -x+ & 3y- & 2z+ & w= & 6 \end{array} \quad \text{or} \quad \begin{array}{rrrrr} 4x+ & 2y+ & 2z- & 2w= & 6 \\ 3x- & 3y- & 3z+ & 6w= & 6 \\ 6x+ & 12y+ & 6z- & 18w= & 6 \\ -x+ & 3y- & 2z+ & w= & 6 \end{array}$$

We must solve the right hand set as the method requires all constants on the right to be identical. Hence if one of the equations has a 0 on the right that equation must be combined with another prior to multiplication to obtain an equivalent set as was done above. From the points $A(4, 2, 2, -2), B(3, -3, -3, 6)$, $C(6, 12, 6, -18), D(-1, 3, -2, 1)$ we form the vectors $[B, A]=e_1+5e_2+5e_3-8e_4$, $[C, A]=-2e_1-10e_2-4e_3+16e_4$, $[D, A]=5e_1-e_2+4e_3-3e_4$ or just as well replace $[C, A]$ by $-\frac{1}{2}[C, A]=e_1+5e_2+2e_3-8e_4$. Thus we have

$$V=\begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 5 & 5 & -8 \\ 1 & 5 & 2 & -8 \\ 5 & -1 & 4 & -3 \end{vmatrix} = 69e_1+111e_2+78e_4.$$

Hence $x=69N$, $y=111N$, $z=0$, $w=78N$. Substituting in the first equation we find $N=1/57$; hence $x=23/19$, $y=37/19$, $z=0$, $w=26/19$.

The author wishes to express his appreciation for the useful suggestions made by the referee.

ON THE PRODUCT OF TWO UNIFORMLY CONTINUOUS FUNCTIONS ON THE LINE

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1. Many properties of continuous functions defined on a closed interval fail to hold for continuous functions defined on the line. For example, continuity may or may not involve uniform continuity, as the functions x and x^2 illustrate. Even when two functions are uniformly continuous on the line, their product need not be uniformly continuous, as for example, x and $\sin x$. It might be of some interest to investigate the question of the product of two uniformly continuous real functions on the line. In doing so we shall restrict our attention to non-negative functions defined on $[1, \infty)$.

2. Let K denote the class of non-negative uniformly continuous functions defined on $[1, \infty)$. The following lemmas will be useful in the sequel.

LEMMA 2.1. *If* $f(x)\in K$, *then* $\limsup_{x\to\infty} f(x)/x<\infty$.

Proof. Suppose on the contrary that $\limsup f(x)/x=\infty$. Then there exists a sequence of reals $\{x_k\}$ such that $x_k\to\infty$, $x_{k+1}-x_k>1$, and if $c_k=f(x_k)/x_k$ then $c_{k+1}\geqq c_k$ and $c_k\to\infty$. Now for each k, $|f(x_{k+1})-f(x_k)|=c_{k+1}x_{k+1}-c_kx_k \geqq c_k(x_{k+1}-x_k)$. Let $\epsilon=1$, and suppose that δ is arbitrary except that $0<\delta<1$.

There exists an index k_0 such that $c_k > 2/\delta$ when $k > k_0$. Let $k > k_0$, and choose $t_1, t_2, \cdots, t_m$ such that $x_k = t_1 < t_2 < \cdots < t_m = x_{k+1}$, where $\delta/2 < t_{i+1} - t_i < \delta$ for $i = 0, 1, \cdots, m-1$. For some index j it must follow that $|f(t_{j+1}) - f(t_j)| \geqq c_k(t_{j+1} - t_j)$; otherwise

$$|f(x_{k+1}) - f(x_k)| \leqq \sum_{i=0}^{m-1} |f(t_{i+1}) - f(t_i)| < c_k(x_{k+1} - x_k),$$

contrary to hypothesis. Hence $|f(t_{j+1}) - f(t_j)| \geqq c_k(t_{j+1} - t_j) > 2/\delta \cdot \delta/2 = 1$, and $f(x)$ is not uniformly continuous.

LEMMA 2.2. *If $xf(x) \in K$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $x|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$.*

Proof. Let $\epsilon > 0$. Since $xf(x) \in K$, there is by Lemma 2.1 a constant $M > 0$ such that $f(x) < M$. Moreover, there exists a $\delta > 0$ such that $\delta < \epsilon/(2M)$ and $|xf(x) - yf(y)| < \epsilon/2$ when $|x - y| < \delta$. Hence $x|f(x) - f(y)| \leqq |xf(x) - yf(y)| + f(y)|x - y| < \epsilon/2 + M \cdot \epsilon/(2M) = \epsilon$, when $|x - y| < \delta$.

LEMMA 2.3. *If $x^2 f(x) \in K$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $x^2|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$.*

Proof. Let $\epsilon > 0$. Since $x^2 f(x) \in K$, there exists a constant $M > 0$ such that $xf(x) < M$. Moreover there exists a $\delta > 0$ such that $\delta < \text{Min}\,(1, \epsilon/(6M))$ and $|x^2 f(x) - y^2 f(y)| < \epsilon/2$ when $|x - y| < \delta$. Hence

$$\begin{aligned} x^2|f(x) - f(y)| &\leqq |x^2 f(x) - y^2 f(y)| + f(y)|x^2 - y^2| \\ &< \epsilon/2 + yf(y)(2 + 1/y)|x - y| < \epsilon/2 + M \cdot (3\epsilon)/(6M). \end{aligned}$$

3. In approaching the question of the product of two functions in K, consider the class of functions K^* defined as follows: K^* is the class of all functions $f(x)$ such that $f(x) \in K$ and $f(x)g(x) \in K$ whenever $g(x)$ is in K. The following theorem will serve to characterize the class K^*.

THEOREM 3.1. *$f(x) \in K^*$ if and only if $xf(x) \in K$.*

Proof. Since $x \in K$ the necessity is trivial. Conversely, suppose $xf(x) \in K$. Let $\epsilon > 0$ and let $g(x) \in K$. By Lemma 2.1 there exist constants M_1 and M_2 such that $f(x) < M_1$ and $g(x)/x < M_2$. By hypothesis on $g(x)$ and by Lemma 2.2 it is possible to choose $\delta > 0$ such that $|g(x) - g(y)| < \epsilon/(2M)_1$ and $y|f(x) - f(y)| < \epsilon/(2M_2)$ when $|x - y| < \delta$. Hence

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leqq f(x)|g(x) - g(y)| + (g(y)/y)y|f(x) - f(y)| \\ &< M_1 \cdot \epsilon/(2M_1) + M_2 \cdot \epsilon/(2M_2) = \epsilon \end{aligned}$$

when $|x - y| < \delta$. It follows that $f(x) \in K^*$.

Pursuing this line, let K^{**} denote the class of functions $f(x)$ such that $f(x) \in K^*$ and $f(x)g(x) \in K^*$ whenever $g(x) \in K$. K^{**} is a proper subset of K^*

since the function $f(x)\equiv 1$ is in K^* but not in K^{**} because $x\in K$ but $x\cdot 1\notin K^*$. The following theorem, analogous to Theorem 3.1, characterizes the class K^{**}.

THEOREM 3.2. *$f(x)\in K^{**}$ if and only if $x^2f(x)\in K$.*

Proof. By definition of K^{**} and K^*, if $f(x)\in K^{**}$, then $xf(x)\in K^*$; hence $x^2f(x)\in K$. Conversely, suppose $x^2f(x)\in K$. Clearly, if $f(x)g(x)h(x)\in K$ whenever $g(x)$, $h(x)\in K$, then $f(x)\in K^{**}$; hence suppose that $g(x)$, $h(x)\in K$. Let $\epsilon>0$. Since $x^2f(x)\in K$, there exists a constant $M_1>0$ such that $xf(x)<M_1$. Since $g(x)$, $h(x)\in K$, there exist constants $M_2>0$ and $M_3>0$ such that $g(x)/x<M_2$ and $h(x)/x<M_3$. Moreover there exists a $\delta_1>0$ such that

$$|\, g(x)h(x)/x - g(y)h(y)/y\,| \;<\epsilon/(3M_1)$$

when $|x-y|<\delta_1$, since $g(x)\in K$, $h(x)/x\in K^*$. By Lemma 2.3, there exists a $\delta_2>0$ such that $y^2|f(x)-f(y)|<\epsilon/(3M_2M_3)$ when $|x-y|<\delta_2$. Take $\delta<\text{Min}\,(1,\delta_1,\delta_2,\epsilon/(9M_1M_2M_3))$. Then

$$\begin{aligned}
&|\, f(x)g(x)h(x) - f(y)g(y)h(y)\,| \leqq xf(x)\,|\, g(x)h(x)/x - g(y)h(y)/y\,| \\
&\quad + xf(x)\,|\, g(y)h(y)/y - g(y)h(y)/x\,| \;+ h(y)g(y)\,|\, f(x) - f(y)\,| \\
&< M_1\cdot\epsilon/(3M_1) + M_1(g(y)/y)(h(y)/y)(y/x)\,|x-y\,| + (g(y)/y)(h(y)/y)y^2\,|\, f(x)-f(y)\,| \\
&< \epsilon/3 + M_1M_2M_3 3\cdot\epsilon/(9M_1M_2M_3) + M_2M_3\cdot\epsilon/(3M_2M_3) = \epsilon
\end{aligned}$$

when $|x-y|<\delta$. Hence $f(x)g(x)h(x)\in K$ and $f(x)\in K^{**}$.

Further properties of the classes K^* and K^{**} are noted in the following obvious theorems.

THEOREM 3.3. *If $f(x)$, $g(x)\in K^*$, then $f(x)+g(x)$ and $f(x)g(x)\in K^*$.*

THEOREM 3.4. *If $f(x)$, $g(x)\in K^{**}$, then $f(x)+g(x)$ and $f(x)g(x)\in K^{**}$.*

5. The definitions of K, K^*, K^{**} naturally suggest a definition of $K^{**\cdots *}$ by induction and it might be of interest to study properties of these extended classes. In addition there is still the question of necessary and sufficient conditions for $f(x)g(x)$ to be in K whenever $f(x)$, $g(x)$ are in K. In connection with this it can be established that $f(x)\in K$ whenever $f^2(x)\in K$ and $f(x)$ is continuous on $[1,\infty)$.

CLASSROOM NOTES

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BOOLEAN ALGEBRA

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Rosenbloom [1] suggests that the factors of 210 form a Boolean algebra if one defines $A \cup B$ as the least common multiple of A and B, and $A \cap B$ as the greatest common divisor of A and B. Andree [2] has a problem indicating the same result for the number 30. The complement of x is $210/x$ and $30/x$, respectively.

The results can be easily generalized. Let $p_1, \cdots, p_n$ be a collection of distinct primes, and consider the set theory algebra of the 2^n subsets. Now associate to each set the product of the numbers in the set (to the null set is associated unity) and consider the algebra which is formed according to the definitions in the first paragraph. The two algebras are isomorphic.

The result is being considered in the analysis of switching circuits.

References

1. P. C. Rosenbloom, The Elements of Mathematical Logic, New York, 1950, p. 10.
2. R. V. Andree, Selections from Modern Abstract Algebra, New York, 1958, p. 76.

MATRIC GROUPS

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There is a commonly-held fallacy that a set of matrices which forms a group under matric multiplication must contain only nonsingular matrices.

The assignment

$$a \to \begin{bmatrix} 0 & 0 \\ -a & a \end{bmatrix} = f(a)$$

establishes an isomorphism between the real field and a subset of the set of all two-by-two matrices, using the usual matric operations. Excluding $a=0$ one obtains a multiplicative group, even though *each* matrix is singular.

The difficulty is that "inversion" has two different meanings. For $a \neq 0$ the matric-inverse of $f(a)$ does not exist while $f(a)$ has the obvious inverse $f(a^{-1})$ relative to the identity element $f(1)$.

The astute undergraduate may observe that the single element $f(1)$ is a group under multiplication and that—even worse!—the single element $f(0)$ is a group.

STRONG CONTINUITY IN TOPOLOGICAL SPACES

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A well known necessary and sufficient condition for $f: S \to T$ to be continuous is that $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset S$. We attempt in this note to investigate the strength of the more restrictive condition $f(\overline{A}) \subset f(A)$ for all $A \subset S$. When S is locally connected we can obtain a characterization of such functions.

DEFINITION 1. *A single valued function $f: S \to T$ will be termed strongly continuous (or called an S.C. function) if and only if $f(\overline{A}) \subset f(A)$ for all $A \subset S$.*

It is quite clear that $f: S \to T$ is *S.C.* if and only if $f(A') \subset f(A)$ for all $A \subset S$ where A' denotes the derived set of A.

THEOREM 1. *$f: S \to T$ is S.C. if and only if $f^{-1}(B)$ is closed for all $B \subset T$.*

Proof. Necessity. Let $p \in (f^{-1}(B))'$. Then $f(p) \in ff^{-1}(B)$ and $ff^{-1}(B) \subset B$ and hence $f(p) \in B$. Thus $p \in f^{-1}(B)$ and $f^{-1}(B)$ is closed. Sufficiency. Let $A \subset S$. Then $A \subset f^{-1}f(A)$ and $A' \subset (f^{-1}f(A))' \subset f^{-1}f(A)$ since $f^{-1}f(A)$ is closed. Thus $f(A') \subset f(A)$ and $f: S \to T$ is *S.C.*

COROLLARY 1. *$f: S \to T$ is S.C. if and only if $f^{-1}(B)$ is open for all $B \subset T$.*

The reader will easily construct a proof using complements.

COROLLARY 2. *$f: S \to T$ is S.C. if and only if $f^{-1}(B)$ is both open and closed for all $B \subset T$.*

The proof follows immediately from Theorem 1 and Corollary 1.

THEOREM 2. *Let $f: S \to T$ be S.C. and A a nonempty connected subset of S. Then $f(A)$ is a single point.*

Proof. Suppose $f(A)$ contains more than one point. Let $p \in f(A)$. Then $f^{-1}(p) \cap A$ is a proper subset of A and by Corollary 2 is both open and closed in A. Thus A is not connected, a contradiction.

The converse of Theorem 2 is false. For, suppose S: 0, 1, $\frac{1}{2}$, $\cdots$, $1/n$, $\cdots$ with the usual topology on the line. Let T be the space of the reals. Define $: S \to T$ as follows: $f(1/n) = 0$ and $f(0) = 1$. Then $f: S \to T$ does take nonempty connected sets in S into points, but it is clearly not continuous, and hence not *S.C.*

We have, however, a partial converse to Theorem 2 in the following theorem.

THEOREM 3. *Let $f: S \to T$ be a single valued transformation and S locally connected. Let $f(B)$ be a single point whenever B is a connected nonempty subset of S. Then $f: S \to T$ is S.C.*

Proof. We will show that $f(A') \subset f(A)$. Let $p \in A'$. Since S is locally connected, we have an open connected set 0 containing p and thus $0 \cap A \neq \phi$. Now $f(p) \in f(0)$ and $f(0)$ is a single point. Also $\phi \neq f(0 \cap A) \subset f(0) = f(p)$. Hence $f(p) = f(0 \cap A)$ and thus $f(p) \in f(A)$.

EXCEPTIONAL EXTREMUM PROBLEMS

C. S. OGILVY, Hamilton College

The elementary calculus method of locating the maximum or minimum points of a function of one variable may fail for either of two reasons. (1) The function is stationary outside the range applicable to the problem, resulting in the well known "end point" maximum or minimum. This situation is usually, but not always, easily recognizable [1]. (2) The failure is caused by the use of what Widder calls an unsuitable independent variable [2]. It is often difficult to predict when this will happen.

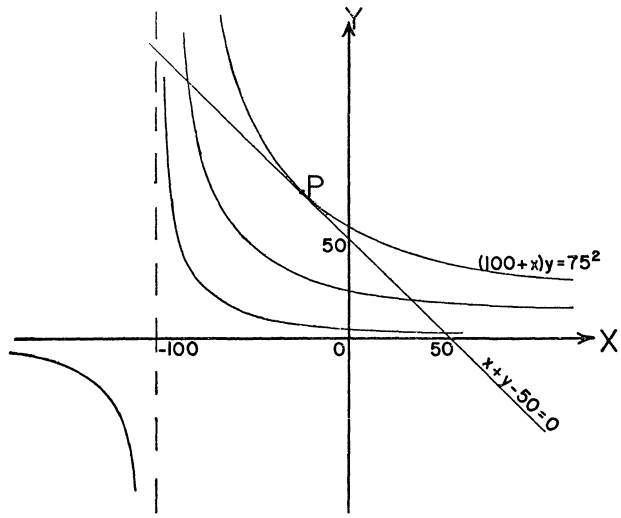


FIG. 1

The method of Lagrange multipliers may or may not get around the difficulty. In the case of an end point maximum or minimum it will frequently be of no help. Consider the following problem. Given a straight fence 100 feet long, we wish by adding 200 feet more to form a rectangular enclosure whose boundary contains all of the original fence. How shall this be done so as to enclose the greatest possible area? [3] If we let x be the length of the new fence which is to be aligned with the original 100 feet and proceed in the usual fashion to maximize the resulting expression for area, x comes out to be -25, which is not permitted by the conditions of the problem. If we use a Lagrange multiplier to maximize the function $f(x, y)=(100+x)y$ subject to the restraint $g(x, y)=100+2x+2y-200=0$, we get the same result. The method of Lagrange must lead to a point or points where one of the curves $f(x, y)=c$ is tangent to $g(x, y)=0$ [4]. Figure 1 indicates what is happening. A member of the family of hyperbolas $(100+x)y=c$ is tangent to the line $x+y-50=0$ at the point P: $(-25, 75)$, the

correct values of (x, y) for absolute maximum area, a square enclosure 75 feet on a side. But in order to bring the solution within the stipulation of the original problem we need $x=0$, an end point maximum. The function is not stationary there, of course; and from this point of view the failure of both the elementary and the Lagrange method is essentially trivial. They could not be expected to succeed.

Consider next Widder's example: to find the shortest distance from the focus $(1, 0)$ to the parabola $y^2=4x$. Here $f(x, y)=(x-1)^2+y^2$ must be minimized subject to the restraint $g(x, y)=y^2-4x=0$. An attempt to eliminate y from $f(x, y)$ and set $f'=0$ yields $x=-1$, a point not on the parabola. Of course the method of Lagrange succeeds, because the circle $f(x, y)=c^2$, c being the distance from the focus, can become tangent to the parabola at the required point $(0, 0)$. We have $u=f(x, y)+\lambda g(x, y)=(x-1)^2+y^2+\lambda(y^2-4x)$,

$$\partial u/\partial x = 2(x-1) - 4\lambda = 0,$$
$$\partial u/\partial y = 2y + 2\lambda y = 0,$$
$$\partial u/\partial \lambda = y^2 - 4x = 0.$$

In the simultaneous solution of these three equations in this particular example one "could not eliminate λ by solving the second equation thereof for λ. For, $\partial g/\partial y=0$ at the very point which yields the minimum" [2]. But actually, the implication that the two conditions in quotations are generally interdependent is unjustified. We shall see that inability to eliminate λ in this way is not a consequence of the vanishing of $\partial g/\partial y$.

What we should like, ideally, is a criterion for judging in advance the "suitability" of an independent variable. Unfortunately this seems rather too much to ask. We might surmise, from the above example, that the difficulty occurs because x and the distance function, say s, behave in the same way at the minimum point. That is, ds/dx becomes meaningless at $(0, 0)$ because, as one moves along the parabola, x and s become stationary at the same time. But this circumstance is in fact neither necessary nor sufficient. If one seeks the extrema of the distance from the point $(2, 0)$ to the hypocycloid whose equation is $x^{2/3}+y^{2/3}=1$, the attempt to maximize s as a function of x fails to yield the local maxima at $(0, \pm 1)$ even though s and x are not "behaving similarly" at these points. (Later we shall find the converse counter example.) This is also an illuminating illustration of the limitations of Lagrange multipliers, which yield only the local maxima at P_1, P_2, where the circle representing the distance function is tangent to the curve, and none of the cusps (Fig. 2).

We digress temporarily to investigate the apparently peculiar state of affairs near the focus in Widder's problem. In order to simplify the details of the necessary algebra, we consider instead a similar example: [5] to find that point on the circle $x^2+y^2=1$ nearest to $(\frac{1}{2}, 0)$. If we let $L^2=s$, the square of the distance from $(\frac{1}{2}, 0)$ to any point (x, y) on the circle, we have $s=(\frac{1}{2}-x)^2+1-x^2$, and setting $ds/dx=0$ yields the absurd result $-1=0$. To be slightly more general,

we can say that if s is the squared distance from a point on the circle to any point (a, b), $a \neq 0$, the elementary method fails when $b=0$. However, for any $b \neq 0$ but *arbitrarily close to zero*, the same method is successful. For instance, if P is a point which is allowed to assume various positions along the line $x=a \neq 0$, then for every position except $(a, 0)$ there is no trouble. What causes this sudden discontinuity?

The expression for s is

$$s = (x - a)^2 + (\pm \sqrt{(1 - x^2)} - b)^2. \tag{1}$$

If $b=0$

$$s = a^2 + 1 - 2ax, \tag{2}$$

a straight line of slope $-2a$ in the (x, s)-plane. Admissible values of x lie in the interval $-1 \leqq x \leqq 1$, and if x equals 1 or -1, $L=|a-1|$ or $|a+1|$, the end point minimum or maximum.

If $b \neq 0$, the curve of (1) is an ellipse. For any fixed a (we take $a=\frac{1}{2}$ in Fig. 3), we have a family of ellipses with b as the parameter, whose envelope is the pair of parallel lines $x=\pm 1$. Each ellipse is tangent to both lines and hence is centered on the s-axis; its maximum and minimum points are the maximum and minimum s for that particular a and b. The major axis of each ellipse is inclined to the horizontal at an angle ϕ whose tangent is negative and numerically larger than $2a$. As b tends toward ∞ or $-\infty$ for any fixed a, $\phi \to \frac{1}{2}\pi$ and the ellipse moves upward, both s(max) and s(min) tending toward $+\infty$. As $b \to 0$, $\tan \phi \to -2a$. The segment of the straight line of (2) lying in the interval $-1 \leqq x \leqq 1$ is a degenerate ellipse, the limiting position of the ellipses of the family. If we differentiate (1) with respect to x and set $ds/dx=0$, and then eliminate the parameter b between this equation and (1), we obtain the locus of the maximum and minimum of s, namely

$$s = (1 - a/x)^2. \tag{3}$$

As $b \to 0$, the ordinary maximum and minimum points of the nondegenerate ellipses approach the end point maximum and minimum points of the degenerate ellipse continuously along the two branches of the curve of (3), thus accounting for the apparent discontinuity mentioned at the outset. This kind of behavior is typically encountered in such problems.

We turn now to our final example: to find the cone of maximum lateral surface area with vertex at the origin and inscribed in the ellipsoid formed by rotating the curve $y^2/a^2+x^2/b^2=1$ around the Y-axis (Fig. 4). If one expresses the lateral surface area S as a function of x and sets $dS/dx=0$, the result yields the true maximum if $a^2>2b^2$. If $a^2=2b^2$, the maximum occurs at the point $(b, 0)$ where the cone has flattened out into a circular disc. The elementary method yields this result even though x and S are stationary at the same time, the converse counter example mentioned earlier. If $a^2<2b^2$, the maximum is again at

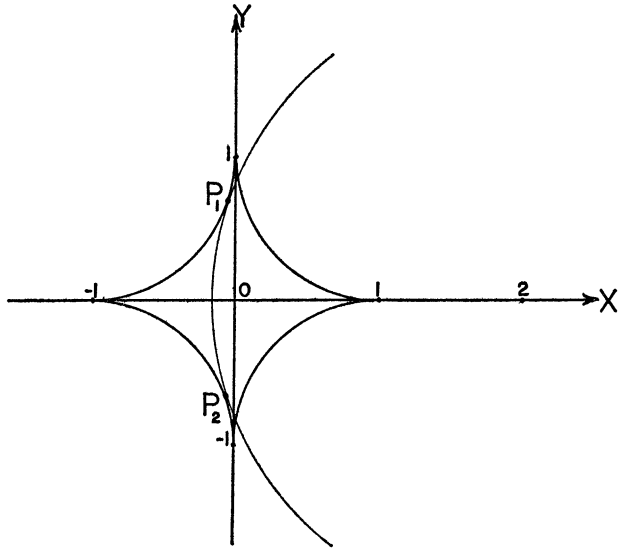

Y
1
P_1
-1
0
1
2
X
P_2
-1

FIG. 2

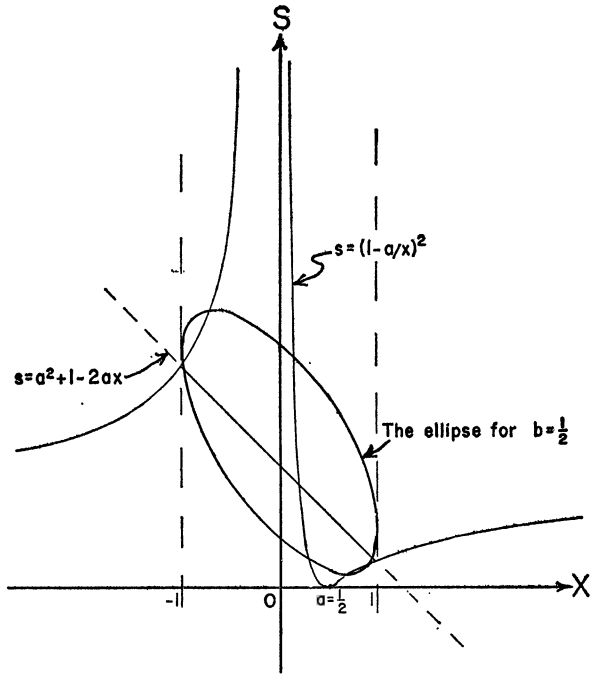

S
$s=(1-a/x)^2$
$s=a^2+1-2ax$
The ellipse for $b=\frac{1}{2}$
X
-1
0
$a=\frac{1}{2}$
1

FIG. 3

$(b, 0)$, but this time it is an end point case and hence not revealed by the elementary attack.

If one uses the Lagrange method to maximize $f(x, y) = \pi x\sqrt{(x^2+y^2)}$ subject to the restraint $g(x, y) = y^2/a^2 + x^2/b^2 - 1 = 0$, one finds: (1) If $a^2 > 2b^2$, the maximum cone is delivered and also a second solution, $y = 0$, overlooked by the first method. (2) If $a^2 = 2b^2$, the circle is the answer. This is the case where $\partial g/\partial y = 2y/a^2 = 0$ at the point where the maximum occurs, yet it is perfectly feasible to eliminate λ in the course of the solution. (3) If $a^2 < 2b^2$, $y = 0$ as in (1).

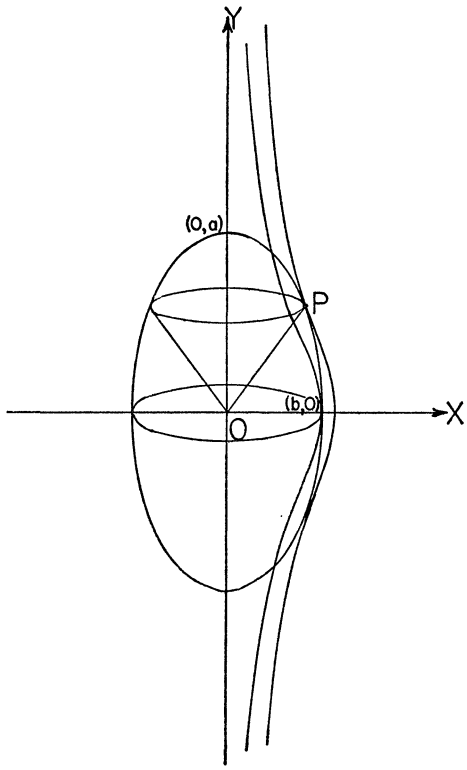


FIG. 4

It should be noted that there is always another extremal at $(0, a)$, the limiting case where the lateral surface area tends to zero. As usual, the reason for the failure of the Lagrange method to produce this minimum is that the curves $f(x, y) = \text{constant}$ can never be tangent to $g(x, y) = 0$ there; they have the line $x = 0$ as a vertical asymptote. On the other hand, one of them can be tangent to $g(x, y) = 0$ at the point $(b, 0)$, which explains why the method picks up the end points in cases (1) and (3) of the previous paragraph. Figure 4 shows an ellipse for which $a^2 > 2b^2$, and two tangent curves $f(x, y) = c$. The lateral surface

area of the inscribed cone has its absolute maximum at the point P and a local minimum at the point $(b, 0)$.

There exists a device which will yield *all* the extrema: a suitable parametrization of the original variables. This is not difficult to effect. The method of choosing the parametrization has been indicated by Oakley [6]. The interested reader can find all six of the extrema in the hypocycloid example by the use of the conventional parametrization, $x=\cos^3\theta$, $y=\sin^3\theta$.

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3. V. L. Klee, Jr., this MONTHLY, vol. 54, 1947, p. 409.
4. See, for instance, Angus E. Taylor, Advanced Calculus, Boston, 1955, Sec. 6.8.
5. Frank Hawthorne, A simple endpoint maximum, this MONTHLY, vol. 58, 1951, p. 188. (The problem is stated but not analyzed.)
6. C. O. Oakley, Endpoint maxima and minima, this MONTHLY, vol. 54, 1947, pp. 407–409.

CURVATURES OF $r^n=\cos n\theta$

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Familiar members of the family of curves with polar equation $r^n=\cos n\theta$ are the *circle* for $n=1$, the *lemniscate* for $n=2$, the *line* for $n=-1$, the *hyperbola* for $n=-2$, the *cardioid* for $n=\frac{1}{2}$, and the *parabola* for $n=-\frac{1}{2}$. For rational n, all members have simple constructions for tangents and centers of curvature if multisection of angles is permitted.

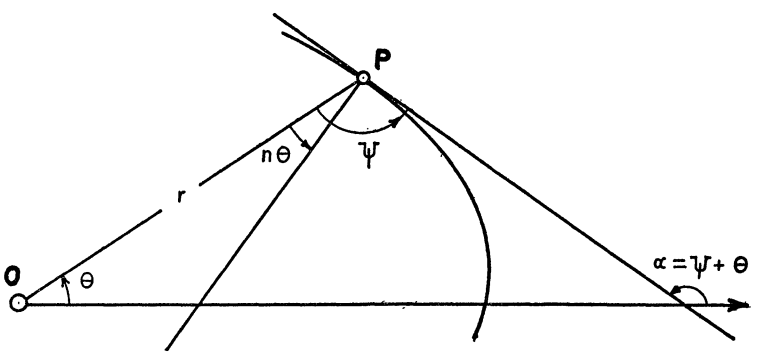


FIG. 1

Let α be the angle formed by a tangent with the polar axis, ψ the angle from OP to the tangent (counterclockwise, positive), and s an arc length of the curve (Fig. 1). Then from $r^n = \cos n\theta$,

$$r^{n-1}r' = -\sin n\theta, \qquad [' \equiv d/d\theta] \tag{1}$$

and

$$\tan \psi = r/r' = -\cot n\theta = \tan (\tfrac{1}{2}\pi + n\theta).$$

Accordingly,*

$$\psi = \tfrac{1}{2}\pi + n\theta \quad \text{and} \quad \alpha = \psi + \theta = \tfrac{1}{2}\pi + (n+1)\theta.$$

Thus the normal at P forms the angle $n\theta$ with OP. From (1) the sum of the squares gives

$$r^{2n-2}(r^2 + r'^2) = 1 \quad \text{or} \quad ds/d\theta = 1/r^{n-1}.$$

The radius of curvature at P is thus

$$R = ds/d\alpha = (ds/d\theta)(d\theta/d\alpha) = (1/r^{n-1})[1/(n+1)],$$

or, since $r^n = \cos n\theta$,

$$R = \{r/(n+1)\} \cdot \sec n\theta.$$

1. *The lemniscate* (Fig. 2): $r^2 = \cos 2\theta$ for $n = 2$ has

$$\psi = \tfrac{1}{2}\pi + 2\theta, \qquad \alpha = \tfrac{1}{2}\pi + 3\theta, \qquad R = \tfrac{1}{3}r \sec 2\theta.$$

The length $\overline{OP} = r$ is trisected to give $\overline{QP} = \frac{1}{3}r$. The perpendicular to OP at Q meets the normal at P in the center of curvature C.

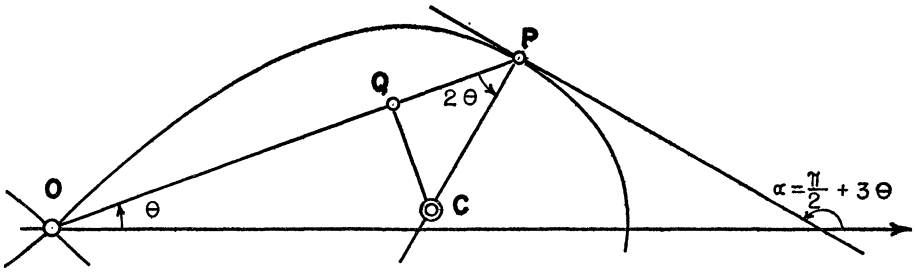


FIG. 2

2. *The hyperbola* (Fig. 3): $1 = r^2 \cos 2\theta$ for $n = -2$ has

$$\psi = \tfrac{1}{2}\pi - 2\theta, \qquad \alpha = \tfrac{1}{2}\pi - \theta, \qquad R = -r \sec 2\theta.$$

OP is extended its own length to Q. The perpendicular to OQ at Q meets the normal in the center of curvature C. Note that the length of the normal from P to the polar axis is r.

3. *The cardioid.* (Fig. 4): $r^{1/2} = \cos \frac{1}{2}\theta$ (or $2r = 1 + \cos\theta$) for $n = \frac{1}{2}$ has

$$\psi = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta, \qquad \alpha = \tfrac{1}{2}\pi + \tfrac{3}{2}\theta, \qquad R = \tfrac{2}{3}\cos \tfrac{1}{2}\theta.$$

* Actually $\psi = \frac{1}{2}\pi + n\theta + k\pi$, k integral. Numbers k other than zero lead to the same tangent line.

The projection of the constant length $\overline{PQ}=\frac{2}{3}$ on the normal is the radius of curvature and locates the center C.

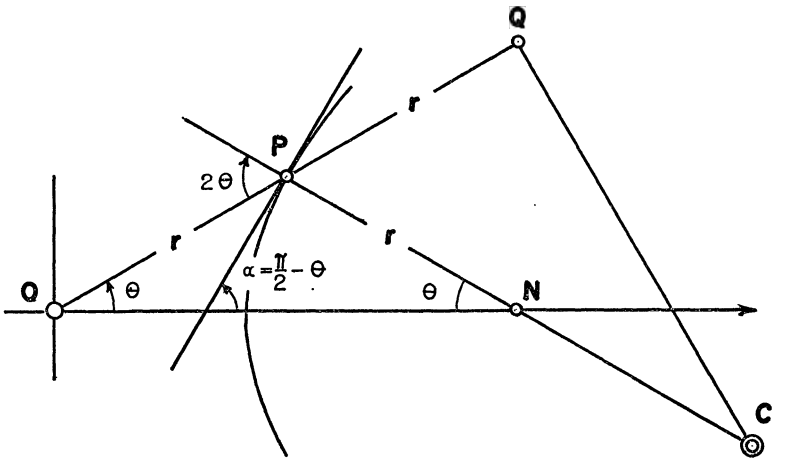


FIG. 3

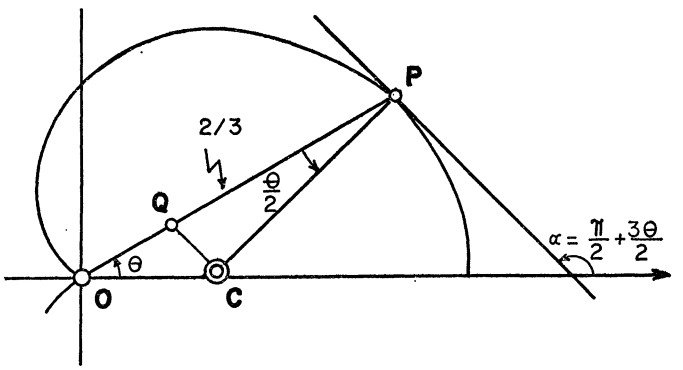


FIG. 4

4. *The parabola.* (Fig. 5): $r^{-1/2}=\cos\frac{1}{2}\theta$ [or $r(1+\cos\theta)=2$] for $n=-\frac{1}{2}$ has

$$\psi=\tfrac{1}{2}\pi-\tfrac{1}{2}\theta,\qquad \alpha=\tfrac{1}{2}\pi+\tfrac{1}{2}\theta,\qquad R=2r\sec\tfrac{1}{2}\theta.$$

PO is extended to Q with $\overline{OQ}=r$. The perpendicular to QP at Q meets the normal in the center of curvature C. Note, since $\alpha=\frac{1}{2}(\pi+\theta)$, that the inclination of the normal is $\frac{1}{2}\theta$. The reflective property with point source at the focus 0 is thus established. Note further since $\overline{ON}=r$ that $\overline{NP}=2r\cos\frac{1}{2}\theta$. Thus the projection of this normal length on the polar axis (a subnormal) is the constant:

$$\overline{PN}\cos\tfrac{1}{2}\theta=2r\cos^2\tfrac{1}{2}\theta=2.$$

Construction is simple: with center at the focus, draw a circle meeting the

parabola in P, the polar axis in N, T, and the extension of PO in Q. The tangent and normal at P are PT and PN. The center of curvature lies on the tangent to the circle at Q.

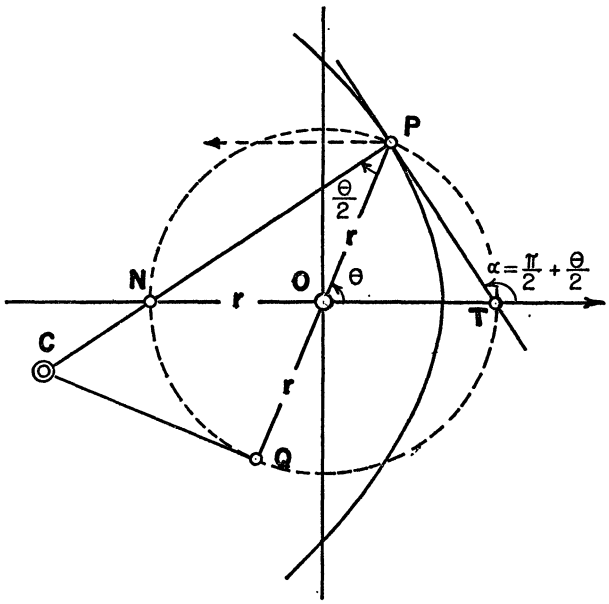


FIG. 5

MOTIVATING THE METHOD OF FROBENIUS

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The following note is believed to complement the interesting article by R. D. Larsson in the August-September 1958 issue of this MONTHLY.

The student is easily convinced that general solutions of linear differential equations may be obtained in the form of power series about an ordinary point at $x=0$ by introducing

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

into the differential equation.

When $x=0$ is a regular singular point he is told to introduce

$$y = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

because (1) may not work. If sufficiently curious he may even try (1). This "device" (2) can be justified, of course, by a theoretical discussion of the method of Frobenius. However, this is not always done, especially in "methods" courses.

The matter can be more successfully dealt with by observing that every linear differential equation (of the second order for simplicity) with a regular singular point at $x=0$ may be written

$$x^2y'' + x(p_0 + p_1x + \cdots)y' + (q_0 + q_1x + \cdots)y = 0. \tag{3}$$

For x sufficiently small we have

$$x^2y'' + p_0xy' + q_0y \approx 0$$

which is an Euler equation whose solution may be found by introducing $y=x^m$ so that

$$m^2 + (p_0 - 1)m + q_0 = 0. \tag{4}$$

Then, near $x=0$

$$y_1 \approx x^{m_1}, \qquad y_2 \approx x^{m_2} \qquad \text{if } m_1 \not\equiv m_2 \tag{5a}$$

and, as in Larsson's note,

$$y_1 \approx x^{m_1}, \qquad y_2 \approx x^{m_1}\log x \equiv y_1 \log x \qquad \text{if } m_1 = m_2. \tag{5b}$$

Thus, the introduction of (1), which in general implies that $y \approx a_0$ near $x=0$, cannot be expected to lead to a series solution of (3). And it is a natural consequence of (5a, b) to assume solutions of the form

$$y_1 = x^{m_1}\sum_{n=0}^{\infty} a_nx^n, \qquad y_2 = x^{m_2}\sum_{n=0}^{\infty} b_nx^n, \qquad m_1 \not\equiv m_2 \tag{6}$$

or

$$\begin{aligned} y_1 &= x^{m_1}\sum_{n=0}^{\infty} a_nx^n, \\ y_2 &= y_1[\log x + c_0 + c_1x + \cdots] \\ &\equiv y_1 \log x + x^{m_1}\sum_{n=0}^{\infty} b_nx^n, \qquad m_1 = m_2. \end{aligned}$$

Incidentally, (4) is the indicial equation for (3) and can always be obtained in this manner, i.e. by grouping the terms which are dimensionally homogeneous in x^m of lowest degree and obtaining the characteristic equation of this expression as if it were an Euler differential equation.

In the event that m_1, m_2 differ by an integer it is well known that (6) will still lead to one solution corresponding to the larger of m_1 and m_2, the second solution being obtained by reduction of order. However, one usually tries the smaller of m_1, m_2 first since both solutions (or none) may result.

A PEDAGOGICAL ASPECT OF THE DEVELOPMENT OF THE REAL NUMBERS

ROBERT S. LEDLEY, The George Washington University

Courses in real variables frequently include some set theory followed by a discussion of infinite sets and the associated cardinals and ordinals. Such courses usually also include as a separate topic the classical development of the real numbers starting from Peano's axioms for the positive integers (see, for example, the text by Thielman [1]). There are two disadvantages to such separate treatments of these two topics: first, it imposes a break in the continuity of the mathematical development of the course; and second, Peano's axioms themselves appear rather arbitrary and perhaps unsatisfying, especially because finite induction reads more like a theorem than a postulate. The purpose of this brief note is to suggest what is in the author's opinion a pedagogically and philosophically more satisfying unified approach to this aspect of the development of the real numbers. This approach would be first to consider infinite sets and ordinals and cardinals; then the *positive integers* can be defined as an *infinite well-ordered set of which every element except the first has an immediate predecessor.* Since the student will have already been introduced to the infinite and to well-ordering through the study of the cardinals and ordinals, such a definition would more naturally appeal to the basic intuitive idea of the positive integers as well as unify the topics of cardinals, ordinals, and positive integers. Finite induction may then be proved as a theorem, and the real numbers then developed from the positive integers in the conventional manner with no other changes.

One procedure might be first to prove the so-called principle of transfinite induction, and from this to prove Peano's finite induction postulate. A standard proof proceeds as follows:

THEOREM (*principle of transfinite induction*). *Let A be a well-ordered set, and let $B \subset A$ such that*

(1) *B includes the first element of A, and*

(2) *If $a \in A$ implies that all elements preceding a are in B, then $a \in B$.*

Then we can conclude that $A = B$.

Proof. Suppose $A \neq B$. Then let $C \subset A$ ($C \neq \phi$) be all elements of A not in B. Then C does not contain the first element of A [because B does by hypothesis (1)]. Hence there exist elements of B preceding all elements of C. Let $c_0 \in C$ be the first element of C. Then all elements preceding c_0 are in B. Then, by hypothesis (2), $c_0 \in B$ contradicting the construction of C. Hence $A = B$. Q.E.D.

The advantage of this proof is that it has only made use of the properties of well-ordered sets, with which the student should now be familiar. Next observe that every element of a well-ordered set A has an immediate successor (*e.g.*, for a' to be the immediate successor of a, we can define a' as the first element of the set $S = \{a^* \mid a^* \in A,\ a^* > a\}$. Also, if the immediate predecessor a of a' exists, then a' is the immediate successor of a.

Now we are ready to prove finite induction, based on our definition of the

positive integers given above.

THEOREM (*Peano's postulate of finite induction*). *Let A be the set of positive integers, and let $B \subset A$ such that*

(1) *B includes the first element of A, and*

(2) *$a \in B$ implies that its immediate successor $a' \in B$.*

Then we can conclude that $A = B$.

Proof. Hypothesis (1) of transfinite induction is the same as hypothesis (1) of finite induction. Next suppose that $a' \in A$ implies that all elements preceding a' are in B. Then, since the immediate predecessor of a' is in B by hypothesis (2), $a' \in B$; thus hypothesis (2) of transfinite induction is satisfied. Thus by the transfinite induction theorem, $A = B$.

The rest of Peano's axioms follow almost trivially, and the development of the real numbers can now proceed in the classical form.

Reference

1. Henry P. Thielman, Theory of Functions of Real Variables, Englewood Cliffs, N. J., 1953, Chs. I and II, pp. 1–51.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE ROLE OF GEOMETRY FOR THE MATHEMATICS STUDENT*

HERBERT BUSEMANN, University of Southern California

The importance of geometry in college mathematics has declined for some time, in fact there are now quite a few colleges which offer no course at all in geometry proper. While this development does not worry all mathematicians, it caused concern to sufficiently many to start the present discussion. Here we first deal with the present status of geometry in college, *i.e.*, with the courses which are actually offered, then try to evaluate these courses in view of the aims of geometry teaching, and will arrive at some definite proposals regarding courses to be eliminated or added. The aims, and consequently the evaluation and the proposals, necessarily depend partly on the speaker's views and taste.

The presentation of the actual situation is principally based on the information furnished by the catalogues of 182 colleges, including all the better known ones. The catalogues usually fail to provide all the relevant facts. Course de-

* Presented at the Fortieth Summer Meeting of the Association, Salt Lake City, September 1, 1959.

scriptions are often inadequate or absent. For example, it would have been important for our present purposes to know whether a course in projective geometry uses an algebraic, a geometric or a mixed approach. There is often no indication, which may have the very good reason that the choice of method is left to the teacher. Also, the frequency with which a course is given is not always listed.

Nevertheless the following two tables extracted from the catalogues are believed to be useful; they give the total number of courses in geometry per college and for each individual course the number of colleges teaching it. However it must first be stated which courses are taken into account. Since this lecture is concerned with undergraduate teaching only, graduate courses in geometry do not figure in the tables unless they are explicitly listed as open to undergraduates. This point principally concerns differential geometry. Vector analysis is not counted although it almost always comprises some solid and some differential geometry. Above all, the standard course: "Analytic Geometry and Calculus" is disregarded. This course originated from the waning interest in geometry, almost always proceeds at the expense of geometry,* and is largely responsible for perpetuating the condition.

TABLE 1

Total number of geometry courses per college

Number n of courses in college	Number of colleges offering n courses
0	13
1	23
2	75
3	39
4	24
5	4
6	4

TABLE 2

Types of Geometry Courses

Course		Number of colleges offering this course
Analytic Geometry		58
Projective Geometry		
undefined	69	
algebraic	20	99
geometric-axiomatic	10	
Foundations of Geometry		9
Noneuclidean Geometry		29
Higher Geometry		33
Modern Euclidean Geometry (triangle and circle)		52
Differential Geometry		61
Other		21

The sum of the numbers in the right column does not equal $1\times23+2\times75+\cdots+6\times4$ because two semester courses in the same subject, like Higher Geometry I, II, appear only once on the right and twice on the left.

Examining the individual courses, we notice that Differential Geometry is in a different position from the rest. Its importance is generally admitted, the subject flourishes. But this leads naturally to the tendency of introducing the student to the now active parts of the field. The course then becomes an advanced graduate course. On the other hand the theory of surfaces in E^3 is a very rich field still quite active, and many of its results are not contained in the

* Since this talk was given Angus E. Taylor's Calculus with Analytic Geometry appeared. It contains most of the material taught in analytic geometry.

type of advanced course just mentioned. Perhaps every undergraduate ought to have a chance (but should not be required) to learn the classical theory, which is also useful in many applications.

There are influential mathematicians doubting the value of any of the remaining courses. We will discuss them from three points of view which may be briefly listed as: factual information, axiomatic thinking, training of the geometric intuition.

Nobody denies that some analytic geometry is necessary, but geometers doubt that "Analytic Geometry and Calculus" suffices. Nevertheless, since saving time for newer developments is essential, this situation may be accepted, provided the student becomes familiar with projective coordinates. Our table shows that projective geometry is by far the most popular course in geometry. The usual objection to the course (which also turned up in the discussion after this lecture) is that the topic is no longer an active subject of research; and that it can easily be introduced (but not adequately treated) as an appendix to the now popular vector spaces. The principles underlying this attitude are highly dangerous. The subject is beautiful, it required major efforts of some of the best mathematicians during and after the last century to understand its structure thoroughly, it still gives a deep insight into geometry. The projective duality principle, besides being constantly used by geometers, is an exciting experience for any interested student. The duality in vector spaces is a very meager substitute. But most of all, do we have the right to completely disrupt historical continuity whenever a subject moves out of the focus of contemporary interests? Do we really expect or agree that our present mathematical efforts will be altogether junked (at least from courses) as soon as the interests change? This disregard for historical continuity obviously leads to dangerous absurdities if taken seriously and must be fought. As far as geometry is concerned the speaker believes that projective geometry is a subject on which to make a stand. A mathematics major should not obtain his degree without knowing projective geometry. There remains the question how the subject should be taught. It will be seen later why the axiomatic approach seems less important now, hence a course should preferably contain both algebraic and geometric arguments. This freedom in method also has the advantage of yielding results most quickly.

Noneuclidean geometry played, historically, the important role of an eye opener. It will still have this effect on the student, moreover it lends itself readily for an exciting undergraduate course and should therefore be offered once in a while either by itself or as a part of another course.

The value of higher geometry as a course depends on its content. It often consists mainly of projective and noneuclidean geometry; then the preceding arguments justify its existence. All too often the content is left to the instructor. Instances are not rare, where the course is assigned to a nongeometer, and then degenerates, for example, into the pure algebra connected with constructions by ruler and compass.

Modern euclidean geometry appears under various synonyms, for example

"Modern (or Advanced) Geometry of Circles and Triangles." It deals with those facts of elementary euclidean geometry which were discovered after Apollonius, so that the definition "modern" used here means "not more than 2100 years old." The course consists of individual niceties entirely unrelated to anything else. It has a slight merit as a training of the intuition and for high school teachers, but we will see later that this can be accomplished far better. The course should therefore disappear.

We now come to our second point, namely axiomatics and in particular to the foundations of geometry as a course, which was passed over above. Not so very long ago this course and axiomatic projective geometry constituted (for historical reasons) the only opportunity for introducing the student to axiomatic thinking, which greatly appeals to many minds with a mathematically critical or a logical bent. Now large parts of mathematics are treated axiomatically, and, what is worse, the axioms for a group, for a field, or for a Hausdorff space, are simpler by far than those for euclidean geometry. Therefore the foundations of geometry no longer satisfy a real need; the same applies to the axiomatic approach to noneuclidean or projective geometry, although the axioms for the latter are simple. Many geometers will find it hard to admit this, because axiomatics is very close to their hearts and is still a rather active field of research. Students interested in the subject turn up now and then, but they can very well be helped by a reading course. In this connection it should be considered whether the modern development of the foundations of geometry on the basis of group theory* might not be preferable to Hilbert's approach.

The shift in mathematics towards the axiomatic approach and the concomitant abstractness make it imperative that some course in geometry strengthen the geometric intuition which is in great danger of being lost, but has always been one of the main inspirations in mathematical creation. The abstract approach not only fails to develop the intuition, but the unexpected phenomena arising in very general situations often completely destroy the confidence in intuitive arguments just in the best students. When "moving a hyperplane in E^n continuously" (in the course of a proof), the author was repeatedly interrupted with the question: "Which topology for hyperplanes do you use?" Needless to say that a student who has qualms regarding this operation is forever lost to geometry and geometric reasoning.

The course in "Modern Euclidean Geometry" does train the intuition to a certain extent, but is objectionable for the reasons already given and, in the present context, because it seems to confirm the rather prevalent feeling that modern mathematics is too advanced or intricate for the intuition.

A glance at *Mathematical Reviews* shows that the theory of convex bodies is quite active. It is applied in many other fields, for example in game theory; its proofs are often purely synthetic. Didactically, convexity has the advantage

* A book on this topic is: F. Bachmann, Aufbau der Geometrie aus dem Spiegelungsbegriff, Berlin, 1959.

over the classical subjects of euclidean geometry that it is defined essentially by an inequality and hence requires much deeper arguments. The proof that convexity can be treated in a form accessible to sophomores is provided by the book of I. M. Yaglom and V. G. Boltyanski on convex figures, which was first published in 1951 in Russia, has since appeared in a second Russian edition and in a German translation (*Konvexe Figuren*, Verlag der deutschen Wissenschaften, 1956), and will soon appear in an English translation (P. J. Kelly and L. F. Walton). It deals with plane convex figures, but leads there to very subtle topics, providing training in continuity arguments, applications of Helly's Theorem, addition of convex figures, maximum and minimum problems with and without solutions, curves of constant width and their generalizations.

The material is divided into theorems or lemmas posed as problems occupying one third, and the solutions occupying the remaining two thirds, of the book, but it can be used as a text.

This book was discussed at some length because there is at present no other book satisfying our requirements, namely, to make strong demands on the intuition, to furnish a useful training for high school teachers as well as to those contemplating research, and to be accessible to sophomores, say. Thus the book constitutes a constructive existence proof, but we do not contend that the problem has only one solution. It is up to the geometers here and elsewhere to discover other possibilities. The more we try the better the chance will be of arriving at a synthesis of ideas which will reverse the present adverse trend in geometry.

TEACHING MACHINES AND LOGIC

JOHN W. BLYTH, Hamilton College

The nearly universal applicability of mathematics and logic stems in large part from their abstract and formal character. Yet this abstractness which is the source of their utility is also the source of widespread difficulty in mastering these subjects. It seems paradoxical that the two subjects with the most thoroughly rational structure of all subjects should prove to be the least intelligible to so many students.

There is now very good reason for believing that this paradox may be eliminated. The reason is the development of instructional materials programmed for presentation by a teaching machine or "learning" machine. This mechanical tutor encourages the student to accept responsibility for his own learning.

The task of programming a lesson bears some analogy to the task of preparing a problem or a set of data for automatic processing in a digital computer. It is necessary to break up a problem into a large number of small successive steps. Processing data requires the performance of many operations such as adding, dividing, printing, etc. But before each operation a question must be answered. Two of the requirements which must be met by each question are illuminating

for our present purpose. In the first place the question must be stated in the unambiguous language of the computer so that one question is clearly discriminated from another. In the second place the question must have an answer that is clearly correct or incorrect. Questions which do not meet these conditions will bring the computer to a halt or send it "looping" around in a vicious circle.

One more illuminating parallel is a characteristic built into a computer. It is designed to double check every step in a calculation. It cannot proceed to the next item in a program until the correctness of the preceding step has been determined.

A program for processing a set of data must be prepared to meet the requirements imposed by the nature of the data and the design of the computer. Similarly a program for self-instruction must satisfy the requirements imposed by the subject matter and by the character of the human learner. The learning process must be analyzed into a large number of small steps to be taken in proper order. Each step is taken in response to a question. Each question must be so presented that the learner can easily discriminate the significant features of the situation. The question must have an answer that the student can recognize as correct or incorrect. Finally, the student must check each step before proceeding. If these conditions are not all satisfied, the learning operation stops temporarily or is at least greatly retarded.

Machines require no motivation and no reward. The learner requires both. Once a student has been motivated to start a learning task success in getting the correct answer may be sufficient reward to keep him working. There is a mounting body of experimental evidence indicating that this reward or reinforcement must follow the response immediately. Since people vary in reaction time, they will be ready to check their answers at different times. This means that the instructional process must be individualized.

The teaching machine is designed for individual use and permits each student to proceed at his own pace. The machine presents a question and provides a space for the student's answer. The correct answer is made available as soon as needed. Only after checking his response is the student able to proceed. The machine presents the steps in a thoroughly tested program in a controlled sequence.

An experiment in the use of such programmed materials in teaching logic was begun at Hamilton College in the fall of 1958 with the aid of a grant from the Fund for the Advancement of Education.* Definitive conclusions cannot yet be drawn, but the preliminary results are most encouraging.

* With an additional grant this experiment is continuing on an enlarged scale. It now includes courses in Mathematics, Psychology, French, and German in addition to Logic. The machine used in this experiment was designed by Professors Blyth, Godcharles and Jacobson and is being manufactured by The Hamilton Research Associates, Inc., Clinton, N. Y. Each lesson of 40 questions and answers is on microfilm in a thin $2'' \times 7\frac{1}{2}''$ plastic card. The student simply inserts the card in a slot and the frames are projected on a reading screen as needed. A companion audio unit is correlated automatically with the visual unit for foreign language.

Even with the programs resulting from the first effort of Professor Jacobson and myself, it was possible to reduce class contact hours by one third, increase the amount of material covered, and simultaneously raise the level of mastery. The average grade on an hour examination on probability was raised by approximately 30 points over the corresponding examination the previous year. The need for private conferences was almost eliminated. Failure for that particular class was entirely eliminated.

The character of classroom work was radically changed. Since the students had mastered no errors through untutored practice, no time was wasted in identifying and removing such difficulties. With no need for routine checking and drill the class time could be devoted to further development and varied applications of the skills and concepts developed in the lessons.

In the summer of 1959 some of the lessons prepared for college students were tried in an experimental course in the Program of Advanced Studies for Gifted Children conducted by the Board of Cooperative Educational Services of the First Supervisory District in Oneida County. It was encouraging to find the younger students, some in the seventh grade, doing work that compared favorably with that of the college students. This suggests that good programs should produce widespread mastery of complex skills at a much earlier age than now thought possible.

Much hard work remains to be done to master this new application of two ancient skills of the expert teacher. Programming combines the Socratic method of teaching by asking questions with the Cartesian method of analyzing a problem into its simplest parts and proceeding from the simple to the complex. The use of the resulting programs in teaching machines will permit superior private tutoring on a massive scale. This will not eliminate the need for human teachers, but it will surely increase their effectiveness.

Mathematics Program of the Arizona Academy of Science

In the November 1959 issue of the MONTHLY a report was given on the mathematics parts of programs of state Academies of Science under the new National Science Foundation Academy of Science project. Since this report was submitted, an additional statement has been received from the Arizona Academy of Science. This statement is as follows:

The Arizona Academy of Science does not have a Mathematics Section, but the Southwestern Section of the Mathematics Association of America is cooperating with us in our Traveling Science Institute. We include mathematics in the program and anticipate considerable activity in this field. We shall be grateful for suggestions from the Committee on Visiting Lecturers to Secondary Schools and glad to cooperate in any way possible.

NSF Summer Institutes for Engineering Teachers

Ten of the 379 Summer Institutes for high school and college teachers being supported by the National Science Foundation in 1960 will be of direct interest to teachers of engineering, pre-engineering, and technical institute subjects. Information about these ten Summer Institutes appeared in the January, 1960, issue of the Journal of Engineering Education. Further information may be obtained from Mr. Clyde C. Hall, National Science Foundation, Washington, D. C.

TOPOLOGICAL IDEAS IN JUNIOR HIGH SCHOOL MATHEMATICS

R. D. ANDERSON, Louisiana State University

The experimental seventh and eighth grade text materials which have been prepared by the School Mathematics Study Group introduce a number of mathematical ideas not found in traditional texts at this level. The experimental books have a greatly increased emphasis on geometric ideas and intuition. A set-theoretic approach to geometry is adopted. Thus the point of view is that of introducing geometry and geometric thought not only for general information and background but also as a vehicle for the introduction of some other significant mathematical concepts. No attempt is made to lift tenth grade geometry as such and to transplant it. There are serious efforts to get the pupils to achieve some elementary understanding of three-dimensional space.

In the trials in the experimental centers during 1958–59, the unit (or chapter) on non-metric geometry was considered on the basis of teacher reports to be rather remarkably successful. This unit dealt with set terminology in the framework of some of those concepts of geometry which do not involve measurement. This experimental unit was the precursor of the present Chapter 4 of Volume I of *Junior High School Mathematics*. In this current chapter on non-metric geometry, there is considerable further emphasis on spatial concepts and relationships. Lines, planes, and space are regarded as sets of points. The possibilities of and properties of intersections of sets of lines or planes are considered. The separation properties of space by planes, of planes by lines, and of lines by points are emphasized. Segments (regarded as sets of points) are used to help identify some of these properties.

After triangles and angles with their interiors and exteriors (all as sets of points) are introduced, there is a brief discussion of one-to-one correspondences among sets of points, sets of lines, and sets of rays. The chapter ends with a discussion of simple closed curves—anticipating applications to circles, quadrilaterals and similar objects. The further chapters of the seventh grade experimental text which are geometrical in nature deal with measurement, parallels and perpendiculars, and the circle. These subjects can all be treated effectively with set terminology and the background of Chapter 4. As the general subject matter of these later chapters is more traditional in character, it will almost certainly be better known to most junior high school teachers.

Volume II is designed basically as an eighth grade program, but for 1959–60 is not considered as a complete text. In the geometric area it has units dealing with congruence relations, similar triangles, non-metric polyhedrons, and spatial volume and area concepts. Set terminology is used throughout. Of the various units, that on non-metric polyhedrons is almost certainly the most unusual and hence the most experimental.

This unit deals with considerations of polyhedrons not involving the concept of measure. Geometric 0-, 1-, 2-, and 3-dimensional simplexes are defined and, using these concepts, 1-, 2-, and 3-dimensional polyhedrons are studied. The

pupils are asked to build several models of tetrahedrons and cubes. The concept of dimension seems to come out rather clearly in this treatment. A solid cube is observed to be a union of 3-simplexes, its surface to be a union of 2-simplexes. The unit leads into the observation that a simple closed polygon separates a simple surface into two pieces. On models, these two pieces may be observed by a coloring process. Finally the Euler formula is developed in some elementary cases with considerable use being made of models and combinations of models.

It is not yet clear how effectively units of this type can be taught. Many mathematicians will agree, though, with the validity of efforts to improve geometric perception and to emphasize the mathematical importance of reducing complicated configurations to simple elements.

We can well ask ourselves whether or not prospective teachers are now being or are likely to be trained in the understanding and appreciation of concepts like those discussed above. Certainly, more and more set theory will continue to permeate the college mathematics courses for teachers. The concepts, terminology, notation, and use of set theory simplify many aspects of mathematics and give a unity to the language of this subject which has been sorely lacking in the recent past. There seem to be clear-cut trends in the teaching of geometry toward the use of set-theoretic definitions and points of view. We might profitably consider the introduction of a two- or three-week program in a seminar or class for (prospective) teachers which would deal with elementary ideas of simplexes, polyhedrons and dimension.

The Mathematics Speaker's Bureau of Metropolitan New York

The Metropolitan New York Section has formed a bureau which provides a convenient means for high schools to obtain speakers. Professor James N. Eastham of The Cooper Union has enlisted the voluntary services of twenty-seven college and five secondary-school mathematicians to speak on a variety of topics not common to the mathematics classroom. The topics are concerned with basic concepts, structures, fields, and applications of mathematics.

The general aim of this endeavor is to stimulate the interest of high school students in mathematics, to make them realize the need for proper mathematics preparation for careers in teaching and in scientific and technical fields, and to acquaint them with the opportunities open to those with an adequate mathematical training. Booklets listing the speakers and their summarized talks were sent to over 235 high schools. A student group can use the directory in the booklet to contact the desired speaker.

Expenses of the speakers are paid through a National Science Foundation grant made to The Cooper Union which is cooperating with the Metropolitan New York Section of the Association in this project.

Correction

There are presently no plans for the production of the IBM 9902 mentioned in *The national contest in high school mathematics in upper New York state*, by Nura D. Turner, this MONTHLY, vol. 67, 1960, pp. 73–74.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1406. *Proposed by Michael Goldberg, Washington, D. C.*

Cut an obtuse triangle into the least number of acute triangles.

E 1407. *Proposed by Leo Moser, University of Alberta*

Prove that, for $p>3$ and prime, ab^p-ba^p is divisible by $6p$.

E 1408. *Proposed by J. L. Selfridge, IBM, Yorktown Heights, N. Y.*

Find the highest power of 2 which divides the numerator of

$$1+\tfrac{1}{3}+\tfrac{1}{5}+\cdots+1/(2k-1).$$

E 1409. *Proposed by Paul Lukacs, Tel-Aviv, Israel*

In the game of bridge North is dealer. (a) What is the number of the different bidding sequences, assuming that West and East are silent? (b) What is the number of the different bidding sequences if all four players may participate in the bidding?

E 1410. *Proposed by Albert Wilansky, Lehigh University*

Is there a field whose additive and multiplicative groups are isomorphic?

SOLUTIONS

A Hula Hoop Problem

E 1362 [1959, 312]. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

Consider a vertical girl whose waist is circular, not smooth, and temporarily at rest. Around the waist rotates a hula hoop of twice its diameter. Show that, after one revolution of the hoop, the point originally in contact with the girl has traveled a distance equal to the perimeter of a square circumscribing the the girl's waist.

Editorial Note. It has been pointed out that Solution I [1959, 918] is faulty. After one revolution of the hula hoop about the girl's waist, the moving point has traveled *half* of a cardioid whose total length is $16a$, where a is the radius of the girl's waist.

Bretschneider's Formula

E 1376 [1959, 588]. *Proposed by V. F. Ivanoff, San Carlos, California*

Show that if A is the area of a quadrilateral having sides a, b, c, d and diagonals e, f then

$$16A^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2.$$

I. *Solution by C. F. Pinzka, University of Cincinnati.* Let e_1, e_2, f_1, f_2 be the segments of the diagonals, with e_1, f_1 adjacent to a. Let θ be the angle between e_1, f_1 and let θ' be its supplement. Then

$$16A^2 = 4e^2f^2 \sin^2\theta = 4e^2f^2 - (2ef\cos\theta)^2.$$

But

$$\begin{aligned} 2ef\cos\theta &= 2(e_1 + e_2)(f_1 + f_2)\cos\theta \\ &= 2e_1f_1\cos\theta - 2e_1f_2\cos\theta' - 2e_2f_1\cos\theta' + 2e_2f_2\cos\theta \\ &= -a^2 + b^2 - c^2 + d^2, \end{aligned}$$

by applying the law of cosines to each term. This gives the desired result.

II. *Solution by Joe Lipman, University of Toronto.* Represent the sides and diagonals a, b, c, d, e, f by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}$ so arranged that $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$, $\mathbf{e}=\mathbf{b}+\mathbf{c}$, $\mathbf{f}=\mathbf{a}+\mathbf{b}$. Then

$$16A^2 = 4(\mathbf{e}\times\mathbf{f})\cdot(\mathbf{e}\times\mathbf{f}) = 4(\mathbf{e}\cdot\mathbf{e})(\mathbf{f}\cdot\mathbf{f}) - 4(\mathbf{e}\cdot\mathbf{f})^2 = 4e^2f^2 - 4(\mathbf{e}\cdot\mathbf{f})^2.$$

But

$$\begin{aligned} 2(\mathbf{e}\cdot\mathbf{f}) &= 2(\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}) = -2\mathbf{b}\cdot(\mathbf{c}+\mathbf{d}) + 2\mathbf{c}\cdot(\mathbf{a}+\mathbf{b}) = 2\mathbf{a}\cdot\mathbf{c} - 2\mathbf{b}\cdot\mathbf{d} \\ &= (\mathbf{a}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{c}) - \mathbf{a}\cdot\mathbf{a} - \mathbf{c}\cdot\mathbf{c} - (\mathbf{b}+\mathbf{d})\cdot(\mathbf{b}+\mathbf{d}) + \mathbf{b}\cdot\mathbf{b} + \mathbf{d}\cdot\mathbf{d} \\ &= a^2 - b^2 + c^2 - d^2. \end{aligned}$$

The result now follows.

Also solved by A. N. Aheart, R. G. Albert, J. W. Baldwin, Leon Bankoff, Shlomo Ben-Adam, D. A. Breault, P. L. Chessin, T. R. Curry, F. J. Duarte, Underwood Dudley, C. E. Franti, George Glauberman, Michael Goldberg, L. D. Goldstone, Larry Grabam, S. H. Greene, G. L. Janusz, Edgar Karst, D. E. Knuth, L. F. Kosinski, T. V. Lakshminarasimhan, D. C. B. Marsh, J. W. Mettler, B. E. Mitchell, E. E. Morrison, F. D. Parker, J. F. Pietenpol, T. A. Porsching, J. M. Rice, Shingu-Tadao, R. Sibson, P. D. Thomas, T. C. Wales, Chung-lie Wang, John Weissman, and the proposer. Late solutions by G. R. Grainger, D. M. J. Hockley, J. M. Kingston, K. N. Kulkarni, and J. P. Phillips.

Editorial Note. The formula of this problem is not new. Pinzka pointed out that it appears as Theorem 8 in an article by G. Dostor entitled "Propriétés nouvelles du quadrilatère en général, etc." in *Archiv der Mathematik und Physik*, vol. 48 (1868), pp. 245–348. It also appears (with proof) in Section 165, pp. 204–205 of Hobson's *Treatise on Plane Trigonometry* (Dover reprint). Another solution appears in "A historically interesting formula for the area of a quadrilateral," by J. L. Coolidge, this MONTHLY, vol. 46, 1939, pp. 345–347.

If the quadrilateral is cyclic, then $ef=ac+bd$ and one easily obtains the Brahmagupta formula

$A^2=(p-a)(p-b)(p-c)(p-d)$, where p is the semiperimeter of the quadrilateral. If the quadrilateral is inscriptible, then $a+c=b+d$ and one obtains the formula $4A^2=e^2f^2-(ac-bd)^2$.

A Simultaneous System of Equations

E 1377 [1959, 588]. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Solve the system of equations

$$u + v = a, \qquad ux + vy = b, \qquad ux^2 + vy^2 = c, \qquad ux^3 + vy^3 = d$$

for u, v, x, y.

Solution by R. Sibson, Sutton, Surrey, England. One finds

$$ac - b^2 = uv(x - y)^2, \qquad bd - c^2 = uvxy(x - y)^2, \qquad ad - bc = uv(x + y)(x - y)^2,$$

whence $(x + y)(ac - b^2) = ad - bc$, $xy(ac - b^2) = bd - c^2$. If $ac=b^2$, then $a/b=b/c=c/d$ and three possible indeterminate solutions arise:

(i) $u=0$, $v=a$, $y=b/a$, x undetermined.

(ii) $v=0$, $u=a$, $x=b/a$, y undetermined.

(iii) $x=y=b/a$, $u+v=a$.

If $ac-b^2\neq 0$, then x and y are the roots of

$$(ac - b^2)t^2 - (ad - bc)t + bd - c^2 = 0,$$

and the values of u and v follow from the first two given equations.

Also solved by A. N. Aheart, Edward Barbeau, Shlomo Ben-Adam, W. J. Blundon, H. S. Bluston, A. P. Boblétt, G. B. Charlesworth, Paul Chessin, G. E. Cooke, F. J. Duarte, Daniel Finkel, George Glauberman, Gerald Goertzel, Michael Goldberg, S. H. Greene, Emil Grosswald, W. O. Hulser, V. F. Ivanoff, G. J. Janusz, M. S. Klamkin, Sidney Kravitz, D. E. Knuth, T. V. Lakshminarasimhan, W. J. Larkin, Joe Lipman, A. E. McGlauchlin, D. C. B. Marsh, C. S. Ogilvy, F. D. Parker, D. J. Persico, C. F. Pinzka, D. C. Stevens, R. G. Wagner, Chung-lie Wang, C. C. Yalavigi, J. W. Young, and the proposer. Late solutions by K. N. Kulkarni and Vencil Skarda.

Klamkin pointed out that this problem, with more variables, appears in *The Collected Papers of S. Ramanujan*, Cambridge, 1927, p. 18. Ramanujan gives a method of solving the system of equations with partial fractions.

The Reunion Problem

E 1378 [1959, 588]. *Proposed by Paul Pargas, Washington, D. C.*

Draw a circle by tracing around a given circular disk D and mark an arbitrary point A on the circle. By drawing circles only with the use of the disk D, where one is permitted to place the rim of D on any two distinct given points, find the point B on the original circle which is diametrically opposite the point A.

Solution by S. H. Greene, RCA, Croydon, Pa. If two equal circles with centers O_1 and O_2 intersect in points P and Q, then $\overrightarrow{O_1P}=\overrightarrow{QO_2}$.

Now let A be any point on circle (O_1) and draw any other circle (O_2) through

A, intersecting (O_1) again at Q. Then $\overrightarrow{O_1A} = \overrightarrow{QO_2}$. Draw any other circle (O_3) through Q intersecting circle (O_2) again at R. Then $\overrightarrow{QO_2} = \overrightarrow{O_3R}$. Draw any other circle (O_4) through R to intersect circle (O_1) at T and circle (O_3) at S. Then $\overrightarrow{O_3R} = \overrightarrow{SO_4}$. Draw the other circle (O_5) through S and T intersecting (O_1) again at B. Then $\overrightarrow{SO_4} = \overrightarrow{O_5T}$ and $\overrightarrow{O_5T} = \overrightarrow{BO_1}$. Thus $\overrightarrow{O_1A} = \overrightarrow{BO_1}$, and B is the point on circle (O_1) diametrically opposite A.

Also solved by D. C. B. Marsh, C. F. Pinzka, R. A. Wiggins, and the proposer. Late solution by L. D. Goldstone.

Editorial Note. Allied to this problem is that of constructing a circle (using the disk D) tangent to a given circle (O_1) (drawn with D) at a given point P. Hans Liebeck remarked that this allied problem was proposed and solved by R. Penrose in the October 1954 issue of *Eureka* (University of Cambridge, England). Problem E 1378 permits us to solve this allied problem. Thus draw any circle (O_2) through P to cut (O_1) again in A. By Problem E 1378, find the point B on (O_2) diametrically opposite A. Now draw the other circle (O_3) passing through P and B. This circle (O_3) is tangent to circle (O_1) at P.

An Abelian Semigroup

E 1379 [1959, 588]. *Proposed by A. A. Mullin, University of Illinois*

Let S be the set of all real numbers and let $a * b$ denote $\max(a, b)$, where $a \in S$, $b \in S$. Show that the system $\{S; *\}$ is an abelian semigroup.

Solution by Underwood Dudley, University of Michigan. First, $a * b$ is a real number, so the system is closed under the operation. Second, $(a * b) * c = a * (b * c) = \max(a, b, c)$, so the operation is associative. Third, since $\max(a, b) = \max(b, a)$, $a * b = b * a$, so $\{S; *\}$ is an abelian semigroup.

Also solved by Edward Barbeau, H. F. Bechtell, D. A. Breault, J. L. Brown, Jr., R. F. Brown, Paul Chessin, R. J. Cormier, R. A. Fleck, L. D. Goldstone, Alfred Gray, S. H. Greene, J. C. Hickman, D. E. Knuth, Gerald Liebowitz, Joel Levy and H. J. Lieberman (jointly), Joe Lipman, D. C. B. Marsh, L. D. Olson, F. D. Parker, D. J. Persico, C. F. Pinzka, P. L. Renz, D. R. Rogers, Benjamin Sims, D. C. Stevens, Chung-lie Wang, C. O. Wilde, and the proposer. Late solutions by F. M. Ali and N. A. Halim (jointly), George Glauberman, Bro. Joseph Heisler, and R. T. Hood.

Editorial Note. If $a \mid b$ denotes $\min(a, b)$, then $\{S; \mid\}$ is also an abelian semigroup, and the system $\{S; *, \mid\}$ obeys both distributive laws. That is, $\{S; *, \mid\}$ is (see, *e.g.*, H. Levi, *Elements of Algebra*) a "number system," in which either operation can serve as "addition" and the other as "multiplication." If $T \subset S$, then $\{T; *, \mid\}$ is also a number system, and if T has a smallest element z and a largest element e, then z and e are the identity elements for $*$ and $\mid$ respectively.

A Matrix Problem

E 1380 [1959, 588]. *Proposed by D. S. Passman, Polytechnic Institute of Brooklyn*

Let A and B be $n \times n$ nonsingular matrices with $A + B = kE$, where k is a scalar and E is a matrix all of whose elements are 1's. If $S(C)$ denotes the sum of the elements of C, show that

$$\{1 - kS(A^{-1})\}\{1 - kS(B^{-1})\} = 1.$$

Solution by Joe Lipman, University of Toronto. Since $ECE = \{S(C)\}E$, we

have $\{1-kS(A^{-1})\}E=\{I-kEA^{-1}\}E=\{I-(A+B)A^{-1}\}E=-BA^{-1}E$. Therefore

$$\{1-kS(A^{-1})\}\{1-kS(B^{-1})\}E=(-BA^{-1})\{1-kS(B^{-1})\}E$$
$$=(-BA^{-1})(-AB^{-1})E=E.$$

The conclusion is immediate.

Also solved by Frank Ayres, Jr., Edward Barbeau, R. F. Brown, Lin-chuan Cha, B. Deaville and J. Hooley (jointly), Seymour Geisser, S. H. Greene, John Jordan, P. G. Kirmser, Alphia E. Knapp, D. E. Knuth, P. S. Landweber, D. C. B. Marsh, F. D. Parker, D. J. Persico, C. F. Pinzka, J. R. Swenson, Chung-lie Wang, J. V. Whittaker, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4893. *Proposed by Joe Lipman, University of Toronto*

Let A, B be positive integers, A odd, B even, and let $A^2+B^2=p$, a prime.

(a) Show that, except perhaps for sign, A and B are respectively the real and the imaginary parts of S, where

$$S=\sum_{x=0}^{\frac{1}{2}(p-1)}\exp\left\{\tfrac{1}{2}\pi i \operatorname{ind}(x^4+a^4)\right\},$$

with a any nonzero, fixed quadratic residue (mod p).

(b) Let R be any fixed quadratic residue (mod p) and N any fixed nonresidue. Prove

$$A=\sum_{x=0}^{\frac{1}{2}(p-1)}\left(\frac{x^4+R}{p}\right),\qquad B=\sum_{x=1}^{\frac{1}{2}(p-1)}\left(\frac{x^4+N}{p}\right),$$

where the summands are Legendre symbols.

4894. *Proposed by Olga Taussky, California Institute of Technology*

Let A, B be two positive definite hermitian matrices which can be transformed simultaneously by unitary transformation to diagonal forms of similarly ordered numbers. Let x be any vector of complex numbers. Show that $(Ax, x)(Bx, x)\leqq(ABx, x)(x, x)$, and discuss the case of equality. Two sets of n

positive numbers, $\{a_i\}$, $\{b_i\}$ are called similarly ordered if $(a_i-a_k)(b_i-b_k)\geqq 0$ for all i, $k=1, \cdots, n$.

4895. *Proposed by E. S. Keeping, University of Alberta*

γ being Euler's constant, prove that

$$\int_0^\infty (\log x)^2 e^{-x}dx = \tfrac{1}{6}\pi^2 + \gamma^2.$$

4896. *Proposed by H. S. Shapiro, New York University*

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$, where the a_n lie in the interior of a Jordan curve Γ in the complex plane, the origin being exterior to Γ. Prove that there exist singularities $z_1, z_2, \cdots, z_k$ $(k\geqq 1)$ on the unit circle, and positive integers $n_1, \cdots, n_k$ such that $z_1^{n_1}\cdots z_k^{n_k}=1$.

4897. *Proposed by D. J. Newman, Brown University*

Let $F(x, y)$ be such that: for any fixed y, $F(x, y)$ is a polynomial in x, and for any fixed x, $F(x, y)$ is a polynomial in y. Prove $F(x, y)$ is a polynomial in x and y.

4898. *Proposed by R. C. Reed, University College of the West Indies*

A sequence of n digits, each digit being a 1, 2 or 3, will be called an A_n-sequence. The six permutations of 123 are the only proper A_3-sequences, and for $n>3$ a proper A_n-sequence is defined recursively as follows: an A_n-sequence $(n>3)$ is proper if, and only if, it can be obtained from some proper A_{n-1}-sequence by deleting a digit of the A_{n-1}-sequence and putting in its place the other two digits (in either order). Thus the sequences 213, 2233, 31233 are examples of proper A_n-sequences.

Show that the number of distinct proper A_n-sequences is $\frac{3}{4}[3^{n-1}-2-(-1)^n]$.

SOLUTIONS

Inverse Functions

4827 [1959; 67, 820]. *Proposed by J. Gallego-Diaz, Vanderbilt University*

If the development of the function $y=f(x)$ is given by

$$y = a_1x + a_3x^3 + a_5x^5 + \cdots,$$

find a function y knowing that if we invert the series we get

$$x = a_1y - a_3y^3 + a_5y^5 - \cdots.$$

III. *Note by Julius Lieblein, David Taylor Model Basin, Washington, D. C.* Previous solutions involve special cases with doubt expressed as to the possibility of a general form.

Starting with the notation and analysis of I, we write (2) p. 821 (November 1959) as $tF(t)=-(1/t)F(1/t)\equiv H(t)$, say, where $t=y/x$. Thus, $H(t)$ is any func-

tion satisfying $H(t)+H(1/t)\equiv 0$. The substitution $t=\exp u$ shows that H is any odd function of u, $u=\log t=\log (y/x)$, say $g(u)$. To avoid singularities, we may limit consideration to odd exponential or hyperbolic functions of $\log t$. Recalling that $F(t)=x^2$, we have $H(t)=tF(t)=xy$. A general form of solution for y as a function of x is therefore given (implicitly) by

$$(*) \qquad xy = g(\log (y/x)) = g(u),$$

where the odd function g is any function subject to the above limitation. A more direct form is $y=xp(x^2)$ where the function p is given by

$$(**) \qquad p^{-1}(t) = (1/t)g(\log t).$$

To see this, note that we have from (*) $g(\log t)=xy=tx^2=tp^{-1}(t)$. Hence $t=y/x=p(x^2)$. This includes the two solutions given previously and leads readily to others.

Solution I, by Kokar, is obtained from

$$g(u) = 2^{2n+1}k^2(\sinh u)^{2n+1}, \qquad y = x(1 - x^2/k^2)^{-1/2} \text{for } n = 0.$$

The latter is, incidentally, the case with the simplest odd hyperbolic function of $\log t$. Solution II, by Carlitz, namely, $y=\operatorname{gd} x$, may be obtained from (**) by defining the function p by $p(z^2)=(\operatorname{gd} z)/z$.

Other interesting special solutions are:

$$(a) \qquad g(u) = \tanh \tfrac{1}{2} u, \qquad xy = (y - x)/(y + x), \qquad p^{-1}(t) = (t - 1)/t(t+1).$$

The series for x or y in terms of the other is obtained from the root of the appropriate quadratic equation.

$$(b) \qquad g(u) = \log \tanh (\tfrac{1}{2} \log u) = \log\left(\frac{\log t - 1}{\log t + 1}\right).$$

It is apparent that explicit solutions for y or x in terms of the other will be very rare.

Definite Integral Evaluation

4830 [1959, 147]. *Proposed by D. B. Larson, Cornell Aeronautical Laboratory, Buffalo, N. Y.*

Evaluate the integral

$$\int_{-1}^{1} e^{iAx}(1 - Bx^2)^{-1}(1 - x^2)^{-1/2}dx,$$

where A and B are positive, real constants, $B<1$ and $i^2=-1$.

Solution by E. J. Scott, University of Illinois. Let $I(A, B)$ be the given integral. In view of the Euler relation $e^{iAx}=\cos Ax+i \sin Ax$ and the fact that $\sin Ax$ and $\cos Ax$ are odd and even functions, respectively, we have the result

$$(1) \qquad I(A, B) = 2\int_0^1 \cos Ax(1 - Bx^2)^{-1}(1 - x^2)^{-1/2}dx.$$

Now,

$$(2) \qquad \cos Ax/(1 - Bx^2) = 1 + \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k B^{n-k}A^{2k}/(2k)!\right]x^{2n},$$

which is convergent in the interval of integration. Therefore,

$$(3) \qquad I(A, B) = \pi + 2\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k B^{n-k}A^{2k}/(2k)!\right]\int_0^1 x^{2n}(1 - x^2)^{-1/2}dx.$$

But,

$$(4) \qquad \int_0^1 x^{2n}(1 - x^2)^{-1/2}dx = \frac{\pi}{2}\cdot\frac{1\cdot 3\cdot\cdots\cdot(2n-3)(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n-2)(2n)}, \quad n = 1, 2, \cdots.$$

Consequently, from (3) we obtain

$$(5) \qquad I(A, B) = \pi\left\{1 + \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(-1)^k \frac{1\cdot 3\cdot\cdots\cdot(2n-3)(2n-1)}{2\cdot 4\cdot\cdots\cdot(2n-2)(2n)}\cdot\frac{B^{n-k}A^{2k}}{(2k)!}\right]\right\}$$

The result (5) holds for nonnegative A and $0 \leqq B < 1$.

Also solved by J. W. Brown, L. Carlitz, Y. L. Luke, O. E. Stanaitis, W. F. Trench, and the proposer and J. L. Lotsoff.

Editorial Note. The final form differed from solution to solution. Most of the solvers preferred to avoid the double series by introducing Bessel's functions or hypergeometric functions. Perhaps the most compact is the proposer's $I(A, B) = \pi\sum_{n=0}^{\infty}(-B)^n(d^{2n}J_0(A)/dA^{2n})$.

Poisson Distribution

4841 [1959, 317]. *Proposed by John Lamperti, California Institute of Technology*

Let N be a random variable whose values are nonnegative integers. Independently of each other, each of N balls is placed either in urn A with probability p $(0 < p < 1)$ or in urn B with probability $1-p$, resulting in N_A balls in urn A and $N_B = N - N_A$ in urn B. Show that the random variables N_A and N_B are independent if and only if N has a Poisson distribution.

Solution by William Kruskal, University of Chicago. By hypothesis,

$$\Pr[N_A = a \quad \text{and} \quad N_B = b \mid N_A + N_B = n] = \binom{n}{a} p^a q^b$$

for $a+b=n$, a, b, n nonnegative integers, and $q = 1-p$. Hence

$$\Pr[N_A = a \quad \text{and} \quad N_B = b] = \frac{(a+b)!}{a!b!}p^a q^b \pi_{a+b},$$

where $\pi_n = \Pr[N=n]$.

If N has a Poisson distribution, *i.e.* if, for some λ, $\pi_n = e^{-\lambda}\lambda^n/n!$, it is immediate that $\Pr[N_A=a \text{ and } N_B=b]$ factors and that N_A and N_B are independent.

If N_A and N_B are independent, then $\pi_{a+b}(a+b)! = f(a)g(b)$ for some functions f and g. Note that neither $f(0)$ nor $g(0)$ can be zero, for there is positive probability that $N_A=0$ and that $N_B=0$. In short there is a function h such that $f(a)g(b) = h(a+b)$ for a, b nonnegative integers. Hence

$$f(1)g(0) = f(0)g(1), \qquad f(2)g(0) = f(1)g(1) = f(0)g(2), \text{ etc.}$$

and, by easy induction,

$$f(a) = f(1)[g(1)/g(0)]^{a-1}, \qquad g(b) = g(1)[f(1)/f(0)]^{b-1}.$$

But the quantities in square brackets are equal. Hence we may write, for some α, α', λ,

$$f(a) = \alpha e^{\lambda a}, \qquad g(b) = \alpha' e^{\lambda b}, \qquad \pi_{a+b} = \alpha\alpha' \frac{e^{\lambda(a+b)}}{(a+b)!}$$

so that N has a Poisson distribution.

Also solved by D. R. Brillinger, Luciano Daboni, N. J. Fine, George Glauberman, Emil Grosswald, D. A. Jones and J. C. Hickman, C. E. Lemke, D. C. B. Marsh, R. S. Pinkham, John Rainwater, G. S. Rogers, W. F. Trench, and the proposer.

Density of Random Variable

4842 [1959, 317]. *Proposed by John Lamperti, California Institute of Technology*

Let X be a nonnegative random variable, and let Y be uniformly distributed on the interval $(0, X)$. Let $Z=X-Y$. Show that Y and Z are independent if and only if X has the density

$$f(x) = \begin{cases} a^2xe^{-ax}, & x \geqq 0 \\ 0, & x < 0 \end{cases}, \qquad \text{for } a \geqq 0.$$

Solution by N. J. Fine, Institute for Advanced Study. Let $F(x) = \Pr[X \leqq x]$, $G(y) = \Pr[Y > y]$. Then

$$G(y) = \int_{y^+}^{\infty} \left(1 - \frac{y}{x}\right) dF(x) \qquad (y \geqq 0).$$

Clearly $\Pr[Z>z] = G(z)$ for $z \geqq 0$. Also, for y, $z \geqq 0$,

$$\Pr[Y > y, Z > z] = \int_{(y+z)^+}^{\infty} \left(1 - \frac{y+z}{x}\right) dF(x),$$

since $Y>y$, $Z>z$ are equivalent to $X = Y+Z > y+z$ and $y < Y < X-z$. Thus the independence of Y and Z is equivalent to $G(y+z) = G(y)G(z)$. It follows that

$G(y)=G(1)^y$ for rationals $y>0$. Since G is monotone, this holds for all $y>0$. If $G(1)=0$, then $G(y)=0$ for all $y>0$, which implies that $\Pr[Y=0]=\Pr[Z=0]=\Pr[X=0]=1$.

Leaving aside this trivial case, we may write $G(y)=e^{-ay}$ for some $a>0$. Now, for $h>0$,

$$G(y+h)-G(y)=-\int_{y^+}^{y+h}\left(1-\frac{y}{x}\right)dF(x)-h\int_{(y+h)^+}^{\infty}\frac{1}{x}\,dF(x).$$

The first integral does not exceed, in absolute value,

$$(h/y)\int_{y^+}^{y+h}dF(x)=(h/y)\Pr[y<X\leqq y+h]=(h/y)o(1).$$

Thus,

$$-ae^{-ay}=\lim_{h\to 0^+}\frac{G(y+h)-G(y)}{h}=-\int_{y^+}^{\infty}\frac{1}{x}\,dF(x)$$
$$=\frac{1}{y}\left(e^{-ay}-\int_{y^+}^{\infty}dF(x)\right)=\frac{1}{y}\left(e^{-ay}-1+F(y)\right).$$

Solving for $F(y)$, we get $F(y)=1-(1+ay)e^{-ay}$, $F'(y)=a^2ye^{-ay}$ $(y\geqq 0)$.

The converse is easily established.

Also solved by D. R. Brillinger, George Glauberman, Emil Grosswald, D. A. Jones and J. C. Hickman, W. S. Lawton, M. D. Mavinkurve, Roger Pinkham, John Rainwater, G. S. Rogers, W. F. Trench, and the proposer.

Generators of the Additive Group of a Field

4847 [1959, 427]. *Proposed by P. T. Bateman, University of Illinois*

Suppose f is a nonconstant polynomial function over a field K. Show that the additive group of K is generated by the values taken on by f provided either (a) K is of characteristic zero or (b) K has prime characteristic p and f has degree less than p. Also show by an example that some restriction such as (a) or (b) is needed.

Solution by N. J. Fine, The Institute for Advanced Study. Suppose $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots$, $a_n\neq 0$. Then the additive group G generated by the values of f contains $\Delta^{n-1}f(x)-\Delta^{n-1}f(0)=n!a_nx$ $(x\in K)$. Either assumption, (a) or (b), guarantees that $n!a_n\neq 0$, so we can replace x by $(n!a_n)^{-1}y$, to get $y\in G$ for all $y\in K$.

The example required is $K=Gf(4)$, with $f(x)=x^2+x+1$. The values assumed by f are 0 and 1, so $G=\{0,1\}\neq K$.

Also solved by the proposer.

A Ratio of Integrals

4848 [1959, 427]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Massachusetts*

Without performing any integration determine the ratio

$$\int_0^1 \frac{dt}{\sqrt{(1-t^4)}} : \int_0^1 \frac{dt}{\sqrt{(1+t^4)}}.$$

Solution by A. B. Farnell, Convair Research Laboratory, San Diego, California. Since the first integral involved is convergent, and

$$\int_0^1 \frac{dt}{\sqrt{(1-t^4)}} = \int_0^{\frac{1}{4}\pi} \frac{d(\sqrt{(\sin 2\theta)})}{\cos 2\theta} = \int_0^{\frac{1}{4}\pi} \frac{d\theta}{\sqrt{(\sin 2\theta)}},$$

$$\int_0^1 \frac{dt}{\sqrt{(1+t^4)}} = \int_0^{\frac{1}{4}\pi} \frac{d(\sqrt{(\tan\theta)})}{\sec\theta} = \frac{1}{\sqrt{2}} \int_0^{\frac{1}{4}\pi} \frac{d\theta}{\sqrt{(\sin 2\theta)}},$$

the desired ratio is $\sqrt{2}$.

Also solved by C. H. Cunkle, R. D. Depew, P. H. Diananda, P. G. Engstrom, H. E. Fettis, N. J. Fine and M. Jerison, Donald Garlock, S. Glusman, H. W. Gould, Cornelius Groenewoud, Emil Grosswald, J. M. Harding, L. S. Kennison, Brockway McMillan, David Moskowitz, C. S. Ogilvy, C. F. Pinzka, G. E. Raynor, E. T. Sheffield, W. C. Teachout, Jr., F. R. Urbanus, Chih-yi Wang, Alan Wayne, C. C. Yalavigi, David Zeitlin, and the proposer.

Editorial Note. Several solvers used contour integration and transformations of the complex plane. In this way Raynor and the proposer obtained the more general result:

$$\int_0^1 dt/(1-t^{2n})^{1/n} : \int_0^1 dt/(1+t^{2n})^{1/n} = \sec(\pi/2n), n = 2, 3, \cdots.$$

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officeers of the Association.

Plane Trigonometry (2nd ed.). By Gordon Fuller. McGraw-Hill, New York, 1959. xiii+281 pp. $4.75.

This edition differs from the original (reviewed in this MONTHLY, vol. 58, 1951, pp. 645–647) primarily in that more emphasis is placed on analytical trigonometry, more exercises are supplied, the order of topics is slightly rearranged (and the price has been increased by $2).

MELVIN HENRIKSEN
Purdue University

College Algebra (4th ed.). By J. B. Rosenbach, E. A. Whitman, B. E. Meserve, P. M. Whitman. Ginn, Boston, 1958. xiv+564 pp. $5.50.

Essentials of College Algebra (2nd ed.). By J. B. Rosenbach, E. A. Whitman, B. E. Meserve, P. M. Whitman. Ginn, Boston, 1958. xii+368 pp. $4.75.

Teachers will be familiar with earlier editions of these texts. Limited changes in subject matter have been made. Elementary notions of statistics, the use of matrices in solving linear systems and summation notation are among the new topics introduced. The shorter of the texts is an abbreviated version of the other. The abbreviation has been effected primarily by omission of certain topics and exercises.

These texts are conventional. On page one the authors remark that the printed symbols 1, 2, 3, · · · are *numerals* rather than *numbers*. However, no serious attempt is made to distinguish between mathematical symbols and the concepts represented by these symbols.

Some of the field postulates for addition and multiplication are stated, *but are not used deliberately to develop computational techniques*. Rather the student is told how to manipulate symbols so as to arrive at approved answers. A remark concerning the fundamental assumptions seems unfortunate. The student is told that these assumptions assert that "*basic properties of numbers in arithmetic hold also for literal number symbols.*"

In the section on functions the authors define a function to be the dependent variable. Since a variable has been defined to be a symbol this leads to logical difficulties.

The format is attractive and exercise lists are adequate. Many teachers who wish to teach traditional college algebra using traditional terminology will find these texts satisfactory.

CHARLES BRUMFIEL
Ball State Teachers College

Intermediate Algebra. By Ross H. Bardell and Abraham Spitzbart. Addison-Wesley, Reading, Massachusetts, 1959. ix+274 pages. $4.75.

This book is written in the traditional style for "the student who begins his college mathematics work with inadequate preparation in mathematics." Such students seem to need (and to profit from) long lists of routine drill problems —this text has long lists of such problems. Many of these inadequately prepared students do not seem to be aware that sometimes an algebraic identity can be used either to multiply or to factor, hence the authors list such identities twice, *e.g.*, $(u+v)(u^2-uv+v^2)=u^3+v^3$ in a list of multiplication formulas and $u^3+v^3=(u+v)(u^2-uv+v^2)$ in a list of factors.

The print on one side of a page shows through on the other side in an annoying fashion.

ROBERT E. GREENWOOD
University of Texas

Analysis and Control of Nonlinear Systems. By Y. H. Ku. Ronald Press, New York, 1958. vii+360 pp. $10.00.

Problems giving rise to nonlinear differential equations derived from consideration of mechanical vibrations and electrical oscillations are here studied in detail using both analytical and topological approaches. By use of the topological approach general system behavior in the neighborhood of singular points is determined, thus largely bypassing the need for numerical solutions of the differential equations. Only brief mention is made of the possibilities for solution of nonlinear systems on analog and digital computers. The last two chapters are devoted to the analysis and control of nonlinear feedback systems. The book was written as a graduate text in nonlinear circuit analysis, and the inclusion of some 120 problems for solution helps in fulfilling this aim.

The bibliography of 735 items in chronological order beginning with the year 1860, spanning periods of 40 years, 20 years, 10 years and yearly from then on seems most adequate. An alphabetical list of authors is lacking.

The author presents striking evidence that nature is nonlinear, that nonlinearities may not only be useful but necessary, and that failure to recognize the reality of nonlinearities may lead to catastrophes such as the tragic collapse of the Tacoma suspension bridge.

The book is well written and illustrated, and is a particularly good book for the mathematician interested in nonlinear problems.

HERBERT A. MEYER
University of Florida

Applied Mathematics for Engineers and Scientists. By C. G. Lambe. Macmillan, New York, 1959. xii+518 pp. $8.50.

This book was apparently written for a specific course of study in British universities. It is hard to see how it can be fitted very snugly to current traditions in American curricula. Largely devoted to the subject of mechanics, its introductions to the various concepts and methods of mechanics will be found too elementary for physicists, too brief for engineers, and too numerous for differential equaters. The principal merit of the book lies in its many thoroughly worked examples, which students and instructors in any of the above mentioned courses would find handy to have on their bookshelves. The authors have, in this respect, done an excellent job of editing their notebooks. There is, of course, a great deal to be said in favor of a long parade of examples as a teaching device. One also suspects that there was difficulty in choosing a title, with an ultimate surrender to the current fad of forming a random combination from the words "applied," "engineer," "methods," "advanced," "mathematics" and "scientist." More informative would have been "Solved: One Thousand Problems in the Mechanics of (Rigid or Deformable) Bodies."

FRANCIS SCHEID
Boston University

Méthodes Numériques. By J. Kuntzmann. Dunod, Paris, 1958. 252 pp. 3,600 F. (About $8.60).

This book provides thorough coverage of the first chapters of numerical analysis: difference calculus, numerical interpolation and differentiation. Aimed toward the practical computer, there are many fine tabular layouts, summaries, computational schemes and tips, graphs, and worked examples. Though the book is, on the whole, elementary, and the bulk of the theory standard, the author is alert to the literature of the past 15–20 years and in addition to many recent references has included a number of new things which are of considerable utility. His treatment of errors via kernels, and interpolation with nonlinear families are particularly noteworthy.

P. J. DAVIS
National Bureau of Standards, Washington, D. C.

Mathematics of Physics and Modern Engineering. By I. S. Sokolnikoff and R. M. Redheffer. McGraw-Hill, New York, 1958. viii+810 pp. $9.50.

This voluminous work is addressed primarily to engineering students and practicing engineers. Both of these groups face a dilemma which confronts most scientific workers, students and practitioners alike—the subjects and topics which they need to know thoroughly are too numerous and extensive to be learned in the time available. An inescapable implication of this predicament is that either depth or extent will have to be sacrificed to a considerable degree. Whether the best solution is to study a few basic subjects intensively or to work entirely toward breadth of scope will depend upon the individual's capacity for the abstract and precise thinking necessary for intensive study. In either case there is a need for easy reading surveys which present, without pretense of precision and completeness in all details, the rather formidable miscellany of topics fundamental and applied which make up the complex of modern industrial mathematics. Although there are facets of this book which run counter to the reviewer's prejudices, it is the reviewer's opinion that this work does present for the engineer who has finished the elementary calculus a comparatively easy and yet satisfactory introduction to a variety of topics which have been judiciously selected.

For the majority of engineers at the junior-senior level, the expositions will be found to tie in closely with their previous mathematical training both with regard to the points of departure and the difficulty of the developments. For the most part, these gradual transitions are definitely in order since the book is designed for readers who must budget their time strictly. However, in one or two instances the reviewer would have preferred more modern formulations than those presented despite the resulting break in the continuity with the typical prerequisite engineering mathematics. Specifically, complex numbers are presented as numbers represented by $a+ib$; the concept infinite series is introduced by the statements: "A series is a sum of terms" $\cdots$ "$a_1+a_2+\cdots+a_n$

is a series consisting of n terms. An infinite series is a series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1.1}$$

which has infinitely many terms." The reviewer is prejudiced in favor of the explicit definition of the concept of complex number as an ordered pair of real numbers subject to the well-known definitions of equality, sum, and product and to the definition of an infinite series as a pair of infinite sequences—a primary sequence $\{a_n\}$ and a derived sequence $\{s_n\}$, $s_n = a_1 + \cdots + a_n$. In addition, there is an occasional recourse to the indefinite "small"—but these matters should be considered with the purpose of the book in mind.

The chapter headings and some of the numerous topics treated are: (1) Ordinary Differential Equations (the usual elementary topics together with an excellent set of applications); (2) Infinite Series (tests for convergence, uniform convergence, Taylor's series, complex series, Fourier series); (3) Functions of Several Variables (partial differentiation, the chain rule, directional derivative, maxima and minima, Taylor's formula, calculus of variations, surface integrals); (4) Algebra and Geometry of Vectors, Matrices (vector algebra including the dot and cross products, differentiation, applications to analytic geometry and mechanics, linear vector spaces, transformations of base vectors); (5) Vector Field Theory (curvilinear coordinates, the metric quantities g_{ab}, the Gauss divergence theorem, Stokes's theorem, applications to heat flow and fluid mechanics); (6) Partial Differential Equations (the vibrating string, solution by series, solution by integrals, elliptic, parabolic, and hyperbolic equations); (7) Complex Variable (analytic functions, integration, Cauchy's integral theorem and integral formula, Taylor's series, Laurent's expansion, residues, conformal mapping, applications); (8) Probability (elementary topics, random variables and expectation, discrete and continuous distributions, probability and relative frequency, the Poisson law, theory of errors, variance); (9) Numerical Analysis (graphical methods, Newton's method, iterative methods, interpolation, empirical formulas, least squares, harmonic analysis, numerical integration of infinite series).

Altogether there are some 214 sections. Considering its size the book is remarkably free of misprints.

HOMER V. CRAIG
Boeing Airplane Company and The University of Texas

Ancillary Mathematics. By H. S. W. Massey and H. Kestelman. Pitman, London, 1959. xxvi+990 pp. 75 s. (about $10.54).

This rather large book is aimed at giving underclassmen college students in physics and chemistry a thorough preparation for the type of mathematics they will need in order to develop an understanding of the physical sciences. In this direction, the authors appear to have especially in mind the requirements of thermodynamics, statistical mechanics, and quantum theory. The book contains thirty-two chapters—starting with the concept of a limit and basic properties of

the real numbers and eventually covering, among other things, most of the topics traditionally given (at American universities) in a four year course on engineering mathematics. The five chapters on mechanics (18, 19, 20, 24, and 25) together with the applications in Chapter 26 (introduction to Fourier series) and Chapter 32 (introduction to the variational calculus) forms the basis of an excellent undergraduate course in mechanics. Since the authors wrote the book with care and precision, it does not have the usual mathematical faults that are typical in books of this nature. The book contains a large number of illustrative examples and exercises.

PASQUALE PORCELLI
Mathematics Research Center
University of Wisconsin

Tauberian Theorems. By H. R. Pitt. Oxford, New York, 1959. 174 pp. $6.00.

Since the classical theorem of Tauber, there has been a continuous and widespread investigation of Tauberian theorems and Tauberian conditions. In the preface the author defines the scope of this book as including the topics and methods which follow most naturally from the work of Hardy, Littlewood and Wiener. These topics and methods have formed a most fruitful part of the theory. Chapter headings are: Introduction, Elementary Tauberian Theorems, Classical Tauberian Theorems, Wiener's Theory, Mercerian Theorems, Tauberian Theorems and the Prime Number Theorem. The chapter on the Prime Number theorem compares the Landeau-Ikehara method, the classical proofs, and the elementary proofs of Selberg and of Erdös. The chapter on Mercerian theorems presents a unified treatment of that subject.

This is clearly a monograph that should be in the hands of anyone interested in the subject of summability, especially those wishing to do research on Tauberian theorems. At the beginning of each chapter is a brief section discussing the results that are to follow. The subject matter of the chapter is illuminated against its proper background and the reader is shown where the material fits in with the rest of the theory. The feasibility of other possible developments is examined so that very soon the reader has an appreciation of where the problems lie and the hazards that will be encountered in attempting their solution. Similar brief remarks appear when appropriate at the beginning of some sections. The reviewer found these brief discussions one of the most instructive and thought provoking features of the book. Nor was the beauty of the formal development diminished, but rather enhanced, by these touches of the author's insight.

This is the second monograph published by the Tata Institute of Fundamental Research, Bombay and it should be congratulated on bringing such a valuable work before the mathematical public.

GORDON M. PETERSEN
University College, Swansea

Mathematics is Easy. By D. S. Watt. Princes Press, London, 48/- and St. Martin's Press, New York. 488 pp. $10.00.

In the foreword, E. H. Lloyd describes the author's purpose as follows: "He sets out to produce a book from which any literate person could teach himself mathematics; that is to say, not just generalities, but real detailed technical mathematics." The author writes that the book "aims at a solid foundation of mathematical knowledge" and belies his title by maintaining: "Hard work and hard thinking are essential to any kind of mathematical knowledge."

The topics covered in the book are rudiments of plane and solid geometry—mostly intuitive—the usual content of a college algebra course, with a smattering of calculus; there is a little on matrices and trigonometry. The style is easy-going and should be intelligible to a determined layman. The foundation laid in elementary algebra is for the most part adequate, but that in calculus is shaky to say the least. For instance, after defining the indefinite integral to be the anti-derivative, he defines the definite integral as follows: "Where the integral has limits or extreme values of the variable, we are said to have a definite integral."

The most serious defect in the book is that it contains no exercises. Though the reader is invited to read carefully, he is not given the opportunity of discovering something himself. But in spite of the title, there is much in this book that is worth working at.

BURTON W. JONES
University of Colorado

Numerical Analysis and Partial Differential Equations. Volume V, *Surveys in Applied Mathematics.* By George E. Forsythe and Paul C. Rosenbloom. Wiley, New York, 1958. x+204 pp. $7.50.

This book consists of two papers, *Contemporary State of Numerical Analysis* by George E. Forsythe, and *Linear Partial Differential Equations* by Paul C. Rosenbloom. One fifth of this book is devoted to a brief survey of numerical analysis with major emphasis on methods devloped for automatic digital computation. The remainder is concerned primarily with advances in the theory of linear partial differential equations in the last ten years and gives many applications to problems in applied mathematics. In both of these reports the status of the work done and to be done is clearly stated and special attention has been given to Russian publications.

LAWRENCE A. WELLER
Monsanto Chemical Company, Miamisburg, Ohio

Intermediate Algebra. By L. M. Kells. Prentice-Hall, Englewood Cliffs, New Jersey, 1959. xiv+296 pp. $5.25.

The author states that "This text is designed primarily for college students who have inadequate training in algebra." Content is so chosen and ordered that the text is adaptable to fairly diverse usage in colleges.

The most distinguishing feature of the book is a precision of statement all too often lacking in such texts. This is particularly welcome on fractional exponents and radicals. The treatment of the "Laws" of algebra is unusually well done. The balance between extreme rigor and intuition is very suitable for this level text. Appropriately for an intermediate algebra text, the elementary processes and signed numbers are treated reasonably briefly but at a more sophisticated level than is customary in elementary texts. Particularly well done is the explanation of operations with signed numbers.

"Topics for discussion" are not necessarily supplementary but represent a "special treatment" that should prove pedagogically useful. Although these topics are generally briefly presented, they are sufficiently clear that many students should be able to follow them easily.

Actual flaws in the text are few. This reviewer does question the desirability of two forms of the quadratic formula (one for roots involving i). The latter form cannot be derived from the author's previous material, although it can be verified by substitution. A slight, but highly desirable, modification in previous definitions would avoid this. This reviewer would have liked to have seen the Factor and Remainder theorems included as well as some problems on reducible trinomials.

The careful exposition and numerous well chosen problems and exercises should serve to provide the student with ample opportunity to practice manipulative skill while learning something of the logical nature of algebra. Illustrative examples are carefully worked out and well explained. This text should make algebra appear interesting and useful in its own right while serving as a stimulating introduction to more mathematics.

JAMES L. SIMPSON
Montana State College

Foundations of Mathematics. By Carl H. Denbow and Victor Goedicke. Harper, New York, 1959. xviii+613 pp. $6.00.

Here is an exciting book with a new approach in pedgagogy. One of the chief objectives is a gradual transition from an empirical approach to a deductive treatment. The authors build the reader's confidence and interest in early sections, and then point up limitations of the experimental approach to pave the way for a logical foundation. Parts I, II, III are entitled respectively Empirical Foundations, Logical Foundations, and Developments and Applications.

As an introduction to Part I, there is an illuminating survey of basic ideas of arithmetic and algebra, including properties of clock numbers, the meaning of associative and distributive laws, and some concepts concerning heat flow in one and two dimensions. Part I begins with modular numbers and proceeds to cover the major part of what is usually styled intermediate algebra. The authors pique the curiosity of the reader by suggesting extensions of given situations. Also, they include some philosophy of mathematics with a view toward inducing

a more flexible and open-minded attitude on the part of the reader.

As an instance of a branch of mathematics in which symbols do not represent numbers, Part II begins with the furniture mover's story which allows introduction of many concepts of group theory. The chapter on rational numbers and fields digs deeper into algebra and throws more light on notions introduced previously. Adequate coverage of logarithms, progressions, and quadratic equations is found in the chapter on real and complex numbers. The final chapter of Part II gives a clear account of logic algebra and set algebra.

In Part III, following a chapter on probability and a brief chapter on statistics, there is a helpful chapter on problem solving with hints on how to transform verbal statements to symbols. There follow 56 pages on trigonometry, a brief but adequate treatment. Some of the usual topics of analytic geometry are treated in 55 pages, and certain topics from the calculus are covered in 83 pages. Here, as elsewhere, the authors point out the need for more powerful mathematics, by means of an example from the experience of the reader. A novel feature here is the introduction of tabular derivatives. The final three chapters consider number theory, "infinity and beyond," and "what is mathematics?" The bibliography for further reading, the tables, and the answers to the odd-numbered exercises add to the usefulness of the book.

This carefully written comprehensive volume is adaptable for use in a wide variety of courses. It should be read by all teachers of mathematics in high school and in the first year of college.

C. E. SPRINGER
University of Oklahoma

Coordinate Geometry with Vectors and Tensors. By E. A. Maxwell. Oxford University Press, London, 1958. xii+198 pp. $4.00.

The author aims to present the minimum amount of coordinate geometry of three dimensions, which he considers necessary for the young mathematician, together with some vectors and tensors. Believing that coordinate methods are essential and that a background should be laid for the application and appreciation of vectors, he presents planes and straight lines first. This preparation permits a presentation of vectors with an algebraic basis. The chapter on tensors is largely confined to the very useful δ_{ij} and ϵ_{ijk}. There follow chapters on spheres, central quadrics, paraboloids, and the general quadric. Such topics as Joachimstal's ratio equation, coaxial spheres, tangents, poles and polars, diameters, centers, and generators are treated. Two treatments of the general quadric are given, one ordinary and the other with the double suffix notation of tensors.

The work throughout is carefully organized. The level is above the elementary. The presentation is clear and rigorous. The exercise lists abound with excellent problems. The author accomplishes his stated aims.

EARL LAFON
University of Oklahoma

Fallacies in Mathematics. By E. A. Maxwell. Cambridge University Press, New York, 1959. 95 pp. $2.95.

By definition, "the fallacy leads by guile to a wrong, but plausible conclusion; the howler leads innocently to a correct result."—This small book is devoted essentially to fallacies. At the end, a few howlers, apparently encountered in class, are included. We mention only one: to find largest angle of triangle with sides 4, 7, 9. Solution: $\sin C = 9/7 = 1.2857$, $\sin 90° = 1$, $\sin 16°36' = .2857$; hence $\sin (90° + 16°36') = 1.0000 + .2857$, $C = 106°36'$ (correct answer). In the discussion the class of triangles is characterized for which this method of solution holds.

The fallacies are taken from the fields of algebra, trigonometry, analytic geometry and calculus. The theory of sets is on the whole disregarded, thus eliminating a fruitful source of good fallacies.

The level of the fallacies, as well as the level of the explanations, varies within wide limits. The fallacy that every triangle is isosceles is taken up in three different forms. The analysis of one of these occupies nine pages and leads to Pasch's axiom, Ptolemy's theorem on four concyclic points, fourth order determinants and the equation in line coordinates of a conic through the vertices of the triangle of reference.

The fallacy that every angle is a multiple of two right angles is derived by assuming an angle (complex) to satisfy $\tan \theta = i$, and using for any angle A, the formula for $\tan (A + \theta)$. Then $\tan (A + \theta) = \tan \theta$, etc. But the explanation given for the existence of i as a Picard exceptional value of the analytic function $\tan x$ is inadequate

$$\left(\tan x = k, \frac{\sin x}{k} = \frac{\cos x}{1} = \frac{1}{\pm\sqrt{(1 + k^2)}},\right.$$

$$\left. k^2 + 1 = 0 \text{ leads to a denominator 0 and } k = \pm i\right).$$

"Thus no (complex) angle exists whose tangent has either of the values $\pm i$."

The circular points at infinity are introduced to explain the fallacies that the four points of intersection of two conics are collinear, and that concentric circles invert into concentric circles with respect to an arbitrary point. The brief explanation given requires a greater degree of knowledge of projective geometry in the complex plane than the preface of the book would seem to assume.

AUBREY J. KEMPNER
University of Colorado

Elements of Modern Mathematics. By K. O. May. Addison-Wesley, Reading, Mass., 1959. 607 pp. $7.50.

This first-rank book represents a major departure from present freshman level texts, even those of the "modern" school.

The book first introduces some of the ideas and notation of logic and sets, and then proceeds to *use* them for the remainder of its 600 pages. One finds no mere lip service to these topics here; they are put to daily use by the student throughout the course. Other material treated includes plane analytic geometry, a start on differential and integral calculus, probability and statistical inference, and an introduction to abstract algebra.

The strong point of this carefully and well-written book is the author's insistence that things be done right. An especially good feature is the notation for and systematic use of substitution. The text contains some innovations in arrangement. For example, each section contains exercises scattered throughout and problems at the end. Answers to the exercises and some of the problems are given at the end of each section.

The following rather halfhearted criticisms are not intended to detract from the overall excellence of the book. There appear to be the usual number of misprints and minor errors. The section on implicit functions fails to point out that an implicit "function" is often not a function. The axiom systems for fields and Boolean algebras are incomplete since the substitution axioms are omitted (an equivalence relation, rather than the preferable equality, is used in each case). Finally, the treatment of equality of vectors is not in accordance with the otherwise consistent use of equality as identity.

The reviewer feels that *Elements of Modern Mathematics* is an excellent book for the good student and the good instructor, but that it may be somewhat ambitious for the average. Certainly any student who masters a course taught from this book will gain far more mathematical maturity than he would from any other text on this level with which the reviewer is familiar.

W. R. Scott
The University of Kansas

Information Theory and Statistics. By Solomon Kullback. Wiley, New York, 1959. xvii+395 pp. $12.50.

Mathematical information theory is a rather new subject, its main motivations coming from communications engineering. The purpose of this book is to apply the basic concepts of "measuring information" to classical problems of statistics. The motif of the book is that the use of goodness criteria for statistical procedures, based on measures of information, tends to unify an otherwise disorganized subject. My own reaction is that the monumental work of Abraham Wald on statistical decision theory has long provided a basis for unification. The writing is on a fairly advanced technical level. The heart of the book lies in the problems of analysis of variance—the usual univariate linear hypothesis, its somewhat less usual multinomial and Poisson population analogues, and multivariate analysis of variance.

Meyer Dwass
Northwestern University

BRIEF MENTION

High School Mathematics, (Units 1, 2, 3, and 4). By the University of Illinois Committee on School Mathematics. University of Illinois Press, 1959. Teachers' edition about 2,000 pp., $6.00. Students' edition, 618 pp. $2.25.

Many readers will be interested to know that the University of Illinois Committee on School Mathematics, Units 1, 2, 3, and 4, are now available for general sale, in both the student and teacher editions. This is a fascinating example of what can be and is being done to revise and revamp high school mathematics.

On Numerical Approximation. Edited by Rudolph E. Langer. University of Wisconsin Press, 1959. x+462 pp. $4.50.

This volume consists of 21 papers which were delivered at the symposium on numerical approximation conducted by the Mathematics Research Center, United States Army at the University of Wisconsin, Madison, April 21–23, 1958. The contributors are some of the top experts in the field, both from the American continent and abroad. The majority of these papers are mathematical in nature, depending upon functional analysis and linear spaces for basic concepts.

Mathematical Programming and Electrical Networks. By J. B. Dennis. Technology Press and Wiley, New York, 1959. vi+186 pp. $4.50.

This is a research monograph rather than a text. It offers an interesting approach to linear programming by use of analogous electrical networks.

Elementary Statistics with Applications in Medicine and the Biological Sciences. By Frederick E. Croxton. Dover, New York, 1953. vii+376 pp. $1.95.

A reprint of Croxton's 1953 volume.

Probability and Statistical Inference for Engineers. By C. Derman and M. Klein. Oxford University Press, New York, 1959. xii+144 pp. $3.75.

An interesting little book designed to be used in conjunction with several other modern books in presenting a course on probability and statistical inference to undergraduate engineering students.

Statistical Methods in Biology. By Norman T. J. Bailey. Wiley, New York, 1959. ix+200 pp. $4.50.

No mathematics here beyond elementary algebra.

Operations Research—Methods and Problems. By Maurice Sasieni, Arthur Yaspan and Lawrence Friedman. Wiley, New York, 1959. xi+316 pp. $10.25.

Essentially devoted to formulating and solving mathematical models which are based on a no more sophisticated background than that usually provided by differential and integral calculus. A number of interesting and life-like problems are presented which should help project the neophytes into this fascinating new mathematical field.

Strategy and Market Structure. By Martin Shubik. Wiley, New York, 1959. xviii+387 pp. $8.00.

This interesting book suggests that mathematics is currently invading the field of economics through the use of a theory of games designed especially to deal with economic problems. Competition, oligopoly and the general strategic interdependence of economic markets are considered.

The Collected Works of George Abram Miller, Volume V. By George Abram Miller. University of Illinois Press, 1959. xii+474 pp. $7.50.

A fascinating collection of Professor Miller's publications on the subject of finite groups. There are some interesting results and near results contained in these papers, many of which may be unfamiliar to the reader.

Algebra Can Be Fun! By William R. Ransom. J. Weston Walch, Portland, Maine, 1958. ix+195 pp. $2.50.

An interesting collection of high school algebra curiosities.

Amusements in Mathematics and *The Canterbury Puzzles*. By H. E. Dudeney. Dover, New York, 1958. 274 and 266 pp. $1.25 each.

Reprints of Dudeney's 1917 and 1919 books, which still provide interesting puzzles for the nation's puzzlers.

Mathematical Puzzles of Sam Loyd. By Martin Gardner. Dover, New York, 1959. xv+181 pp. $1.00.

This certainly is an all-time puzzle favorite and belongs on every puzzle fan's bookshelf.

Introduction to Symbolic Logic and Its Applications. By Rudolf Carnap. Dover, New York, 1959. xiv+241 pp. $1.85.

An excellent English translation of Carnap's 1954 volume.

Individual Choice Behavior. By R. Duncan Luce. Wiley, New York, 1959. xii+153 pp. $5.95.

This is a research monograph, a probabilistic analysis of choice, by an author who is not afraid of mathematics. Another worthwhile example of modern mathematical inroads into the fields of social science.

On Mathematics and Mathematicians. By Robert Edouard Moritz. Dover, New York, 1958. vii+410 pp. $1.95.

A reprint of Moritz's 1914 *Memorabilia Mathematica*, a collection of anecdotes and quotations concerning famous mathematicians.

Scientific Russian. By George E. Condoyannis. Wiley, New York, 1959. xii+225 pp. $3.50.

An admirable adjunct to the similar volumes on scientific German and scientific French available from the same publisher. Contains a concise description of the structural elements of scientific and technical Russian.

Studies in Mathematical Learning Theory. By R. R. Bush and W. K. Estes. Stanford University Press, Stanford, California, 1959. viii+432 pp. $11.50.

This is a collection of research papers including the most recent developments in the analysis and application of mathematical learning theory. An interesting and worthwhile book of the Stanford Mathematical Studies in the Social Sciences.

Intermediate Algebra, Revised. By J. R. Britton and L. C. Snively. Rinehart, New York, 1959. x+353 pp. $3.00.

A well-known text.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor C. T. Bumer, Clark University, represented the Association at the Academic Convocation of the formal opening of Haberlin Hall at the College of the Holy Cross, on December 8, 1959.

University of Minnesota: Associate Professors W. S. Loud and Bjarni Jonsson have been promoted to Professors; Assistant Professor Steven Orey has been promoted to Associate Professor; Mr. L. R. Harper, Jr. has been promoted to Assistant Professor; Professor M. D. Donsker is spending a sabbatical leave at the University of Aarhus in Denmark.

Dr. Iain Adamson, Queen's University, Belfast, Ireland, has been appointed lecturer at Queen's College, Dundee, Scotland.

Mr. Mansur Arbabi, Seattle University, has accepted a position as Scientist for Booz-Allen Applied Research, Bethesda, Maryland.

Associate Professor Ralph Beatley, Harvard University, has retired with the title Associate Professor of Education, Emeritus and is now teaching half-time in the College of Liberal Arts at Boston University.

Dr. C. R. Cassity, New Mexico Institute of Mining and Technology, has accepted a position as Shock Hydrodynamicist with the General Electric Company, Philadelphia, Pennsylvania.

Dr. K. G. Clemans, USNOTS, China Lake, California, has been appointed Professor at Southern Illinois University, Southwestern Campus, East St. Louis.

Mr. G. R. Costello, Arma Division of American Bosch Arma Corporation, has been promoted to Chief Engineer.

Mrs. Harriet B. Dietz, Auburn High School, Massachusetts, has been appointed Teacher of Mathematics at McArthur High School, Hollywood, Florida.

Mr. E. L. Eagle, Martin Company, Orlando, Florida, has accepted a position as Reliability Manager with the Bendix Aviation Corporation, Ann Arbor, Michigan.

Mr. Herbert Ginsberg, Stanford University, has accepted a position as Statistician with the Fairchild Semiconductor Corporation, Palo Alto, California.

Mr. W. A. Golomski, Oscar Mayer and Company, has been appointed Vice-President of H. J. Mayer and Sons Company, Chicago, Illinois.

Dr. Irwin Guttman, Princeton University, has been appointed Associate Professor at McGill University.

Mr. D. W. Hinkkanen, Boeing Airplane Company, Seattle, Washington, has accepted a position as Research Mathematician with the Chrysler Corporation, Missile Division, Detroit, Michigan.

Mr. D. F. Jordan, Texas Technological College, has been appointed Instructor.

Dr. I. N. Katz, Massachusetts Institute of Technology, has accepted a position as Senior Mathematician with Avco Research and Advanced Development, Wilmington, Massachusetts.

Mr. A. J. Leino, Convair Astronautics Corporation, has accepted a position as Engineer with Lockheed Missile Systems Division, Sunnyvale, California.

Dr. R. D. Mayer, University of Washington, has been appointed Assistant Professor at Idaho State College.

Dr. Fred Meyer, Wayne State University, has accepted a position as Research Scien-

tist with the Boeing Airplane Company, Seattle, Washington.

Dr. J. A. Navarro, General Electric Company, Ithaca, New York, has accepted a position as Statistician with I.B.M. Research, Yorktown Heights, New York.

Mr. C. A. Oster, California Institute of Technology, has accepted a position as Analyst in the Data Processing Division of General Electric Company, Richland, Washington.

Mr. R. L. Page, University of Maine, has been appointed Assistant Professor at Nasson College, Springvale, Maine.

Mr. R. O. Pan, Case Institute of Technology, has accepted a position as Programmer with the Service Bureau Corporation, New York.

Mr. R. R. Parker, General Electric Company, Ithaca, New York, has accepted a position as Senior Engineer with H. R. B. Singer, Inc., State College, Pennsylvania.

Miss Nancy L. Parr, University of Kentucky, has been appointed Instructor at Hanover College, Hanover, Indiana.

Mrs. Jean J. Pedersen, University of Utah, has been appointed Instructor.

Dr. R. P. Peterson, Jr., Matson Navigation Company, has accepted a position as Senior Mathematician with the Burroughs Electrodata Division, Pasadena, California.

Mr. M. B. Richins, Bell Telephone Laboratories, New Jersey, has accepted a position as Senior Operations Research Analyst with the System Development Corporation, Santa Monica, California.

Mr. Paul Roethel, Wisconsin State College, has been appointed Teacher at Berlin High School, Berlin, Wisconsin.

Mr. F. D. Roscoe, Westinghouse Electric Corporation, has accepted a position as Senior Programmer at Lawrence Radiation Laboratories, Livermore, California.

Mr. Sam Rosenberg, Harvard University, has accepted a position as Mathematician with the Texaco Research Laboratory, Bellaire, Texas.

Mr. J. W. Royal, University of Maine, has been appointed Instructor at Merrimack College.

Mr. Yomei Sawanobori, I.B.M. Corporation, New York City, has accepted a position as Senior Mathematician with the International Business Machines Corporation, Washington, D. C.

Associate Professor H. W. E. Schwerdtfeger, McGill University, has been promoted to Professor.

Mr. H. A. Seebald, Sperry Rand Corporation, has accepted a position as Computer Specialist with the System Development Corporation, Santa Monica, California.

Sister Ruthmary, F.S.P.A., Viterbo College, has been appointed Mathematics and Physics Instructor at DePadua High School, Ashland, Wisconsin.

Assistant Professor W. A. Spivey, on leave from the University of Michigan, has been appointed Visiting Associate Professor at the Institute of Basic Mathematics, Harvard Graduate School of Business.

Dr. M. D. Springer, Technical Operations, Inc., Fort Monroe, Virginia, has accepted a position as Systems Operations Analyst with the Allison Division of General Motors Corporation, Indianapolis, Indiana.

Dr. C. E. Stewart, Illinois Institute of Technology, has accepted a position as Research Scientist with the Aeronutronic Division of the Ford Motor Company, Newport Beach, California.

Mr. R. W. Stewart, Nortronics, Anaheim, California, has accepted a position as Electrical Engineer with the Hughes Aircraft Company, Fullerton, California.

Associate Professor W. G. Stokes, Northwestern State College, has been appointed Associate Professor at East Texas State College.

Mr. J. E. Strang, University of Arizona, has accepted a position as Mathematical Programmer at the United States Army Electronic Proving Ground, Fort Huachuca, Arizona.

Associate Professor Takashi Terami, College of St. Thomas, has been promoted to Professor.

Dr. R. J. Warne, University of Tennessee, has been appointed Assistant Professor at Louisiana State University, New Orleans Branch.

Professor A. L. Whiteman, University of Southern California, has been appointed a Member of the Institute for Advanced Study, Princeton, New Jersey, for the year September 1959 to September 1960.

Miss Annie J. Williams, Julian S. Carr High School, has been appointed Supervisor of Mathematics, State Department of Public Instruction, Raleigh, North Carolina.

Dr. C. R. B. Wright, University of Wisconsin, has been appointed Research Fellow at California Institute of Technology.

Professor R. C. Yates, College of William and Mary, has accepted an appointment as Director of the Mathematics Program at the new University of South Florida, Tampa, effective in July, 1960.

Miss Anne Motycka, Proctor High School, Utica, New York, died in July, 1959. She was a member of the Association for six years.

Assistant Professor W. C. Ross, Jr., Knox College, died November 9, 1959. He was a member of the Association for four years.

Professor J. C. Smith, High Point College, High Point, North Carolina, died October 27, 1959. He was a member of the Association for sixteen years.

Professor Emeritus R. D. Stone, Purdue University, died February 27, 1959. He was a member of the Association for forty-one years.

Meeting of Mathematics Division of ASEE

The annual meeting of the Mathematics Division of the American Society for Engineering Education will be held June 20–24 at Purdue University, Lafayette, Indiana. Chairman Sam Selby, University of Akron, has included in the program panel discussions on "The New Role of Undergraduate Mathematics in Engineering" and "The Impact of Digital Computers on Engineering Education" as well as a general session featuring a lecture by Colonel Wernher von Braun.

For further information regarding details of the Annual Meeting write to W. E. Restemeyer, Department of Mathematics, University of Cincinnati.

National Council of Teachers of Mathematics

The Thirty-eighth Annual Meeting of the National Council of Teachers of Mathematics will be held at the Hotel Statler Hilton, Buffalo, New York, April 21–23, 1960. Registration will begin Wednesday afternoon, April 20. The opening general session address "Mathematics and Human Knowledge" will be given by Dr. Carroll V. Newsom, President of New York University, on Thursday evening, April 21.

Additional information may be obtained from Dr. Irvin H. Himmele, NCTM Publicity Chairman, 702 City Hall, Buffalo 2, N. Y. or Louis F. Scholl, Chairman of Local Arrangements, 728 City Hall, Buffalo 2, N. Y.

AMS Summer Institute on Finite Groups

With the support of the National Science Foundation, a Summer Institute on Finite Groups will be held at the California Institute of Technology from August 1 to August 28, 1960. Two distinguished group theorists have been invited from abroad and have accepted the invitation. These are Professor Graham Higman of Oxford University and Professor Helmut Wielandt of the University of Tübingen. Participation in the Summer Institute will be by invitation, but the lectures and seminars will be open to interested mathematicians who happen to be in the area. The Program Committee consists of Professors Richard Brauer, Richard Bruck, H. S. M. Coxeter, Robert Dilworth, Herbert Ryser, and Marshall Hall, Jr., Chairman.

Southern Regional Graduate Summer Session in Statistics

The 1960 Southern Regional Graduate Summer Session in Statistics will be held at the University of Florida at Gainesville from June 20 to July 29, 1960. The University of Florida, North Carolina State College, Virginia Polytechnic Institute and Oklahoma State University have agreed to operate a continuing program of graduate summer sessions in statistics to be held at each institution in rotation. The first such session was held at Virginia Polytechnic Institute in the summer of 1954.

It is the purpose of this program to serve: (1) teachers of introductory statistical courses and college teachers of mathematics who want formal training in modern statistics; (2) research and professional workers who want intensive instruction in basic statistical concepts and modern statistical methodology; (3) professional statisticians who wish to keep informed about advanced specialized theory and methods; (4) prospective candidates for graduate degrees in statistics; and (5) graduate students in other fields who desire supporting work in statistics.

The session will last six weeks and courses will carry three semester hours of credit. Not more than two courses may be taken for credit at any one session. The summer work in statistics may be applied as residence credit at any one of the cooperating institutions, as well as certain other universities, in partial fulfillment of the requirements for a graduate degree. The program may be entered at any session, and consecutive courses will follow in successive summers.

The courses to be offered in statistics in 1960 at the University of Florida are as follows: Statistical Methods I, II and III, through Sample Survey Methods; Statistical Theory I, II and III, including Probability, Inference and Least Squares; Statistical Problems; Advanced Statistical Inference; and Response Surfaces.

Requests for application blanks should be addressed to Dr. Herbert A. Meyer, Statistical Laboratory, University of Florida, Box 3568, Gainesville, Florida.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. L. Alder, Secretary, announces that the following 150 persons have been elected to membership by the Board of Governors on applications duly certified.

VIJA ABELE, B.A. (Ohio Northern) Asst. Research Physicist, Mound Laboratory

ROCHELLE ABEND, M.A. (Illinois) Instr., Kansas State University

PFC. ALFRED ADAMSKI, B.A. (Ripon Coll.) Math. Asst., United States Army

JOHN E. ALLEN, B.S. (Louisiana Polytech. Inst.) Grad. Asst., Oklahoma State University

LEE W. ANDERSON, Ph.D. (Tulane) Asst. Prof., University of Oregon

RICHARD J. ANDRES, Student, College of the Holy Cross

HUBERT AUSTIN, M.A. (Ball S.T.C.) Asst. Prof., Tri State College

ABDUL K. AZIZ, Ph.D. (Maryland) Asst. Prof., Georgetown University

RICHARD F. BAKER, Student, University of Oklahoma

MARGARET G. BANTA, A.M. (Columbia) Chairman of Dept., Butler High School, New Jersey

JOHN F. BARTH, B.A. (Willamette) Grad. Asst., Michigan State University

JOHN H. BIGGS, A.B. (West Virginia) Grad. Asst., West Virginia University

GORDON D. BJONERUD, B.S. (Northern S.T.C.) Teacher, High School, Bound Brook, New Jersey

WYLAND L. BLANCHARD, Student, Amherst College

RAYMOND E. BOCHE, B.S. (Calif. State Polytech.) Programmer, Lockheed Missiles & Space Division

RONALD L. BOHUSLOV, M.A. (California, Berkeley) Instr., Oakland City College

THOMAS B. BOONE, JR., M.A. (Northwest S.C. of Louisiana) Instr., Northwestern State College

BROTHER ROBERT N. COSTA, B.S. (St. Mary's Coll., Calif.) Chairman of Dept., Chaminade High School, Los Angeles, California

STEVEN J. BRYANT, Ph.D. (Missouri) Asst. Prof., Fresno State College

JACK W. CARLYLE, M.S. (Washington) Asso., Electrical Engineering, University of California

DAVID C. CARRAD, Student, Montclair Academy, New Jersey

WARREN P. CASH, JR., B.S. (Texas) Grad. Student, University of Texas

DAVID J. CAVENY, B.A. (Western Carolina Coll.) Grad. Asst., University of Kentucky

SHU-KWAN CHAN, Student, Massachusetts Institute of Technology

HERBERT H. CHU, M.S. (Oklahoma S.U.) Instr., American University

HENRY C. CLARKE, M.S. (California) Teacher, Beverly Hills High School, California

WILLIAM F. CLARY, M.D. (Tennessee) Psychiatrist, Springfield, Missouri

RAY W. CLEVELAND, M.A. (Columbia T.C.) Teacher, Rahway High School, New Jersey

ROBERT L. COOLEY, M.S. (Purdue) Part-time Instr., Purdue University

EDWARD J. CREAGER, JR., Student, Xavier University

DOUGLAS A. DAVIDS, M.S. (Newark Coll. Engineering) Engineer, Applied Physics Lab., Johns Hopkins University

WILLIAM E. DEMALIGNON, M.A. (Wisconsin) Asst. Prof., State University of South Dakota

ROBERT J. DIAMOND, Ph.D. (Calif. Inst. Tech.) Asst. Prof., Los Angeles State College

ALPHONSO J. DIPIETRO, Ph.D. (George Peabody Coll.) Asst. Prof., Eastern Illinois University

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HENRY C. DIXON, JR., A.B. (Bowdoin) Grad. Student, Boston College

JAMES J. DODD, M.S. (Illinois) Instr., Whitewater State College

EDWIN D. ECKER, M.S. (Illinois) Instr., MacMurray College

MAGDI EL-KAMMASH, M.P.H. (North Carolina) Research Asst., Duke University

ARNOLD ESTEP, Student Asst., University of Kentucky

JANE L. EVANS, M.A. (Alabama) Instr., St. Petersburg Junior College

HAROLD E. FASSBERG, M.A. (Pittsburg) Deputy Director, Corp. for Economic & Industrial Research

WYMAN G. FAIR, M.A. (Missouri) Instr., University of Missouri

WILLIAM M. FITZGERALD, M.S. (Michigan) Teaching Fellow, University High School, Ann Arbor, Michigan

RICHARD A. FLECK, B.S. (Eastern Kentucky S.C.) Teaching Fellow, University of Kentucky

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MARTIN GARDNER, B.A. (Chicago) Editor, Scientific American

SEYMOUR GEISSER, Ph.D. (North Carolina) Mathematician, National Institute of Mental Health

ALLAN GEWIRTZ, B.S. (Brooklyn) Lecturer, Brooklyn College; Engineer, Waldorf Indus. Controls

LARRY K. GILCHRIST, B.A. (Knox Coll.) Grad. Asst., Oklahoma State University

RICHARD P. GOBLIRSCH, Ph.D. (Wisconsin) Asst. Prof., University of Rochester

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LARRY C. GROVE, Student, University of Minnesota

DAVID G. HALMSTAD, A.B. (Harvard) Grad. Student, University of Wisconsin

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HARVEY JACKINS, President, Personal Counselors Inc., Seattle, Washington

MARC Q. JACOBS, Student, University of Oklahoma

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BJARNI JONSSON, Ph.D. (California) Prof., University of Minnesota

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CARL D. KEEFER, B.S. (Millersville S.T.C.) Grad. Student, Bucknell University

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PAUL KROLL, A.M. (Columbia) Senior Manager, New York State Division of Employment, Brooklyn

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STEPHEN LANGE, A.B. (Connecticut) Teacher, Storm King School, Cornwall-on-Hudson, New York

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JAMES D. WAGNER, Student, Madison College
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JAMES E. WINCHESTER, A.B. (Alabama) Applied Science Representative, I.B.M., Washington, D. C.
LT. GEORGE C. WOLF, M.A. (Dartmouth) United States Army
HERBERT H. WOOD, Student, University of Oklahoma
CATHERINE E. WORMLEY, M.A. (N.Y.S.C. for Teachers, Albany) Teacher, Colonie Central High School, Albany, New York
EDWIN A. YERMAN, A.A. (Pasadena City Coll.) Student, Los Angeles State College
RONALD ZARSKI, B.A. (Tulsa) Asst. Instr., University of Kansas

MAA BECOMES MEMBER OF ISF

The Mathematical Association of America has accepted the invitation of the International Science Foundation to become a participating member. As such, the Association agrees to exchange information and publications with the Foundation and cooperate in activities that may be of mutual interest. The Foundation provides certain facilities and services to visiting scientists when they are in the San Francisco Bay area or the Los Angeles metropolitan area. Services in other U. S. metropolitan areas will be available in the near future. In addition, the Foundation's service organization for research, the Cooperative Research Institute, is available to work with Foundation members on cooperative research projects. The ISF is located at the World Trade Center, San Francisco 6, California.

ITINERARIES OF VISITING LECTURERS, 1960

J. Sutherland Frame

Mar. 28–29	University of Alberta, Edmonton, Alta. (E. S. Keeping)
Mar. 31–Apr. 1	University of British Columbia, Vancouver, B. C. (R. D. James)
Apr. 4–5	Western Washington College of Education, Bellingham, Wash. (Harvey M. Gelder)
Apr. 6	St. Martin's College, Olympia, Wash. (Rev. John Raymond, O.S.B.)
Apr. 7–8	Seattle University, Seattle 22, Wash. (T. S. Chihara)
Apr. 11–13	Washington State University, Pullman, Wash. (S. G. Hacker) and University of Idaho, Moscow, Ida. (K. A. Bush)
Apr. 15–19	Whitman College, Walla Walla, Wash. (W. L. Hutchings) and Walla Walla College, College Place, Wash. (L. G. Schlotthauer)
Apr. 21–23	College of Idaho, Caldwell, Ida. (Boyd Henry)
May 2–3	Eastern New Mexico University, Portales, N. Mex. (E. A. Propes)

May 5–6	New Mexico State University, University Park, N. Mex. (Earl Walden)
May 9–10	California State Polytechnic College, San Luis Obispo, Calif. (M. E. Whitson)
May 12–13	Fresno State College, Fresno 26, Calif. (G. Don Alkire)
May 16–17	Humboldt State College, Arcata, Calif. (H. D MacGinitie)
May 19–20	Chico State College, Chico, Calif. (Lloyd M. Cook)
May 23–24	Sacramento State College, Sacramento, Calif. (Stanley P. Hughart)
May 25	University of Utah, Salt Lake City, Utah (C. R. Wylie, Jr.)
May 26–27	Brigham Young University, Provo, Utah (H. J. Fletcher)

THE OCTOBER MEETING OF THE IOWA SECTION

Members of the Iowa Section of the Mathematical Association of America met jointly with Teachers of Mathematics at the 29th Annual Conference of Teachers of Mathematics at the State University of Iowa, October 16, 1959. The morning meeting included an address by Lenore John, University of Chicago, who spoke on "The Recommendations of the School Mathematics Study Group for Grades 7 and 8." Mr. Henry Swain of the New Trier Township High School, Winnetka, Illinois also gave an address entitled "The Recommendations of the School Mathematics Study Group for Grade 9."

At the afternoon meeting, the first topic, "The Minnesota National Laboratory for the Improvement of Secondary Mathematics," was presented by Frank Wolf, Carleton College. A general discussion period followed this address.

E. L. CANFIELD, *Secretary*

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual fall meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Delaware on Saturday, November 28, 1959, with Professor Marguerite Lehr, Bryn Mawr College, presiding. There were 54 present at the meeting, including 44 members of the Association.

At the Business Meeting the following officers were elected: Chairman, Professor W. S. Lawton, Temple University; Third member of the Executive Committee, Professor R. F. Jackson, University of Delaware; Professor John Brown, University of Delaware, was appointed chairman of a committee to act as Liaison agent to State Authority.

Professor Lawton reported on the progress of the High School contest conducted in the Philadelphia area. Professor Lehr reported on the Professional Standards Conference.

The following program was presented:

1. *Probabilistic models in psychology for the study of choice behavior*, by Professor R. D. Luce, University of Pennsylvania.

An organism making choices from a finite set of alternatives can, sometimes, be characterized as a probability distribution over the alternatives. The heart of the psychological problem, then, is to ascertain the laws that relate these distributions to one another, both in a static and dynamic sense. One general static law, several specific static laws, and one general dynamic law are described, and their application to problems in psychophysics, gambling behavior, and learning are sketched. A number of unresolved mathematical problems are outlined that arise in attempting to test these results empirically.

2. *On some extremal problems in geometry*, by Professor A. S. Besicovitch, University of Pennsylvania. (By invitation.)

Three extremal problems are considered: (1) The classical isoperimetric problem about the maximum area of a domain with fixed length of the boundary. Steiner's solution is given and analyzed. (2) The problem on the maximum area of a set of a given diameter. An elegant solution of W. Feller is given. (3) The problem on the minimum area of a set of constant width is treated by expressing the area bounded by an arc and the tangents at the end points in terms of curvature.

3. *College mathematics for the prospective graduate student*, by Professor Bernard Epstein, University of Pennsylvania.

Many beginning graduate students are handicapped by the lack of understanding of the fundamental ideas of real analysis. In too many cases "advanced calculus," as a mere continuation of "elementary calculus," is not meeting this need. A "little real-variable" course is needed containing, in particular, material dealing with the topology of the reals, including the Bolzano-Weirstrass and Heine-Borel Theorems. A course dealing with some phases of higher algebra is also highly desirable. These courses could replace, for the prospective graduate student, traditional courses which are largely devoid of mathematical content—in particular, the usual manipulative courses in differential equations and statistics.

4. *Work of the School Mathematics Study Group at Boulder and Ann Arbor*, by Professor V. H. Haag, Franklin and Marshall College.

The School Mathematics Study Group at Boulder and Ann Arbor in the summer of 1959, consisting of more than eighty mathematicians and teachers, not only prepared preliminary editions of sample texts for grades seven through twelve, but found themselves learning a great deal about each other's problems. It is believed that this liaison between the two groups, the mathematicians and the teachers, was a valuable contribution to the solution of problems of mathematics education.

5. *Liaison problems in collegiate mathematics today—with the high school*, by Mr. M. A. Linton, Jr., William Penn Charter School.

Liaison between high school and college teachers of mathematics appears to be stronger today than at any previous time. Most, if not all of the agencies now at work in reshaping the elementary and secondary school mathematics curricula depend upon the cooperative efforts of members of both groups. As a result, not only are the new courses sound in mathematical content, but they are also teachable. An important by-product of this cooperation is increased awareness by both college and school teachers of the problems of each other. Ultimately, the reforms being wrought in school mathematics courses will affect the college curriculum as well.

F. L. DENNIS, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-D.C.-VIRGINIA SECTION

The winter meeting of the Maryland-D.C.-Virginia Section of the Mathematical Association of America was held on December 5, 1959, at American University, Washington, D. C. Professor E. E. Floyd, of the University of Virginia, presided. The total attendance was 151 including 107 members.

At the business meeting, following the program, the registration fee for meetings was increased from twenty-five cents to fifty cents to cover rising costs of postage and occasional expense for outside invited speakers. Mr. W. H. Norris, of Norfolk, described the progress made in improving the high school contest program.

The following papers were presented:

1. *New contributions to educational research from operations research*, by Professor J. M. Long, College of William and Mary.

The paper discussed four points: (1) the nature and development of operations research; (2) several aspects of the educational system which might be susceptible to operations research methods; (3) queueing theory was briefly described and analyzed in relation to the educational process; (4) the use for education of methods associated with the optimum solution problem was considered.

2. *On the proof of Gödel's incompleteness theorem*, by Professor D. B. Lloyd, D. C. Teachers College.

In 1931 Kurt Gödel proved an outstanding theorem to the effect that any broad mathematical system, such as arithmetic, is an incomplete theory in the sense that there are theorems therein which are true but which cannot be proved within the framework of its own axiomatic structure

Gödel mapped the metamathematical logic into an arithmetical model, converting the proof into a number-theoretic one. The paper sketched the techniques of this unique proof and the resulting impacts upon the modern understanding of the inherent nature of mathematics.

3. *Elementary combinatorial problems related to a mail-sorting device*, by Dr. A. J. Goldman, National Bureau of Standards.

Mail to any of r destinations enters the device k letters at a time. After each k letters enter, the device "asks" to which destination it contains most letters, and drops out all letters to this destination. It is shown that a capacity of $(r-1)(k-1)+k$ letters (but no fewer) suffices to prevent overflow, and that any capacity in excess of rk letters will be wasted; for realistic values (*e.g.*, $r=200$, $k=3$) these numbers are much lower than the upper bound of r^2k obtained by another investigator. These results were obtained jointly with Mrs. B. K. Bender, as part of a project supported by the Post Office Department, Office of Research and Engineering.

4. *A note on polynomial interpretation*, by Dr. C. H. Frick, Naval Weapons Laboratory, Dahlgren, Virginia.

By assigning errors at the data points, Chebychev polynomials were used without the inconvenience of normalizing arguments or establishing data points with irrational arguments.

5. *The calculus of relations*, by Dean W. L. Duren, University of Virginia.

This was an elementary treatment of the calculus of relations of the form $F(x, y)=C$. Without regarding the relation as an implicit function it leads to the tangent relation $F_x dx+F_y dy=0$, where dx and dy are local coordinates with respect to axes centered at (x, y). Conversely, the integral calculus gives integrals of differential equations of the form $M(x, y)dx+N(x,y)dy=0$. The uniqueness of the integral is proved but only in the case of separable variables is the existence of the integral an elementary problem. It is pointed out that the uniqueness problem for integrals is essentially different from that for solutions.

6. *Some examples of completely semiassociative and noncommutative loops and algebras*, by Professor Volodymyr Bohun-Chudyniv, Morgan State College.

An algebra we call *completely semiassociative* if the number α of nonassociative cases in 6 products in all ordered triplets u_α, u_β, u_γ, of the basal units, not belonging either to the center or to the same subalgebra, equal the number β of *associative cases*. The aims of this paper are to determine some examples of completely semiassociative algebras and loops and consider some of their properties.

7. *Hypercomplex methods in the theory of partial differential equations*, by Professor J. M. Horvath, University of Maryland.

Given a homogeneous partial differential operator with constant coefficients D, an infinite dimensional algebra Q can be associated with it, which permits to construct a complete set of polynomials Y, solutions of $DY=0$ (Proc. Amer. Math. Soc., vol. 9, 1958, pp. 569–575). If D is elliptic, then every solution of $Df=0$ can be approximated by linear combinations of the Y. If the number of variables is two, a finite-dimensional quotient algebra A of Q and D-analytic of functions with values in A can be defined. The theory of the D-analytic functions yields further results concerning the approximation of solutions of $Df=0$.

8. *An algorithm for the determination of divisibility of one natural number by another*, by Professors Herta T. and A. H. Freitag, Hollins College, Virginia.

A new criterion for determining divisibility of one natural number by another is developed. In case of divisibility, this algorithm constructs the quotient without the necessity of estimating trial divisors. The operations in this procedure are less involved than those in ordinary division. Certain number properties become apparent.

9. *Curvature of* $r^n=\cos n\theta$, by Professor R. C. Yates, College of William and Mary.

(See pp. 275–278 of this issue.)

10. *What is the mathematical theory of knots?* by Professor R. H. Fox, Princeton University. (Invited address.)

Knot theory deals with the topological classification of the ways in which a simple closed curve can be imbedded in 3-dimensional space. To each such imbedding is associated the fundamental group of the complementary space, and thereby an integral polynomial $\Delta(t)$. Thus to the trivial knot is associated $\Delta(t)=1$, to the overhand knot $\Delta(t)=1-t+t^2$, to the figure-of-eight $\Delta(t)=1-3t^2$, to the square knot or to the granny knot $\Delta(t)=(1-t+t^2)^2$. There are many other invariants, some of greater power and subtlety. With these it is possible to distinguish, for example, between the square knot and the granny. However, a complete description of all the imbedding types is very far from being realized.

D. B. LLOYD, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April 30, 1960.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK, City College, New York, April 2, 1960.

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA, St. Olaf College, Northfield, May 7, 1960.

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, University of Oklahoma, Norman, April 15–16, 1960.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

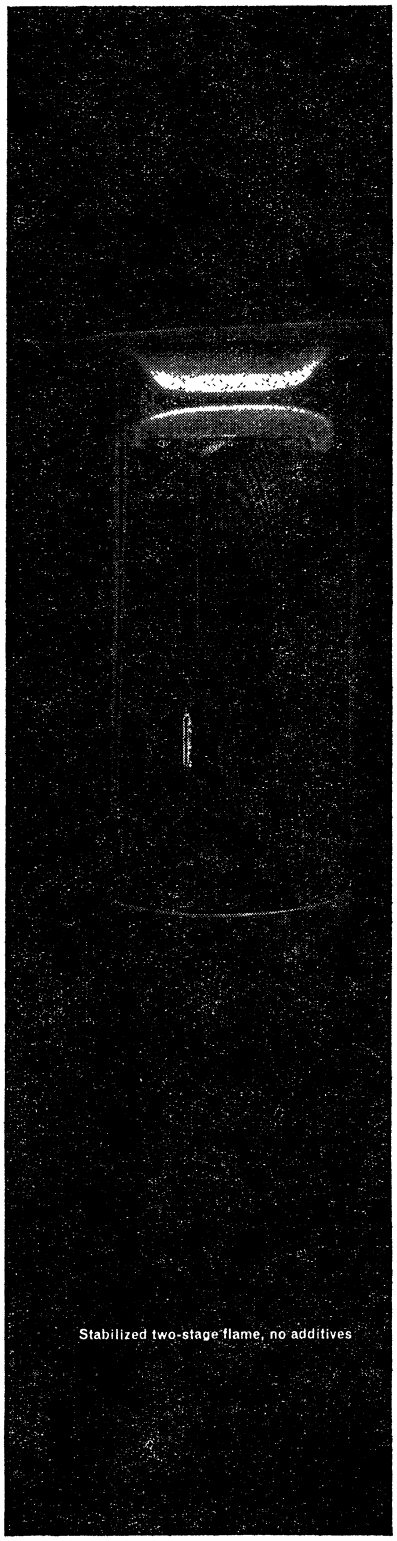
SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April 8–9, 1960.

TEXAS, San Antonio College, April 8–9, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.



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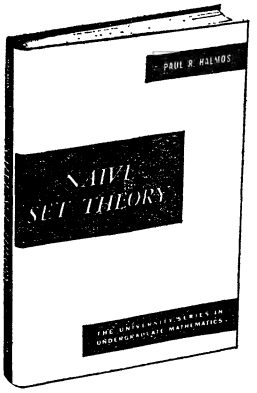
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ON MATHEMATICAL INDUCTION*

LEON HENKIN, University of California, Berkeley

Introduction. According to modern standards of logical rigor, each branch of pure mathematics must be founded in one of two ways: either its basic concepts must be *defined* in terms of the concepts of some prior branch of mathematics, in which case its theorems are deduced from those of the prior branch of mathematics with the aid of these definitions, or else its basic concepts are taken as *undefined* and its theorems are deduced from a set of axioms involving these undefined terms.

The natural numbers, 0, 1, 2, 3, · · · are among those mathematical entities about which we learn at the earliest age, and our knowledge of these numbers and their properties is largely of an intuitive character. Nevertheless, if we wish to establish a precise mathematical theory of these numbers, we cannot rely on unformulated intuition as the basis of the theory but must found the theory in one of the two ways mentioned above. Actually, both ways are possible. Starting with pure logic and the most elementary portions of the theory of sets as prior mathematical sciences, the German mathematician Frege showed how the basic notions of the theory of numbers can be defined in such a way as to permit a full development of this theory. On the other hand the Italian mathematician Peano, taking *natural number*, *zero*, and *successor* as primitive undefined concepts, gave a system of axioms involving these terms which were equally adequate to allow a full development of the theory of natural numbers. In the present paper we shall examine the concept of *definition by mathematical induction* within the framework of Peano's ideas. In this development we shall presuppose only logic and the most elementary portions of the theory of sets; however, we shall find that our subject is greatly illuminated by the introduction of some of the terminology of modern abstract algebra, even though we do not presuppose any of this algebraic material as a basis of our proofs or definitions.

1. Models and the axioms of Peano. It will be convenient here to use the word *model* to refer to a system consisting of a set N, an element 0 of N, and a unary operation† S on N. A model $\langle N, 0, S\rangle$ will be called a *Peano model* if it satisfies the following three conditions (or axioms).

P1. *For all* $x\in N$, $Sx\neq 0$.

P2. *For all* $x, y\in N$, *if* $x\neq y$ *then* $Sx\neq Sy$.

P3. *If* G *is any subset of* N *such that* (a) $0\in G$, *and* (b) *whenever* $x\in G$ *then also* $Sx\in G$, *then* $G=N$.‡

* This article was translated into Russian and published in Matematicheskoe Prosveshchenie, No. 6, 1959. This English version appears with the consent of the editors of the Soviet publication.

† A unary operation on N is a function having N as its domain and having a range which is a subset of N. For any $x\in N$, we let Sx be the element of N which is the value obtained by operating on x with S.

‡ In the terminology of set-theory, P1 and P2 respectively express the conditions that 0 is not in the range of S, and that S is one-one. A subset G of N which satisfies condition (b) of P3 is said

If we consider the case when N^* is the set of natural numbers as we know them intuitively, 0^* is the number zero, and S^* is the successor operation (*i.e.*, the operation such that for any natural number x, S^*x is the next following number), then $\langle N^*, 0^*, S^* \rangle$ is an example of a Peano model. Of course it was this example which suggested consideration of the Peano axioms in the first place. Note, however, that there are many other Peano models, for example the system $\langle N', 0', S' \rangle$ where N' is the set of all positive even integers, $0'$ is the number two, and S' is the operation of adding two.

Condition P3 is called the *axiom of mathematical induction.* It will be useful to introduce the term *induction model* to refer to any model which satisfies this axiom. Thus we see at once that every Peano model is an induction model. But the converse is far from true.

For example, let N'' be a set containing a single element, let $0''$ be this element, and let S'' be the only possible unary operation on N'', *i.e.*, the operation such that $S''0'' = 0''$. Then clearly $\langle N'', 0'', S'' \rangle$ is an induction model, but it is not a Peano model since it does not satisfy Axiom P1. For another example let a_0 and a_1 be two distinct objects, and let N''' be the pair $\{a_0, a_1\}$ (*i.e.*, the set having a_0 and a_1 as its only elements). Let S''' be the unary operation on N''' with constant value a_1 (*i.e.*, $S'''x = a_1$ for all $x \in N'''$). Then $\langle N''', a_0, S''' \rangle$ is an induction model, but it is not a Peano model since it does not satisfy Axiom P2.

The reader may notice that the model $\langle N'', 0'', S'' \rangle$ satisfies P2, while $\langle N''', a_0, S''' \rangle$ satisfies P1. It then becomes natural to ask whether there are induction models which satisfy *neither* P1 nor P2. As it happens, there are none: any model which satisfies P3 must also satisfy either P1 or P2. A direct proof of this fact, using only the laws of logic and the elements of set theory, is rather troublesome to find—it is a task we leave for the enterprising reader. Presently we shall see that after the theory of definition by mathematical induction is established, the result can be obtained quite simply.

Now consider the following two statements.

P4. *If y is any element of N such that $y \neq Sx$ for all $x \in N$, then $y = 0$.*
P5. *For all $x \in N$, $x \neq Sx$.*

We recognize that each of these is true for the Peano model $\langle N^*, 0^*, S^* \rangle$ of natural numbers as we know them intuitively, and in fact we can easily show that each one is true of *all* Peano models by deriving it from Axioms P1–P3. But there is an important difference between the two: the proof of P4 requires

to be *closed* under S. If we regard $\langle N, S \rangle$ as an algebraic system, a subset G of N which is closed under S would be called a *subalgebra* of the system. Thus P3 expresses the condition that the only subalgebra of $\langle N, S \rangle$ which contains the element 0 is N itself. In algebraic terminology this condition is expressed by saying that the element 0 *generates* the algebra $\langle N, S \rangle$. It is also possible to regard the system $\langle N, 0, S \rangle$ itself as an algebraic system. In that case, in order to qualify as a subalgebra a subset G of N must contain 0 as well as be closed under S. Thus P3 expresses the condition that the only subalgebra of $\langle N, 0, S \rangle$ is N itself. Hence the system $\langle N, 0, S \rangle$ is generated by any one of its elements.

only Axiom P3, so that P4 holds for all induction models, but the proof of P5 requires Axiom P1 as well as P3, and our example $\langle N'', 0'', S'' \rangle$ shows that P5 does not hold for all induction models.

2. Operations defined by mathematical induction. The statements P4 and P5 are examples of true theorems about natural numbers, but we generally regard their mathematical content as quite trivial. In order to develop a richer theory it is essential to introduce additional concepts beyond the primitive notions of number, zero, and successor; in particular, we must define such central concepts as addition, multiplication, exponentiation, prime number, *etc.* Let us consider addition first, and inquire how it can be defined.

Addition is a binary operation on natural numbers; *i.e.*, it is a function $+$ which may act on any ordered pair $\langle x, y \rangle$ of natural numbers, the result of the action being again a natural number, $x+y$. Peano's idea was to define $+$ by means of the pair of equations

$$x + 0 = x, \tag{1.1}$$

$$x + Sy = S(x + y). \tag{1.2}$$

Of course on the basis of our intuitive knowledge of the operation of addition we recognize that these equations are true for all natural numbers x and y. But in what sense do the equations constitute a *definition* of addition? In particular, does the definition hold only for natural numbers, or for arbitrary Peano models as well? In order to get a clear answer to these questions we first consider a related but more general problem.

The introduction of an operation by means of the pair of equations 1.1 and 1.2 is an example of what is called *definition by mathematical induction.* To describe this concept in general terms we must consider a Peano model $\langle N, 0, S \rangle$ and in addition a second model $\langle N_1, 0_1, S_1 \rangle$ which, however, is not required to be a Peano model (or even an induction model). Being given these two models we say that the pair of equations

$$h0 = 0_1, \tag{2.1}$$

$$h(Sy) = S_1(hy), \tag{2.2}$$

defines (by mathematical induction) a function h: a function which maps N into N_1 and satisfies 2.1 and 2.2 for all $y \in N$. Again we may raise the question: In what sense do these equations define a function? The answer is provided by the following theorem.

THEOREM I. *No matter what Peano model* $\mathfrak{N} = \langle N, 0, S \rangle$ *we have, and no matter what model* $\mathfrak{N}_1 = \langle N_1, 0_1, S_1 \rangle$ *we start with, there exists a unique homomorphism of* $\mathfrak{N}$ *into* $\mathfrak{N}_1$; *that is, there exists one and only one function* h *mapping* N *into* N_1 *which satisfies* 2.1 *and* 2.2 *for all* $y \in N$.†

† As remarked in footnote ‡, pp. 323–324, a model $\langle N, O, S \rangle$ is an induction model if and only if 0 is a generator of the algebraic system $\langle N, S \rangle$. In algebraic terminology the content of Theorem I

Before attempting to prove this theorem let us see how it applies to the case of 1.1 and 1.2.

Let $\mathfrak{N}$ be an arbitrary Peano model $\langle N, 0, S\rangle$, and for each $x \in N$ let $\mathfrak{N}_x$ be the model $\langle N, x, S\rangle$. Applying Theorem I to the models $\mathfrak{N}$ and $\mathfrak{N}_x$ we see that for each $x \in N$ there will be a unique function h_x mapping N into itself such that the equations

$$3.1 \qquad h_x 0 = x,$$

$$3.2 \qquad h_x(Sy) = S(h_x y),$$

hold for all $y \in N$. From the existence and uniqueness of these functions h_x we shall show that there exists a unique binary operation of addition on $\langle N, 0, S\rangle$; for later purposes the reader should note that the argument by which we infer the existence and uniqueness of addition from the existence and uniqueness of the functions h_x is purely set-theoretical in character, and does not depend in any way on Axioms P1–P3. Let f be the binary operation on N whose value for any $x, y \in N$ is determined by the equation

$$4 \qquad fxy = (h_x y).$$

Using 4, 3.1, and 3.2 we see that f satisfies the equations

$$5.1 \qquad fx0 = x,$$

$$5.2 \qquad fx(Sy) = S(fxy),$$

for all $x, y \in N$. Furthermore, f is the *only* binary operation on N with this property. For suppose g is any binary operation on N satisfying

$$6.1 \qquad gx0 = x,$$

$$6.2 \qquad gx(Sy) = S(gxy),$$

for all $x, y \in N$. Then for each $x \in N$ let g_x be the unary operation on N such that

$$7 \qquad g_x y = gxy$$

for all $y \in N$. From 7, 6.1, and 6.2 we infer that for any $x \in N$ the equations

$$8.1 \qquad g_x 0 = x,$$

$$8.2 \qquad g_x(Sy) = S(g_x y),$$

hold for all $y \in N$. Comparing 3.1, 3.2 and 8.1, 8.2 we see that for each $x \in N$ we have $g_x = h_x$, since Theorem I assures us that the function h_x defined by 3.1 and 3.2 is unique. But if $g_x = h_x$ for each $x \in N$, 4 and 7 imply that $fxy = gxy$ for all $x, y \in N$. Since f and g are binary operations on N, we see that $f = g$ by the principle of extensionality.

We have thus inferred from Theorem I that if $\langle N, 0, S\rangle$ is any Peano model,

would be expressed by saying that in the class of all algebraic systems $\langle N, S\rangle$ the systems $\langle N, S\rangle$ derived from Peano models $\langle N, 0, S\rangle$ are *free*, and in fact are *freely generated by* 0.

there is a unique binary operation f on N satisfying 5.1 and 5.2 for all $x, y \in N$. This binary operation we call addition; if we denote it by the symbol "+", we see that 1.1 and 1.2 are simply another way of writing 5.1 and 5.2. In this way the definition of addition by 1.1 and 1.2 is explained by Theorem I.

We now take up the problem of proving this theorem.

Suppose, then, that $\langle N, 0, S\rangle$ is any Peano model, and that $\langle N_1, 0_1, S_1\rangle$ is an arbitrary model. We wish to show the existence of a unique function h mapping N into N_1 which satisfies 2.1 and 2.2 for all $y \in N$. Putting aside for a moment the question of uniqueness, an argument to show the existence of such a function h is sometimes given along the following lines.

Clearly (the argument goes), h is defined for 0, since $h0 = 0_1$ by 2.1. Furthermore, if h is defined for an element y of N then h is also defined for Sy since $h(Sy) = S_1(hy)$ by 2.2. Thus if we let G be the set of all those $y \in N$ for which h is defined, we see that (a) $0 \in G$, and (b) whenever $y \in G$ then also $Sy \in G$. Applying Axiom P3 we conclude that $G = N$. Thus h is defined for all $y \in N$.

At first sight this argument may seem convincing, but a moment's reflection will suffice to raise doubts. For in this argument we refer to a certain function h. But what is h? Apparently it is a function which satisfies 2.1 and 2.2. Recall, however, that the argument is designed to establish the existence of such a function; clearly, then, it is incorrect to assume in the course of the argument that we have such a function.

This objection may be thought at first to be simply a verbal matter which can be avoided by some minor rewording of the argument. Actually, however, there is something fundamentally wrong with the argument—for the only property of the model $\langle N, 0, S\rangle$ which it employs is Axiom P3! If this argument were essentially correct, it would follow that for any *induction* model $\langle N, 0, S\rangle$, and an arbitrary model $\langle N_1, 0_1, S_1\rangle$, there exists a function h, mapping N into N_1, which satisfies 2.1 and 2.2 for all $y \in N$. But this statement is simply false, as the following example shows.

Let $\mathfrak{N}'' = \langle N'', a_0, S''\rangle$ be the induction model considered in Section 1, in which N'' is the pair of two distinct objects a_0 and a_1, and $S''x = a_1$ for all $x \in N''$. Let T be the unary operation on N'' such that $Ta_0 = a_1$ and $Ta_1 = a_0$, and let $\mathfrak{N}_1$ be the model $\langle N'', a_0, T\rangle$. Now if Theorem I could be applied to the induction model $\mathfrak{N}''$ and the model $\mathfrak{N}_1$, there would exist a mapping h of N'' into itself such that

$$9.1 \qquad ha_0 = a_0,$$

$$9.2 \qquad h(S''y) = T(hy),$$

for all $y \in N''$. From 9.2 we compute that $h(a_1) = h(S''a_0) = T(ha_0)$, and so by 9.1, $h(a_1) = Ta_0 = a_1$. On the other hand, $h(a_1) = h(S''a_1) = T(ha_1)$ by another application of 9.2, and since we have already computed $ha_1 = a_1$ this shows that $h(a_1) = Ta_1 = a_0$. From $h(a_1) = a_0$ and $h(a_1) = a_1$ we get $a_0 = a_1$, contrary to our hypothesis that a_0 and a_1 are distinct. This contradiction shows that Theorem I does

not apply to the induction model $\mathfrak{N}''$, and shows *a fortiori* that any proof of this theorem must employ either Axiom P1 or P2, as well as P3; actually, as we shall see later, all three axioms must be employed. We may phrase this observation by saying that the *axiom* of mathematical induction does not itself justify *definitions* by mathematical induction.†

Proof of Theorem I. Let $\mathfrak{N}=\langle N, 0, S\rangle$ be any Peano model and let $\mathfrak{N}_1=\langle N_1, 0_1, S_1\rangle$ be an arbitrary model. A subset H of N is called a *segment* if $0\in H$ and if, whenever $Sx\in H$ then also $x\in H$. By a *partial function* let us mean a function j whose domain is some segment H, which has values in N_1, and which is such that

$$10.1 \qquad j0 = 0_1,$$

$$10.2 \qquad j(Sx) = S_1(jx)$$

for all x such that $Sx\in H$.

LEMMA 1. *Every element of N is in the domain of some partial function.*

Proof. Let G be the set of those elements of N which are in the domain of some partial function. It is clear that the set $\{0\}$ whose only element is 0 is a segment, since there is no x such that $Sx\in\{0\}$ according to Axiom P1. Furthermore, the function j whose domain is $\{0\}$ and whose value $j0$ is 0_1 is a partial function for the same reason. Hence $0\in G$.

Now suppose that y is any element of G, and let j be a partial function having y in its domain. Let H be the domain of j. If $Sy\in H$ then Sy too is in G; so let us consider the case where $Sy\notin H$. In this case let H' be the subset of N obtained from H by adding Sy as an additional element, and let j' be the function whose domain is H' and whose values are given by the following rule: if $x\in H$ then $j'x=jx$, and if $x=Sy$ then $j'x=S_1(jy)$. We shall show below that j' is a partial function, and hence that $Sy\in G$ in the case that $Sy\notin H$ (as well as in the contrary case considered above). Since G contains 0 and is closed under S, it will follow by Axiom P3 that $G=N$. By definition of G the lemma will therefore be proved.

Thus to complete the proof of the lemma it remains only to show that j' is a partial function. To this end, consider first the domain H' of j'; we have $H'=H\cup\{SY\}$. Since H is the domain of a partial function it is a segment, and hence $0\in H$, so that $0\in H'$. Further, if x is any element of N such that $Sx\in H'$, then also $x\in H'$ as we see by cases: if $Sx\in H$ then $x\in H$ because H is a segment, and if $Sx=Sy$ then $x=y$ by Axiom P2 so that $x\in H$ again (since we know $y\in H$). These considerations show that H' is a segment. Now $j'0=j0=0_1$, and $j'(Sx)=S_1(j'x)$ whenever $Sx\in H'$; the last equation is again established by cases, for if $Sx\in H$ then also $x\in H$ so that $j'(Sx)=j(Sx)$ and $j'x=jx$, while if $Sx=Sy$ then $x=y$ and so $j'(Sy)=S_1(j'y)$. Since j' has a domain which is a seg-

† This fact was clearly brought out by Dedekind in his famous book: Was sind und was sollen die Zahlen? (See "Bemerkung," paragraph 130, section 9.)

ment, and since it satisfies 10.1 and 10.2, it is a partial function. This completes the proof of Lemma 1.

LEMMA 2. *If j_1 and j_2 are partial functions and x is in the domain of each, then $j_1x = j_2x$.*

Proof. Let G be the set of those elements x of N such that $j_1x = j_2x$ whenever j_1 and j_2 are partial functions each of which has x in its domain. Clearly $0 \in G$, since $j0 = 0_1$ for any partial function j. Now suppose x is an arbitrary element of G, and let j_1 and j_2 be any partial functions each of which contains Sx in its domain. Then $j_1(Sx) = S_1(j_1x)$ and $j_2(Sx) = S_1(j_2x)$; but since $x \in G$, $j_1x = j_2x$ and hence $j_1(Sx) = j_2(Sx)$. Therefore $Sx \in G$. Since $0 \in G$ and G is closed under S, it follows by Axiom P3 that $G = N$. Referring to the definition of G we see that Lemma 2 is proved.

By combining Lemmas 1 and 2 we see that for any $x \in N$ there is one and only one $z \in N$ such that $z = jx$ for some partial function j. We let h be the function, with domain N, such that for any $x \in N$ the value hx is this unique $z \in N$. We claim that this function h satisfies 2.1 and 2.2 for all $y \in N$, and that it is the only function with domain N which has this property.

Clearly $h0 = 0_1$ since $j0 = 0_1$ for any partial function j. Furthermore, for any $y \in N$ there is a partial function j such that Sy is in the domain of j (by Lemma 1), whence we see that $h(Sy) = j(Sy) = S_1(jy) = S_1(hy)$, so that h satisfies 2.1 and 2.2 as claimed. Now if h_1 is any other function with domain N which satisfies 2.1 and 2.2 for all $y \in N$, then $h_1 = h$. The reason is that N is clearly a segment, so that h and h_1 are both partial functions, whence by Lemma 2 we have $hx = h_1x$ for all $x \in N$, and therefore $h = h_1$ by the principle of extensionality. This completes the proof of Theorem I.

This proof is rather more involved than the simple argument sketched at first, but it has the advantage of being correct. The reader will notice that all of the Axioms P1–P3 were employed in the proof (in connection with Lemma 1).

The construction of h by means of partial functions, as described in this proof, is not the only means we know of obtaining this function. There is another method of constructing h which leads to a different proof of Theorem I. We will outline this other process of construction briefly, leaving the reader to supply details of the proof.

Being given a Peano model $\langle N, 0, S\rangle$ and an arbitrary model $\langle N_1, 0_1, S_1\rangle$, we consider subsets A of the product set $N \times N_1$, *i.e.*, sets A all of whose elements are ordered pairs $\langle x, y\rangle$ where $x \in N$ and $y \in N_1$. We call such a set A *regular* if $\langle 0, 0_1\rangle \in A$ and if, whenever $\langle x, y\rangle \in A$ then also $\langle Sx, S_1y\rangle \in A$. Clearly there are regular sets, for example $N \times N_1$ itself. It is easy to see that the intersection A^* of all regular sets A is itself regular. Now using Axioms P1–P3, which hold for $\langle N, 0, S\rangle$, we can show that for every $x \in N$ there is one and only one $y \in N_1$ such that $\langle x, y\rangle \in A^*$: this is the part of the proof where the reader will have to supply some detail. Having shown this, we define h to be the function with domain N such that for any $x \in N$, hx is the unique $y \in N_1$ for which $\langle x, y\rangle \in A^*$.

From the fact that A^* is regular it then easily follows that h is a homomorphism of N into N_1, as required by Theorem I. Finally, to show the uniqueness of h, we consider any homomorphism h_1 of N into N_1. Letting B be the subset of $N \times N_1$ such that $\langle x, y\rangle \in B$ if and only if $y = h_1 x$, we easily see that B is regular and hence $A^* \subseteq B$. From the fact that for each $x \in N$ there is only one $y \in N_1$ such that $\langle x, y\rangle \in B$, we then infer that $A^* = B$; and from this we easily conclude that $h_1 = h$. This completes our outline of the second proof of Theorem I.

3. Addition and multiplication in arbitrary induction models. As we have previously noted, we can infer from Theorem I the existence, in every Peano model, of a unique operation of addition (*i.e.*, a binary operation f which satisfies 5.1 and 5.2 for all $x, y \in N$). Actually, something more is true.

THEOREM II. *In every induction model there is a unique operation of addition.*

We cannot hope to prove this theorem by means of Theorem I, for the latter is not true of all induction models, as we have seen in Section 2. Instead, we proceed by means of the following lemma.

LEMMA. *If $\langle N, 0, S\rangle$ is any induction model, then for every $x \in N$ there is a unique unary operation h_x on N such that* 3.1 *and* 3.2 *hold for all $y \in N$.*

Proof. We first observe that for any $x \in N$ there can be at most one operation h_x satisfying 3.1 and 3.2. For suppose h_x and h_x' are both operations which satisfy these equations, and let G be the subset of N such that $y \in G$ if and only if $h_x y = h_x' y$. Clearly $0 \in G$, since $h_x 0 = x = h_x' 0$. Also G is closed under S, for if $y \in G$ (so that $h_x y = h_x' y$), we see that $h_x(Sy) = S(h_x y) = S(h_x' y) = h_x'(Sy)$ whence $Sy \in G$. Since Axiom P3 holds for $\langle N, 0, S\rangle$ we infer that $G = N$. By the principle of extensionality it follows that $h_x = h_x'$.

Now let H be the subset of N consisting of those elements x for which an operation h_x exists. Taking h_0 to be the identity operation on N (such that $h_0 y = y$ for all $y \in N$), we see that $h_0 0 = 0$ (thus satisfying 3.1), and for any $y \in N$, $h_0(Sy) = Sy = S(h_0 y)$ (thus satisfying 3.2). Hence $0 \in H$. Furthermore, H is closed under S. For suppose $x \in H$, so that an operation h_x exists. Let h_{Sx} be the operation on N such that $h_{Sx} y = S(h_x y)$ for all $y \in N$. We see that $h_{Sx} 0 = S(h_x 0) = Sx$ (thus satisfying 3.1 for Sx), while for any $y \in N$, $h_{Sx}(Sy) = S(h_x(Sy)) = S(S(h_x y)) = S(h_{Sx} y)$ (thus satisfying 3.2 for Sx); whence $Sx \in H$. Applying Axiom P3, which holds for $\langle N, 0, x\rangle$ by assumption, we infer that $H = N$, which proves the lemma.

Using this lemma we can complete the proof of Theorem II by exactly the same argument used earlier to infer the existence of an addition operation from Theorem I. The reader will notice that in this earlier argument of Section 2 we used Theorem I to obtain the statement of our present lemma; as noted there, the remainder of the argument was of a general set-theoretical character independent of the axioms P1–P3, and so can be applied to the induction model $\langle N, 0, S\rangle$ of Theorem II.

When we consider multiplication we find a situation entirely analogous to the one we have just encountered for addition. By a multiplication operation for an induction model $\langle N, 0, S\rangle$ we mean a binary operation $\cdot$ on N such that the equations

$$11.1 \qquad x\cdot 0 = 0,$$

$$11.2 \qquad x\cdot(Sy) = (x\cdot y) + x,$$

hold for all $x, y\in N$, where $+$ is the addition operation for the model. Employing a proof closely patterned after the one given above for Theorem II, we can show that in every induction model there is one and only one multiplication operation. We leave the reader to supply the details of such a proof.

4. Operations in Peano models obtained by primitive recursion. When we come to the operation of exponentiation the situation changes. By an exponential operation for an induction model $\langle N, 0, S\rangle$ is meant a binary operation exp such that the equations

$$12.1 \qquad x \exp 0 = S0,$$

$$12.2 \qquad x \exp (Sy) = (x \exp y)\cdot x,$$

hold for all $x, y\in N$, where $\cdot$ is the multiplication of the model. This time there is no analogue of Theorem II, for it is simply not true that every induction model has an exponential operation; a counter-example is provided by the model $\langle N'', a_0, T\rangle$ which we have considered in Section 2, where a_0 and a_1 are distinct objects, $N''=\{a_0, a_1\}$, and $Ta_0=a_1$, $Ta_1=a_0$. It is a simple matter to show that if exp were a binary operation on N'' satisfying 12.1 and 12.2 for all $x, y\in N''$, then we would have $a_0 \exp z=a_0$ (for any $z\in N''$) from 12.2 (since $w\cdot a_0=a_0$ for all $w\in N''$), while from 12.1 we get $a_0 \exp a_0=a_1$. This contradiction shows that $\langle N'', a_0, T\rangle$ possesses no exponential operation.

However, by an application of Theorem I we can show that every *Peano* model possesses a unique exponential operation. In fact, we can get a general result of which this is a special case.

THEOREM III. *Let $\langle N, 0, S\rangle$ be any Peano model, let f be a unary operation on N, and let g be a ternary operation on N. Then there exists one and only one binary operation j on N such that the equations*

$$13.1 \qquad jx0 = fx,$$

$$13.2 \qquad jx(Sy) = gxy(jxy),$$

hold for all $x, y\in N$.†

† This function j is said to be obtained by *primitive recursion* from f and g. More generally, one allows for the possibility of obtaining an n-ary operation j from an $(n-1)$-ary operation f and $(n+1)$-ary operation g, for any $n=1, 2, \cdots$; we have selected the case $n=2$ merely to simplify notation, since the idea of the proof is the same for any n.

(If we take the particular case where f is the unary operation on N such that $fx=S0$ for all $x\in N$, and where g is the ternary operation on N such that $gxyz=z\cdot x$ for all x, y, z, then clearly the function j obtained will be the exponential operation.)

Proof. As remarked in the course of our proof of Theorem II, it is sufficient to show that for each $x\in N$ there is a unique unary operation j_x on N such that

$$14.1 \qquad i_x0 = fx,$$

$$14.2 \qquad i_x(Sy) = gxy(j_xy),$$

for all $y\in N$; for then we can infer Theorem III by a general set-theoretical argument which holds for all models (*i.e.*, which does not involve any of the Axioms P1–P3). To this end, let N_1 be the set $N\times N$ of all ordered pairs $\langle y, z\rangle$ for all $y, z\in N$. For each $x\in N$ let 0_x be the element $\langle 0, fx\rangle$ of N_1, and let S_x be the unary operation on N_1 such that $S_x\langle yz\rangle=\langle Sy, gxyz\rangle$ for all $y, z\in N$. Applying Theorem I to the Peano model $\langle N, 0, S\rangle$ and the model $\langle N_1, 0_x, S_x\rangle$, we see that for each $x\in N$ there is a unique mapping h_x of N into N_1 such that the equations

$$15.1 \qquad h_x0 = 0_x,$$

$$15.2 \qquad h_x(Sy) = S_x(h_xy),$$

hold for all $y\in N$.

Next let L and R be the mappings of N_1 into N such that $L\langle x, y\rangle=x$ and $R\langle x, y\rangle=y$ for all $x, y\in N$, and for each $x\in N$ let j_x and k_x be the unary operations on N such that $j_xy=R(h_xy)$ and $k_xy=L(h_xy)$ for all $y\in N$. We shall show that j_x satisfies 14.1 and 14.2 for all $y\in N$, but first it is necessary to prove that $k_xy=y$ for all $y\in N$. To do this, let G be the subset of N consisting of those elements y such that $k_xy=y$. Since $k_x0=L(h_x0)=L0_x=0$, we see that $0\in G$. Next, let y be any element of G. Then $k_x(Sy)=L(h_x(Sy))=L(S_x(h_xy))$ by 15.2. By definition of S_x we have

$$\begin{aligned} S_x(h_xy) &= \langle S(L(h_xy)), gx(L(h_xy))(R(h_xy))\rangle \\ &= \langle S(k_xy), gx(k_xy)(j_xy)\rangle, \end{aligned}$$

so that $k_x(Sy)=S(k_xy)$. Since $y\in G$ it follows that $k_x(Sy)=Sy$, whence $Sy\in G$. We have thus shown that G is closed under S, and since Axiom P3 holds for $\langle N, 0, S\rangle$ we conclude that $G=N$, *i.e.*, that $k_xy=y$ for all $y\in N$. Returning to the formula

$$S_x(h_xy) = \langle S(k_xy), gx(k_xy)(j_xy)\rangle,$$

derived above, we see that it can now be simplified to

$$S_x(h_xy) = \langle Sy, gxy(j_xy)\rangle.$$

Since $j_x(Sy)=R(h_x(Sy))=R(S_x(h_xy))$, we get at once that $j_x(Sy)=gxy(j_xy)$,

showing that j_x satisfies 14.2. On the other hand $j_x0 = R(h_x0) = R0_x = fx$, so that j_x also satisfies 14.1. To complete the proof of Theorem III it merely remains to show that j_x is the *only* unary operation on N which satisfies these two equations; this can be done along the same lines of argument which we used when we inferred from Theorem I that there was only one addition operation for any Peano model—as we leave the reader to verify.

5. The relation between Peano models and induction models. Why is it that the operations of addition and multiplication exist in every induction model, while the existence of an exponential operation can be guaranteed only for Peano models? To answer this, we must first understand the relation which holds between Peano models and more general induction models. It turns out that from the algebraic viewpoint this relation is a very simple and natural one.

Let $\mathfrak{N} = \langle N, 0, S\rangle$ and $\mathfrak{N}_1 = \langle N_1, 0_1, S_1\rangle$ be any two models. We say that $\mathfrak{N}_1$ is a *homomorphic image* of $\mathfrak{N}$ if there exists a homomorphism h of $\mathfrak{N}$ *onto* $\mathfrak{N}_1$, *i.e.*, a function h whose domain is N, and whose range is the whole of N_1, such that $h0 = 0_1$ and $h(Sx) = S_1(hx)$ for all $x \in N$.

THEOREM IV. *Let $\mathfrak{N} = \langle N, 0, S\rangle$ be a Peano model and $\mathfrak{N}_1 = \langle N(, 0_1, S_1\rangle$ an arbitrary model. A necessary and sufficient condition that $\mathfrak{N}_1$ be a homomorphic image of $\mathfrak{N}$ is that $\mathfrak{N}_1$ be an induction model.*

Proof of necessity. Suppose that $\mathfrak{N}_1$ is a homomorphic image of $\mathfrak{N}$, and let h be a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_1$. Let G_1 be any subset of N_1 such that $0_1 \in G_1$ and such that G_1 is closed under S_1. To show that $\mathfrak{N}_1$ is an induction model, it suffices to show that $G_1 = N_1$. To this end we consider the subset G of N consisting of just those elements x such that $hx \in G_1$. Since $h0 = 0_1$ we see that $0 \in G$. Furthermore, G is closed under S. For suppose $x \in G$, so that $hx \in G_1$. Now $S_1(hx) \in G_1$ since G_1 is closed under S_1. But $h(Sx) = S_1(hx)$ since h is a homomorphism. Thus $h(Sx) \in G_1$ and so $Sx \in G$; hence G is closed under S. Since $\langle N, 0, S\rangle$ satisfies Axiom P3, we see that $G = N$. This means that $hx \in G_1$ for all $x \in N$. Since h has domain N and range N_1 we infer that $G_1 = N_1$.

Proof of sufficiency. Suppose that $\langle N_1, 0_1, S_1\rangle$ is an induction model. Since $\langle N, 0, S\rangle$ is a Peano model we can apply Theorem I to infer the existence of a (unique) homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}_1$. To complete the proof of our theorem it is only necessary to show that the range of h is the whole of N_1. But clearly the range of h contains 0_1, since $h0 = 0_1$, and it is closed under S_1 since for any element z in the range of h there must be an $x \in N$ for which $hx = z$, whence $S_1z = S_1(hx) = h(Sx)$ so that S_1z is also in the range of h. But $\mathfrak{N}_1$ satisfies Axiom P3, so the range of h is N_1 as was to be shown.

From Theorem IV there follows an important and well-known corollary.

THEOREM V. *Any two Peano models are isomorphic.*†

Proof. Let $\mathfrak{N}$ and $\mathfrak{N}_1$ be any Peano models. By Theorem IV there is a homo-

† That is, there is a homomorphism of one onto the other which is one-one.

morphism h of $\mathfrak{N}$ onto $\mathfrak{N}_1$ and a homomorphism h_1 of $\mathfrak{N}_1$ onto $\mathfrak{N}$. Clearly the composed function (h_1h), which is a unary operation on N such that $(h_1h)x = h_1(hx)$ for all $x \in N$, is a homomorphism of $\mathfrak{N}$ into $\mathfrak{N}$. But by Theorem I there is only one homomorphism of $\mathfrak{N}$ into $\mathfrak{N}$, and obviously the identity operation on $\mathfrak{N}$ is such a homomorphism. Hence (h_1h) is the identity operation: $(h_1h)x = x$ for all $x \in N$. It easily follows that h is one-one, and hence h is an isomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_1$.

The principal significance of Theorem V is a metamathematical consequence to the effect that for a certain large set of sentences containing the symbols "N", "0", and "S", any sentence of the set which is true of one Peano model is also true of every other Peano model; hence if a sentence of this set is true of some particular Peano model such as the system $\langle N^*, 0^*, S^* \rangle$ of natural numbers as we know them intuitively, it will be a logical consequence of the Axioms P1–P3. The set of sentences to which this metamathematical result applies contains all those sentences about models which have any interest for us in the present work. Accordingly, from now on, instead of speaking of an arbitrary Peano model we may speak of the system $\langle N^*, 0^*, S^* \rangle$ of natural numbers.

In Theorem II we have seen that every induction model $\mathfrak{N} = \langle N, 0, S \rangle$ possesses a unique operation of addition, $+$. Now that we see from Theorem IV that $\mathfrak{N}$ bears a close relation to the system of natural numbers $\mathfrak{N}^* = \langle N^*, 0^*, S^* \rangle$, it is natural to inquire into the relation between the operation $+$ of $\mathfrak{N}$ and the addition operation $+^*$ of the system $\mathfrak{N}^*$. The answer to this inquiry is given by the following result.

THEOREM VI. *Let $\mathfrak{N} = \langle N, 0, S \rangle$ be any induction model and $+$ its operation of addition. Let h be the unique homomorphism of $\mathfrak{N}^*$ onto $\mathfrak{N}$. Then $h(x +^* y) = hx + hy$ for all $x, y \in N^*$.*

Proof. Let x be any element of N^*, and let G be the subset of N^* such that $y \in G$ if and only if $h(x +^* y) = hx + hy$. Since $h(x +^* 0^*) = hx = hx + 0 = hx + h0^*$ we see that $0^* \in G$. Now let y be any element of G. Then $h(x +^* y) = hx + hy$, so that $h(x +^* S^*y) = h(S^*(x +^* y)) = S(h(x +^* y)) = S(hx + hy) = hx + S(hy) = hx + h(S^*y)$, and hence $S^*y \in G$. Since G is closed under S^* we apply Axiom P3 (which holds for $\mathfrak{N}^*$) to conclude that $G = N^*$; and this proves the theorem.

In the terminology of set-theory and algebra we express the content of Theorem VI by saying that the operation $+$ on N is the image (under the homomorphism h) of the operation $+^*$ on N^*. The corresponding theorem for multiplication is equally true, and can be proved in essentially the same way. Now there is a well-known result for general algebraic systems to the effect that if an equation is satisfied identically in one system, the same equation will be satisfied in any homomorphic image of the system if each operation of the original system which enters into the equation is replaced by an operation (of the second system) which is its image under the homomorphism. It follows that such identities as the associative, commutative, and distributive laws which hold for $+^*$ and

$\cdot^*$ in the system $\mathfrak{N}^*$ will also hold for the operations $+$ and $\cdot$ of any induction model. Actually this fact can be shown directly, without appeal to any result which holds for general algebraic systems; for upon inspecting the usual derivation of these laws for Peano models, proceeding from 1.1, 1.2, 11.1, and 11.2, it will be found that only Axiom P3, never P1 or P2, is used.

6. Congruence relations. Another fact about homomorphisms which is well known for general algebraic systems, and which can be established directly in a simple manner for the models under investigation here, is their close connection with congruence relations. By a *congruence relation* for a model $\mathfrak{N}=\langle N, 0, S\rangle$ is meant an equivalence relation R on N† such that whenever xRy then also $(Sx)R(Sy)$. If h is any homomorphism of $\mathfrak{N}$ into some other model, then the relation R_h such that for any $x, y\in N$ we have xR_hy if and only if $hx=hy$, is a congruence relation. Conversely, for every congruence relation R of $\mathfrak{N}$ there is a homomorphism h of $\mathfrak{N}$ onto some other model $\mathfrak{N}_R$, such that $R_h=R$. To construct $\mathfrak{N}_R$ we take N_R to be the class of all equivalence sets x_R for all $x\in N$, we let S_R be the operation on N_R (whose existence follows from the fact that R is a congruence relation of $\mathfrak{N}$) such that $S_Rx_R=(Sx)_R$ for all $x\in N$, and we set $\mathfrak{N}_R=\langle N_R, 0_R, S_R\rangle$. If we define h to be the function mapping N onto N_R such that $hx=x_R$ for all $x\in N$, we easily see that h is a homomorphism of $\mathfrak{N}$ onto $\mathfrak{N}_R$ and that $R_h=R$.

If h_1 and h_2 are homomorphisms of $\mathfrak{N}$ onto models $\mathfrak{N}_1$ and $\mathfrak{N}_2$ respectively, and if $R_{h_1}=R_{h_2}$, then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are isomorphic. It follows that every homomorphic image of a model $\mathfrak{N}$ is isomorphic to one of the models $\mathfrak{N}_R$ determined (in the manner described above) by some congruence relation R of $\mathfrak{N}$. In view of Theorems IV and V we therefore see that every induction model is isomorphic to a model $\mathfrak{N}_R^*$ determined by some congruence relation R of the system $\mathfrak{N}^*$ of natural numbers.

As it happens, we can give an explicit description of all congruence relations on $\mathfrak{N}^*$ in terms of the familiar ordering relation $<$ on N^*.‡ Namely, let m, n be any elements of N^*. We define the relation $R_{m,n}$ on N^* by the rule that $xR_{m,n}y$ if and only if one of the following two conditions holds: (i) $x, y<n$ and $x=y$, (ii) $x, y\geqq n$ and for some $z\in N^*$ either $x=y+^*(z\cdot^* m)$ or $y=x+^*(z\cdot^* m)$.

THEOREM VII. *A binary relation R on N^* is a congruence relation of $\mathfrak{N}^*$ if and only if it is the identity relation on N^* or there exist numbers $m, n\in N^*$ such that $R=R_{m,n}$.*

† An equivalence relation on N is a binary relation R, having N both for its domain and range, which is reflexive, symmetric, and transitive. It is an elementary fact of set-theory that for each such relation R there is a partition of N into disjoint subsets such that two elements are in the same subset if and only if these elements are in the relation R. The subset containing an element x is called the *equivalence set of x under R*; we will denote it x_R.

‡ This ordering relation can be introduced into the axiomatic theory of $\mathfrak{N}^*$ (*i.e.*, into the theory of Peano models), by the definition $x<y$ if and only if there is an element $z\neq 0^*$ such that $x+^*z=y$. Since every induction model possesses an addition function, we can use this definition to define a

We shall leave the proof of this theorem to the reader, contenting ourselves with giving the following hint. If R is a congruence relation other than the relation of identity, there must be an $x \in N^*$ such that xRy for some $y \neq x$. Choose n to be the least of these numbers x. It follows from this choice of n that there are numbers $z \neq 0$ such that $nRn +^* z$; choose m to be the least of these numbers z. It can then be shown that $R = R_{m,n}$.

The congruence relations $R_{m,0}$ are the so-called modular congruences which have been extensively studied in the theory of numbers.§ The induction model $\mathfrak{N}^*_{Rm,0}$ corresponding to such a relation is simply the system of residue classes modulo m, and the operations of addition and multiplication of this model are the familiar $+(\text{mod } m)$ and $\cdot \ (\text{mod } m)$. The congruence relations $R_{m,n}$ for $n \neq 0$ do not seem to have received much attention in the literature,# but a little reflection will suffice to give the reader a clear intuitive picture of the models $\mathfrak{N}^*_{Rm,n}$, as well as of the corresponding operations of addition and multiplication, in this case also.

Incidentally, it is evident that if R is one of the modular congruence relations $R_{m,0}$, then S^*_R is a permutation of the elements of N^*_R so that $\mathfrak{N}^*_R$ satisfies Axiom P2 in this case. On the other hand, if R is one of the congruence relations $R_{m,n}$ for $n \neq 0$, it is clear that $0^*_R \neq S^*_R x$ for all $x \in N_R$, so that $\mathfrak{N}^*_R$ satisfies Axiom P1 in this case. Thus every induction model satisfies either P1 or P2, as mentioned in Section 1.

Let f be any operation on $\mathfrak{N}^*$—let us say a binary operation for definiteness. If R is an arbitrary congruence relation on $\mathfrak{N}^*$, there is in general no binary operation g on N^*_R which is a homomorphic image of f under the homomorphism h which corresponds to R. It is not hard to see that the necessary and sufficient condition for the existence of such an image operation g is that for all $x, x_1, y, y_1 \in N^*$ such that xRx_1 and yRy_1 we have $(fxy)R(fx_1y_1)$. If this condition holds then g is the operation on N^*_R such that $gx_Ry_R = (fxy)_R$ for all $x, y \in N^*$.

For example, although $2 \equiv 2 \pmod 3$ and $0 \equiv 3 \pmod 3$ we have $2^0 \not\equiv 2^3 \pmod 3$. Hence the exponential operation of N^* has no homomorphic image in $N^*_{R_{3,0}}$. On the other hand, $+^*$ and $\cdot^*$ are examples of what we may call *universal operations* on N^*; that is, they are operations f with the property that for *any* congruence relation R of N^*, $(fxy)R(fx_1y_1)$ whenever x, y, x_1, y_1 are elements of N^* such that xRx_1 and yRy_1.† It is for this reason that every induction model possesses operations of addition and multiplication.

relation $<$ in every *induction* model; but in general the relation so obtained will not be an ordering relation.

§ It is customary, in works on number-theory, to write $x \equiv y \pmod m$ instead of $xR_{m,0}y$.

A brief reference may be found in a note by H. S. Vandiver, Bull. Amer. Math. Soc. vol. 40, 1934, pp. 914–920.

† We may call f a *modular operation* if it has this property for all *modular* congruence relations R (and not necessarily for the other congruence relations). An interesting characterization of the modular operations has been given by N. G. de Bruijn. (Cf. Proc. Kon. Ned. Ak. Wetensch. Amsterdam, series A, 58 (Indagationes Math. 17), 1955, pp. 363–367).

Suppose that an operation j on N^* is obtained by primitive recursion from operations f and g which are universal; is j necessarily universal? The example of the exponential operation shows that this is not the case. However, if j happens to be commutative then it *will* be universal: this can be shown by a straightforward generalization of the proof of Theorem II. Thus it is because the exponential operation on N^* is noncommutative that we cannot extend the proof of Theorem II to show the existence of an exponential operation in every induction model.

Of course the condition of commutativity, although sufficient to guarantee universality under the stated conditions, is by no means necessary. For example, the operation j such that $jxy = x^2 \cdot y$ for all $x, y \in N^*$ is a universal operation, and it is obtained by primitive recursion from the universal functions f, g such that $fx = 0$ and $gxyz = z + x^2$ for all $x, y, z \in N^*$; but j is noncommutative.

7. A characterization of Peano models. In Section 2 we have seen that the justification for a definition by mathematical induction in a model $\mathfrak{N}$ is the existence of a unique homomorphism of $\mathfrak{N}$ into some other model; and thus Theorem I constitutes a justification of *all* definitions by mathematical induction in Peano models. As it happens, this property is characteristic for Peano models: these are the *only* models in which all definitions by mathematical induction are justified.

THEOREM VIII. *Let $\mathfrak{N}$ be a model such that, for any model $\mathfrak{N}_1$ there is a unique homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}_1$. Then $\mathfrak{N}$ is a Peano model.*

Proof. Let $\mathfrak{N} = \langle N, 0, S \rangle$, and suppose that $\mathfrak{N}$ satisfies the hypothesis of the theorem. We shall show first that $\mathfrak{N}$ satisfies Axiom P3. To this end let G be any subset of N which contains 0 and is closed under S. Let H be the complement of G (with respect to N), and assume that H is not empty. Let k be a one-one mapping of H onto a set P which is disjoint from N. Let M be the union of N and P. Define a unary operation T on M, as follows: If $x \in N$ then $Tx = Sx$; if $x \in H$ and $Sx \in H$ then $T(kx) = k(Sx)$; if $x \in H$ and $Sx \in G$ then $T(kx) = Sx$. Let $\mathfrak{M}$ be the model $\langle M, 0, T \rangle$. It is clear that the mapping h_1 of N into M such that $h_1 x = x$ for all $x \in N$ is a homomorphism of $\mathfrak{N}$ into $\mathfrak{M}$. On the other hand, consider the following mapping h_2 of N into M: If $x \in G$ then $h_2 x = x$; if $x \in H$ then $h_2 x = kx$. It is not difficult to see that h_2 is also a homomorphism of $\mathfrak{N}$ into $\mathfrak{M}$, and that it is distinct from h_1. But this contradicts the hypothesis of our theorem, and hence shows that it was incorrect to assume H nonempty. H is empty, and so $G = N$. That is, $\mathfrak{N}$ must be an induction model.

From Theorem IV we can now infer that there is a homomorphism h' of $\mathfrak{N}^*$ onto $\mathfrak{N}$. On the other hand the hypothesis of our theorem assures us that there is a homomorphism h of $\mathfrak{N}$ into $\mathfrak{N}^*$. As in the proof of Theorem V we consider the composed function (hh') which is a homomorphism of $\mathfrak{N}^*$ into itself, and so (by Theorem I) must be the identity operation on $\mathfrak{N}^*$. It follows that h' is one-one, and hence is an isomorphism of $\mathfrak{N}^*$ onto $\mathfrak{N}$. This proves the theorem.

It is clear from Theorem VIII that any proof of Theorem I must employ all of the Axioms P1–P3, as claimed in Section 2.

8. Conclusion. We have now finished our discussion of the concept of definition by mathematical induction in the theory of numbers, based upon Peano's axiomatic foundation for this theory. In the literature of mathematics there are various other types of inductive definitions employed, *e.g.*, definition by transfinite induction, or by induction in certain types of partially ordered systems. Many of the ideas of this paper can be generalized to cover these other types of induction.

While the method of presenting the material in this paper may have some claim to originality, several of the proofs which were given are well known to mathematicians. In particular, this is true of the two proofs (one given in detail, the other outlined) of Theorem I. According to Professor Alonzo Church, the origin of the first proof goes back to the Hungarian mathematician L. Kalmár, while the idea of the second proof should be credited to P. Lorenzen, and D. Hilbert and P. Bernays, who discovered the proof independently and published their work nearly simultaneously.† The proof of Theorem II is given by E. Landau‡ who credits it to L. Kalmár. However, Landau fails to note the significance of the fact that the proof does not use Axioms P1 and P2.§

† Kalmár's article appears in Acta Sci. Math. (Szeged), vol. 9, No. 4, 1950, pp. 227–232. Lorenzen's work appears in Monatsh. Math. und Phys., vol. 47, 1938–39, pp. 356–358, and the proof of Hilbert and Bernays appears in the Appendix to Volume 2 of their book Grundlagen der Mathematik.

‡ See E. Landau, Foundations of Analysis.

§ (Added October 2, 1958.) There has just come to my attention an article with several ideas closely related to those of this paper: H. Lenz, Zur Axiomatik der Zahlen, Acta Math. Acad. Sci. Hungar., vol. IX, 1958, pp. 33–44.

GLOBAL EQUILIBRIUM THEORY OF CHARGES ON A CIRCLE*

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1. Introduction. Recently electronic calculations were made [1] to see if there is some "brute force" solution to an old problem of finding equilibrium positions of charged particles constrained to lie on a sphere and acted upon by mutual Newtonian repulsion (see Föppl [2]). While some new stable minima seemed plausible on the basis of the calculation, the problem of verifying unstable equilibrium positions was too formidable for conclusive results from numerical data at present.

Surprisingly enough the analogous two-dimensional unstable equilibrium problem is far from trivial and yet can be analyzed completely; but it never seems to have made its way into the literature [6]. Here we would consider n

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point charges, not necessarily equal, constrained to lie on a circle and acting under a fairly general law of repulsion which permits particles to coincide for "closure." We find all critical configurations, stable and otherwise.

The conditions are then perfect for the application of the Poincaré-Morse [3] global theory. Here we calculate s, the number of negative squares of the quadratic form that gives the potential near each critical configuration. These values of s are interrelated by a formula ((4.3) below) of the "Euler characteristic" type.

As intuition suggests, there is only one stable critical configuration (excluding rigid rotation) for every ordering of noncoincident charges (for example the obvious regular n-gon for equal charges). The Poincaré-Morse theory reduces to identities among partition "tallies" that are not trivial in nature.

2. Symbols. We describe each of the n positive charges q_i by an angular variable θ_i, $0 \leqq i \leqq n-1$, and use a potential function

$$V(\theta_0, \cdots, \theta_{n-1}) = \sum_{i>j} V_{ij}, \tag{2.1}$$

where we introduce the function of "distance" $f(\phi)$ and

$$V_{ij} = q_i q_j f(\theta_i - \theta_j). \tag{2.2}$$

Here $f(\phi)$ is subject to the properties that $f(\phi)$ is even, $f(\phi)$ has period 2π, $f''(\phi) > 0$, and $f'(\epsilon)/\epsilon \to -\infty$ as $\epsilon \to 0$ (clearly satisfied by the logarithmic potential, $f(\phi) = -\log \left| 2 \sin \tfrac{1}{2}\phi \right|$). Under these conditions $f(0)$ may be infinite, making a *global* theory inapplicable as the set of angles θ_i would not be permitted to range over a *closed* manifold. We, therefore, always modify $f(\phi)$ to a new function $f_\epsilon(\phi)$ defined for ϵ which is always taken "as small as the context requires":

$$\begin{aligned} f_\epsilon(\phi) &= f(\phi), \qquad \epsilon \leqq \left|\phi\right| \leqq \pi; \\ f_\epsilon(\phi) &= f(\epsilon) + f'(\epsilon)(\phi^2 - \epsilon^2)/2\epsilon, \qquad 0 \leqq \left|\phi\right| \leqq \epsilon; \end{aligned} \tag{2.3}$$

so as to preserve continuity and differentiability. We modify V according to this new function $f_\epsilon(\phi)$, and drop the subscript ϵ for convenience. If ϵ is chosen small enough none of the nonsingular configurations ($\theta_i \neq \theta_j$ when $i \neq j$), are affected locally, while $V(\theta_0, \cdots, \theta_{\epsilon-1})$ is continuously differentiable on a closed manifold.

To see the closed manifold more clearly, we take advantage of the circular symmetry by setting $\theta_0 = 0$ and

$$V(0, \theta_1, \cdots, \theta_{n-1}) = W(\theta_1, \cdots, \theta_{n-1}). \tag{2.4}$$

Then W is defined on a closed manifold $\mathfrak{M}$ consisting of the $(n-1)$-dimensional torus or the product of the $n-1$ intervals modulo 2π

$$0 \leqq \theta_i < 2\pi, \qquad i = 1, \cdots, n-1. \tag{2.5}$$

The *critical* points (of stability, metastability, etc.), are all characterized by

the conditions at $\theta_i=\theta_i^{(0)}$,

$$\partial W/\partial\theta_i = 0, \qquad i = 1, \cdots, n-1. \tag{2.6}$$

Each critical point of $\mathfrak{M}$ corresponds to a configuration of $n-1$ values of $\theta_i^{(0)}$ and $\theta_0^{(0)}(\equiv 0)$. These configurations form k clusters of m_t coincident values of $\theta_i^{(0)}$, $0\leqq i\leqq n-1$,

$$1 \leqq k \leqq n, \qquad 1 \leqq m_t \leqq n \ (t = 1, \cdots, k), \qquad \sum_{t=1}^{k} m_t = n. \tag{2.7}$$

We define m_t-1 as the degree of *degeneracy of each cluster* and

$$D = \sum_{t=1}^{k} (m_t - 1) = n - k \tag{2.8}$$

as the degree of *degeneracy of the configuration* $\theta_i^{(0)}$.

3. Positive and negative eigenvalues. The potential energy W in the neighborhood of a critical point is denoted by a quadratic form in $\alpha_i=\theta_i-\theta_i^{(0)}$, $0\leqq i\leqq n-1$, so that, $\alpha_0\equiv 0$ as before. The basic lemma used here concerns the form

$$Q^*(\alpha_i) = \sum_{i>j}^{*} \frac{1}{2} p_{ij}(\alpha_i - \alpha_j)^2, \tag{3.1}$$

where the * superscript refers to summation over a previously designated subset of the indices $\{0, 1, \cdots, n-1\}$, which may or may not include 0.

LEMMA 3.1. *If there are m indices present, and $p_{ij}>0$, the form $Q^*(\alpha_i)$ has $m-1$ positive eigenvalues.*

Proof. For convenience we can take the case where the subscript 0 is included in the set of m indices

$$Q^*(\alpha_i) = \sum_{i>j}^{*} \alpha_i\alpha_j P_{ij}, \qquad (i \neq 0, j \neq 0);$$

$$P_{ij} = -p_{ij} \ (i \neq j), \qquad P_{ii} = \sum_{j}^{*} p_{ij} + p_{i0} \qquad (j \neq i, i \neq 0, j \neq 0); \tag{3.2}$$

so that for the $m-1$ indices i (excluding "0," of course),

$$P_{ii} > \sum_{j}^{*} |P_{ij}| \qquad (j \neq i, j \neq 0). \tag{3.3}$$

This inequality is sufficient for the positiveness of eigenvalues by a well-known theorem [4]. To use this lemma we need two further results:

LEMMA 3.2. *If we consider the family of quadratic forms $Q(\alpha_i)=Q_1(\alpha_i)+\eta Q_2(\alpha_i)$, then as $\eta\to\infty$ the form $Q(\alpha_i)$ will have at least as many positive (negative) eigenvalues as $Q_2(\alpha_i)$.*

LEMMA 3.3. *If we project the Euclidean space α_i onto a subspace of lower dimension by imposing linear conditions on the α_i, then we obtain no greater number of positive (negative) eigenvalues in the form $Q(\alpha_i)$ after projection than before projection.*

These lemmas are very simple to prove, and naturally are quite classical [5]. Generally, $W(\theta_i) - W(\theta_i^{(0)})$ is approximated by

$$(3.4) \qquad Q(\alpha_i) = \sum_{i>j} f''(\theta_i^{(0)} - \theta_j^{(0)}) \tfrac{1}{2} q_i q_j (\alpha_i - \alpha_j)^2, \qquad (\alpha_i = \theta_i - \theta_i^{(0)}).$$

For the coincident points ($\theta_i^{(0)} = \theta_j^{(0)}$), of the configuration, from (2.3), $f''(\theta_i^{(0)} - \theta_j^{(0)}) = f'(\epsilon)/\epsilon$, which is infinitely negative compared with the derivatives that occur when $\theta_i^{(0)} \neq \theta_j^{(0)}$, as $\epsilon \to 0$.

Thus asymptotically, as $\epsilon \to 0$, $Q(\alpha_i)$ becomes the sum of k forms of type (3.1), (but with negative coefficients), formed by dropping the terms of (3.4) in which i and j belong to different clusters. By Lemma (3.1), the form $Q^*(\alpha)$ has exactly $\sum(m_t - 1) = D$ *negative* eigenvalues, and by Lemma (3.2), $Q(\alpha_i)$ has at least as many.

On the other hand, if we make all θ_i equal for those subscripts i belonging to each one of the k clusters, we can apply the argument of the preceding paragraph directly and see the existence of $k-1$ *positive* eigenvalues. Since $D + k - 1 = n - 1$, we have accounted for all eigenvalues in (3.4), on the assumption that ϵ be sufficiently small, depending on the charges q_i. Thus the number s of negative eigenvalues is D, the degeneracy, *i.e.*,

$$(3.5) \qquad D = n - k = s.$$

4. Global theory. The critical points are next classified according to the number s of negative eigenvalues in the quadratic form. We saw $s = D$ in this problem.

LEMMA 4.1. *To every ordered partition of subscripts $\{0, 1, \cdots, n-1\}$ into $k (\leqq n)$ clusters (each cluster being unordered), a unique critical configuration is determined, for ϵ small enough.*

We shall prove this theorem by induction on k. First of all it is clear that at least one such critical configuration is determined since we can identify all m points of each cluster and have k charges, each of value $\sum q_i$ (summed over each cluster). Then W is a continuous function of $k-1$ variables (exception being made of the fixed zero cluster) and must achieve a minimum.

To see that the minimum is unique we use induction. For instance, when $k = 2$ the fact that $f''(\phi) > 0$ for $|\phi| < \epsilon$ is sufficient. Let us assume the truth of Lemma 4.1 for all values of k up to n (inclusive). We then see that the number of critical points M_s corresponding to s negative eigenvalues is exactly the number of partitions of n points into k clusters keeping one fixed. Thus

$$M_s = \sum (n-1)!/(m_1-1)!m_2!\cdots m_k!, \qquad n-1 \geqq s \geqq 0, \tag{4.1}$$

subject to $k=n-s$, and

$$\sum_{t=1}^{k} m_t = n, \qquad m_t \geqq 1. \tag{4.2}$$

The global theory calls for the relation [3]

$$-\sum_{s=0}^{n-1} (-)^s M_s = \chi \tag{4.3}$$

for χ the Euler characteristic of $\mathfrak{M}$. But in our case $\chi=0$, and relation (4.3) is a combinatorial identity that can be proved by induction on n. (This is left to the reader.) It should be noted that the identity is *not* equivalent to the triviality,

$$\chi = -(1-1)^{(n-1)}. \tag{4.4}$$

In fact, $M_s \geqq \binom{n-1}{s}$, with equality only for $s=n-1$.

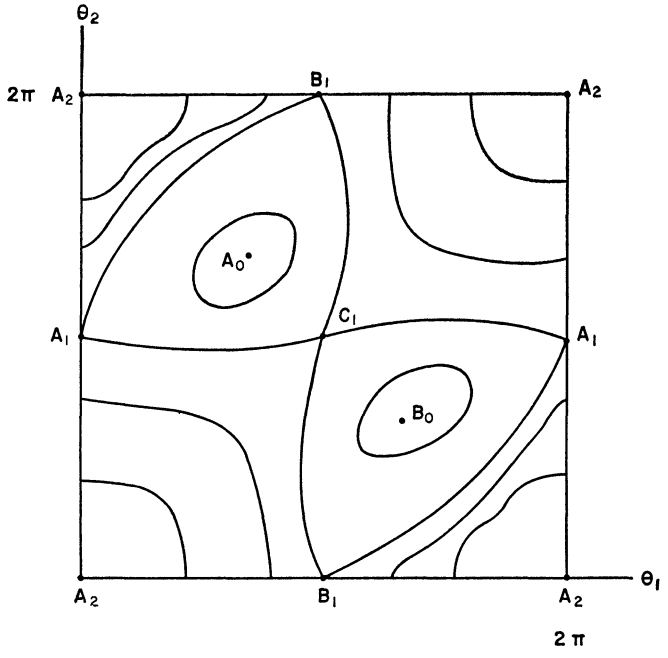


FIG. 1

We are now in a position to resume the proof of Lemma 4.1. If we replace n by $n+1$, the only value of s for which (4.1) does not apply is $s=0$, $k=n+1$. Then M_0 would be *greater* than the summation if some partition failed to produce

a unique critical point. But by (4.3), independently verified, M_0 is determined by the other M_s and the uniqueness could not fail for $k=n+1$.

5. Illustration. (Fig. 1.) We now take $n=3$ and sketch the level lines for $W(\theta_1, \theta_2)$. We note that the range of θ_i determines a square of side 2π with opposite sides identified to form the conventional "torus." For convenience we take equal charges at $\theta_0(\equiv 0)$, θ_1, and θ_2. We find the partitions of $n=3$ with the critical points as noted.

MINIMA $\{A_0, B_0\}$

$3 = 1 + 1 + 1; \quad k = 3, s = 0; \quad M_0 = 2!/0!1!1! = 2.$

SADDLE POINTS $\{A_1, B_1, C_1\}$

$3 = 2 + 1; \quad k = 2, s = 1; \quad M_1 = 2!/1!1!$

$3 = 1 + 2; \quad k = 2, s = 1; \quad + 2!/0!2! = 3.$

MAXIMUM $\{A_2\}$

$3 = 3; \quad k = 1, s = 2; \quad M_2 = 2!/2! = 1.$

6. Final conclusion. If we go back to the stage where no coincident charges could occur, we find the following result:

THEOREM. *Let n positive point charges q_i be placed at angles θ_i on a circle and be subject to a potential of mutual repulsion*

$$V = \sum_{i>j} f(\theta_i - \theta_j) q_i q_j,$$

for which $f(\phi) = f(-\phi)$, $f(\phi+2\pi) = f(\phi)$, $f''(\phi) > 0$ and $f'(\epsilon)/\epsilon \to -\infty$ as $\epsilon \to 0$. Then ignoring rotations, there is exactly one configuration for which V is a relative minimum, and in which the θ_i occur in a preassigned order clockwise around the circle.

The proof is immediate if we use Lemma 4.1. A more direct proof might be difficult to find.

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ON BERNOULLI PLAY WITH LIMITED RESOURCES

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Let $\{x_k\}$ be a sequence of mutually independent Bernoulli random variables. Suppose that at each trial a nonnegative quantity of a limited resource is forfeited; say σ_1, if the trial outcome is zero, and $\sigma_2 > \sigma_1$, if the outcome is 1. Every play will then terminate after a finite number of trials, N.

Introducing the concept of a "stop" function we shall derive simple exact expressions for the probabilities

$$\begin{aligned} P(j) &= \Pr\{x_1 + \cdots + x_N = j\}, \\ Q(N) &= \Pr\{\text{number of trials in a play} = N\}, \end{aligned} \tag{1}$$

under the assumption that trials are continued until depletion of the resource. This is a typical compound distribution problem ([2], p. 221), however, the classical formula is not applicable since N and j here are correlated random variables; a large j is usually accompanied by a small N.*

Construct a "stop" function, $s(n, j)$, for $n \geqq j$, which indicates the remaining amount of the resource after the nth trial, the cumulative number of successes being j. After the nth trial no further trials are permitted if $s(n, j) \leqq 0$, otherwise at least one more trial is allowed. The function $s(n, j)$ is such that

$$s(n+1, j) < s(n, j), \tag{2}$$

$$s(n, j+1) < s(n, j). \tag{3}$$

Let $f = \max j$ for the set $\{j \mid s(j, j) > 0\}$; then a play cannot terminate with fewer than $(f+1)$ trials, and $(f+1)$ is the maximum number of attainable successes. For each j, $0 \leqq j < f+1$, determine integer N_j defined by

$$s(N_j, j) > 0, \qquad s(N_j + 1, j) \leqq 0. \tag{4}$$

It follows that $N_{j-1} \geqq N_j$. Also, no one play of the game may have more trials than (N_0+1). Define $N_{f+1} = f$, since for every $n \geqq (f+1)$, $s(n, f+1) \leqq 0$; also define $N_{-1} = N_0$.

In what follows we shall use the function G (with $q = 1 - p$) defined by

$$\begin{aligned} G(N, j, p) &= p^j \sum_{c=0}^{N-j} \binom{j+c-1}{j-1} q^c = \sum_{i=j}^{N} \binom{N}{i} p^i q^{N-i}, \qquad j \leqq N, \\ &= 0, \qquad j > N. \end{aligned}$$

† The author thanks D. Y. Barrer of the Institute for Defense Analyses for his helpful comments.

* The author's first thoughts on this type of problem in the form of application to missile firing doctrines were presented in A Comparison of Firing Doctrines for a Sequentially Fired Weapon *vs.* a Non-Saturation Attack (U), George W. Morgenthaler, Report 58-1, Institute for Air Weapons Research, University of Chicago (Air Force Contract No. 33(616)-3274, Weapons Guidance Lab., Wright Air Development Center).

The latter expression is tabled ([1], p. 251, or [4]), and the equality of these two expressions for G is well known ([3], p. 40).

THEOREM 1. *For* $j=0, 1, \cdots, (f+1)$

$$P(j) = \binom{N_j}{j} p^j q^{N_j-j+1} + G(N_{j-1}+1, j, p) - G(N_j, j, p), \tag{5}$$

where p *is the probability of success and* $q=1-p$.

Proof. The first term corresponds to the probability of j successes in N_j trials, with the one additional permissible trial resulting in failure. Conditions (3) and (4) guarantee that in order to attain j successes, at least (N_j+1) trials will occur.

If $(j-i)$ successes occur in the first N_j trials, the remaining i successes must occur by trial $(N_{j-1}+1)$. If not, at that or any trial thereafter at which a total of $(j-1)$ successes have occurred, trials would cease according to the stop rule and the jth success would never occur. Thus from trial (N_j+1) to trial $(N_{j-1}+1)$ the play would cease whenever the jth success occurs. The probability that the jth success occurs at the (N_j+1+c)th trial, $0 \leqq c \leqq (N_{j-1}-N_j)$ is given by the Pascal distribution ([2], p. 218), as

$$\binom{N_j+c}{j-1} p^j q^{N_j+1-j+c}.$$

The sum of these probabilities is easily shown to be

$$[G(N_{j-1}+1, j, p) - G(N_j, j, p)],$$

upon writing down the sum and transforming the variable of summation from c to, say, $k=(N_j+1-j+c)$.

The distribution of N is most easily obtained by writing down the expressions $P(j)$ and then setting $Q(N)$ equal to the sum of all terms among the expressions $P(j)$ for which the exponents of p and q sum to N. However, explicit formulae for $Q(N)$ can also be written.

THEOREM 2. *Given the sequence* $N_{f+1}+1 \leqq N_f+1 \leqq \cdots \leqq N_j+1 \leqq N_{j-1}+1 \leqq \cdots \leqq N_0+1$, *the distribution of* N, $f+1 \leqq N \leqq N_0+1$, *is given by*

$$Q(N) = \begin{cases} \binom{N-1}{j-1} p^j q^{N-j}, & \text{if } j,\ 0 < j \leqq f+1, \text{ is an integer such that} \\ & N_j + 1 < N < N_{j-1} + 1. \\ \sum_{i=0}^{k} \binom{N}{j-i} p^{j-i} q^{N-j+i} + \binom{N-1}{j} p^{j+1} q^{N-1-j}, & \text{if } j,\ 0 \leqq j \leqq f+1, \\ & \text{and } k,\ k \leqq j, \text{ are the largest integers such that} \\ & N = N_j + 1 = N_{j-1} + 1 = \cdots = N_{j-k} + 1. \end{cases} \tag{6}$$

Proof. When j successes are obtained there will not be fewer than (N_j+1)

trials, nor more than $(N_{j-1}+1)$. Thus to each integer j we associate the integers of the interval $I_j=[N_j+1, N_{j-1}+1]$. Then $I_{j+1}\cap I_j=N_j+1$; $I_j\cap I_{j-1}=(N_{j-1}+1)$. If N is such that for some j, $(N_j+1)<N<(N_{j-1}+1)$, then $N\in I_j$ only. Then N trials can occur only if the jth success occurs on the Nth trial. This probability is given in the first line of (6).

Alternatively, N may be of the form (N_j+1) for certain consecutive j's. If j and k are as stated in the theorem, N can occur when $(j-i)$ successes occur in N_{j-i} trials and the remaining trial is a failure, or when $(j-i-1)$ successes occur in N_{j-i} trials, the remaining trial being a success. The sum of the probabilities of these two events is

$$\binom{N_{j-i}+1}{j-i}p^{j-i}q^{N_j-i-j+i+1}, \qquad i=0, 1, \cdots, k,$$

and we may replace $(N_{j-i}+1)$ by N.

The only other way in which $N=N_j+1$ could occur is if the $(j+1)$st success was obtained on trial (N_j+1). This probability is the final term in the second line for $Q(N)$.

As a first example of such processes, consider a machine on a production line which produces one unit every T seconds. Suppose that imperfect or broken units occur with probability p at each trial and suppose that T_s additional seconds must then be allotted to eject the defective unit from the machine (possibly machine clean-up or temporary shutdown is necessary). The distribution of stoppages, and hence the expected number of seconds of shutdown per production day, may be calculated using (5). The stop function would be

$$s(n, j) = (\text{Total sec. per production day}) - nT - jT_s.$$

Secondly, if a particle of given initial kinetic energy is followed in a random walk in which more energy is lost for a step to the right than for movement to the left, (6) will give the distribution of the number of steps up to standstill.

In some processes it is of interest to invoke a second "stop" rule which terminates trials when a total of $B<(f+1)$ successes have occurred. Then (5) is valid only for $j<B$.

THEOREM 3. *Under the additional "stop" rule which terminates trials after B successes,*

$$P(B) = G(N_{B-1}+1, B, p). \tag{7}$$

Proof. The probability of the Bth success occurring at the $(B+i)$th trial is given by $\binom{B+i-1}{i}p^Bq^i$. If the Bth success has not occurred by trial $(N_{B-1}+1)$, there will not be sufficient of the resource remaining to achieve B successes. Summing these probabilities results in (7).

THEOREM 4. *Under the additional "stop" rule which terminates trials after B successes, the distribution of N, $B \leqq N \leqq N_0+1$, is given by*

$$Q(N) = \begin{cases} \binom{N-1}{B-1} p^B q^{N-B}, & \text{for } N \leqq N_{B-1}, \\ \text{same as (6)}, & \text{for } N > N_{B-1}+1, \end{cases} \tag{8}$$

$$Q(N_{B-1}+1) = 1 - \sum_{N \neq (N_{B-1}+1)} Q(N).$$

Proof. If $N > (N_{B-1}+1)$, then fewer than B successes occurred in the N trials and the probabilities of such N are treated in (6). If $N \leqq N_{B-1}$, necessarily the Bth success occurred on the Nth trial, thereby halting the trials according to the stop rule. Thus, by the Pascal distribution, $Q(N)$ has the value given in the first line of (8). For $N = N_{B-1}+1$, some plays which would have had $(N_{B-1}+1)$ trials may have been stopped because B successes were attained. This is true, for example, when $N_{B-1} = N_B$.

A final example concerns a process which terminates after B successes. Consider a defensive weapon which is sequential-fire rather than salvo-fire. Suppose that the depth of the zone of effective fire is d. A line-abreast wave of B attackers travelling at speed V assaults the defensive position.

Suppose that the time required for each shot is T and the time to observe the result of fire is T_0. Denote the switching time, *i.e.*, the time required to reorient fire-control equipment to a new attacker, by T_s.

Of the many possible firing doctrines, consider one in which the first target is fired upon and the result is observed. If the target survives, a second shot is fired at it and the result observed, etc. When the target is destroyed, the second target is fired upon and the result observed, etc., until it too is destroyed. Fire is then switched to the third target, etc., until either all targets are destroyed or the attackers have reached the defensive position. Suppose that the attack is nonsaturating, *i.e.*, if but one shot were directed at each attacker, all attackers could be fired upon before they reached the defensive position. Under this fire-observe-switch doctrine the number of shots fired and the number of attackers destroyed are correlated random variables. Theorems 3 and 4 apply.

The limited resource here is time, and an appropriate stop function is

$$s(n, j) = (d/V) - n(T + T_0) - jT_s + T_0. \tag{9}$$

This particular military example has alternative treatments at this point. Stop function (9) together with (4) defines N_j as the integers

$$N_j < \frac{(d/V) - jT_s + T_0}{T + T_0} \leqq N_j + 1. \tag{10}$$

This permits firing (N_j+1) shots even though there is not a full T seconds of

firing time left for the last shot. This also requires that if the last shot fired is a successful shot there must be sufficient time for switching even though firing will cease. It is possible to change these assumptions and, *e.g.*, to define N_j using the greatest integer notation, "[]," *i.e.*,

$$N_j + 1 = \left[\frac{(d/V) - jT_s + T_0}{T + T_0}\right], \tag{11}$$

thus insuring sufficient time for all (N_j+1) shots. Each alternative rule of this sort corresponds to a slightly different game and the corresponding probability distributions are different.

If N_j is defined by (10) or by (11), and $T_s \leqq T+T_0$ (a reasonable engineering assumption) then $0 \leqq (N_{j-1} - N_j) \leqq 1$. Hence for $j<B$, (5) gives

$$P(j) = \begin{cases} \binom{N_j+1}{j} p^j q^{N_j+1-j}, & \text{if } (N_{j-1} - N_j) = 0, \\ \binom{N_j+1}{j} p^j q^{N_j+1-j} + \binom{N_j+1}{j-1} p^j q^{N_j+2-j}, & \text{if } N_{j-1} - N_j = 1, \end{cases} \tag{12}$$

and $P(B)$ is given by (7).

Consider the specific instance of B aircraft attacking a missile-defended target and suppose $d=52.5$ n.m., $T=64$ sec., $T_s=20$ sec., $T_0=11$ sec., $V=596$ n.m./hr., $B=3$. Then by (10), $N_0=4$, $N_1=4$, $N_2=3$, $N_3=3$. By (7), (8), and (12),

$$P(0) = q^5, \qquad P(1) = 5pq^4,$$

$$P(2) = \binom{4}{2} p^2q^2 + \binom{4}{1} p^2q^3, \qquad P(3) = \sum_{j=3}^{4} \binom{4}{j} p^j q^{4-j};$$

$$Q(3) = p^3, \qquad Q(4) = 3p^3q + 6p^2q^2, \qquad Q(5) = 5pq^4 + 4p^2q^3 + q^5.$$

On the other hand, if (11) is used, $N_0=3$, $N_1=3$, $N_2=2$, $N_3=2$. By (7), (8), and (12),

$$P(0) = q^4, \qquad P(1) = 4pq^3,$$

$$P(2) = \binom{3}{2} p^2q + \binom{3}{1} p^2q^2, \qquad P(3) = p^3;$$

$$Q(3) = p^3 + 3p^2q, \qquad Q(4) = 3p^2q^2 + 4pq^3 + q^4.$$

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ON THE ENUMERATION OF POLYGONS

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1. Introduction. Given n equally spaced points on a circle, one may pick a first vertex in n ways, a second vertex in $(n-1)$ ways, . . . , an nth vertex in 1 way, and return to the starting point in 1 way, for a total of $n!$ polygonal paths. Two polygonal paths which differ only in starting point or orientation will be called *identical polygons.* If, besides possible difference in starting point and orientation, two polygons differ only by a plane rotation, they will be termed *equivalent.* If, in addition to possible differences of these three types, two polygons differ only by a reflection through some axis, they will be called *similar.* Using a combinatorial formula of Pólya, it has been possible to obtain explicit expressions for the number of classes $E(n)$ of equivalent n-gons, and for the number of classes $S(n)$ of similar n-gons.

2. The formulas. Let n exceed 2. It is convenient to separate the even from the odd values of n. In all cases, summation is extended over the divisors d of n, and $\phi(a)$ is Euler's totient function.

$$E_{\text{odd}}(n) = \frac{1}{2n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d\right) \tag{1}$$

$$E_{\text{even}}(n) = \frac{1}{2n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{n/2}\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)!\right) \tag{2}$$

$$S_{\text{odd}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{(n-1)/2}\cdot n^2\cdot\left(\frac{n-1}{2}\right)!\right) \tag{3}$$

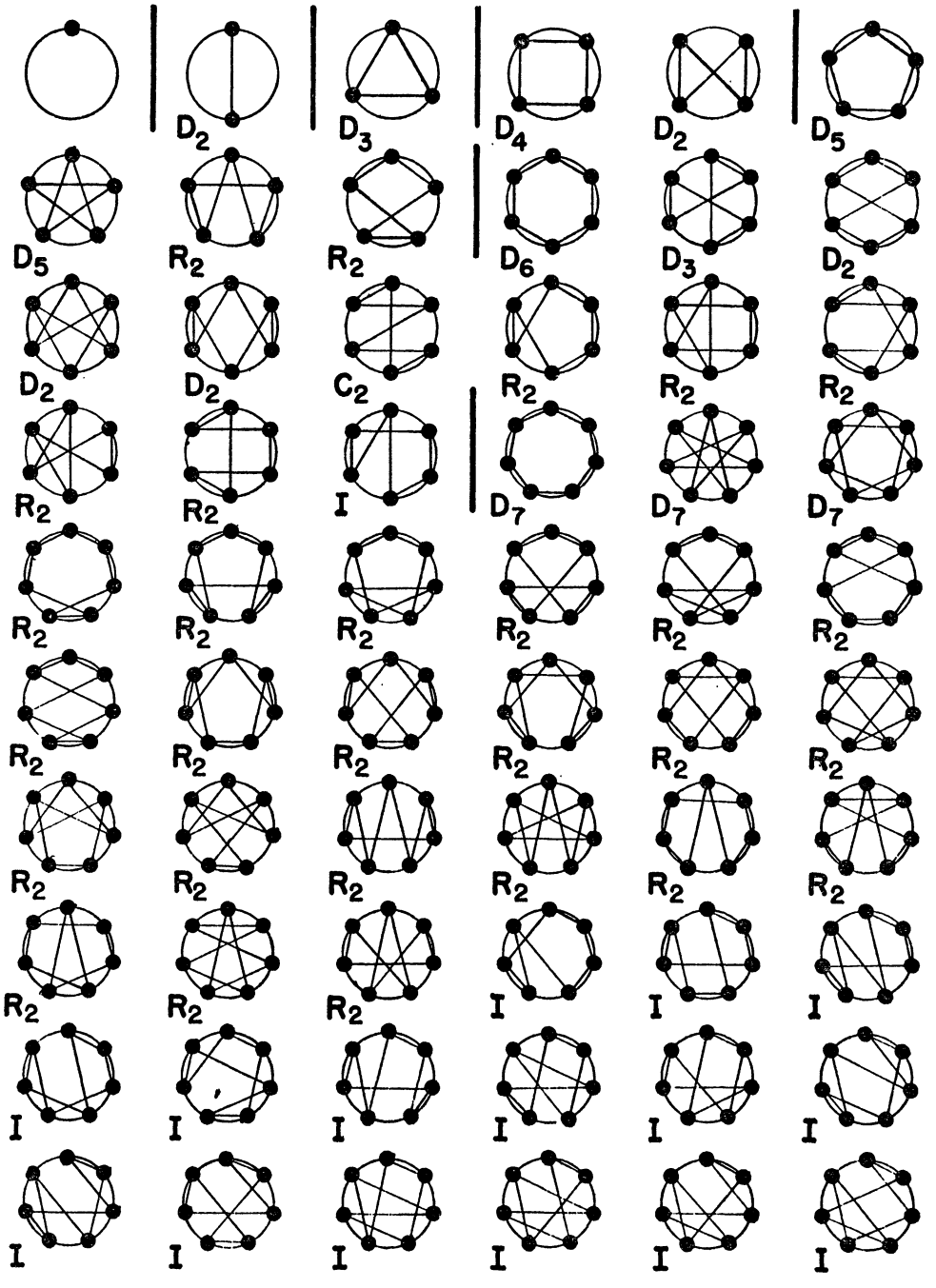
$$S_{\text{even}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{n/2}\cdot\frac{n(n+6)}{4}\left(\frac{n}{2}\right)!\right). \tag{4}$$

These results are tabulated for $n \leqq 11$ in Table I, and the actual polygons are shown for $n \leqq 7$ in Table II.

TABLE I. The number of equivalent and similar n-gons for $n \leqq 11$.

n	$E(n)$	$S(n)$
3	1	1
4	2	2
5	4	4
6	14	12
7	54	39
8	332	202
9	2,246	1,219
10	18,264	9,468
11	164,950	83,435

TABLE II. All polygons with $\leqq 7$ sides, with their symmetry groups. The groups I and C_2 correspond to polygons with reflections which are not rotations.



3. Deriving the formulas. The basic combinatorial formula [1], [2] for the number C of equivalence classes established in a set S by the operation of a group G of order N is

$$C = \frac{1}{N} \sum_{g \in G} I(g), \tag{5}$$

where $I(g)$ is the number of objects s in S left fixed by the operator g in G, *i.e.* for which $g(s) = s$.

The original set S for present purposes is the set of all $n!$ polygonal paths. The "identity group" contains $2n$ elements, for $n > 2$, based on the n starting points and two orientations of path. Whenever e is the unity element of the group G in (5), $I(e) =$ total size of set S. For the "identity group," none of the polygonal paths remain invariant under operations other than the unity. Hence the trivial result

$$I(n) = \frac{1}{2n}(n!) = \frac{1}{2}(n-1)!. \tag{6}$$

The "equivalence group" is the direct product of the identity group with the group of cyclic permutations of the n-gon, and has $(2n)(n) = 2n^2$ elements. For the purposes of equation (5), only the n cycling operators contribute to the sum, provided that polygons which are changed into identical polygons are counted in $I(g)$. Cycling by an angle $\theta_k = 2\pi k/n$, $1 \leqq k \leqq n$, leaves one or more polygons fixed. The set of invariant polygons under this rotation clearly depends only on the greatest common divisor $(n, k) = d$. Thus

$$E(n) = \frac{1}{2n^2} \sum_{k=1}^{n} I(\theta_k) = \frac{1}{2n^2} \sum_{d \mid n} \phi\left(\frac{n}{d}\right) I(\theta_d), \tag{7}$$

where ϕ is Euler's function, and θ_d also depends on n.

To obtain $I(\theta_d)$, we must determine the number of n-gons left fixed by the rotation $2\pi d/n$. Unless $d = n/2$ (in the case of even n), all invariant n-gons for θ_d can be constructed as follows:

Number vertices consecutively from 1 to n, and distinguish residue classes modulo d. A first vertex is picked in any of n ways. The next vertex is picked arbitrarily, but *outside* the residue class of the first point—thus, in $(n - n/d)$ ways. The next vertex is picked from a new residue class, in $(n - 2n/d)$ ways, and this process continues until all d residue classes are represented once. This "first loop" of the polygon can thus be formed in

$$n\left(n - \frac{n}{d}\right)\left(n - \frac{2n}{d}\right) \cdots \left(n - \frac{(d-1)n}{d}\right)$$

$$= n^d \left(\frac{d-1}{d}\right)\left(\frac{d-2}{d}\right) \cdots \left(\frac{1}{d}\right) = \left(\frac{n}{d}\right)^d \cdot d!$$

ways. The first point on the "second loop" must belong to the same residue class as the first point on the "first loop," and can be chosen in $\phi(n/d)$ ways. The future history of this polygon is now completely determined by the pattern within each loop, and the transition from one loop to the next. This transition does not lead to an early termination of the polygon, since it was restricted to be one of the $\phi(n/d)$ nondegenerate possibilities. (See particularly the second through fifth hexagons in Table II.)

For the case of odd n, this completes the derivation of

$$E_{\text{odd}}(n) = \frac{1}{2n^2} \sum_{d|n} \phi^2\left(\frac{n}{d}\right) \cdot d! \cdot \left(\frac{n}{d}\right)^d. \tag{1}$$

For even n, there is also the possibility that a rotation by π will reproduce the polygon with its orientation reversed. This can happen as follows:

There are $n/2$ pairs of antipodal points. A diameter can thus be drawn in $n/2$ locations. A choice of one member from each antipodal pair can be made in $2^{n/2}$ ways. The first members can be arranged sequentially in $(n/2)!$ ways. This leads to $2^{n/2}(n/2)(n/2)!$ ways of drawing a polygon which inverts upon itself. (See especially the sixth hexagon in Table II.) Hence the formula

$$E_{\text{even}}(n) = \frac{1}{2n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right) \cdot d! \cdot \left(\frac{n}{d}\right)^d + 2^{n/2}\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)!\right). \tag{2}$$

The "similarity group" contains reflections as well as rotations, for a total of $4n^2$ operations. Hence the relations

$$S_{\text{odd}}(n) = \frac{1}{2} E_{\text{odd}}(n) + \frac{1}{4n^2} \cdot n \cdot I(R_2), \tag{8}$$

where R_2 is the operation of reflection in any one of the n axes of the polygon, and

$$S_{\text{even}}(n) = \frac{1}{2} E_{\text{even}}(n) + \frac{1}{4n^2}\left(\frac{n}{2} I(R_2') + \frac{n}{2} I(R_2'')\right), \tag{9}$$

where R_2' and R_2'' refer to reflection in the vertex diagonals and edge diagonals, respectively.

To compute $I(R_2)$, there are n ways to pick a starting point. With respect to the symmetry axis, there is one solitary vertex and $(n-1)/2$ pair of matched vertices. There are $2^{(n-1)/2}$ ways to select first members of the pairs; and $((n-1)/2)!$ ways to order these first members. Hence

$$I(R_2) = n \cdot 2^{(n-1)/2} \cdot \left(\frac{n-1}{2}\right)!. \tag{10}$$

Substituting (10) into (8) yields the expression

$$(3) \qquad S_{\text{odd}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{(n-1)/2}\cdot n^2\cdot\left(\frac{n-1}{2}\right)!\right).$$

By an analogous counting process,

$$(11) \qquad I(R_2') = n\cdot 2^{n/2}\cdot\left(\frac{n}{2} - 1\right)!$$

and

$$(12) \qquad I(R_2'') = n\cdot 2^{n/2-1}\cdot\left(\frac{n}{2}\right)!$$

Equations (9), (11), (12), and (2) combine to yield

$$(4) \qquad S_{\text{even}}(n) = \frac{1}{4n^2}\left(\sum_{d|n} \phi^2\left(\frac{n}{d}\right)\cdot d!\cdot\left(\frac{n}{d}\right)^d + 2^{n/2}\frac{n(n+6)}{4}\cdot\left(\frac{n}{2}\right)!\right).$$

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ON THE THEORY OF LÖBELL ON TRANSFORMATION OF SURFACES

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1. Introduction. Frank Löbell [1] has given a discussion of the theory of transformation of surfaces. He pointed out that, for certain cases as the nature of the problem requires, the consideration of the mixed fundamental magnitudes of a pair of surfaces is opportune. But he did not give any example for this. In the present paper we propose to give such an example.

2. Parallel surfaces. A surface S' which is at a constant distance along the normal from another surface S is said to be parallel to S. As the constant distance may be chosen arbitrarily, the number of such parallel surfaces is infinite. Parallel surfaces have the property that the normals to the two surfaces at corresponding points are parallel [2]. This property agrees with the common requirement of some of the important formulas such as (12), (13), (15b), (15c), of Löbell. Therefore we may use these formulas to investigate parallel surfaces and so give our illustrative examples of Löbell's theory, as follows.

If $\mathbf{x}$ is the current point on the surface S, $\mathbf{c}$ the unit normal to that surface, and α the constant distance, the corresponding point on the parallel surface S' is

$$\mathbf{h} = \mathbf{x} + \alpha\mathbf{c}. \tag{1}$$

Let E, F, G and E', F', G' denote the fundamental magnitudes of the first order of S and S', respectively, and $\overline{E}$, $\overline{F}_1$, $\overline{F}_2$, $\overline{G}$, the mixed fundamental magnitudes of the first order of the surface pair $S \rightleftarrows S'$, that is, let

$$\begin{gathered} E = \mathbf{x}_u\cdot\mathbf{x}_u,\ F = \mathbf{x}_u\cdot\mathbf{x}_v,\ G = \mathbf{x}_v\cdot\mathbf{x}_u; \qquad E' = \mathbf{h}_u\cdot\mathbf{h}_u,\ F' = \mathbf{h}_u\cdot\mathbf{h}_v,\ G' = \mathbf{h}_v\cdot\mathbf{h}_v; \\ \overline{E} = \mathbf{x}_u\cdot\mathbf{h}_u,\ \overline{F}_1 = \mathbf{x}_u\cdot\mathbf{h}_v,\ \overline{F}_2 = \mathbf{x}_v\cdot\mathbf{h}_u,\ \overline{G} = \mathbf{x}_v\cdot\mathbf{h}_v. \end{gathered} \tag{2}$$

Then, by (1), we have

$$\begin{aligned} \overline{E} &= \mathbf{x}_u\cdot(\mathbf{x}_u + \alpha\mathbf{c}_u) = E - \alpha L, \quad \overline{F}_1 = \mathbf{x}_u\cdot(\mathbf{x}_v + \alpha\mathbf{c}_v) = F - \alpha M, \\ \overline{F}_2 &= \mathbf{x}_v\cdot(\mathbf{x}_u + \alpha\mathbf{c}_u) = F - \alpha M, \quad \overline{G} = \mathbf{x}_v\cdot(\mathbf{x}_v + \alpha\mathbf{c}_v) = G - \alpha N, \end{aligned} \tag{3}$$

where L, M, N are the fundamental magnitudes of the second order of S.

In the first place, we find from (3) that

$$\overline{E} - E = -\alpha L, \quad \overline{F}_1 - F = \overline{F}_2 - F = -\alpha M, \quad \overline{G} - G = -\alpha N. \tag{4}$$

It then follows that

$$\begin{aligned} \alpha^2(LN - M^2) &= (\overline{E} - E)(\overline{G} - G) - (\overline{F}_1 - F)(\overline{F}_2 - F) \\ &= (\overline{E}\overline{G} - \overline{F}_1\overline{F}_2) + (EG - F^2) - [\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)]. \end{aligned}$$

Dividing by $EG - F^2$, we obtain

$$\frac{\alpha^2(LN - M^2)}{EG - F^2} = \frac{\overline{E}\overline{G} - \overline{F}_1\overline{F}_2}{EG - F^2} - \frac{\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)}{EG - F^2} + 1. \tag{5}$$

The term on the left side of (5) is equal to α^2 times the Gaussian curvature of S and the first term on the right is the ratio of the Gaussian curvature of S to that of S'.* The second term on the right is equal to $(m_1+m_2)\cos\delta$ by [1], (15c), where m_1, m_2 are, respectively, the lengths of the orthogonal vectors $a\mathbf{h}_u+b\mathbf{h}_v$, $c\mathbf{h}_u+d\mathbf{h}_v$ on S' corresponding to the orthogonal unit vectors $a\mathbf{x}_u+b\mathbf{x}_v$, $c\mathbf{x}_u+d\mathbf{x}_v$ on S, and δ is the angle of rotation from $m_1(a\mathbf{x}_u+b\mathbf{x}_v)$ to $a\mathbf{h}_u+b\mathbf{h}_v$. In order to determine the value of the second term on the right side of (5), we note that in (3), $\overline{F}_1=\overline{F}_2$. Therefore by [1], (15b), we have

$$W(m_1 + m_2)\sin\delta = 0, \tag{6}$$

where $W=\sqrt{(EG-F^2)}$. Since m_1+m_2 is, in general, different from zero (and the case $m_1+m_2=0$ is excluded throughout the paper), then $\delta=0$ or π. Hence we obtain

* This was pointed out without proof by Löbell ([1], II). A proof may be given as follows: Write $W=\mathbf{c}\cdot(\mathbf{x}_u\times\mathbf{x}_v)$, $W'=\mathbf{c}'\cdot(\mathbf{h}_u\times\mathbf{h}_v)$, where $\mathbf{c}'$ is the unit normal to S' at $\mathbf{h}(u, v)$, and $T^2=LN-M^2$, $T'^2=L'N'-M'^2$. Then we must prove that the first term on the right side of (5) is equal to $(T^2/W^2)\div(T'^2/W'^2)$. By a well-known formula of differential geometry, $(T^2/W)\mathbf{c}=\mathbf{c}_u\times\mathbf{c}_v=\mathbf{c}_u'\times\mathbf{c}_v'=(T'^2/W')\mathbf{c}'=(T'^2/W')\mathbf{c}$, so that $T^2/W=T'^2/W'$. The required result now follows from this and the fact that, by [1], (12), $(\overline{E}\overline{G}-\overline{F}_1\overline{F}_2)/(EG-F^2)=W'/W$.

$$\frac{\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)}{EG - F^2} = \pm (m_1 + m_2). \tag{7}$$

Such being the case, (5) takes the form

$$\alpha^2 K = (K/K') \mp (m_1 + m_2) + 1. \tag{8}$$

If the Gaussian curvature K of S is constant and equal to $1/a^2$, and we take $\alpha = \pm a$, we have

$$\pm (m_1 + m_2) = K/K'. \tag{9}$$

Thus we get the first property of parallel surfaces:

THEOREM 1. *With every surface S of constant Gaussian curvature $1/a^2$ there are associated two surfaces parallel to S and distant $\pm a$ from it, on which the sum of the moduli of the orthogonal tangential vectors corresponding to the orthogonal tangential unit vectors of S is equal to the ratio of the Gaussian curvature of S to that of the associated surface at corresponding points.*

Secondly, since* $K/K' = m_1 m_2$, it follows from (9) that

$$\pm \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = 1. \tag{10}$$

We then obtain the second property of parallel surfaces:

THEOREM 2. *With every surface S of constant Gaussian curvature $1/a^2$ there are associated two surfaces parallel to S and distant $\pm a$ from it, on which the sum of the reciprocals of the moduli of the orthogonal tangential vectors corresponding to the orthogonal tangential unit vectors of S is unity.*

Finally, it can be shown that the well-known theorems of Bonnet may be found in a similar way. Let the lines of curvature on S be taken as parametric curves, so that $F = M = 0$. Then if ρ_a, ρ_b denote the principal radii of curvature on S and K_a, K_b, the principal curvatures of S, we have

$$\mathbf{h}_u = \mathbf{x}_u + \alpha \mathbf{c}_u = [(\rho_a - \alpha)/\rho_a]\mathbf{x}_u, \quad \mathbf{h}_v = \mathbf{x}_v + \alpha \mathbf{c}_v = [(\rho_b - \alpha)/\rho_b]\mathbf{x}_v.$$

Hence

$$\begin{aligned} \overline{E} &= \mathbf{x}_u \cdot \mathbf{h}_u = (1 - \alpha K_a)E, & \overline{F}_1 &= \mathbf{x}_u \cdot \mathbf{h}_v = (1 - \alpha K_b)F = 0, \\ \overline{G} &= \mathbf{x}_v \cdot \mathbf{h}_v = (1 - \alpha K_b)G, & \overline{F}_2 &= \mathbf{x}_v \cdot \mathbf{h}_u = (1 - \alpha K_a)F = 0. \end{aligned} \tag{11}$$

But from (7),

$$\pm (m_1 + m_2) = \frac{\overline{E}G + E\overline{G} - F(\overline{F}_1 + \overline{F}_2)}{EG - F^2} = \frac{\overline{E}G + E\overline{G}}{EG},$$

* Since $m_1(a\mathbf{x}_u + b\mathbf{x}_v) \times m_2(c\mathbf{x}_u + d\mathbf{x}_v) = (a\mathbf{h}_u + b\mathbf{h}_v) \times (c\mathbf{h}_u + d\mathbf{h}_v)$, we have $m_1 m_2 (ad - bc)\mathbf{x}_u \times \mathbf{x}_v = (ad - bc)\mathbf{h}_u \times \mathbf{h}_v$. But ([1], II) $(K/K')(\mathbf{x}_u \times \mathbf{x}_v) = \mathbf{h}_u \times \mathbf{h}_v$, so that $K/K' = m_1 m_2$.

which, by virtue of (11), may be written $\pm(m_1+m_2)=2-\alpha(K_a+K_b)$. On substituting this in (8), we have

$$\alpha^2K = (K/K') + \alpha J - 1, \tag{12}$$

where J denotes the mean curvature of S.

Now, if the mean curvature J is constant and equal to $1/a$ and we take $\alpha=a$, we find that $a^2K=K/K'$, *i.e.*, $K'=1/a^2=$const. Consequently we get the important theorem of Bonnet on parallel surfaces:

THEOREM 3. *With a surface of constant mean curvature* $1/a$ *there is associated a parallel surface of constant Gaussian curvature* $1/a^2$ *at a distance* a *from it.*

Again, for a pseudospherical surface of curvature $-1/a^2$, we may take $\alpha=\pm a$, so that $\alpha^2K=-1$. Then from (12), we have $K/K'=\mp J$, that is, $K'=\pm 1/(a^3J)$. From this we have:

THEOREM 4. *With a pseudospherical surface* S *of curvature* $-1/a^2$ *there are associated two surfaces parallel to* S *and distant* $\pm a$ *from it, of which the Gaussian curvature is equal to the reciprocal or its negative of* a^3J, *according as the distance is* $+a$ *or* $-a$, *where* J *is the mean curvature of* S.

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REMARKS ON THE INTRINSIC EQUATIONS OF TWISTED CURVES

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1. Introduction. The shape of a twisted curve is defined by two functions of its arc length s, the curvature $k_1(s)$ and the torsion $k_2(s)$. The equations $k_1=k_1(s)$, $k_2=k_2(s)$ are known as "intrinsic equations." Their solution would give us the Cartesian coordinates of the curve but for an arbitrary rigid motion.

In the case of a plane curve, $k_2(s)=0$, the solution reads

$$x(s) = \int \cos\left(\int k_1(s)ds\right)ds; \qquad y(s) = \int \sin\left(\int k_1(s)ds\right)ds. \tag{1}$$

There is not much known in the general case. The problem can be reduced to the solution of a certain Riccati differential equation. But this type can be solved in closed form only under exceptional conditions. The main significance of the method is to assure us of the existence of a solution. Even if the Riccati

MATHEMATICAL NOTES

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A NOTE ON A CONVOLUTION

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Consider the integral transform $f(s)$ defined by

$$f(s) = \mathfrak{D}\{F(x)\} = s\int_1^\infty x^{-s-1}F(x)dx,$$

which converges for Re $s>\alpha$ if $F(x)=O(x^\alpha)$ and $F(x)$ is sectionally continuous. This is, of course, closely related to the Laplace transform since

$$f(s) = \mathfrak{D}\{F(x)\} = s\mathfrak{L}\{F(e^t)\}.$$

If we consider the transform $s^{-1}f(s)g(s)$ we have the convolution relation

$$s^{-1}f(s)g(s) = \mathfrak{D}\left\{\int_1^x F(x/u)G(u)u^{-1}du\right\} = \mathfrak{D}\left\{\int_1^x F(u)G(x/u)u^{-1}du\right\},$$

and the special case

$$s^{-1}f(s) = \mathfrak{D}\left\{\int_1^x u^{-1}F(u)du\right\}.$$

Another important relation is

$$\sum_{n=1}^\infty a_n n^{-s} = \mathfrak{D}\left\{\sum_{n\leq x} a_n\right\}$$

in which case the integral is convergent for Re $s>\max(0, \sigma)$ where σ is the abscissa of convergence of the Dirichlet series.

Now since

$$\sum_{n=1}^\infty a_n n^{-s}\sum_{n=1}^\infty b_n n^{-s} = \sum_{n=1}^\infty\left(\sum_{d|n} a_d b_{n/d}\right)n^{-s},$$

we have from the convolution relation

$$\mathfrak{D}\left\{\int_1^x\left(\sum_{n\leq u} a_n\right)\left(\sum_{n\leq x/u} b_n\right)u^{-1}du\right\} = \mathfrak{D}\left\{\int_1^x\sum_{n\leq u}\left(\sum_{d|n} a_d b_{n/d}\right)u^{-1}du\right\},$$

so that

$$I = \int_1^x\left(\sum_{n\leq u} a_n\right)\left(\sum_{n\leq x/u} b_n\right)u^{-1}du = \int_1^x\sum_{n\leq u}\left(\sum_{d|n} a_d b_{n/d}\right)u^{-1}du.$$

These integrals can be converted into various forms involving sums,

$$I = \sum_{n\leqq x} a_n \int_n^x \Big(\sum_{k\leqq x/u} b_k\Big) u^{-1}du = \sum_{n\leqq x} a_n \int_1^{x/n} \Big(\sum_{k\leqq u} b_k\Big) u^{-1}du$$
$$= \sum_{n\leqq x}\sum_{k\leqq x/n} a_n b_k \log (x/kn) = \sum_{n\leqq x} \log (x/n) \sum_{d\mid n} a_d b_{n/d},$$

in any of which the a's and b's may be interchanged.

As an example consider $a_n = n^k$, $b_n = 1$, then

$$\int_1^x \Big(\sum_{n\leqq u} n^k\Big)\Big[\frac{x}{u}\Big]\frac{du}{u} = \int_1^x \Big(\sum_{n\leqq u} \sigma_k(n)\Big) u^{-1}du = \sum_{n\leqq x} \log (x/n)$$

in which $[x]$ denotes the greatest integer not exceeding x and $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n.

For a second example to illustrate relations between interesting integrals and sums, let $a_n = n$, $b_n = \mu(n)$, the Möbius function; then

$$\int_1^x \sum_{n\leqq x/u} \mu(n)\, \frac{[u][u+1]}{2u}\, du = \int_1^x \sum_{n\leqq u} \phi(n) u^{-1}du$$

in which $\phi(n)$ is Euler's ϕ-function.

A thorough discussion of the convolution (Faltung) for the Laplace transformation is given by G. Doetsch, *Handbuch der Laplace-Transformation*, vol. 1, Basel, 1950.

CONCERNING THE FOURIER COEFFICIENTS OF A NONNEGATIVE FUNCTION

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In vol. 66, pp. 223–224 of this Monthly, J. D. Weston established the inequality

$$c_0^2 + c_0 |c_{2n}| \geqq 2|c_n|^2 \tag{1}$$

for the Fourier coefficients of a nonnegative function $g(\theta)$:

$$g(\theta) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta) e^{-in\theta}d\theta. \tag{2}$$

The proof does not seem to point out the basic and simple background of this inequality.

First, let us apply the operation of "symmetrization":

$$g_n(\theta) = \frac{1}{n}\sum_{\nu=0}^{n-1} g\Big(\theta + \frac{2\pi\nu}{n}\Big).$$

It produces a function $g_n(\theta)$ whose harmonics are precisely those of $g(\theta)$ which have a frequency divisible by n:

$$g_n(\theta) \sim \sum_{m=-\infty}^{\infty} c_{nm} e^{inm\theta}.$$

Thus it suffices to prove (1) in the special case $n=1$:

$$c_0^2 + c_0 |c_2| \geqq 2|c_1|^2. \tag{3}$$

We consider the Toeplitz form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta) |u_0 + u_1 e^{i\theta} + u_2 e^{i\theta}|^2 d\theta \tag{4}$$

with the matrix

$$\begin{pmatrix} c_0 & c_1 & c_2 \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 \end{pmatrix}, \qquad c_{-m} = \bar{c}_m. \tag{5}$$

Since (4) is positive definite (unless $g(\theta)$ is a zero-function), we have

$$c_0 > 0, \quad \begin{vmatrix} c_0 & c_1 \\ c_{-1} & c_0 \end{vmatrix} > 0, \quad \begin{vmatrix} c_0 & c_1 & c_2 \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 \end{vmatrix} > 0. \tag{6}$$

We have the identity

$$c_0 \begin{vmatrix} c_0 & c_1 & c_2 \\ c_{-1} & c_0 & c_1 \\ c_{-2} & c_{-1} & c_0 \end{vmatrix} = \begin{vmatrix} c_0 & c_1 \\ c_{-1} & c_0 \end{vmatrix} \cdot \begin{vmatrix} c_0 & c_1 \\ c_{-1} & c_0 \end{vmatrix} - \begin{vmatrix} c_1 & c_2 \\ c_0 & c_1 \end{vmatrix} \cdot \begin{vmatrix} c_{-1} & c_0 \\ c_{-2} & c_{-1} \end{vmatrix},$$

so that (6) can be written as follows:

$$c_0^2 - |c_1|^2 > |c_1^2 - c_0 c_2|, \qquad c_0 > 0. \tag{7}$$

But $|c_1^2 - c_0 c_2| \geqq |c_1^2| - |c_0 c_2| = |c_1|^2 - c_0 |c_2|$, and this yields (3).

We note that (6) represents the necessary and sufficient condition for the initial Fourier-Stieltjes coefficients of a nonnegative measure $d\alpha(\theta)$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} d\alpha(\theta). \tag{8}$$

THE INCIDENCE EQUATION $AA^T=aA$

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We say that a matrix is an incidence matrix if each of its elements is either 0 or 1. We denote the zero-matrix of order n by O_n and the one-matrix (the matrix with 1 in every position) of order n by J_n. We shall prove the

THEOREM. *An incidence matrix A satisfies $AA^T=aA$ for some nonzero a if and only if there exists a permutation matrix P such that PAP^T is the direct sum of a zero-matrix and one-matrices of order a.*

The equation $AA^T=aA$ implies that A is square (denote its order by n), and symmetric.

Let $A=(a_{ij})$ and $a_i=\sum_{k=1}^{n} a_{ik}$. The defining equation is equivalent to

$$\sum_{k=1}^{n} a_{ik}a_{jk} = aa_{ij} \qquad i,j = 1, 2, \cdots, n. \tag{1}$$

If we let $i=j$ and note that $a_{ik}^2=a_{ik}$, we have $a_i=aa_{ii}$. Therefore, if the ith row is not a zero row (*i.e.*, if $a_i>0$) then $a_{ii}=1$ and $a_i=a$ or, in words, the nonzero rows of A have row sum a and 1 on the main diagonal.

If $A=0$ we are finished. Suppose $A\neq 0$. We can simultaneously renumber the rows and columns of A (*i.e.*, we can find a permutation matrix Q so that QAQ^T effects the renumbering) in such a way that the first a elements of the first row are 1 and the rest 0. It is important to note that such a renumbering permutes the diagonal elements, so that we can achieve this only because the diagonal element of a non-zero row is 1. Such a renumbering also leaves invariant the defining equation $AA^T=aA$.

Therefore, suppose $a_{11}=\cdots=a_{1a}=1$ and $a_{1k}=0$ for $k>a$. Let $i=1$ and let j lie between 1 and a in equation (1). Then $a_{ij}=1$ and we have

$$\sum_{k=1}^{a} a_{kj} = a. \qquad j = 1, \cdots, a.$$

Since each a_{kj} is either 0 or 1 it follows that $a_{kj}=1$ for $k,j=1,\cdots,a$. Because the row, and by symmetry the column, sums of the nonzero rows (and columns) are equal to a we have $a_{kj}=0$ if $1\leqq k\leqq a$ and $j>a$ or if $1\leqq j\leqq a$ and $k>a$.

Therefore A is the direct sum of J_a and a matrix of order $n-a$ with the same defining equation. We now repeat the above process on this matrix to prove the necessity half of the theorem.

The sufficiency is clear because $O_mO_m^T=aO_m$ for any m, and $J_aJ_a^T=aJ_a$. This completes the proof of the theorem.

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THE PROBABILITY THAT $(n, f(n))$ IS r-FREE

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Let f be a function mapping the set of positive integers into the set of non-negative integers. J. Lambek and L. Moser [1] have shown that with certain conditions* placed on f, the probability that n and $f(n)$ are relatively prime is $6/\pi^2$. This result complements the fact that if the integers m and n are chosen at random then the probability that they are relatively prime is $6/\pi^2$. We have shown ([2], Th. 4) that if m and n are chosen at random then the probability that (m, n) is r-free, *i.e.*, is not divisible by the rth power of any prime, is $1/\zeta(2r)$. In this note we shall show that an analogue to the Lambek-Moser result holds when $r>1$.

THEOREM. *Let $r>1$. Assume that*

(i) *f is nondecreasing,*

(ii) *f^* is finite and nondecreasing, where $f^*(n)$ is the number of m such that $f(m)=n$,*

(iii) $f(n)=o(n)$,

(iv) $f^*(f(n))=o(n)$.

Then, if the integer n is chosen at random, the probability that $(n, f(n))$ is r-free is $1/\zeta(2r)$.

Let $Q(t)$ be the number of positive integers $n\leqq t$ such that $(n, f(n))$ is r-free. We shall show that if f satisfies (i)–(iv) then

$$\lim_{t\to\infty} t^{-1}Q(t) = 1/\zeta(2r).$$

Let $S(t, a, b)$ be the number of multiples of a, not exceeding t, which are mapped onto multiples of b by f. Let μ be the Möbius function. As we have pointed out in [2],

$$\sum_{d^r\mid n}\mu(d) = \begin{cases}1 & \text{if } n \text{ is } r\text{-free,}\\ 0 & \text{otherwise.}\end{cases}$$

Therefore,

$$Q(t) = \sum_{\substack{n\leqq t\\ (n,f(n)),r-\text{free}}} 1 = \sum_{n\leqq t}\sum_{d^r\mid(n,f(n))}\mu(d) = \sum_{d=1}^{k}\mu(d)S(t, d^r, d^r),$$

where we have denoted $[f(t)^{1/r}]$ by k. Lambek and Moser have shown in [1] that as a consequence of (i) and (ii),

$$S(t, a^r, b^r) = (ab)^{-r}t + O(b^{-r}f(t)) + O(a^{-r}f^*(f(t))).$$

Hence,

* Conditions considerably weaker than those of Lambek and Moser have been obtained by P. Erdös and G. G. Lorentz in a paper to appear in Acta Arithmetica.

$$Q(t) = t\sum_{d=1}^{k}\mu(d)d^{-2r} + O\left(f(t)\sum_{d=1}^{k}d^{-r}\right) + O\left(f^*(f(t))\sum_{d=1}^{k}d^{-r}\right).$$

Now,

$$\sum_{d=1}^{k}\mu(d)d^{-2r} = 1/\zeta(2r) + O\left(\sum_{d=k+1}^{\infty}d^{-2r}\right),$$

and it is a simple exercise to show that

$$\sum_{d=k+1}^{\infty}d^{-2r} = O(k^{-r}) \quad \text{and} \quad \sum_{d=1}^{k}d^{-r} = O(1).$$

Therefore,

$$Q(t) = t/\zeta(2r) + O(tf(t)^{-1}) + O(f(t)) + O(f^*(f(t))).$$

It is a consequence of (i) and (ii) that $nf(n)^{-1} = o(n)$, and therefore, using (iii) and (iv) we have $\lim t^{-1}Q(t) = 1/\zeta(2r)$ as $t \to \infty$.

Example. Let $f(n) = [n^{1/s}]$, where s is an integer greater than 1. Then

$$f^*(n) = \sum_{i=1}^{s}\binom{s}{i}n^{s-i},$$

and so f satisfies all the conditions of the theorem.

Examples can be given, similar to those in section 5 of [1], which show that none of the conditions of the theorem are superfluous.

References

1. Joachim Lambek and Leo Moser, On integers n relatively prime to $f(n)$, Canad. J. Math., vol. 7, 1955, pp. 155–158.

2. Paul J. McCarthy, On a certain family of arithmetic functions, this MONTHLY, vol. 65, 1958, pp. 586–590.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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ON SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

D. A. KEARNS, Merrimack College

The extent to which the exponential function enters into the study of linear differential equations is familiar to all students of the subject. We point out here that the functional equation satisfied by e^x also can be applied in a certain way

A NOTE ON THE BASE OF NATURAL LOGARITHMS

ROBERT WEINSTOCK, University of Notre Dame*

Ernest Leach† has exhibited an elegant device for proving that $(1+t)^{1/t}\to e$ as $t\to 0$ *directly* from the definition of e through the equation

$$\int_1^e t^{-1}dt = 1. \tag{1}$$

It is the purpose of the present note to show how Leach's proof can be modified so as to avoid his use of the uniform-continuity concept. (As an inessential alteration we consider here the sequential limit and prove the equivalent $\lim_{n\to\infty}(1+n^{-1})^n = e$.)

We require the following elementary results from the integral calculus of *continuous* functions:

(i) If $f(t) < g(t)$ for $a < t \leqq b$, then $\int_a^b f(t)dt < \int_a^b g(t)dt$.

(ii) If $a<b$, $a<c$, and $0<f(t)<g(t)$ for $t>a$, then $\int_a^b f(t)dt = \int_a^c g(t)dt$ implies $a<c<b$.

(iii) If $a<b$, $a<c$, and $0<f(t)$ for $t>a$, then $\int_a^b f(t)dt > \int_a^c f(t)dt$ if and only if $b>c$.

We require also the fact (iv) that if $p>1$, then for each $u>0$ there is an $N(u)$ such that $p^{-1/n}>1-u$ whenever $n>N(u)$.

Defining (after the manner of [1]) the sequence $\{s_n\}$ by the equation

$$\int_1^{s_n} t^{-1+1/n}dt = 1, \tag{2}$$

we have from (1) and (ii) that $1<s_n<e$ (and incidentally that $s_{n+1}>s_n$) for all natural numbers n. From (1) and (iii) we have, for arbitrary ϵ satisfying $0<\epsilon<e-1$,

$$\int_1^{e-\epsilon} t^{-1}dt = 1 - u(\epsilon), \quad \text{with} \quad u(\epsilon) > 0. \tag{3}$$

From (2) and (i), since $1<s_n<e$—whence $t^{-1+1/n}<t^{-1}e^{1/n}$ for $1\leqq t\leqq s_n$—we obtain with the subsequent aid of (iv)

$$1 < e^{1/n}\int_1^{s_n} t^{-1}dt, \text{ whence } \int_1^{s_n} t^{-1}dt > e^{-1/n} > 1 - u(\epsilon) \text{ for } n > N(u(\epsilon)) = N_0(\epsilon).$$

We then conclude from (3) and (iii) that $s_n > e-\epsilon$ for $n>N_0(\epsilon)$; with $s_n<e$, we therefore have $\lim_{n\to\infty} s_n = e$. Since direct computation in (2) yields $s_n = (1+n^{-1})^n$, we have achieved the desired result.

* On leave at Oberlin College, 1959–60.

† Ernest Leach, A note on the base of natural logarithms, this MONTHLY, vol. 60, 1953, pp. 622–623.

AN EXTENSION OF INTEGRATION BY PARTS

JAMES W. BROWN, Stanford University

The undergraduate mathematics student spends much of his time learning basic formulas which become familiar tools in applications. An example is the method of Integration by Parts which he must often apply a number of times in the same problem. There is a useful extension of this method which can be simply stated and easily remembered.

The basic integration formula is given in the operational notation as

$$D^{-1}(fg') = fg - D^{-1}(gf').$$

If $g = D^{-1}F$, an indefinite integral of the function fF is given by

$$D^{-1}(fF) = fD^{-1}(F) - D^{-1}[D(f)D^{-1}(F)]. \tag{1}$$

It follows by mathematical induction that

$$D^{-1}(fF) = \sum_{i=0}^{n} (-1)^i D^i(f) D^{-(i+1)}(F) - (-1)^n D^{-1}[D^{n+1}(f) D^{-(n+1)}(F)]. \tag{2}$$

Equation (1) is the case for $n=0$, equation (2) is the inductive hypothesis, and the conclusion is that

$$\begin{aligned} D^{-1}(fF) &= \sum_{i=0}^{n} (-1)^i D^i(f) D^{-(i+1)}(F) \\ &\quad - (-1)^n \{ D^{n+1}(f) D^{-(n+2)}(F) - D^{-1}[D^{n+2}(f) D^{-(n+2)}(F)] \} \\ &= \sum_{i=0}^{n+1} (-1)^i D^i(f) D^{-(i+1)}(F) \\ &\quad - (-1)^{n+1} D^{-1}[D^{n+2}(f) D^{-(n+2)}(F)]. \end{aligned}$$

If f is a polynomial of degree n, $D^{n+1}(f) = 0$ and (2) reduces to the integration formula:

$$D^{-1}(fF) = \sum_{i=0}^{n} (-1)^i D^i(f) D^{-(i+1)}(F). \tag{3}$$

This result may be thought of as an algorithm where the polynomial f is successively differentiated, the function F is successively integrated, alternating signs are affixed to the terms, and an arbitrary constant is added for generality. For example,

$$\begin{aligned} \int (x^2 + x + 1)e^{2x} dx &= D^{-1}[(x^2 + x + 1)e^{2x}] \\ &= (x^2 + x + 1)(\tfrac{1}{2}e^{2x}) - (2x + 1)(\tfrac{1}{4}e^{2x}) + 2(\tfrac{1}{8}e^{2x}) + C. \end{aligned}$$

ERRORS RESULTING FROM A MANIPULATIVE APPROACH TO PROVING THE ORTHOGONALITY OF TWO CURVES

KENNETH MAY, Carleton College

If asked to prove that the two curves defined by $y = -\sqrt{(2x)}$ and $y = e^{-x}$ are orthogonal, the thoughtless manipulator quickly finds $y' = 1/y$ for the first and $y' = -y$ for the second. Citing something about "negative reciprocals" he announces that the proof is complete. Protesting that it is really a waste of time, he draws the curves—and discovers that they do not meet at all! The experience may help convince him that formal calculations are not enough. It is easy to find other examples by experiment. A routine procedure (not available to the student in beginning calculus) is to construct the family of orthogonal trajectories of a family of curves and then pick a nonintersecting pair from the two families.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

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AAAS STUDIES IN TEACHER EDUCATION

The educational programs of the National Science Foundation and of scientific societies in cooperation with NSF, with private foundations, or with their own resources, indicate widespread interest and progress in the improvement in offerings in science and mathematics for those preparing to teach these subjects in elementary and secondary schools. On the other hand, not much information is available about studies of the professional education requirements for science teachers. As a consequence, one of the important projects in the new Science Teaching Improvement Program (STIP) of the American Association for the Advancement of Science is entitled "Studies in Teacher Education." The project is intended to encourage experimentation with professional requirements for science teachers. An advisory board has chosen seven colleges and universities for cooperation in the project. These colleges and universities, with a brief description of the projects they are undertaking, are listed below.

For the most part, the studies will be carried out independently by the seven colleges, but there may be advantages in the cooperation of two or more of the institutions of higher education on a problem of common interest. A part of the STIP contribution will be to make this possible. A first conference of representatives of the cooperating institutions was held at the University of Michigan on July 20–21. A feature of that conference was a report from the West Virginia

State Department of Education on the plans in that state for teacher certification by examination. A brochure announcing the new studies, containing information about them and the cooperating institutions is available for distribution on request.

The cooperating colleges and their projects are as follows:

Bucknell University and Emory University—Use of examinations to satisfy teacher-certification requirements.

Emory University and the University of Tennessee—Use of seniors in college laboratories as part of the student-teaching assignment and as assistants in research projects in science.

Hunter College and Emory University—Development of a course in science (Hunter) and a course in mathematics (Emory) for elementary-school teachers, to be offered jointly by science and education staffs.

Oklahoma State University—An experimental study in the education of elementary teachers, under which elementary teachers will do their student teaching during the second semester of the junior year with the advice and counsel of science staff members. During the senior year science staff members will further assist them in making appropriate choices of content courses to supplement the four-year bachelor's degree program.

San Francisco State College—The establishment of a laboratory of materials for teaching science and mathematics, and participation, as a part of the professional education requirements for prospective teachers, in keeping the laboratory up to date and making use of the materials.

University of Arizona—A part of requirement for preservice science (including mathematics) teachers will be a seminar of scientists and preservice teachers, at which papers will be presented and discussed both from the scientific and pedagogical points of view.

GEOMETRY FOR ELEMENTARY SCHOOL TEACHERS

LENORE JOHN, The Laboratory School, University of Chicago

Among the proposals for revision of the arithmetic curriculum in the elementary grades is the suggestion that study of some aspects of mathematics other than arithmetic be begun in those grades. In particular, it is proposed that some geometry be included. Studies of the development of young children indicate that some space concepts are developed early. The everyday experiences of children contribute to further development, and suggest the possibility of a more systematic treatment of space relations than is currently found in the curriculum. Below the seventh grade, the geometry content is frequently limited to study of the perimeter and the area of a rectangular region and the volume of a rectangular solid, with perhaps some study of the triangle. Exploratory studies now in progress seem to indicate that other topics can be taught with success even in the primary grades, and it seems probable that new courses of study will include more geometry than has been the case in recent years.

Exactly what aspects of geometry should become part of the curriculum for Grades One to Six is a question which must be answered in the light of evidence obtained from experimental teaching in classrooms, as well as other considerations. However, successful teaching of geometric content on any widespread scale will depend on adequate preparation of teachers. Courses designed to provide such preparation may well include at least the following topics and points of emphasis.

1. Consideration of points, lines, planes, and space as sets of points, and their intersections and unions. Conditions determining a point, a line, a plane. Separation of a line, a plane, and space. Simple closed curves and their intersections; non-metric properties of common polygons and polyhedrons. Concepts associated with projection in a plane and in three dimensions.

2. Correspondence between the set of points on a line and the set of real numbers, between the points in a plane and ordered pairs, between points in space and ordered triples. Geometric and algebraic description of geometric figures, including the line, circle, ellipse, and parabola.

3. Assumptions underlying the measurement of length, area, volume, and angle. The concept of precision of a measure.

4. Angle relationships associated with intersecting and parallel lines and planes. Congruence and similarity of geometric figures. Metric properties of common two- and three-dimensional figures, including the triangle, quadrilateral, circle, prism, pyramid, cylinder, cone, and sphere. The special properties of the right triangle.

5. Physical representations of geometric figures in the environment; description of familiar objects in terms of their geometric properties; application of geometric relations in solving practical problems.

6. Presentation of the geometric content in such a way as to make clear the use of inductive reasoning in drawing a probable inference from observed data; the use of deductive reasoning in drawing a necessary inference from stated postulates; and the need for undefined terms and for precisely stated postulates and definitions.

7. Brief overview of the history of the development of geometry.

The amount of the content outlined above which is new to the prospective teacher will depend in part upon the amount of mathematics studied in the secondary school. One who has studied two or three years of college preparatory mathematics should be familiar with much of the material, but even in such cases a review of the topics from a new point of view would probably be desirable.

While adequate mathematical background is the most important element in preparation of teachers to introduce geometry in the elementary grades, assistance is also needed in developing effective methods of teaching, especially in the selection and construction of suitable models. If a mathematics course and a course in methods of teaching are taught by different instructors, close collaboration between instructors is essential for effective preparation of the prospective teachers.

NSF Program in Mathematics for High-Ability Secondary School Students at the University of Texas

During the summer of 1959 the University of Texas sponsored a program in mathematics for high-ability secondary school students. Thirty participants were selected from a total of 548 applications. The six-week program consisted of daily group meetings and work sessions.

In the report of the Director, H. J. Ettlinger, Professor of Mathematics, the contents were described as follows:

> "The objective of the program was to provide a maximum opportunity for the participants to broaden and deepen their knowledge of mathematics. In the main the assumption was made that this could be accomplished best not by reading or listening to lectures, but by *working problems* and *doing mathematics*. In the main a definite attempt was made to avoid those mathematical subjects which the pupils would be studying the following school year. As evidenced by the summary and notes in the appendix, attention was concentrated on the basic idea of number in arithmetic and algebra, and developing fundamental mathematical patterns by working with theorems and exercises. This was also followed in geometry, which was developed by means of sets of points, definitions and axioms, the very essence of modern mathematics."

And he further reported that:

> "The pupils responded to this program far more enthusiastically and successfully than could have been expected. It was very heartening to witness the presentation of mathematical ideas and proofs of this high-ability group. This they did with equal success in algebra and in geometry. They handled the patterns of modern mathematics with great skill. . . . This program has been a most enjoyable and exciting experience for the staff. It is hoped that other opportunities to work with groups of pupils of this type will be provided."

GREATER CLEVELAND MATHEMATICS PLAN

GEORGE H. BAIRD, The Educational Research Council of Greater Cleveland

In September 1959, 20 public, private and parochial schools from the Greater Cleveland area (Cleveland City Public Schools not included) started implementing a new mathematics curriculum called the Greater Cleveland Mathematics Program (G.C.M.P.). This mathematics curriculum project was initiated by the Educational Research Council of Greater Cleveland and is directed by Dr. B. H. Gundlach of Bowling Green State University. The G.C.M.P. provides a five year program with new curricula introduced gradually in all grades of elementary and secondary schools after transitional and enrichment programs and in-service teacher training.*

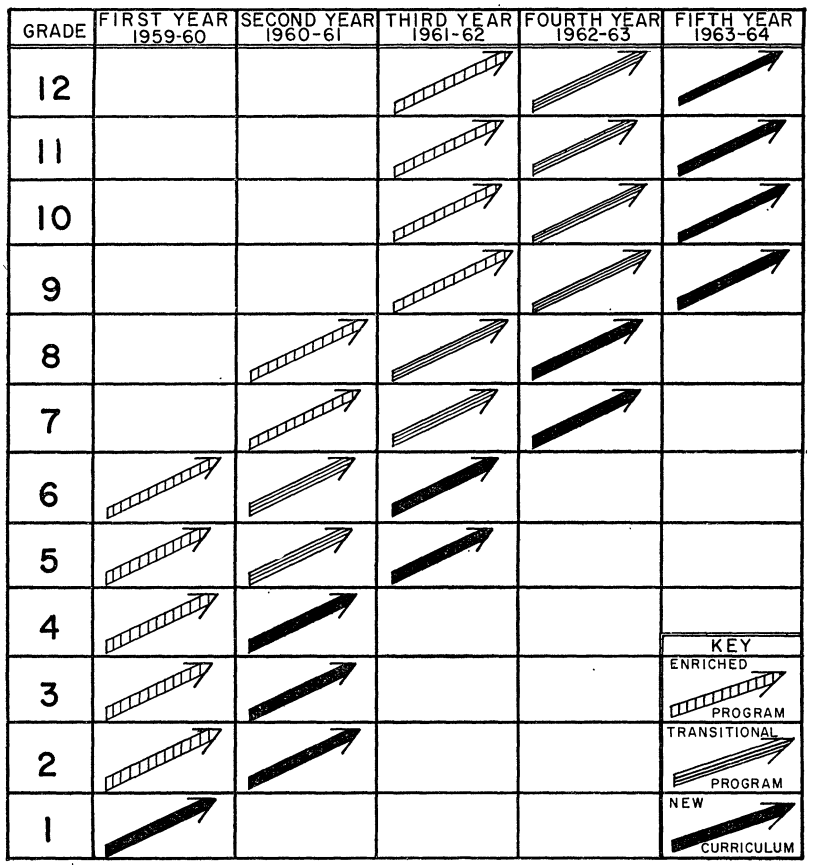
For the first school year, 1959–60, during the first semester, an intensive 1st-grade-teacher retraining is provided by Dr. Gundlach and the Council staff. The new 1st-grade-curriculum is being introduced to the pupils in the second semester. Meanwhile teachers from kindergarten through grade 6 were given in-

* Added in proof: The completed G.C.M.P. will result in one continuous mathematics course, starting in kindergarten and going through grade 12, supplemented by an advanced placement program. The schedule of implementation shown in the diagram is now somewhat out of date, since it does not include the kindergarten grade and certain other modifications.

service training so that they could introduce in their classrooms an enriched course of study. In the following four years the new curriculum will reach all 13 grades, kindergarten through grade 6.

The Council has received the collaboration of Dr. J. G. Kemeny, Dartmouth College; Dr. Max Beberman, University of Illinois; Dr. C. F. Brumfiel, Ball State Teachers College; and Dr. P. S. Jones, University of Michigan, to insure a first hand combination of the best of all recent research in the field of mathematics.

The prompt implementation of the G.C.M.P. by a compact group of 20 schools, with a school enrollment of 100,000 pupils, will offer an opportunity for close supervision and evaluation of the whole program, so that if successful in this limited area, it could serve as a pattern for other schools in the nation to follow.



PROPOSED SCHEDULE OF IMPLEMENTATION

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1411. *Proposed by H. Y. Shee, Hualien, Taiwan, China*

Let $ABCDE$ be any pentagon inscribed in a circle and let P, Q, R, S, T be intersections of the diagonals such that P and Q lie on AC, Q and R on BD, R and S on CE, S and T on DA. Prove that

$$\frac{AB\cdot BC\cdot CD\cdot DE\cdot EA}{AC\cdot BD\cdot CE\cdot DA\cdot EB}=\frac{AP\cdot BQ\cdot CR\cdot DS\cdot ET}{CP\cdot DQ\cdot ER\cdot AS\cdot BT}.$$

Generalize.

E 1412. *Proposed by H. E. Chrestenson, Reed College*

Let $\{m_k\}$, $k\geqq 0$, be a strictly increasing sequence of integers with $m_0=0$. Under what conditions does the series $\sum_{k=1}^{\infty}(m_k-m_{k-1})/m_k$ converge?

E 1413. *Proposed by B. K. Harrison, University of California, Los Alamos Scientific Laboratory*

Consider an integer, written in the usual form to the base r as

$$N = a_0a_1a_2\cdots a_n, \qquad 0\leqq a_i<r.$$

We require that $r\geqq n+1$. Find all N, for any n, such that a_k is equal to the number of times that k appears in the sequence $a_0, a_1, a_2, \cdots, a_n$.

E 1414. *Proposed by George Glauberman, Polytechnic Institute of Brooklyn*

Suppose g is a nonnegative and integrable function on $[0, 1]$. Let $A_r=\int_0^1[g(x)]^r dx$ for r a nonnegative integer. Prove that $A_r\geqq A_1^r$.

E 1415. *Proposed by Michael Herschorn, McGill University*

It began snowing in the morning; at noon a snowblower set out to clear a street of constant width. Later on a second blower followed, and, still later, a third. Each blower cleared at a constant rate (volume per unit of time), the spacing between blowers remaining constant with the first separation twice the second. At any given time during the day the rate of snowfall was the same throughout the town. A total of four inches had fallen when the second blower started out, and another four inches fell before the third began on the route. How deep was the snow at noon?

SOLUTIONS

A Polynomial Equation with no Real Roots

E 1381 [1959, 724]. *Proposed by H. E. Hart, City College of New York*

Show that

$$1 + x + x^2/2! + x^3/3! + \cdots + x^{2n}/(2n)! = 0$$

has no real roots.

I. *Solution by Joe Lipman, University of Toronto.* If there is a real root, it must be negative, say $-y$. But

$$1 - y + y^2/2! - y^3/3! + \cdots + y^{2n}/(2n)! > e^{-y} > 0.$$

II. *Solution by H. F. Bechtell, Grove City College.* Note that if $f(x) = f'(x) + x^{2n}/(2n)! = 0$, then $f'(x) < 0$; *i.e.*, at each root the function is decreasing. Since $f(x)$ is continuous and positive for all $x \geqq 0$, a contradiction is reached. Thus $f(x)$ has no real roots.

III. *Solution by C. F. Pinzka, University of Cincinnati.* Denoting the left member of the equation by $f(x)$, we note that $f(x) = f'(x) + x^{2n}/(2n)!$. Since $f(x)$ is of even degree with a positive leading coefficient, it has an absolute minimum at, say, $x = a$, where $a \neq 0$ and $f'(a) = 0$. Thus $f(a) = a^{2n}/(2n)! > 0$ and $f(x)$ is never zero for real x.

Also solved by R. G. Albert, Tom Bagby, Edward Barbeau, Merrill Barnebey, P. T. Bateman, W. R. Becker, Shlomo Ben-Adam, Marvin Blum, D. A. Breault, Brother Joseph Heisler, J. L. Brown, Jr., P. R. Chernoff, C. H. Cunkle, A. E. Danese, Underwood Dudley, G. W. Erwin, Jr., N. J. Fine, Evelyn Frank, Stanley Franklin, W. C. G. Fraser, J. D. Gilbert, T. A. Gitlin, George Glauberman, Gerald Goertzel, Michael Goldberg, L. D. Goldstone, S. H. Greene, Emil Grosswald, D. W. Henderson, J. C. Hickman, J. H. Hodges, Richard Holt, I. M. Isaacs, Lawrence Isenecker, J. N. Issos, Walter James, G. J. Janusz, C. M. Joiner, J. P. Jordan, Seymour Kass, Irving Katz, P. G. Kirmser, D. E. Knuth, Andrew Korsak, A. M. Krall, W. E. Lawrence, W. J. Larkin III and L. J. Schneider (jointly), Joel Levy and H. J. Lieberman (jointly), R. J. Lewyckyj, Viktors Linis, Shen Lui, R. F. McDermot, W. M. McKeeman, Wallace Manheimer, R. H. Marquis, D. C. B. Marsh, D. S. Mitrinović, Morris Morduchow, T. J. Morse, Amos Nannini, Hugh Noland, C. S. Ogilvy, Henry Palczewski, D. J. Persico, J. L. Pietenpol, Thomas Porsching, P. L. Renz, R. T. Sandberg, Donna Jean Seaman, Harold Shulman, E. L. Spitznagel, Jr., D. C. Stevens, J. P. Tull, K. Venkannayah, J. E. Vinson, Julius Vogel, R. J. Wagner, Dale Woods, Kenneth Williams, C. Winston, R. J. Wisner, J. D. M. Wright, C. C. Yalavigi, Catherine Yamada, David Zeitlin, and the proposer. Late solutions by M. S. Klamkin, Nathaniel Queen, B. E. Rhoades, and Jonathan Sondow.

This problem was located, with solution, in Polyá-Szego, *Aufgaben und Lehrsätze aus der Analysis*, Zweiter Band, Prob. 74, p. 48, with a further reference to J. J. Sylvester, *Mathematical Papers*, vol. 2, p. 516. It also appears, with solution, in E. A. Maxwell, *An Analytical Calculus*, vol. 1, p. 65, and in D. S. Mitrinović, *Zbornik matematičkih problema*, t. 1, 3° ed. (in press), and as part of Prob. 9, p. 118, of J. V. Uspensky, *Theory of Equations*.

An Unusual Sequence

E 1382 [1959, 724]. *Proposed by Ian Connell, University of Manitoba*

Let u_n be the nth term of the sequence

$$1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, \cdots,$$

where one odd number is followed by two evens, then three odds, etc. Prove that

$$u_n = 2n - [(1 + \sqrt{8n - 7})/2],$$

where square brackets denote the integral part.

Solution by Andrew Korsak, University of Toronto. Writing $u_n = 2n - t_n$, we see that the sequence $\{t_n\}$ is

$$1; 2, 2; 3, 3, 3; 4, 4, 4, 4; 5, 5, \cdots.$$

For a given t_n, it follows that the value of $n-1$ cannot be less than the sum of the first $t_n - 1$ integers, hence $t_n(t_n-1)/2 \leqq n-1$, from which we obtain $t_n \leqq (1+\sqrt{8n-7})/2$ and, t_n being the largest integer to satisfy this, we conclude $t_n = [(1+\sqrt{8n-7})/2]$.

Also solved by R. G. Albert, Edward Barbeau, Shlomo Ben-Adam, Julian Braun, J. L. Brown, Jr., Leonard Carlitz, G. B. Charlesworth A. G. Clark, Underwood Dudley, N. J. Fine, George Glauberman, H. W. Gould, Alfred Gray, S. H. Greene, Emil Grosswald, J. H. Hodges, J. Hooley, John Jordan, P. G. Kirmser, D. E. Knuth, Sidney Kravitz, H. R. Leifer, R. J. Lewyckyj, Joe Lipman, Shen Liu, William McKeeman, Wallace Manheimer, D. C. B. Marsh, N. S. Mendelsohn, Amos Nannini, Walter Penney, C. F. Pinzka, C. M. Sandwick, Sr., J. K. Siberz, C. M. Sidlo, Philip Smedley, D. C. Stevens, J. P. Tull, K. Venkannayah, J. E. Vinson, Chung-lie Wang, C. Winston, J. D. M. Wright, C. C. Yulavigi, and the proposer. Late solutions by M. S. Klamkin and Nathaniel Queen.

Mendelsohn pointed out that if v_n is the nth term of the sequence "complementary" to $\{u_n\}$, i.e., the sequence of integers in increasing order of magnitude which do not appear in $\{u_n\}$, namely, 3, 6, 8, 11, 13, 15, 18, 20, 22, 24, $\cdots$, then, like $\{u_n\}$, $\{v_n\}$ consists of one odd, two even, three odd numbers, etc. The only difference in rule is that in changing parity the first available integer is omitted. Also, $u_n + v_n = 4n$.

Gould noted the connection of the problem with that of the countability of the rational numbers, and he explored some generalizations of the problem.

A Problem with a Unique Solution

E 1383 [1959, 724]. *Proposed by E. J. Burr, University of New England, N.S.W., Australia*

Let P_n be the problem: "Find a set of $n>2$ consecutive positive integers, the greatest of which is a divisor of the least common multiple of the remaining $n-1$." Show that precisely one of the problems P_n has a unique solution.

Solution by N. J. Fine, Institute for Advanced Study. For fixed n, let M_b be the least common multiple of $b-1, b-2, \cdots, b-(n-1)$. Since every prime power which divides b and some j, $1 \leqq j < n$, also divides b and $b-j$, and con-

versely, $(b, M_b) = (b, M_n)$. In particular, $b \mid M_b$ is equivalent to $b \mid M_n$. Thus, the solutions of P_n are precisely those b which divide M_n and are not less than n. For $n=3$, $M_n = 2 < n$, so P_3 has no solution. For $n=4$, $M_4 = 6$, and P_4 has a unique solution. For $n \geqq 5$, both $(n-1)(n-2)$ and $(n-1)(n-2)/2$ are divisors of M_n which are not less than n, hence solutions of P_n.

Also solved by R. G. Albert, Tom Bagby, Edward Barbeau, G. B. Charlesworth, Underwood Dudley, Merritt Elmore, Michael Goldberg, Donald Knuth, Andrew Korsak, Sidney Kravitz, Emma Lehmer, Joe Lipman, Wallace Manheimer, D. C. B. Marsh, Hugh Noland, J. L. Pietenpol, R. T. Sandberg, B. L. Schwartz, Donna Jean Seaman, D. C. Stevens, J. P. Tull, J. E. Vinson, and the proposer.

Circle Through Two Points Cutting a Given Circle at Least Angle

E 1384 [1959, 724]. *Proposed by J. H. Butchart, Arizona State College*

Construct a circle through two given points, separated by a given circle, which shall cut the given circle at the smallest possible angle.

Solution by D. C. Stevens, Alabama College. Designate the given points by P_1, P_2 and the given circle by C_1. Let C_2 be the circle orthogonal to C_1 and on P_1, P_2, and let C_3 be the circle orthogonal to C_2 and on P_1, P_2. Invert the figure about P_1, designating the images of elements with primes. Now C_2' is a diameter of C_1', and C_3' is the line through P_2' making minimum angle with C_1', since C_3' is perpendicular to C_2'. Angles are preserved under inversion, and so C_3 makes minimum angle with C_1.

Also solved by J. Basile, Michael Goldberg, L. D. Goldstone, Andrew Korsak, R. J. Lewyckyj, D. C. B. Marsh, Beckham Martin, Vincent Mead, C. F. Pinzka, and the proposer. Late solution by J. W. Baldwin.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4899. *Proposed by R. C. Buck, Institute for Advanced Study*

As $(x, y) \to (0, 0)$ let $\lim F(x, y) = 0$. Suppose that $\{s_n\}$ is a real sequence such that $0 \leqq s_{n+m} \leqq F(s_n, s_m)$ for all n, $m = 1, 2, \cdots$. Suppose also that $\lim (s_1 + \cdots + s_n)/n = 0$. Show that $\lim s_n = 0$.

4900. *Proposed by S. P. Lloyd, Bell Telephone Laboratories*

For every measurable subset E of the unit interval let $\{C_n(E)\}$ denote the sequence of numbers

$$C_n(E) = \int_E e^{2\pi i n x} dx, \qquad n = 1, 2, \cdots.$$

Show that a sequence $\{C_n(E)\}$ may tend to zero arbitrarily slowly, in the following sense: for every positive nondecreasing sequence $\{a_n\}$, if $\sup_n a_n |C_n(E)|$ is finite for every E then $\sup_n a_n$ is finite. (Problem 4749 [1958, 453], treats the case $a_n = n$.)

4901. *Proposed by Albert Wilansky, Lehigh University*

Let the class of subsets of measure zero of the interval $[0, 1]$ be partially ordered by inclusion and let M be a maximal chain. Is the cardinal number of M greater than that of the continuum? Is the union of the sets in M measurable?

4902. *Proposed by R. A. Melter, University of Missouri*

At most how many different values may an nth order determinant have, if its elements are a given set of n^2 different, nonzero real numbers?

4903. *Proposed by Victor Klee, University of Washington and University of Copenhagen*

A Darboux map of one topological space onto another is one under which the image of a connected set is always connected. With R denoting the real number space, show that there is a map f of R onto R such that f is a Darboux map but the product g of f with the identity map is not. (Here g is a map of the plane $P = R \times R$ onto itself, given by $g(x, y) = (fx, y)$.)

4904. *Proposed by D. J. Newman, Brown University*

Suppose a set, S, of functions continuous on $(0, 1)$ has the property that any linear combination of them has a zero in $(0, 1)$. Prove that there exists a nondecreasing function $\alpha(x)$ such that

$$\int_0^1 f(x) d\alpha(x) = 0 \qquad \text{for all } f \in S.$$

SOLUTIONS

A Diophantine Equation

4807 [1958, 633]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the integral solutions of the equation

$$(m - 1)x^2 + my^2 = (m + 1)z^2.$$

Editorial Note II. The proposer's solution [1959, 518] is not complete as mentioned in the

earlier *Note*. This is shown by the example: $m=1322$, $x=105$, $y=21$, $z=107$. Nevertheless, since $1322=31^2+19^2$ only, there exist no values of x, y, z such that $x^2+z^2=m$, $x^2+y^2-z^2=1$.

The Ring of p-adic Integers

4820 [1958, 779]. *Proposed by K. Mahler, the University, Manchester, England*

Let p be a prime, K the p-adic field, and I the ring of p-adic integers. Further, let $f(x)$ be a function defined for all $x\in I$ with $f(x)\in K$, such that $f(n)=n!$ $(n=0, 1, 2, \cdots)$. Show that the points of discontinuity lie dense in I.

Solution by Leonard Carlitz, Duke University. Let $P=(0, 1, 2, \cdots)$, the set of nonnegative integers. We show first that $f(x)$ is discontinuous for every $x\in P$. Indeed, for every such x, consider the sequence

$$x_n = x + p^n \qquad (n = 0, 1, 2, \cdots).$$

Then, if $v(x)=v_p(x)$ denotes the usual p-adic valuation for K, we have $v(x_n-x)=v(p^n)\to 0$ $(n\to\infty)$. On the other hand, since $f(x_n)=(x+p^n)!$ it follows that $v(f(x_n))\to 0$ as $n\to\infty$. But since $v(f(x))\neq 0$, we conclude that $f(x)$ is discontinuous at x.

Finally, if $\alpha=a_0+a_1p+a_2p^2+\cdots$ $(0\leqq a_i<p)$ is any number of I, and we put $\alpha_n=a_0+a_1p+\cdots+a_np^n$, then clearly $v(\alpha-\alpha_n)\to 0$ as $n\to\infty$. Hence P is dense in I.

Also solved by P. T. Bateman, Harley Flanders, and the proposer.

Normal Completion of Matrices

4821 [1958, 779]. *Proposed by M. P. Drazin, RIAS, Baltimore, Md.*

A square matrix B is called normal if its elements are in the complex field and if it commutes with its transposed conjugate matrix B^* (*i.e.*, if $BB^*=B^*B$); more generally, call a given $n\times n$ matrix A m-normal if A can be imbedded, as leading $n\times n$ submatrix, in some normal $m\times m$ matrix B. Find the greatest value of n for which every (complex) $n\times n$ matrix is $(n+1)$-normal, and show that every square matrix is m-normal for all sufficiently large m.

What are the corresponding results when A, B are restricted to be real?

Solution by the proposer. We show first that, for each integer $n\geqq 3$, a real $n\times n$ matrix A exists having no normal $(n+1)\times(n+1)$ completion

$$B = \begin{pmatrix} A & x^* \\ y & z \end{pmatrix}$$

(where x, y denote arbitrary complex row n-vectors and z can be any complex number), *i.e.*, an A exists such that

$$(1)\quad BB^* - B^*B = \begin{pmatrix} AA^* - A^*A + x^*x - y^*y, & Ay^* - A^*x^* + x^*\bar{z} - y^*z \\ yA^* - xA + zx - \bar{z}y, & yy^* - xx^* \end{pmatrix}$$

That every square matrix A is m-normal for large m follows at once from the identity

$$\begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}\begin{pmatrix} A^* & A \\ A & A^* \end{pmatrix} = \begin{pmatrix} A^* & A \\ A & A^* \end{pmatrix}\begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}.$$

Finally, just the same results hold good over the real field; for each U_n is real, and every B we have constructed is real when the corresponding A is.

Hilbert Space, Identity Operator as a Sum

4825 [1959, 66]. *Proposed by J. J. Schäffer, University of Uruguay*

Prove that in any n-dimensional (real or complex) Euclidean space, $n>1$, and indeed in any Hilbert space, every point x, $\|x\| \leqq 1$ may be written $x = \sum_{i=1}^{m} f_i(x)$, where $f_1(x), \cdots, f_m(x)$ is a fixed finite set of functions which are defined and uniformly continuous in $\|x\| \leqq 1$ and satisfy $\|f_i(x)\| = 1$ for $i = 1, \cdots, m$ and for all $\|x\| \leqq 1$.

From this result it follows at once that if $\mathfrak{C}$ is the Banach space of all bounded continuous functions $\phi(t)$ of a real variable t with values in any Hilbert space of dimension >1 (with the norm of the supremum), the set of $\phi(t) \in \mathfrak{C}$ with $\|\phi(t)\| = 1$ identically contains a linear basis of $\mathfrak{C}$.

Solution by the proposer. We prove that the theorem indeed holds with $m=4$ in every case. Choose two mutually orthogonal unit vectors e_1, e_2 (this requires dimension >1). For every x, $\|x\| \leqq 1$, set $y(x) = x - (x, e_1)e_1$. Thus $(y(x), e_1) = 0$ and $\|y(x)\|^2 + |(x, e_1)|^2 = \|x\|^2 \leqq 1$. $y(x)$ is a projection, hence uniformly continuous. The functions

$$\begin{aligned} f_1(x) &= \tfrac{1}{2}\{(x, e_1)e_1 + \sqrt{(4 - |(x, e_1)|^2)} \cdot e_2\}, \\ f_2(x) &= \tfrac{1}{2}\{(x, e_1)e_1 - \sqrt{(4 - |(x, e_1)|^2)} \cdot e_2\}, \\ f_3(x) &= \tfrac{1}{2}\{y(x) + \sqrt{(4 - \|y(x)\|^2)} \cdot e_1\}, \\ f_4(x) &= \tfrac{1}{2}\{y(x) - \sqrt{(4 - \|y(x)\|^2)} \cdot e_1\}, \end{aligned}$$

then clearly satisfy the requirements (uniform continuity is implied by the inequalities $4 - |(x, e_1)|^2 \geqq 3$, $4 - \|y(x)\|^2 \geqq 3$.)

It would be interesting to know if m can ever be chosen <4. For $m=2$, continuity cannot be assured at the origin; for n odd and $m=2$, continuity cannot even be assured if a neighborhood of the origin is excluded.

Also solved by R. D. Gordon.

Convolution, Inner Product

4849 [1959, 427]. *Proposed by L. A. Rubel, University of Illinois*

Suppose that $\phi(t)$ is a bounded measurable real function defined for $t>0$, and let $f(x) = (\phi * \phi)(x) = \int_0^x \phi(t)\phi(x-t)dt$. Must $f(x)$ be nonnegative for all sufficiently small positive x?

Solution by Brockway McMillan, Bell Telephone Laboratories. The answer "no" is indicated by the following example. Define $\phi(t)$ by

$$\phi(t) = (-1)^n \text{ for } 2^{-n-1} < t \leqq 2^{-n}, \qquad n = 0, 1, 2, \cdots.$$

With $f(x)$ as defined in the proposal, it is easy to establish the following three results in the order given:

$$\phi(\tfrac{1}{2}t) = -\phi(t), \qquad 0 < t \leqq 1; \tag{1}$$

$$f(\tfrac{1}{2}x) = \tfrac{1}{2}f(x), \qquad 0 \leqq x \leqq 1; \tag{2}$$

$$f(1) = 2\int_0^{1/2} \phi(t)\,dt = -\tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots = -\tfrac{1}{3}. \tag{3}$$

It follows from (2) and (3) that $f(2^{-n}) < 0$ for $n = 0, 1, \cdots$. Therefore any interval $0 < x \leqq t$ contains points where $f(x) < 0$.

Also solved by N. J. Fine and M. Jerison, Leopold Flatto, E. N. Gilbert, W. M. Gilbert, L. S. Kennison, R. H. Mihalek, John Rainwater, E. D. Ramer, I. J. Schoenberg, B. L. Schwartz, and the proposer.

Editorial Note. Schoenberg obtains similar results using the *continuous* function $\phi(t) = \sin(1/t)$.

Closed Convex Sets

4850 [1959, 427]. *Proposed by Ward Cheney and Allen Goldstein, Convair-Astronautics, San Diego, California*

Let $\mathfrak{A}$ and $\mathfrak{B}$ denote two closed convex sets in Hilbert space. For a point x, let Ax and Bx denote respectively the points of $\mathfrak{A}$ and $\mathfrak{B}$ closest to x. Prove that A satisfies the Lipschitz condition $\|Ax - Ay\| \leqq \|x - y\|$. Furthermore, if x is a fixed point of AB, then $\|x - Bx\| = \text{dist}\,(\mathfrak{A}, \mathfrak{B})$.

Solution by Robert C. James, Harvey Mudd College. Let T be a normed linear space. Then there is a nearest point Ax for each x and closed convex set A if and only if T is reflexive, since this is true for each separable subspace of T. [James, *Reflexivity and the supremum of linear functionals*, Annals of Math., 66 (1957), Theorem 3.] However, these propositions have meaning for T whether Ax necessarily exists and even if it need not be unique. Let $x \perp y$ mean that $\|x + ky\| \geqq \|x\|$ for each k. The first proposition is true if and only if

(i) $x + y \perp z$ whenever $x \perp z$ and $y \perp z$, and

(ii) $y \perp x$ whenever $x \perp y$.

Assume (i) and (ii) and let C be the entire line passing through Ax and Ay. Then two-dimensional considerations of convexity show that $\|Ax - Ay\| \leqq \|Cx - Cy\|$. But $x - Cx \perp Cx - Cy$ and $Cy - y \perp Cx - Cy$. Hence $(x - y) - (Cx - Cy) \perp Cx - Cy$ and $Cx - Cy \perp (x - y) - (Cx - Cy)$. Then

$$\|x - y\| = \|Cx - Cy + [(x - y) - (Cx - Cy)]\| \geqq \|Cx - Cy\| \geqq \|Ax - Ay\|.$$

If T is a Hilbert space, (i) and (ii) are satisfied and the first proposition is established. This was done by Phelps in a different way [*Convex sets and nearest points*, Proc. A.M.S., 8 (1957), Theorem 5.1]. Phelps also showed that (ii) is a

necessary condition [Lemma 5.4], but (ii) is also a sufficient condition for T to be a Hilbert space if the dimension of T is greater than 2. If T is two-dimensional, (i) is equivalent to strict convexity (each is equivalent to the set of all x with $x \perp y$ being one-dimensional when $y \neq \theta$) and the necessity then follows from a result of Phelps [Lemma 5.3].

The second proposition is true if and only if

(iii) $x \perp y+z$ whenever $x \perp y$ and $x \perp z$.

It is known that (iii) is equivalent to the uniqueness of supporting hyperplanes at points of the unit sphere and to the existence of $\lim_{s \to 0} (\|u+sv\| - \|u\|)/s$ when $u \neq \theta$ [James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc., vol. 61, 1947, Ths. 4.1 and 5.1]. Let x be such that $ABx = x$. Suppose there exist elements $u \in \mathfrak{A}$ and $v \in \mathfrak{B}$ with $\|u-v\| < \|x-Bx\|$. Then for $0 \leqq s \leqq 1$,

$$\begin{aligned} \|x - Bx\| &\leqq \left\| (x - Bx) + \frac{s}{1+s}(Bx - v) \right\| \\ &= \left\| \frac{s}{1+s}(u - v) + \frac{1}{1+s}[(x - Bx) - su - x)] \right\| \\ &\leqq \frac{s}{1+s}\|u - v\| + \frac{1}{1+s}\|(x - Bx) - s(u - x)\|. \end{aligned}$$

Then $\|(x-Bx)-s(u-x)\| - \|x-Bx\| \geqq s(\|x-Bx\| - \|u-v\|)$ and

$$\lim_{\bar{s} \to 0^-} \frac{\|(x - Bx) + \bar{s}(u - x)\| - \|x - Bx\|}{\bar{s}} \leqq \|u - v\| - \|x - Bx\| < 0.$$

But $\|(x-Bx)+s(u-x)\| \geqq \|x-Bx\|$, since $ABx = x$. Hence

$$\lim_{s \to 0^+} \frac{\|(x - Bx) + s(u - x)\| - \|x - Bx\|}{s} \geqq 0.$$

Thus (iii) is not satisfied. Conversely, if some point of the unit sphere does not have a unique supporting hyperplane, then there is a two-dimensional subspace whose unit sphere has a point x with two supporting lines. Clearly $\mathfrak{A}$ and $\mathfrak{B}$ can be taken as intersecting supporting lines at x and at $-x$, respectively, so that $Bx = -x$ and $A(-x) = x$. But dist $(\mathfrak{A}, \mathfrak{B}) = 0$.

Also solved by C. C. Farrington and W. G. Strang, D. M. Friedlen, D. S. Greenstein, John Rainwater, Peter Treuenfels, and the proposers.

An Enumeration of the Rationals in (0, 1)

4851 [1959, 515]. *Proposed by Z. A. Melzak, McGill University, Montreal*

Let X be the set of all rational numbers of the open interval (0, 1). Prove that for some enumeration $\{x_n\}$ of X, the series $\sum_{n=1}^{\infty} \prod_{i=1}^{n} x_i$ diverges.

Solution by S. J. Einhorn, University of Pennsylvania. Let $\{b_n\}$ be the sequence 1/2, 2/3, 3/4, $\cdots$, let $\{a_n\}$ be any enumeration of the rationals in (0, 1) excluding those of the form $n/(n+1)$, and let $\sum c_n$ be an arbitrary divergent series of positive terms.

Then, because of the divergence of the harmonic series, an integer k_1 can be chosen large enough so that

$$d_1 = a_1 + a_1b_1 + a_1b_1b_2 + \cdots + a_1b_1b_2 \cdots b_{k_1} \geqq c_1.$$

k_2 can now be chosen large enough so that

$$d_2 = a_1b_1b_2 \cdots b_{k_1}(a_2 + a_2b_{k_1+1} + \cdots + a_2b_{k_1+1}b_{k_1+2} \cdots b_{k_2}) \geqq c_2.$$

Similarly let k_3, k_4, $\cdots$ be chosen.

Then the sequence a_1, b_1, b_2, $\cdots$, b_{k_1}, a_2, b_{k_1+1}, $\cdots$, b_{k_2}, a_3, $\cdots$ possesses the desired property since the series of the proposal contains the terms d_n and these already diverge by comparison with $\sum c_n$.

Also solved by N. J. Fine, Leopold Flatto, Fritz Herzog, S. S. Page, John Rainwater, P. L. Renz, J. T. Rosenbaum, R. T. Sandberg, Robert Spira, and the proposer.

Editorial Note. The proposer indicates a slightly more general result, similarly proved: If X is any countable set, dense in (0, 1), and $\sum a_n$ is a divergent series of positive terms, then there is an enumeration $\{x_n\}$ of X, such that the series $S = \sum_{n=1}^{\infty} a_n \prod_{j=1}^{n} x_j$ diverges.

Renz relates that Professor Besicovitch gave one of his classes the similar problem: Let $\{r_i\}$ be an enumeration of the rationals in (0, ∞). $\{r_i\}$ can be so chosen that $\sum_{i=1}^{\infty} \prod_{j=1}^{i} r_i$ either converges or diverges.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Elements of Calculus and Analytic Geometry. By George B. Thomas. Addison-Wesley, Reading, Mass., 1959. x+580 pp. $7.50.

This book is a revised and reorganized version of the author's successful *Calculus and Analytic Geometry* (see this MONTHLY, vol. 61, 1954, pp. 435–437) intended for a year course rather than a three-semester one. The main topics omitted are partial differentiation, multiple integration, solid geometry and vectors, differential equations, and infinite series. The same level of rigor has been kept and the understandability has, if anything, been increased.

ROY DUBISCH
Fresno State College

Elements of the Theory of Functions and Functional Analysis, Volume 1. *Metric and Normed Spaces*. By A. N. Kolmogorov and S. V. Fomin. Translated by L. F. Boron. Graylock Press, Rochester, N. Y. 1957, 129 pp., $3.95.

The book being reviewed has been derived from courses at the Moscow State University. The book is addressed to those with a background which may seem strange to us. For, while it begins with elementary concepts of sets and cardinal numbers, it supposes (for proofs left to the reader) that the reader knows the Weierstrass approximation theorem and the limitations of the Riemann integral. On the whole, the book is elegantly, concisely, and yet understandably written. The topics covered include, in addition to those mentioned, the standard basic material on metric spaces, contraction mappings and their application to existence theorems of differential and integral equations, and continuous curves in metric spaces. The reader is also introduced to normed linear spaces, Banach spaces, and their conjugate spaces, to linear operators and the spectrum of an operator, and to the Fredholm theorems using operator equations. There is an addendum on generalized functions.

The treatment throughout is mathematically satisfying. At the same time, its importance to applications is noted. A considerable amount of material has been compressed into a small book by the omission of topics such as Hilbert space, the L_p spaces, and others which require Lebesgue integration. A large number of examples are included in spite of these limitations. For the most part, the terminology and notation is consistent with general use in this country. One difference is that the definition of a compact set does not require that the set be closed. In a few places, for example, Section 18, Theorem 7, a few more words would greatly simplify the reading. The translator has done a good job. The "in other words" following the definition of continuity is the only place noticed where a little roughness resulted in an inaccuracy. The publisher has attended to the matters of format, printing, figures and binding in an excellent fashion.

James C. Bradford
Abilene Christian College and
Chance-Vought Aircraft, Inc.

Modern Geometry. By C. F. Adler. McGraw-Hill, New York, 1958. xiv+215 pp. $6.00.

This text participates in the recent trend to reduce Euclidean Geometry to selected topics (Ceva-Menelaus, Harmonic, Inversion) to make room for notions from projective and non-Euclidean geometries. The material included may serve to whet the appetite of the beginner, but the treatment is inadequate to give the student confidence in any of the new notions. There are a few exercises and topical bibliographies.

Arthur Bernhart
University of Oklahoma

Introductory College Mathematics. By Thomas L. Wade. Wiley, New York, 1959. xiv+319 pp. $5.50.

This text is designed for a three-semester-hour precalculus course for freshmen who have mastered high school algebra and trigonometry. Essentially, it is a brief course in plane analytic geometry and elementary differentiation, developed and presented in the terminology and notation of sets and relations. The book is well organized and well written, with rigor suitable to the level of instruction being maintained throughout.

Approximately the first third (Chs. 1, 2) is devoted to basic theory in deductive reasoning, sets, relations and real numbers; nearly half (Chs. 3–6) is concerned with basic analytic geometry through conic sections, and including polar coordinates; and the remainder (Chs. 7, 8) deals with functions and limits, differentiation and tangents. Inequalities are stressed throughout.

All chapters have a generous number of appropriate exercises. There is an exceptionally large number of well-drawn and clearly labeled figures, but there is no discernible difference between the "heavily drawn" and "lightly drawn" lines of Figures 145, 146.

H. S. KALTENBORN
Memphis State University

Fundamentals of Freshman Mathematics. By C. B. Allendoerfer and C. O. Oakley. McGraw-Hill, New York, 1959. 475 pp. $6.50.

This reviewer is inclined to say that this book is a new version of the authors' *Principles of Mathematics* written for a mathematically more conservative audience—teachers of potential scientists or engineers, and their students. Comparing it with *Principles* one might say it is three steps backward and two steps forward. On the whole the result is a book better suited to its intended audience. It retains the flavor of modern mathematics while ruthlessly suppressing proofs and omitting parts of *Principles*.

Many of the chapters from *Principles* are reprinted with the addition of routine drill problems (Ch. 2 on the number system, 10 on algebraic functions, 11 on exponents and logarithms, and 14 on analytic geometry). Others are rewritten versions of chapters in *Principles*, notably Chapter 1 which incorporates much of the material on logic and methods of proof, Chapter 6 on sets (Boolean Algebra and switching circuits are omitted), Chapter 8 on inequalities which was an appendix to the chapter on fields in the earlier version, and Chapter 9 on functions and relations. There are two chapters on trigonometry. Chapter 12 is new and is an impatient bow to old-fashioned trigonometry. For example, it includes a section *The Law of Tangents* which consists of one paragraph which does not even state the law of tangents. Chapter 13 is a cut-and-paste version of the trigonometry chapter in *Principles*.

Chapters 3, 4 and 5 are essentially review and drill on manipulative skills of "intermediate algebra" (such things as factoring, fractions, exponents and radicals).

Chapter 7 is a treatment of the application of matrices to the solution of simultaneous systems of linear equations.

Chapters 15 and 16 on "intuitive calculus" are also new and are especially well written for their intended audience. Chapter 17 on hyperbolic functions is regrettably more confusing than helpful. Many new ideas are mentioned casually with little real attempt at explanation; for instance, infinite series and complex exponents.

The spacing in Chapter 2 was apparently handled by editors who do not read copy. Bold faced headings such as **Closure Law of Addition** are separated from the paragraphs which explain them, and not separated from the following paragraph which is concerned with a different topic.

One disappointing lack of revision was the passage on the axiom of mathematical induction, page 32. The induction axiom is stated, and a rule of thumb is given which purports to test whether a given statement meets the conditions of the axiom, but which actually says nothing about the set of numbers mentioned in the axiom. This should have been rewritten.

JEAN M. CALLOWAY
Carleton College

A Modern Introduction to College Mathematics. By I. H. Rose, Wiley, New York, 1959, 530 pp. $6.50.

According to the author, this text book is intended for students who have had one and one half years of high school algebra and one year of high school geometry. Its intent seems to be to give the young student an understanding of the basic mathematical concepts and to acquire a "feel" for what mathematics is. The first part deals with the elements of set theory, natural numbers, real numbers and a little elementary algebra. The second part is concerned with analytic geometry and has a fairly extensive treatment of trigonometry. The third part has a very little of the elements of calculus, probability and statistics. Throughout the text there is a good deal of historical material either briefly mentioned or extensively treated. This book suffers—in common with many "modern" introductory texts—from attempts at complete rigor on the basis of fuzzy intuitive notions. Thus at one place the author relies on a figure and the properties of similar triangles which are defined twenty-four pages later, while at another he disparages the given proof because it relies on a figure and gives a "more rigorous" proof which also relies on a figure. Angle is defined in a rather bizarre fashion while area, which is used in the proof of the Pythagorean theorem, is not defined. Despite many such faults the book achieves its main purpose; the young student will not fail to see the power and the beauty of mathematics. The book is highly readable, the type is clear, the figures well drawn and the one or two misprints are very minor.

M. S. KNEBELMAN
Washington State University

Intermediate Algebra for the College Student. By J. Vincent Robison. Van Nostrand, Princeton, N. J., 1959. vii+286 pp. $4.25.

The author claims some 15 years' experience teaching intermediate algebra to college students, and has found ways of explaining points which should be satisfactory to most students. Proofs are given (or stated to be beyond the scope of the text), and ideas are discussed. However, in 43 places there are boxes containing "GUIDE TO . . . ," and giving step-wise procedures for accomplishing 43 different objectives. Thus, it would be possible to use the text as a cook book by ignoring the explanations and concentrating on the guides. It might be more difficult to persuade students to ignore the guides and concentrate on the explanations. On pages 48–69, 10 different types of word problems are discussed, one type for each set of exercises. The first set of word problems of various (mixed) types appears on pages 114–116. There is no review set of problems on factoring—each set of exercises is concerned with only one type of factoring. Topics beyond quadratic equations which are discussed include logarithms, variation and proportion, progressions and binomial theorem.

The typography is pleasing, and few misprints were noted. The book contains some statements to which a purist might object: (p. 6) The statements of the commutative and associative laws in words may create the wrong impression. (p. 99) "Fractions cannot be added unless they have the same denominator." (p. 129) "The left member [of $3x-4y=12$] states that $3x-4y=12$." This is a better-than-average text for teaching manipulation.

B. H. Arnold
Oregon State College

Einführung in Theorie und Andwerdung der Laplace-Transformation. By Gustav Doetsch. Birkhäuser Verlag, Basel, 1958. 301 pp. SFr. 39.40. (About $9.25).

This book, with German text, contains the majority of the topics to be found in the author's earlier book on the same topic. The style of the present work is considerably more direct, however, even though the motivations remain obscure. There has been a definite effort toward appealing to the interest of engineers and physicists, as well as mathematicians, and the book is designed as a text, rather than a reference. The exposition is clear and concise, and the development of the material follows in easy steps.

On the whole, one obtains the impression of a rigorous treatment of a classical tool applied to classical problems. The Laplace transform is presented as an operator and as a transformation. Among the topics considered are differential equations, both ordinary and partial, integral equations and difference equations, asymptotic developments, and some Tauberian theorems.

For use as a text book the only prerequisite would be a first course in complex variables, and the level would be senior year, or first year graduate.

Gordon Latta
Stanford University

Freshman Mathematics. By Paul K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1959. 285 pp. $5.25.

The preface states "This book is intended primarily for prospective elementary teachers and nonscience majors in arts college; however, it can be used by those students who need a refresher course in algebra or who must study algebra as a background for work in the mathematics of finance." To the reviewer this is a rather ambitious program.

The author has succeeded in producing a book for students with little or no background in high school mathematics. Ten of the seventeen chapters deal with junior high school mathematics or elementary high school algebra. The other seven chapters are on such topics as trigonometry, straight line, simple annuities, statistics, permutations and combinations. The reviewer believes that this book would be a suitable one for a refresher course. The few new topics might be sufficient to hold the interest of this type of student.

The reviewer *does not* agree that this is a book for elementary teachers and nonscience majors in arts college. These people should be taught more of the concepts, rather than a collection of techniques and manipulations.

JOHN C. KNIPP
University of Pittsburgh

Integral Equations. By I. G. Petrovskii. Graylock Press, Rochester, N. Y., 1957. 97 pp. $2.95, cloth, $1.95, paper.

As the size of this book readily indicates, Petrovskii's aims in his discussion of integral equations are of a more modest nature than those published by his Russian colleagues during the last decade or so. His primary purpose is to prove the existence of the solutions to linear integral equations of the second kind under sufficiently general conditions. No attempt has been made to consider the numerous applications to which this theory has been put; for this reason, this fine work may be less attractive to the physicist, engineer, and technician than to the mathematician. With the exception of one section of the appendix, no knowledge of Lebesque integration theory is required. The material is presented with great care and in accordance with present standards of mathematical rigor.

In Chapter 1 the author formulates the Fredholm theorems from known theorems of linear algebraic equations (by means of an analogy between linear integral and linear algebraic equations), and then proceeds to prove these theorems for certain classes of equations. Since the number of independent variables plays no essential role here, all proofs are carried out for an arbitrary number of independent variables. The procedure used by Petrovskii parallels the one originated by E. Schmidt. Firstly, the theorems are verified for the class of integral equations with the so-called degenerate kernels—the Pincherle-Goursat kernels—followed in order by an extension to continuous kernels of sufficiently small absolute bound, almost degenerate kernels, and to uniformly continuous kernels. The chapter concludes with a consideration of kernels of the form

$K(P, Q)/PQ^{\alpha}$ where points P and Q belong to a closed and bounded domain, K is a continuous function of (P, Q), PQ is the distance between P and Q and $\alpha < n$ (dimension of the domain). Chapter 2 consists of a three page discussion of the Volterra integral equations, while the last chapter is devoted to the Hilbert-Schmidt theory of integral equations with symmetric kernels. Here all functions are assumed to be real-valued and square integrable in every bounded region where they are defined.

The material is presented in a very readable and systematic manner, for which a good share of the credit should go to the translators Kamel and Komm. In the opinion of the reviewer, this book (actually a series of lectures) is not suitable as a text for normal course work; however, the graduate student in mathematics, who is primarily interested in the theory rather than application of integral equations, will find this an excellent presentation.

D. TRIFAN
University of Arizona

Calculus with Analytic Geometry. By Angus E. Taylor. Prentice-Hall, Englewood Cliffs, N. J., 1959. xv+762 pp. $8.50.

This is an excellent text for use in a first course in analytic geometry and the calculus. It is successful in presenting a union of the two subjects without slighting either.

A feature which will please those who believe that a course in analytic geometry and the calculus should be more than a study of techniques and type problems is the group of "Review Questions and Problems" which appears at the ends of every two or three chapters. These groups consist of thought-provoking questions concerning fundamental concepts and definitions, theory, and additional problems involving topics in the preceding chapters. These questions will keep the student (and the instructor) from forgetting that this is a course in mathematics.

Such a usually passed-over concept as the completeness of the real number system is treated by means of a sufficiently rigorous for its purpose but elementary discussion of Dedekind sections.

Physics and engineering departments will like the early introduction of the derivative (p. 36) and the antiderivative (p. 73). They also will like the emphasis on the applications of the calculus to theoretical mechanics as well as the use of vector ideas and notation, including an introduction to vector algebra and calculus.

The author has been successful in presenting a substantial course in the subject matter in a digestible but not predigested form.

LLOYD L. LOWENSTEIN
Arizona State University

An Introduction to the Theory of Groups. By P. S. Alexandroff. (Translated by Hazel Perfect and G. M. Petersen from the German edition, a translation from the original Russian book.) Hafner, New York, 1959. 112 pp. $3.25.

The author states in the foreword that "Every pupil in a senior class of a grammar school who enjoys doing mathematics is capable of grasping the idea of a group if he is interested and industrious. And so this book has been written in the first place for the mathematically inclined pupils in the senior classes in the grammar school, but also for those who teach mathematics to the senior or to the advanced level." This is a clear and well-motivated book on elementary group theory. The author takes great pains in defining each new concept and illustrating its meaning. Quite naturally, there are many interesting examples of groups of symmetries taken from geometry. An unusual feature of the book is that the group operation is designated by "+." Topics covered go through difference groups.

R. E. JOHNSON
Smith College

Three Pearls of Number Theory. By A. Ya. Khinchin. (Translated by F. Bagemihl, H. Komm, W. Seidel.) Graylock Press, Rochester, N. Y., 1952. 64 pp. $2.00.

This 64-page booklet was written in 1945 in the form of a letter to a wounded soldier recuperating in a Russian hospital. Of the recipient, not much is known except that he listened to Khinchin's lectures for one year before the war. The author was, of course, one of the most eminent Russian mathematicians. (He died in 1957.) His principal field of interest was mathematical probability, but he also made significant contributions in metrical number theory, among other subjects, and the present booklet attests to his interest in arithmetical questions.

The three pearls of the title are three theorems the author selected as especially suited for study by a partially trained young man, both because of the elementary nature of their proofs and because they were all first proved by young men after strenuous efforts on the part of more experienced mathematicians.

The first is van der Waerden's theorem on arithmetic progressions: given any two positive integers k and l, there exists an integer $n=n(k, l)$ such that, if an arbitrary segment of length n of the sequence of integers is divided in any manner into k classes, then an arithmetic progression of length l appears in at least one of the classes. The proof given here is due to M. A. Lukomskaya.

For an arbitrary increasing sequence A of positive integers $a_1, a_2, \cdots$, let $A(n)$ be the number of the a_i not exceeding n. The quantity g.l.b. $A(n)/n$ is called the Schnirelmann density of the sequence A, and is designated by $d(A)$. If A and B are two such sequences, the sum $A+B$ is the increasing sequence of all integers which can be written in one of the forms a_i, a_i+b_j, b_i. The second pearl is the theorem that in all cases, $d(A+B) \geqq \min(1, d(A)+d(B))$. This was

conjectured by L. G. Schnirelmann and E. Landau in 1931, and after many partial results on this so-called "$\alpha+\beta$ hypothesis" (including a proof in the special case $A=B$ by Khinchin himself) it was finally proved in 1942 by H. B. Mann. The proof exposed here is the simplification by E. Artin and P. Scherk.

The third pearl is the elementary proof discovered by E. Linnik in 1942 of Hilbert's theorem on Waring's problem: to every positive integer n there corresponds an integer k such that every natural number m can be represented as a sum of k nth powers of nonnegative integers, $m=x_1^n+\cdots+x_k^n$. Hilbert proved this theorem analytically in 1909, and estimates were obtained later for the size of k as a function of n, by analytic methods, but Linnik succeeded in constructing a completely elementary proof, based on an idea of Schnirelmann.

Almost no formal background is required for reading this book, and the author has taken great pains to make both the details and the over-all structure of the proofs transparent. On the other hand, there are intrinsic difficulties in the problems discussed, and the reader must add his own efforts to those of the author. The final paragraph of the book is worth quoting: "This proof [Linnik's], so exquisitely elementary, will undoubtedly seem very complicated to you. But it will take you only two or three weeks' work with pencil and paper to understand and digest it completely. It is by conquering difficulties of just this sort, that the mathematician grows and develops."

W. J. LeVeque
University of Michigan

Analytical Elements of Mechanics, Vol. I. By Thomas R. Kane. Academic Press, New York, 1959. xv+250 pp. $4.75.

The book is the first part of an introduction to classical mechanics, dealing only with the concepts of statics. An attempt is made to present the subject matters on a general basis and encourage the student to deduce the more special cases from this. Following a detailed exposition of elementary vector algebra, vector methods are used throughout the text. The theoretical aspects are more rigorously developed than usually found, including proofs and derivations in considerable detail. The emphasis then centers on transmitting a thorough understanding of the principles rather than proficiency in routine problem solving. Complemented by classroom communication, the book should prove a valuable help for the teaching of elementary mechanics.

H. W. Bergmann
University of Oklahoma

Engineering Statistics. By A. H. Bowker and G. J. Lieberman. Prentice-Hall, Englewood Cliffs, N. J. 1959. 585 pp. $8.25.

This book is a welcome addition to the material available for teaching statistics to physical scientists. It presents the topics in a manner which is more acceptable to statisticians than some other recent publications have done. The authors bring the statistical concepts forward rather than attempting to skirt

them, as so many books do that are aimed at "nonstatisticians." The book, however, is meant as a first course in statistics.

In certain places, the authors have presented the theory of a topic in a section immediately following the general discussion. This should allow more flexibility in the level of the class presentation.

Each chapter has ample problems. Most problems emphasize applications of theory to numerical examples, but a few theoretical problems are included.

The first four chapters cover the introductory concepts of random variables and distribution theory, including the normal distribution and its related distributions. The operating characteristic curve is introduced in chapter five. The next chapters deal with tests of hypotheses, estimation, fitting of straight lines and the analysis of variance. The chapter on analysis of variance includes discussion of fixed and random models for the one- and two-way classification. The *OC* curve and the computational procedures for these models are presented in detail. The book does not attempt to present additional material on experimental designs; latin squares, etc., do not appear, nor do the recent developments of response surface exploration.

Although this reviewer does not like the heuristic argument presented for the analysis of variance (it is easy to dislike such arguments) and feels that more should have been said about randomization (one could still get the impression that the validity of experimental results depend upon a model), I feel that this is a good book and well worth considering as a text for a course in engineering statistics.

W. H. WILLIAMS
McMaster University

Statistics: An Introduction. By D. A. S. Fraser. Wiley, New York, 1958. ix +398 pp. $6.75.

The preface states: "This book was prepared as a text for a two-semester course in mathematical statistics for those with the background of a full year of calculus." Such students could handle comfortably the first 35 pages which present an excellent introduction to probability. But the demands upon mathematical maturity become progressively more severe. On page 38 the reader is called upon to understand the distinction between a countable set of values and a continuum. On page 135 (a chapter on vectors and matrices comes later) during an incidental discussion of degrees of freedom one reads: "Geometrically, if we represent a sequence as a vector in N space, the degree of freedom is the dimension of the space the vector is free to move in." On page 365 we are told: "The theory of Galois fields can be used to construct orthogonal Latin squares whenever the number of levels is a prime or a power of a prime." The treatment of statistical inference, based largely on the work of R. A. Fisher, stresses analysis of variance, linear models, orthogonal transformations, pivotal solutions, and pivotal quantities. Each chapter ends with a bibliography and a wide assortment of problems many of which introduce essential background material before

posing a numerical problem. Problem 6, page 127, introduces coding; problem 22, page 130, contains the only discussion of quality control in the book. By sprinkling bits of theory and partially defined terms among the problems the author has included a surprising amount of material in a book of moderate size.

Mathematics majors should find the book stimulating and rewarding but in the opinion of the reviewer most engineering students would find it frustrating.

V. V. LATSHAW
Lehigh University

The Growth of Mathematical Ideas, Grades K–12. (Twenty-fourth Yearbook, The National Council of Teachers of Mathematics) Phillip Jones, Editor. Washington, D. C., 1959. x+507 pp. $5.00.

This yearbook has three basic aims: to highlight "the essential elements of those basic mathematical understandings which should be continually developed and extended throughout the entire mathematics curriculum, grades K–12 and beyond," "to define and illustrate some classroom procedures and their psychological bases," and "to assist teachers and supervisors to extend and apply the ideas of the book to their own situations."

The "basic mathematical understandings" are included in Chapters 2–8 inclusive. The chapter titles are in order: Number and Operation, Relations and Functions, Proof, Measurement and Approximation, Probability, Statistics, and Language and Symbolism in Mathematics. These topics were selected by the Yearbook Committee as "the most basic mathematical themes which should be central to the entirety of a modern mathematics curriculum." The Committee recognized that there are other ideas and concepts which belong in the mathematics programs of the schools, but lack of space did not permit including them.

The topics and concepts are discussed as continuing themes and hence each chapter is designed so that teachers from the elementary school through the high school will find useful material for their particular level of teaching. It seems to this reviewer that the authors have accomplished this purpose well and that this feature of the yearbook is probably its more significant contribution.

Only three chapters are devoted to the second and third aims of the yearbook. Their titles are: Mathematical Modes of Thought, Implications of the Psychology of Learning, and Promoting Continuous Growth of Mathematical Concepts. Since these chapters deal with procedures as contrasted to concepts and ideas, it seems probable that this part of the book will be less useful to teachers than the first part. This point of view is based on the assumption that the most urgent need of present day teachers of mathematics at all levels of the schools is more knowledge of mathematics, especially of the more modern concepts.

As stated in the preface it is inevitable that this yearbook "should be compared with and related to the twenty-third yearbook, *Insights Into Modern*

Mathematics." The yearbooks did not have the same purpose, but they should supplement each other very well. The twenty-third yearbook stressed theory and modern concepts, while this yearbook stresses continuing themes of basic mathematical ideas so that these may be recognized by pupils and teachers as a part of growth and development of mathematical understandings.

JAMES H. ZANT
Oklahoma State University

Japanese Chess, The Game of Shogi. By E. Ohara. Bridgeway (Tuttle) Press, 1958. 182 pp. $2.75.

Many western mathematicians have been fascinated by the game of Go but Japan's other great strategic game, Shogi, seems almost unknown here. This little manual should help remedy this imbalance. Shogi, unlike Go, is clearly a form of Chess and similarities to western chess as well as basic differences are clearly pointed out. Chess players particularly should find Shogi, as here presented, both easy to learn and interesting to play.

R. C. SIMPSON
St. Lawrence University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

New Mexico State University: Dean Earl Walden, Head of the Mathematics Department and Dean of the Graduate School, will become full-time Dean of the Graduate School on July 1. Professor R. B. Crouch will become Head of the Mathematics Department.

Oregon State College: Associate Professors H. E. Goheen and A. R. Poole have been promoted to Professors; Mr. Helmut Groemer has been promoted to Assistant Professor; Associate Professor W. M. Stone is on leave at the Boeing Airplane Company, Seattle, Washington.

Dr. E. B. Allen, Head of the Department of Mathematics at Rensselaer Polytechnic Institute, has been appointed Dean of the Graduate School, to succeed Dr. C. O. Dohrenwend.

Professor Cristobal de Losada y Puga, Catholic University of Peru, has been incorporated as "miembro de numero" in the Academia Peruana correspondent of the Real Academia Espanola. He has also been elected Director of Publications in the Academia Nacional de Ciencias Exactas, Fisicas y Naturales de Lima.

Miss Leah Fine, Lockheed Aircraft Corporation, Marietta, Georgia, has accepted a position as Programmer-Analyst with the System Development Corporation, Paramus, New Jersey.

Professor R. M. Foster, Polytechnic Institute of Brooklyn, was awarded the honorary degree of Doctor of Science by Fairleigh Dickinson University on January 15, 1960.

Mr. J. W. Fothergill, Jr., Aerojet General Corporation, Nimbus, California, has accepted a position as Senior Programmer with the Thiokol Chemical Corporation, Brigham City, Utah.

Associate Professor Herta T. Freitag, Hollins College, has been promoted to Professor.

Dr. H. D. Friedman, General Electric Company, DeWitt, New York, has accepted a position as Mathematician with Technical Operations, Inc., Burlington, Massachusetts.

Mr. W. G. Hazlett, System Development Corporation, Santa Monica, California, has accepted a position as Scientific Computer Analyst with Temco Aircraft, Garland, Texas.

Associate Professor I. N. Herstein, Cornell University, has been promoted to Professor.

Professor M. W. Keller, Purdue University, has been promoted to Assistant Head of the Department of Mathematics and Statistics in charge of Extension and graduate teacher programs.

Associate Professor R. E. Lee, The University of Missouri School of Mines and Metallurgy, has been promoted to Professor.

Mr. Parviz Morewedge, Douglas Aircraft, Santa Monica, California, has accepted a position as Logical Designer with Litton Industries, Beverly Hills, California.

Mr. M. J. Schrader, Army, Redstone Arsenal, Alabama, has accepted a position as Mathematician with McDonnell Aircraft Corporation, St. Louis, Missouri.

Mr. D. B. Singer, Armour Research Foundation, Illinois, has accepted a position as Research Specialist with the Missiles and Space Division of Lockheed Aircraft Corporation, Van Nuys, California.

Mrs. Dorothy P. Smith, Kansas State College, has been appointed Instructor at New Mexico Highlands University.

Mr. Arthur Sullivan, Army Research and Development Laboratory, Fort Monmouth, New Jersey, has accepted a position as Senior Microwave Engineer with the Canoga Division of the Underwood Corporation, Van Nuys, California.

Mr. D. H. Taranto, International Business Machines Corporation, Flushing, New York, has been appointed Instructor at Carleton College, Northfield, Minnesota.

Mr. Neal Bradley, Georgia Institute of Technology, died December 20, 1959. He was a member of the Association for three years.

Mr. G. B. Glover, Rutgers College of South Jersey, died December 3, 1959. He was a member of the Association for five years.

Mr. C. E. Harrington, W. H. Smith, Inc., Springville, New York, died January 7, 1960. He was a member of the Association for thirty-eight years.

Professor T. W. Moore, U. S. Naval Academy, died December 17, 1959. He was a member of the Association for thirty-five years.

Mr. V. E. Roberts, University of Alaska, died January 2, 1960. He was a member of the Association for seven years.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1960:

University of Buffalo, June 27 to August 6: Professor Schneckenburger, higher algebra, functions of a complex variable. NSF Institute: Professor Montague, introduction to modern algebra; Professor Olson, probability and statistics, fundamental concepts of calculus; Mr. Montzingo, demonstration class and workshop.

University of California, Berkeley, June 20 to July 29: Professor Namioka, foundations of analysis; Professor Dubisch, foundations of mathematics. August 1 to September 9: Professor Godemont, introduction to the theory of functions of a complex variable; Professor Freudenthal, Lie groups.

University of Colorado, June 20 to August 26: Dean Crawford (Mount Allison University) and Professor Zirakzadeh, higher algebra and analysis for teachers; Professor Halfar (University of Nebraska), foundations of analysis and modern geometry; Professor Nering (University of Arizona) theory of numbers and probability theory. June 20 to July 22: Professor Briggs, infinite processes. July 25 to August 26: Professor Emeritus Kempner, history of mathematics.

Columbia University, Teachers College, July 5 to August 12: Dr. Allegri, teaching of arithmetic, history of mathematics; Professor Fehr, supervision and problems in teaching secondary school mathematics; Professor Felix and Dr. Steiner, elements of modern mathematics; Professor Rosskopf, logic in elementary mathematics; Mr. Rourke, introduction to analysis, professionalized subject matter for senior high school mathematics; Dr. Zlot, teaching of algebra, elements of algebra.

Cornell University, June 27 to August 6: Professor Robison, determinants and matrices; Mrs. Hertzig, advanced calculus.

De Paul University, (Day): Professor DeCicco, theory of numbers, solid analytic geometry; Dr. O'Neil, Taylor series. (Evening): Dr. Yao, theory of rings.

University of Florida, June 21 to August 13: Professor Blake, introduction to mathematical thought; Professor South, mathematical statistics; Professor Lang and Professor Patterson, advanced mathematics for engineers and physicists; Professor Hadlock, advanced topics in calculus; Professor Moore, introduction to topology; Professor Pirenian, vector analysis; Professor Hutcherson, synthetic projective geometry; Professor Phipps, foundation of geometry, history of elementary mathematics; Professor Cowan, partial differential equations.

Indiana University, June 17 to August 12: Professor Wolfe, elementary statistics, projective geometry; Professor MacKenzie, introduction to modern algebra, topics for mathematics teachers; Staff, history of mathematics, introduction to probability theory, mathematical reading and research.

University of Kansas, June 13 to August 6: Professor Himmelberg, elementary number theory, topological groups; Professor Kruse, applied math I, applied math II; Mr. Pulley, programming for a digital computer.

Kent State University, June 20 to July 23: Selected topics for classroom teachers; solid analytic geometry; introduction to modern algebra; history of mathematics. July 25 to August 27: Vector analysis; mathematical statistics I; the teaching of junior high school mathematics.

Michigan State University, June 21 to September 2: Professor Herzog, sets and abstract spaces, boundary behavior problems in complex variables; Professors Larcher and Stewart, matrices and groups; Professor J. H. Powell, theory of polynomials; Professor Reid, boundary value problems; Professors Reid and Sheedy, differential equations; Professor Stelson, theory of equations. June 21 to July 28: Professor Campbell, theory of numbers; Professor Grove, projective geometry; Professor Hall, advanced calculus I and II; Professor Hill, concepts in algebra; Professor Larcher, complex variables I; Professor Norman, elements of complex variables, vector analysis; Professor Oehmke, group theory; Professor J. E. Powell, concepts in calculus; Professor J. H. Powell, differential equations. July 29 to September 2: Professor Hocking, advanced calculus II, topology III; Professor Kelly, set theory and axiomatic systems, college geometry; Professor Nordhaus, concepts in geometry, advanced calculus III; Professor J. H. Powell, introduction to matrices; Professor Stewart, rings and algebras; Professor Wells, partial differential equations, complex variables II.

University of Minnesota, College of Science, Literature, and the Arts, June 13 to July 16: Professor Engeler, elementary set theory; Professor Gil de Lamadrid, differential equations, topics in topology; Dr. Joichi, advanced algebraic theory I, solid analytic geometry. July 18 to August 20: Professor Harper, advanced algebraic theory II; Professor Hatfield, probability; Dr. Miracle, intermediate differential equations, theory of numbers. An institute for high school teachers of mathematics will be conducted from June 13 to August 6.

University of Nebraska, July 8 to July 31: Professor Abel, topics in geometry; Professor Basoco, Laplace transforms; Professor Harris, topics in probability and statistics; Professor Miller, topics in algebra.

University of North Carolina, June 9 to July 19: Professor Cameron, fundamental concepts of mathematics with special reference to algebra; Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Professor Jones, advanced calculus I; Professor Linker, differential equations. Professor Mac Nerney, seminar in analysis. July 20 to August 27; Professor Carr, introduction to modern algebra I; Professor Hoyle, advanced calculus II; Professor Lasley, analytic geometry from a higher standpoint; Professor Mackie, theory of equations; Professor Pettis, topics in analysis; Professor Wells, seminar in analysis.

Northwestern University, June 21 to August 13 and June 21 to July 30: Engineering mathematics 1-T; numerical methods; digital computer programming 1 and 2; statistics for teachers; probability; algebra for teachers; geometry for teachers; the history of mathematics I; foundations of calculus for teachers; complex variables for applications; topics in modern mathematics for teachers; modern algebra; introduction to topology.

University of Oklahoma, June 6 to August 6: Professor Bernhart, college geometry; Professor Brixey, principles of mathematical statistics; Professor LaFon, vector analysis; Professor Levy, foundations of analysis; Professor Nicol, theory of numbers; Professor Pan, elementary differential equations; Professor Springer, analytic projective geometry, ordinary and partial differential equations.

University of Pennsylvania, June 27 to August 5: Professor Schub, introduction to real analysis, matrix theory and numerical calculations; Professor Abian, numerical methods and differential equations; Professor Gottschalk, theory of sets.

Syracuse University, July 5 to August 12: Professor Gould, higher mathematics for engineers; Professor Cargo, foundations of geometry; Professor Hemmingsen, topology; Professor Exner, logic for secondary teachers; Professor Pollak, analysis and application I (differential equations).

University of Washington, June 20 to July 20, July 21 to August 19: Differential equations; advanced calculus, linear algebra; introduction to modern algebra; fundamental concepts of analysis; topics in applied analysis; foundations of geometry; projective geometry; complex variable.

West Virginia University, June 6 to July 15: Professor Vest, differential equations, theory of probability; Professor Bragg, topology, history of mathematics; Professor Cochran, advanced calculus. July 18 to August 26: Professor Cunningham, foundations of algebra and analysis, topology; Professor Peters, advanced calculus, algebraic surfaces; Mr. Lowenberg, modern geometry for teachers.

University of Wisconsin, June 20 to August 13: Professor Kleene, modern algebra, differential geometry; Professor Bicknell, introduction to mathematical statistics; Professor Van Engen, survey in the foundations of arithmetic; Professor Spragens (University of Louisville), survey of the foundations of algebra; Professor Fox (Princeton University), theory of numbers, advanced topics in algebraic topology; Mr. Howard, applied differential equations; Staff, topics in geometry, applied mathematical analysis, introduction to the theory of probability, convex figures and inequalities, theory and operation of computing machines, modern views of mathematics, topics in the history of mathematics, foundations of mathematics.

University of Wyoming, June 13 to July 15: Professor Walsh, introduction to higher algebra, advanced calculus; Professor W. N. Smith, ordinary differential equations; Miss Guerin, college geometry; Professor Varineau, fundamental concepts of mathematics, numerical analysis. July 18 to August 19: Professor DeZur, theory of matrices and determinants; Professor S. R. Smith, vector analysis; Professor Schwid, partial differential equations, Fourier series and boundary value problems; Professor Barr, seminar in geometry.

INTERNATIONAL CONFERENCE

The Mathematics Research Center, United States Army, will conduct at the University of Wisconsin, June 7–15, 1960, an INTERNATIONAL CONFERENCE ON

PARTIAL DIFFERENTIAL EQUATIONS AND CONTINUUM MECHANICS. A considerable number of invited addresses will be given and there will be sessions for contributed papers. The invited addresses and abstracts of the contributed papers will be published as the proceedings of the Conference. Further information may be obtained from Professor R. E. Langer, Mathematics Research Center, U. S. Army, University of Wisconsin, Madison 6, Wisconsin.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1959

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1959. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the Buffalo office of the Association.

There was a surplus of $16,499 in the Current Fund of the Association for 1959, of which $15,000 has been transferred to the General Fund. The Treasurer is pleased to report gifts from two members of the Association amounting to $210, which have also been added to the General Fund, which serves as an endowment fund for the Association.

ASSETS OF THE ASSOCIATION	JANUARY 1, 1959	DECEMBER 31, 1959
M & T Trust Co., Buffalo	$ 19,687.37	$ 25,081.05
Savings Accounts	108,363.39	111,699.22
Securities	169,620.23	275,542.00
	$297,670.99	$412,322.27
FUNDS OF THE ASSOCIATION		
Current Fund	$ 1,027.14	$ 2,526.07
Carus Fund	26,706.76	28,798.31
Chace Fund	8,760.58	9,545.72
Houck Fund	13,832.45	12,028.00
Chauvenet Fund	1,620.95	1,642.31
Dunkel Fund	19,451.65	19,744.78
General Fund	51,428.73	64,265.57
	$122,918.26	$138,550.76
Visiting Lecturers Fund	38,285.38	78,423.03
Fund for Committee on Undergraduate Program	88,691.79	54,923.28
Fund for Committee on High School Contests	427.34	Deficit 77.33
Fund for Survey of Non-Teaching Mathematical Employment	1,218.43	10,383.55
Secondary School Lecturers Fund	16,054.92	73,883.15
Fund for Committee on Production of Films	27,827.45	15,437.08
Fund for Washington Office	—	34,154.39
Fund for Survey of European Mathematical Education	—	6,644.36
	$297,670.99	$412,322.27

THE FORTY-THIRD ANNUAL MEETING OF THE ASSOCIATION

The forty-third annual meeting of the Mathematical Association of America was held at the Conrad Hilton Hotel, Chicago, Illinois, from Thursday to Saturday, January 28 to 30, 1960, in conjunction with meetings of the American Mathematical Society, the National Council of Teachers of Mathematics, and the Society for Industrial and Applied Mathematics. The Saturday sessions of the Association were joint sessions with the National Council of Teachers of Mathematics. There were registered 1322 persons, including 816 members of the Association.

Sessions of the Association were held on Thursday morning and on Friday morning in the Boulevard Room of the Conrad Hilton Hotel and on Saturday morning and afternoon in the Eighth Street Theater. Professor P. T. Bateman presided at the lectures on Friday morning. President C. B. Allendoerfer presided at the Annual Business Meeting and the general session on Saturday morning. The Program Committee for the meeting consisted of P. T. Bateman, Chairman; H. Flanders, P. S. Jones, and M. Kline.

FIRST SESSION OF THE ASSOCIATION

"Exterior Differential Forms in the Undergraduate Program." Moderator: Harley Flanders, University of California, Berkeley. "Introductory Remarks," by Carl B. Allendoerfer, University of Washington; "Exterior Differential Calculus," by Lawrence Markus, University of Minnesota; "Applications to Geometry," by Albert Nijenhuis, University of Washington; "Concluding Remarks," by H. Flanders, University of California, Berkeley.

SECOND SESSION OF THE ASSOCIATION

"The Four and Eight Square Problem and Division Algebras," by Charles W. Curtis, University of Wisconsin.

Annual Business Meeting and Award of the Chauvenet Prize.

"Entropy in Ergodic Theory," by Paul R. Halmos, University of Chicago.

JOINT MEETING OF MAA AND NCTM

General Session

"Short and Long Range Improvement of the School Mathematics Curriculum," by Paul C. Rosenbloom, University of Minnesota; "Better Curriculum or Better Pedagogy?" by Morris Kline, New York University.

Session on Advanced Placement

Moderator: Harold P. Fawcett, Ohio State University. "Collegiate Views and Experiences," by E. P. Vance, Oberlin College and W. T. Scott, Northwestern University; "High School Views and Experiences," by Edwin C. Douglas, The Taft School, Watertown, Connecticut, and Henry Swain, New Trier Township High School, Winnetka, Illinois.

Session on the Nature and Role of Geometry in High School

Moderator: Max Beberman, University of Illinois. "Collegiate Views," by Kenneth B. Leisenring, University of Michigan, and Robert R. Christian, University of British Columbia and University of Illinois; "High School Views," Henry W. Syer, Kent School, Kent, Connecticut, and Roderick McLennon, Arlington Heights High School, Arlington Heights, Illinois.

Session on Calculus in High School

Moderator: Phillip S. Jones, University of Michigan. "Collegiate Views," by Albert A. Blank, New York University, and J. H. Neelley, Carnegie Institute of Technology;

"High School Views," by W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts, and Hubert Davis, Cranbrook School, Bloomfield Hills, Michigan.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday evening in Private Dining Room 18 of the Conrad Hilton Hotel with twenty-six members present. Among the more important items of business transacted were the following:

The Board approved the appointment by President Allendoerfer of the following Nominating Committee for 1960: F. A. Ficken, Chairman; E. P. Vance, and Paul B. Johnson.

The Board approved the following schedule of future meetings: Michigan State University, August 29—September 1, 1960; Willard Hotel, Washington, D. C., January 25–27, 1961; Oklahoma State University, August 28–31, 1961; Kansas City, Missouri, January 1962; University of Colorado, August 26–29, 1963; University of Michigan, August, 1964.

The Board voted that the list price of all Carus Monographs be set at $2 per copy for members and $4 per copy for nonmembers for all orders received after April 1, 1960.

The Board also gave its approval to the recommendation of the Committee on Sections whereby each section will receive $60 per year plus an additional $10 per year for each 100 members or fraction thereof in excess of 200.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, January 29, 1960 in the Boulevard Room of the Conrad Hilton Hotel, Chicago, Illinois. President C. B. Allendoerfer presided.

The Secretary announced the results of the balloting for officers in which 1886 votes were cast: Dr. A. S. Householder, Oak Ridge National Laboratory, was elected First Vice-President for the two-year term 1960–1961, and Professors R. C. Buck, University of Wisconsin, and J. G. Kemeny, Dartmouth College were elected Governors for the three-year term 1960–1962.

The membership of the Association was 9113 as of January 26, 1960, a net gain of 480 since August 17, 1959.

Professor G. B. Price reported for the Committee on the Undergraduate Program in Mathematics.

AWARD OF THE CHAUVENET PRIZE

The 1960 Chauvenet Prize was awarded to Professor Cornelius Lanczos of the Dublin Institute for Advanced Studies for his paper "Linear Systems in Self-Adjoint Form," published in this MONTHLY, vol. 65 (1958), pp. 665–679.

The 1960 Chauvenet Prize is awarded for a noteworthy expository paper published in English during the three-year period 1956–1958 by a member of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. This is the twelfth award of the Chauvenet Prize since its institution by the Association in 1925.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Wednesday, January 27, through Friday, January 29. The thirty-third Josiah Willard Gibbs Lecture, entitled "Quantum Field Theory," was given by Professor Julian Schwinger of Harvard University. Invited addresses were delivered by Professor A. S. Besicovitch of Cambridge University and the University of Pennsylvania, and Professor Paul Rosenbloom of the University of Minnesota.

The Society for Industrial and Applied Mathematics met on Friday, January 29, at which time Dr. David Slepian and Dr. H. O. Pollak of the Bell Telephone Laboratories, Inc., gave invited addresses.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: L. R. Wilcox, Chairman; H. L. Alder, H. J. Curtis, E. H. C. Hildebrandt, G. L. Walker, J. W. T. Youngs.

Registration headquarters was located in the third floor corridor of the Conrad Hilton Hotel. On Saturday only, registration took place in the lobby of the Eighth Street Theater. Sleeping accommodations were available in the Conrad Hilton Hotel and in other hotels. Meals were available in the Hotel and at numerous nearby restaurants and cafeterias. The Mathematical Sciences Employment Register was located in Parlor Dining Room 2 and textbook exhibits were located on the fourth floor of the Conrad Hilton Hotel.

Since the opportunities for entertainment and recreation throughout Chicago in general, and the Conrad Hilton Hotel in particular, are so extensive, it was decided not to hold an annual banquet for this occasion.

HENRY L. ALDER, *Secretary*

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The fourth annual meeting of the New Jersey Section of the Mathematical Association of America was held at Princeton University, Princeton, New Jersey, on November 7, 1959. Dr. H. O. Pollak, Chairman of the Program Committee, presided at the morning session and the business meeting. Professor A. W. Tucker of Princeton University presided at the afternoon session. About 115 persons were present, including 81 members of the Association.

The following officers were elected: Chairman, Dr. H. O. Pollak, Bell Laboratories, Murray Hill, New Jersey; Secretary-Treasurer, Professor I. L. Battin, Trenton State College; Member of the Executive Committee, Dr. G. Y. Cherlin, Mutual Benefit Life Insurance Company, Newark, New Jersey. Continuing members of the Executive Committee are Dr. J. D. Daugherty and Professor E. P. Starke.

Dr. Cherlin reported for the Contest Committee that 4,462 students from 133 schools had participated in the 1959 contest.

The following papers were presented, all by invitation:

1. *Atoms, trees, and algebraic numbers*, by Professor R. C. Buck, University of Wisconsin.

An observer stands at the origin in a forest of trees located at the lattice points (k, m). The radius of each tree is $F(r)$ where $r=(k^2+m^2)^{1/2}$. In this context, many of the classic problems in Diophantine approximation are embraced by the question: How far can one expect to see into the forest? Various solutions to this, ranging from Khintchine's theorem to Roth's theorem, were discussed, and results extended to the three-dimensional case.

2. *The transportation problem*, by Professor Murray Gerstenhaber, University of Pennsylvania.

In the transportation problem, a homogeneous commodity is available at m "producers" in amounts $a_1, \cdots, a_m$ and required at n "consumers" in amounts $b_1, \cdots, b_n$ where $\sum a_i = \sum b_j$. The cost of transporting a unit quantity from the ith producer to the jth consumer is given, and the problem is to determine the amount that should be delivered from the ith producer to the jth consumer to minimize the total cost of transportation. By transforming the problem into a model of a competitive economic situation, it has been possible to program UNIVAC I to solve problems with $m \leqq 180$, $n \leqq 2100$. With more recent machines, values of m and n up to 40,000 are both possible and feasible.

3. *The work of the School Mathematics Study Group*, by Professor E. G. Begle, Yale University.

The objectives and accomplishments of SMSG were reviewed. Teaching materials which the Group has developed for each of grades seven through twelve were described in detail. Attention was called to the work done in preparing reference materials for teachers, and special materials for gifted students. The combination of university faculty members, secondary school teachers, and mathematicians from business and industry in the writing groups has proven to be very productive.

I. L. BATTIN, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The fifth annual meeting of the Northeastern Section of the Mathematical Association of America was held at Boston College, Chestnut Hill, Massachusetts, on November 28, 1959. Professor N. H. McCoy, Chairman of the Section, presided at the morning and afternoon sessions. There were 115 persons registered for the meeting, including 95 members of the Association.

Officers chosen for 1959–60 were Professor J. G. Kemeny, Dartmouth College, Chairman; Professor D. E. Christie, Bowdoin College, Vice-Chairman; Mr. R. S. Pieters, Phillips Academy, Secretary-Treasurer. Professor R. A. Rosenbaum of Wesleyan University was appointed liaison officer to deal with mathematical matters within the state of Connecticut. W. E. Ferguson of Newton (Mass.) High School gave a report of the secondary school visiting lectureship program.

The executive committee of the Section proposed a motion that a one-dollar registration fee be charged to all members attending the annual meeting of the section, for the year 1960. The motion was carried unanimously.

By invitation of the Executive Committee of the Section, the following hour talks were presented:

1. *On some problems of group theory*, by Professor Richard Brauer, Harvard University.

Some unsolved problems in the theory of groups of finite order are discussed. These deal with the finite collineation groups of a given degree, relations between the representations of the group by linear transformations and abstract properties of the group, the structure of group algebras, certain types of generalizations of the group concept, the simple finite groups, and related topics.

2. *The mean value property of harmonic functions*, by Professor Hans Rademacher, University of Pennsylvania. (By invitation).

3. *Some aspects of the theory of knots*, by Professor R. H. Crowell, Dartmouth College, introduced by the Secretary.

For any tame knot K in the 3-sphere S^3, there exists a tame orientable surface in S^3 whose boundary is K. The *genus* of K is the minimum of the genera of such surfaces. The following is due to H. Schubert (S.-B. Heidelberger Akad. Wiss. Math., Nat. Kl., 1949, pp. 57–104). For any two oriented tame knot types, an associative commutative product is defined, and the trivial type is an identity. The genus of a product is the sum of the genera of the factors. The existence of prime factorization follows. Prime factorization is unique and, since the identity is the only unit, the cancellation law holds.

4. *Decision making*, by Professor Howard Raiffa, Harvard University.

R. S. PIETERS, *Secretary*

CARUS MONOGRAPHS

In April, Carus Monograph No. 13 will be published. It is entitled "A Primer of Real Functions." The author is Professor Ralph P. Boas, Jr., of Northwestern University. A brief description of the Monograph is as follows:

This Monograph develops parts of the theory of sets and real functions for the sake of their intrinsic interest. The subject, which originally grew out of detailed examinations

of the concepts of continuity, differentiability, and the like, startled the mathematicians of the nineteenth century with nowhere-differentiable continuous functions and space-filling curves. Although it is now more usually studied as background material for complex analysis or for the theory of integration, it continues to produce surprises even today. The first part of the Monograph presents selected topics from the theory of sets in metric spaces and illustrates the power of the theory in applications to analysis. The second part applies the tools developed in the first part to a more systematic study of increasingly specialized classes of functions: continuous, differentiable, monotonic, convex and infinitely differentiable. Some of the results are old and familiar, at least by reputation; some are old and undeservedly forgotten; and some are quite recent.

Because of increases in the cost of printing and distribution, the Board of Governors has voted that the list price of all Carus Monographs will be set at $2 per copy for members and $4 per copy for non-members, for all orders received for Carus Monographs 1–4, 6–12, after April 1, 1960, and for all orders for Carus Monograph No. 13. Members should send orders to the Buffalo office of the Association.

CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA, Earlham College, Richmond, May 7, 1960.

IOWA

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, May 7, 1960.

METROPOLITAN NEW YORK

MICHIGAN

MINNESOTA, St. Olaf College, Northfield, May 7, 1960.

MISSOURI

NEBRASKA

NEW JERSEY

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, University of Oklahoma, Spring, 1960.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6-7, 1960.

SOUTHEASTERN

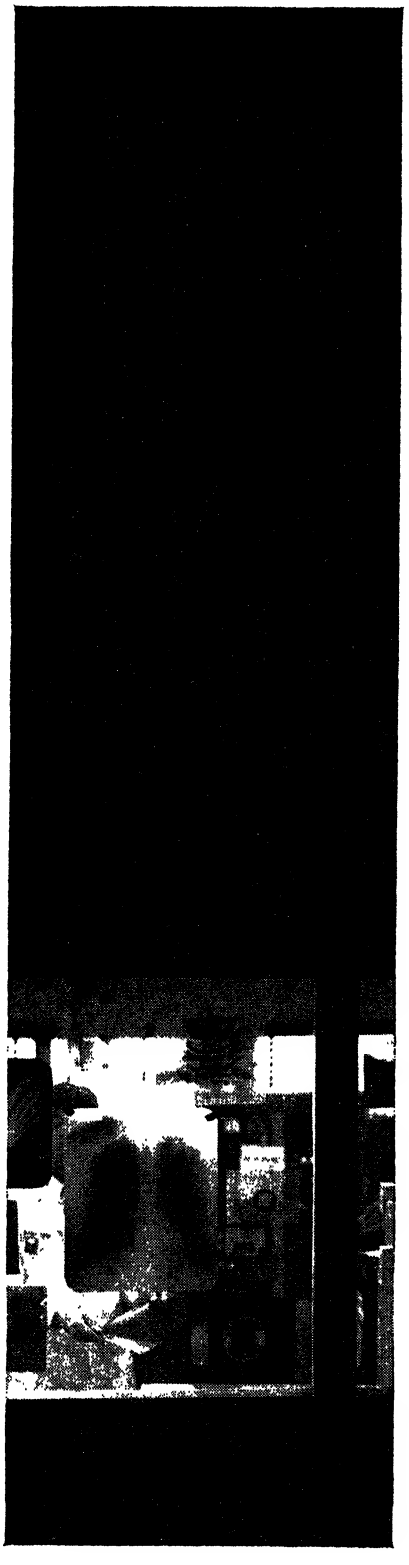
SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.



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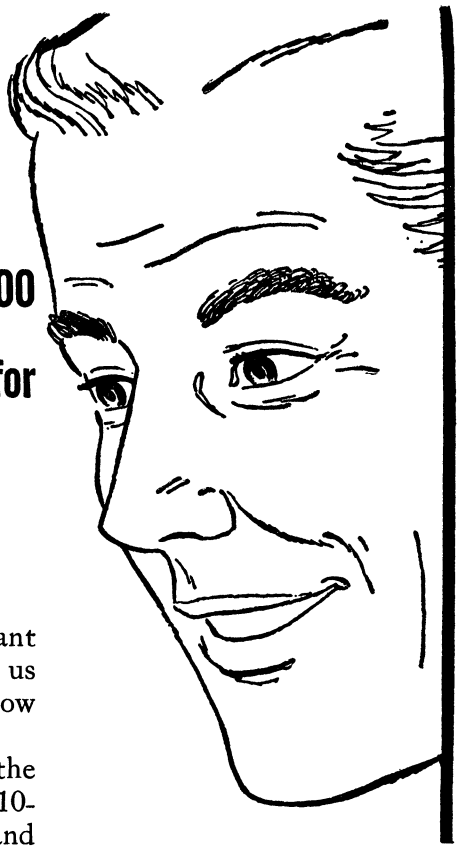
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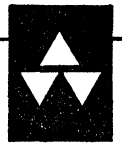
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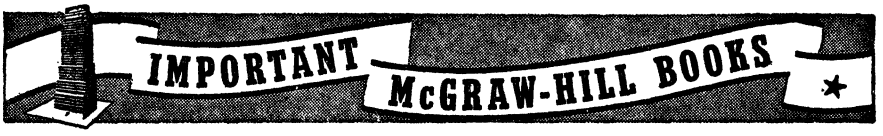
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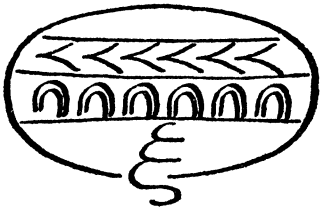
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PASCAL AND THE INVENTION OF PROBABILITY THEORY

OYSTEIN ORE, Yale University*

Most textbooks on probability feel obliged to include a brief account of the history of the theory. Their descriptions of this process of initiation usually run somewhat in the following vein: "In the year 1654 a gambler named de Méré proposed to Pascal two problems which he had run across in his experiences at the gaming table."

It is likely that the distinguished Antoine Gombaud (or Gombauld), chevalier de Méré, sieur de Baussay, would turn in his grave at such a characterization of his main occupation in life. He certainly considered himself a model of courtly behavior and taught his esthetic principles elegantly to the *haut monde* as one may see from the frontispiece of his collected works. His writings appear today a little humorless and pedantic, but parts are still sufficiently entertaining to be readable, and they have secured him a permanent niche in the French literature of the seventeenth century.

De Méré (1607–1684) had received a good classical education and had served briefly in the army. His time was divided about equally between his small estate in Poitou and the court at Paris. His works show him to be a philosopher who in popular form expounded the ethics of a noble life, with particular emphasis upon the *agrêments* and the pleasant considerations for others which are essential for the *honnêtes hommes* of high society. He rapidly became a prominent figure at the court of Louis XIV where he was an adviser in delicate situations and an arbiter in conflicts. His charm, good taste, art in conversation, and correspondence made him an attractive guest in the salons and a friend of many of the important figures of his period. He recalled with particular pride his assistance to Madame de Maintenon before she became the favorite of the king. As time wore on he seems occasionally to have had rather exaggerated ideas of his own importance.

Sociability was his ideal; the criterion for good conversation was that it should be pleasant. Specialists were an abomination to him: "Most of them do not instruct at all since they rely upon obscure, sometimes even false, principles, and instead of seeking the truth to clarify it they aim to embarrass each other, also by terms which they do not even themselves understand, and by chimerical distinctions." It may even seem a little *mal à propos* for a mathematician to lecture on him, in view of de Méré's contempt for the scholars. He describes some categories of them as follows: "The best mathematicians who do not know how to entertain us except by numbers and figures, those knowing history by heart without ever having reflected upon it, or those who have a curious knowledge of many languages without having anything to say in them."

There is a little of this disdain in de Méré's description, in one of his letters, of how he made the acquaintance of Pascal: "I once made a trip with the duke

* The present article is based upon a lecture given on February 25, 1959, at Colorado College on the occasion of the centennial of the birth of Florian Cajori (Feb. 28, 1859–Aug. 14, 1930).

of Roannez, who used to express himself with good and just sense and whom I found good company. Monsieur Mitton, whom you know and who is liked by all at court, was also with us, and because the trip was supposed to be a promenade rather than a voyage, we only thought of enjoying ourselves and we discussed everything. The duke was interested in mathematics, and in order to relieve tedium on the way he had provided a middle-aged man, who then was very little known, but who later certainly has made people talk about him. He was a great mathematician who knew nothing but that. These sciences give little sociable pleasure, and this man, who had neither taste nor sentiment, could not refrain from mingling into all we said, but he almost always surprised us and often made us laugh." De Méré goes on to tell that Pascal also carried strips of paper which he brought forth from time to time to write down some observations. After a few days Pascal came to enjoy the company and talked no more of mathematics.

This trip to Poitou probably took place in 1651 or 1652, but de Méré's account was written many years later and it seems that, in reminiscing, the chevalier's memory must have failed him to some extent. At the time, Blaise Pascal (1623–1662) was not yet thirty years old and could hardly be called a middle-aged man. Of course, he was almost constantly ill and may have aged prematurely. But he was already a well-known scientist. As a child prodigy he had accompanied his father to the meetings of the *Académie libre* in Paris, and when sixteen years old he had published a remarkable treatise on conic sections. At eighteen he had caused a great stir through the invention of his calculating machine, this *machine arithmétique* by means of which " . . . one could not only do all sorts of reckoning without feather or casters, but they could be done infallibly even if one did not know any of the rules of arithmetic." Pascal's demonstration of the weight of the atmosphere by barometric measurements at the base and summit of the Puy de Dôme peak had placed him in the forefront among contemporary physicists. Yet, in spite of all these achievements, Pascal's renown in Paris was still small in comparison with the fame he was later to acquire as the author of the *Lettres provinciales*. Perhaps, in retrospect, this was what the chevalier had in mind.

Pascal was at this time at the beginning of his so-called "worldly period," and, in his further account, de Méré even takes some credit for having brought it on. Certainly the two became well acquainted, and Pascal was influenced by de Méré's ideas on literary style. It has been indicated that de Méré advised Pascal on strategy in his attacks on the Jesuits in some of the *Lettres provinciales*.

The extent to which Pascal participated in the pleasures of life in the *haut monde* has been much argued by his biographers. His pious sisters feared that he was on the path to perdition. His interest in gaming and gambling questions have been cited as evidence of his dissipation in this period. Probably both Pascal and his friend de Méré spent some time at play; it was the fashionable pastime. But there is no indication that they did so with any passion; on the

contrary, in their writings they both express themselves rather contemptuously against gambling. De Méré wrote to their common acquaintance Mitton about the pleasures of the countryside—two months is all he can stay in Paris before he gets homesick. "But I confess also that on my part I deplore you who are confined to gambling, longing for nothing but luck, without eyes for anything but this artificial world, almost like the courtesans to whom the great beauties of nature are unknown."

Pascal on his side tries to analyze the gambler's soul in the *Pensées*. "But you may say, what is his object? To boast tomorrow to his friends that he has played better than another? Such a man relieves the tedium of his life by playing every day. If you were to give him in the morning the money he might win during the day on condition that he should not gamble, you would make him unhappy. One may think possibly that he seeks the entertainment only of the game and not the gain. But let him play for nothing and he is bored and does not warm up to it. Therefore, it is not only the amusement which he seeks—languishing play without passion he finds tedious. He must become excited and deceive himself into believing that he would be happy to win that which he would not even accept were it not for the play. He must form an object for his passion to excite his desire, his anger, his fear, just like children who are scared of their own faces when they have blackened them."

Next let us turn to the two probability problems for which Pascal actually found solutions. It has often been stated that they were based upon de Méré's personal gambling experiences. This, as we shall see, seems very unlikely. While he was the first to call Pascal's attention to them, they were old and well-known questions. We shall take up first the so-called *dice problem*: When one throws with two dice, how many throws must one be allowed in order to have a better than even chance of getting two sixes at least once? Games of this kind were evidently popular in the middle ages; Cardano had dealt with them more than a century earlier, and there are still French dice games of a similar nature.

Let us recall briefly how the usual solution is obtained. It is convenient to determine first the probability of not obtaining any sixes. If one throws once there are thirty-six different possible throws with two dice and thirty-five of these do not give two sixes. Thus the probability of not getting two sixes in one throw is $q_1 = 35/36$. If one throws twice, there are 36×36 cases and 35×35 of them do not give two sixes either time. Thus $q_2 = (35/36)^2$. In the same way one finds that in n throws the probability of not getting any two sixes is $q_n = (35/36)^n$. Hence the opposite event, that of getting two sixes at least once, has the probability

$$p_n = 1 - q_n = 1 - \left(\frac{35}{36}\right)^n.$$

To have a better than even chance one must have $p_n > \frac{1}{2}$ and one finds

$$p_{24} = .4914, \qquad p_{25} = .5055.$$

Thus, only if one has 25 or more throws is it an advantageous proposition.

De Méré believed that the smallest advantageous number of throws should be 24. As the matter has been presented, he turned to Pascal because his own experiences had shown him that 25 throws were required. This is an unreasonable explanation. The difference between the probabilities for 24 and 25 throws is so small, as we have just seen, that to decide experimentally that one of them is less than $\frac{1}{2}$ would, according to modern statistical standards, require at least 100 sequences of trials, which in turn would involve several thousand individual throws with the two dice. Besides, the dice would have to be specially made in order to show no bias; the usual bone cubes turned out by the diciers of Paris would be much too inaccurate. To prepare special equipment of this kind and to keep the tedious records involved was evidently contrary to the chevalier's temperament.

However, Pascal's letters on probability to Pierre de Fermat (1601–1665), the learned jurist in Toulouse, throw light on the subject. In reply to an earlier letter from Fermat, Pascal writes, on July 29, 1654, "I admire much more your method for the division problem than that for the dice problem. I know that several persons have found the solution of the dice problem, as for instance, Monsieur le chevalier de Méré, who was the one who proposed these questions to me, and also M. de Roberval. But M. de Méré has never been able to determine the correct value in the division problem, nor the method to solve it, so that I found myself to be the only one who knew this proposition."

A little later in the same letter Pascal reports further on de Méré's views:

"He told me that the figures were wrong for the following reason: If one wants to throw a six with one die one has an advantage in four throws, as the odds are 671 to 625. If one shall throw two sixes with two dice there is a disadvantage in having only 24 throws. However, 24 to 36 (the number of cases for two dice) is as four to six (the number of cases on one die).

This was a great scandal which made him proclaim loudly that the theorems were not constant and Arithmetic belied herself. But you can easily see the reason for this result by the principles you possess."

Pascal does not understand de Méré's reasoning, and the passage also has been unintelligible to the biographers of Pascal. However, de Méré bases his objection upon an ancient gambling rule which Cardano also made use of: One wants to determine the *critical number* of throws, that is, the number of throws required to have an even chance for at least one success. If in one case there is one chance out of N_0 in a single trial, and in another one chance out of N_1, then the ratio of the corresponding critical numbers is as $N_0 : N_1$. That is, we have

$$n_0 : N_0 = n_1 : N_1.$$

This immediately gives the proportion stated by de Méré. The rule was first proved by Abraham de Moivre (1667–1754) in his *Doctrine of Chances* (1716). If the chances are one in N_0 in a single trial, then the critical number is, with good approximation when N_0 is not too small,

$$n_0 = (\text{nat log } 2) \times N_0 = .69 \times N_0.$$

De Moivre applied this to the so-called Royal Oaks lottery in London; here there was one chance in 32, and so the critical number of trials by this rule is found to be $n_0 = 22.08$. The actual value is $n_0 = 22.135$, and, in general, de Moivre's rule gives very good values when N_0 is fairly large. However, de Méré made the error of believing that his gambling proportion was an absolute rule; for a small value, like $N_0 = 6$, the approximation to the actual value is not good enough.

These observations all indicate strongly that de Méré did not turn to Pascal with an actual gambling experience; rather, he was confused by the fact that a seemingly well-established gambling rule did not conform with the theoretical calculations which had been made. This tends to confirm the hypothesis that probability theory was at this time not in the state of absolute nonexistence that one is often led to believe. Cardano, around 1525, had already discovered certain rules which made it possible to solve the dice problem exactly, for one die. More than fifty years before Pascal, Galileo had given a complete table of probabilities for all throws with three dice. It appears likely that also in Pascal's circle in the mathematical academy, the simplest probability considerations were known. Pascal states that in addition to Fermat and himself, also de Méré and the mathematician Roberval could solve the dice problem.

In the preserved letters in the correspondence with Fermat, Pascal never refers to his own solution of the dice problem. Nor is it mentioned in Pascal's treatise on the *Arithmetic Triangle*, which was composed at this time and which includes the solution of the division problem as well as a few general probability principles. In regard to all other scientific achievements, Pascal always appeared anxious to receive proper recognition; in his pleasant correspondence with Fermat he insists on the importance of his own method in the division problem. In his investigations on the vacuum, he engaged in considerable argument to establish his priority, and the same is true in his later dispute regarding the cycloid or roulette curve. Thus there seems to be reason to believe that had Pascal had any feeling that this was an important discovery he would have expressed himself quite explicitly in regard to his priority rights.

The second problem which de Méré proposed was the *problème des parties*, commonly called the *division problem*. The question is how one shall divide equitably the prize money in a tournament in case the series for some reason is interrupted before it is completed. As we would say now, it is a question of determining the probability to win for each contestant, at a stage where each has a certain number of games or points to go. This is a problem of such difficulty that its solution by Pascal may well be considered a decisive break-through in the history of probability theory.

In this case it is still more evident that de Méré was not proposing a question from his own experience. For three centuries it had been a standard problem in mathematical texts. The first printed version may perhaps be found in Fra

Luca Paciuoli's *Summa* (1494), where, among the amusement questions, he proposes the following. "A team plays ball such that a total of 60 points is required to win the game, and each inning counts 10 points. The stakes are 10 ducats. By some incident they cannot finish the game and one side has 50 points and the other 20. One wants to know what share of the prize money belongs to each side. In this case I have found that opinions differ from one to another, but all seem to me insufficient in their arguments, but I shall state the truth and give the correct way."

His second example is a shooting match. "Three compete with the cross bow and the one who first obtains six first places wins; they stake 10 ducats among themselves. When the first has four best hits, the second three, and the third two, they do not want to continue and decide to divide the prize fairly. One asks what the share of each should be." Fra Luca also warns against gamblers who play "morra" to five points for five ducats and when they are behind, for instance four wins to three, " . . . they say, we come back . . . ," and want the prize divided three to two. Otherwise there is a noticeable avoidance of formulating the problem for a straight gambling game, presumably to avoid criticism for dealing with objectionable pastimes.

Most accounts of the division problem take their starting point in Fra Luca's examples. However, the problem is a much older one. I have found it in Italian mathematical manuscripts as early as 1380. It seems likely that it is of Arabic origin. It does not appear in Leonardo Fibonacci's *Liber abaci* (1202), which brought many Arabic puzzles to Italy, but the form of the problem is reminiscent of the distribution and inheritance problems of the Arabs.

The Renaissance mathematicians made only trivial contributions to the division problem, although those who deal with it make great claims for their own methods and are liberal in their criticisms of others. Cardano, for instance, says about the solution by Paciuoli, "And there is an evident error in the determination of the shares in the game problem as even a child should recognize, while he (Paciuoli) criticizes others and praises his own excellent opinion."

Cardano's arch enemy, Tartaglia, feels himself on swaying ground when he deals with the division problem in his *General Trattato* (1556). The margin displays the warning, "*Error di Fra Luca dal Borgo*," and Tartaglia gives his own rule, but with the reservation: "Therefore I say that the resolution of such a question is judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation." Toward the end of the chapter, Tartaglia considers these matters as having "*poco sugo*" and giving rise "to great dispute and so it appears to me better not to speak more of this matter, although some people like such facetious questions in order to have an occasion to create an argument."

The division problem remained in the arithmetic texts until well into the seventeenth century. It can be found also in French books, and de Méré may have read it in his own school. The form changed somewhat with time; let me cite a couple of examples from an arithmetic by Forestani (Venice 1603). "An

elderly nobleman, staying at his country house, was extremely fond of watching ball games, and so he called in two young farmhands, saying, 'Here are four ducats for which you may play; the one who first takes eight games is the winner.' So they began to play, but when one had five games and the other three games, they lost the ball and were unable to finish. The question is how the prize should be divided."

Another runs as follows: "Three soldiers garrisoned at a fortress take a walk and find a scudo. Each of them claims it; however at last they agree that they will play a match of pallatella to fourteen games, and he who wins shall have the scudo." When they have won respectively ten, eight, and five games, they are called to guard duty and the proper shares shall again be determined.

Pascal probably found the solution of the division problem early in the spring of the year 1654. The method depends on binomial coefficients, and Pascal was undoubtedly greatly aided by his studies of the *Arithmetical Triangle*, a table of such coefficients. The solution was based upon a somewhat artificial approach. When the two players needed a and b games respectively to win, Pascal lets them play altogether $a+b-1$ games, regardless of whether the series might have been decided before this many games. The argument is quite correct and is still in common use in textbooks on probability. However, the mathematician Roberval objected strenuously, and a discussion with Roberval was usually not pleasant. One of his contemporaries called him "the greatest mathematician in Paris, and in conversation the most disagreeable man in the world."

These criticisms by Roberval seem to have been the immediate reason why Pascal sought the opinion of Fermat, the recognized grand master of mathematics in France at the time. Carcavy, the royal librarian, also a member of the scientific circle in Paris, acted as an intermediary; he had formerly been Fermat's colleague as a judge at the parliament court in Toulouse.

It will carry us too far to give a detailed account of this correspondence, which lasted through the summer and fall of 1654. Fermat was delighted to come into closer contact with the young Pascal; he had previously been on friendly terms with his father. Fermat had begun to feel the scientific isolation in Toulouse and hoped that Pascal might assist him in publishing his mathematical results. In one of his letters to Carcavy he writes, "I have been delighted to have my own sentiments conform to those of M. Pascal, for I admire his genius infinitely and believe he is capable of achieving anything he may undertake. The friendship which he offers to me is so precious and so considerable that I believe it will not be disturbed if I should make some use of it in the printing of my treatises."

Pascal, on the other hand, was cheered by finding that Fermat's results were in complete agreement with his own: "I see that the truth is the same in Toulouse and Paris." Pascal at this time was absorbed in an intense scientific production; among other things he completed his *Traité du Triangle Arithmétique* with an extensive discussion of the division problem. After his death

this work was found fully printed, but unpublished, among his posthumous papers. The reason is well known. On November 23, 1654, in Paris, the crisis occurred which has been called Pascal's definitive conversion. From now on his life and thoughts revolved almost exclusively around the questions of his Christian faith. He joined his sister as a member of the Jansenist group for whom scientific studies were but illusions serving to distract from the final purpose of life, the salvation of the soul.

But Pascal never quite relinquished his interest in the newly created field of probability. In his famous *Pensées*, one of the most curious sections is "*Le pari*," the wager, a dialogue about the existence of God. It is at first difficult to understand and it has been widely discussed. But if one recognizes that Pascal has a definite mathematical probability formula in mind, the passage becomes quite lucid. It has been conjectured that Pascal conceived of "*Le pari*" as a dialogue between himself and his unbelieving friend de Méré:

Pascal: God exists or he does not. Which side shall we take. Reason can decide nothing. An infinite chaos separates us. A game is being played where a decision, heads or tails, will be made at the end of this infinite distance. On what do you place your bet? By reason you cannot take one or the other; by reason you can defend neither choice. Therefore, do not blame the error of those who have made a choice, since you know nothing about it.

De Méré: No, but I blame them for having made a choice at all, not for their particular choice, for they are equally at fault, both he who chooses heads and he who chooses tails. The correct attitude is not to bet at all.

Pascal: Yes, but one is compelled to wager, it is not voluntary, you are in the game. Which side do you take? Let us see. Since you must wager, let us find out which alternative is the least profitable.

Pascal goes on to argue, on the principle of mathematical expectation, that the value of a game is the prize to be won times the probability for winning it. This should be compared to the amount risked times the probability for losing.

Let us see: since there is an equal chance of gain or loss, and if you were to win only two lives for one, you should still bet. But if there were three to win you would also play—since you are compelled to—and you would be imprudent not to risk your life to win three others in a game with such chances to win or lose. But there is an eternity of life and happiness. And when this is so, if there should be an infinite number of chances with only a single favorable to you, it would still be right to bet one to obtain two; you would act with bad judgment if, when obliged to play, you would refuse to stake one life against three, even if there is an infinity of chances and but one for you, provided there is an infinite life of infinite happiness to be gained. But here, actually, there is such an infinite life of infinite happiness to be won, one chance of winning against a finite number of possibilities for a loss, and that which you risk is finite. This eliminates all choice; whenever an infinite gain is involved and there is not an infinite number of losing chances against the winning ones, there is nothing to weigh, one must give all. And so, when forced to play, one must sacrifice reason to win life rather than to stake it against the infinite profit which may accrue just as easily as the loss: annihilation.

Pascal continues in the same vein, and in the end de Méré seems to be convinced, but finally he asks somewhat irreverently, "I confess, this I admit, but then is there no way of looking at the underside of the cards which have been dealt?" To this Pascal replies, "Yes, the Holy Scriptures and the rest."

Much criticism has been expressed against Pascal's logic, and one must admit that this is hardly a field for applied mathematics. But even Pascal himself cannot have believed that his argument should carry a convincing mathematical weight; however, if his design was to create a thought-provoking parable, he succeeded admirably.

It was well known in Paris that Pascal and Fermat had disccovered a new branch of mathematics, but no one knew much about the details. The young Dutch genius Christiaan Huygens arrived in Paris less than a year later. He had worked on a couple of problems concerning games of chance and was anxious to consult with one of the principals in probability theory. Huygens was well received in Paris. He was introduced to the members of the informal academy, and made the acquaintance of Roberval and the lawyer and amateur mathematician Mylon. But Fermat was far away in Toulouse, and after his conversion Pascal admitted no visitors. Huygens did not receive the information he desired, but after his return to Holland he began drafting his own little treatise on probability, *Calculations in Games of Chance*. But he was concerned about the correctness of his own results, and to obtain a check on them he sent one of his problems to Roberval, Mylon, and Carcavy. Mylon replied with an erroneous answer which Huygens politely corrected, and from Roberval we have no report. Carcavy, however, consulted with his close friend Pascal, and also forwarded the problem to Fermat. Fermat promptly confirmed Huygens' solution and included for Huygens' consideration a series of five problems, which are reproduced at the end of the little treatise.

Encouraged by this contact, Huygens wrote another letter to Carcavy and a little later, much to his surprise, he received a letter from Mylon stating that Pascal had found his principle admirable and in conformity with his own procedure. Mylon explained that "although it is very difficult to meet Pascal since he has retired completely to give himself entirely to devotion, he has not lost his mathematics from view. When M. de Carcavy can visit him and propose some problem to him, he does not refuse to give the solution, particularly in the field of games of chance which he was the first to bring under discussion. Since I do not possess the same goodness as these gentlemen, I have all the difficulties in the world to meet them, since they are entirely absorbed in religious affairs and I only rarely visit those places."

Equally surprising was the fact that Pascal had given Carcavy a gambling problem to transmit. "*A* and *B* play at hazard with three dice and fixed points fourteen and eleven respectively. Each has twelve pennies and receives one penny from the other every time his own point turns up. What are the odds for one player to ruin the other?"

This problem Huygens included as the last in his collection of exercises for the readers of his booklet on probability. It is far more difficult than the rest, and it embodies, in spite of its innocuous form, the beginnings of a whole field of probability, the theory of random walks, Brownian motion, and other questions from the kinetic gas theory.

Pascal's moral views evidently did not permit him to propose a plain gambling proposition, so that in his original version as reported by Carcavy he only lets the players make pencil marks on paper. Pascal was keenly aware of the difficulties involved and he had the hope that he had invented a problem which might stymie even the formidable Fermat. In the correspondence between the two, one has a feeling that Pascal is a little dismayed at the apparent ease with which the old master tackled his problems. But also this time Fermat immediately returned a solution which agreed with the one found by Pascal.

Huygens was elated at the confirmation of his methods, but regretted deeply his failure to meet Pascal: "If one had not assured me while I was in Paris that he had abandoned the study of mathematics entirely I should have tried by every means to make his acquaintance." Later Huygens corresponded with Pascal on other matters, and when Huygens returned to Paris in 1660, a couple of years before Pascal's death, the two met on several occasions.

Huygens also became friendly with the duke of Roannez and in his diary he relates that he was once entertained at dinner in the ducal palace in the company of the chevalier de Méré, "inventor of the division in games." It is noteworthy that in all the discussion about probability problems no one thought it worth while to consult with de Méré. Pascal, in one of his letters to Fermat, made the comment upon him, "He is a good wit, but not a mathematician."

Nevertheless, the fact that de Méré had been a figure, albeit a minor one, in the creation of a new mathematical field seems to have gone to his head. To the consternation of contemporary scientists he wrote a letter to Pascal in the following vein.

"Do you remember you once told me that you were no longer convinced of the excellence of mathematics? You write to me this time that I have disillusioned you completely and also that I have discovered things which you would never have perceived if you had not known me. I don't know, however, Monsieur, if you are as obliged to me as you may think. You still have the habit, which you have gathered from this science, not to judge anything except from your demonstrations, which are often false. These long reasonings drawn from line to line prevent you from obtaining the higher point of view which never deceives."

Later on he admonishes Pascal as follows:

"You know that I have discovered such rare things in mathematics that the most learned among the ancients have never discussed them and they have surprised the best mathematicians in Europe. You have written on my inventions, as well as Monsieur Huygens, Monsieur de Fermat, and many others who have admired them. You may conclude from this that I do not propose to anyone to scorn this science and truly, it may be of service provided one does not attach oneself too closely to it, for ordinarily, that which one seeks with so much curiosity appears useless to me and the time spent at it could be better employed."

De Méré's outburst seems to have amused the court and a wit proposed epigrammatically that the chevalier believed "he could teach Madame de Maintenon courtly behavior and Pascal mathematics."

Peculiarly enough, the whole letter was printed verbatim in Boyle's important encyclopedia *Dictionaire historique et critique.* When Leibniz came across

the article he commented:

"I almost laughed at the airs which the chevalier de Méré takes on in his letter to Pascal which Boyle reports in the same article. But I notice that the chevalier knew that Pascal's genius also had its weak sides which sometimes made him susceptible to the influence of too extravagant spiritualists and even at times made him lose the taste for solid knowledge.

M. de Méré takes advantage of this to talk down to Pascal. It seems to me that he makes a little fun of him, as men of the world often do when they have an abundance of esprit, but mediocre knowledge. They want to convince us that those things which they do not sufficiently understand are but of small value; one should send them to school with Roberval. It is true, nevertheless, that the chevalier was unusually gifted even in mathematics."

Leibniz goes on with a brief mention of some of the men who had worked with probability problems. He himself often showed an interest in the subject but never made any contributions of consequence. However, he concludes his epistle with the judgment which has never been more true than at present, "So also the games in themselves merit to be studied and if some penetrating mathematician meditated upon them he would find many important results, for man has never shown more ingenuity than in his plays."

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INTEGRATION BY PARTS FOR STIELTJES INTEGRALS

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There are several formulas for integration by parts. The most familiar of these, as found in the usual calculus text, states that

$$\int_a^b f(t)g'(t)dt + \int_a^b f'(t)g(t)dt = f(b)g(b) - f(a)g(a). \tag{1}$$

Clearly (1) does not hold without some restriction on the functions f and g. (We shall point out below the class of functions to which it is appropriate to apply (1).)

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For Riemann-Stieltjes integrals, there is a reasonable formula for integration by parts. Let f and g be finite real-valued functions defined on the closed interval $[a, b]$ of the real number system R†. Let $\int_a^b f(t)dg(t)$ denote the Riemann-Stieltjes integral of f with respect to g over $[a, b]$. Then, if $\int_a^b f(t)dg(t)$ exists, the integral $\int_a^b g(t)df(t)$ also exists, and

$$\int_a^b f(t)dg(t) + \int_a^b g(t)df(t) = f(b)g(b) - f(a)g(a). \tag{2}$$

For a proof of (2), see for example [1], p. 195. Formula (2) suffers from the inherent defect of Riemann-Stieltjes integrals: even for quite ordinary functions f and g, the integral $\int_a^b f(t)dg(t)$ may fail to exist (for example, if f and g have a common left or right discontinuity).

The appropriate vehicle for Stieltjes integration appears to be the Lebesgue-Stieltjes integral, discussed in detail in [4, Ch. 3] and in [3, Ch. IV]. For this integral there is a perfectly general formula for integration by parts. The formula for integration by parts is usually stated with certain restrictions on the integrand and integrator (see [4, p. 102, Theorem (14.1)] and [3, p. 161, Satz 1]). These restrictions are in fact unnecessary. We shall give what appears to us the proper general form of the formula for integration by parts.

We recall first a few notions from measure theory. Consider a closed interval $[a, b]$ on the line R, and the family $\mathfrak{F}$ of all closed subsets of $[a, b]$. There are families of subsets of $[a, b]$ that contain $\mathfrak{F}$ and also are closed under the formation of countably infinite unions and complements relative to $[a, b]$. (The family of *all* subsets of $[a, b]$ is such a family.) Let $\mathfrak{B}([a, b])$, the Borel sets in $[a, b]$, be defined as the *smallest* family of subsets of $[a, b]$ that contains $\mathfrak{F}$ and is closed under the formation of countable unions $\bigcup_{n=1}^{\infty} B_n$ and complements $[a, b] \cap B'$. In this note, we shall define a *measure* on $\mathfrak{B}([a, b])$ as a nonnegative, finite, real-valued function, say μ, defined for all sets in $\mathfrak{B}([a, b])$, such that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{3}$$

if $A_1, A_2, A_3, \cdots$ are sets in $\mathfrak{B}([a, b])$ and $A_n \cap A_m = \varnothing$ for $n \neq m$ ($\varnothing$ denotes the void set).

For every number t in $[a, b]$, the intervals $[a, t]$ and $[a, t[$ are obviously Borel sets, and hence $\mu([a, t])$ and $\mu([a, t[)$ are defined for all t, $a \leqq t \leqq b$, if μ is a measure on $\mathfrak{B}([a, b])$.

Now let μ and ν be any two measures on $\mathfrak{B}([a, b])$. For $a \leqq t \leqq b$, let us write

$$\begin{aligned} M(t) &= \tfrac{1}{2}\{\mu([a, t]) + \mu([a, t[)\}, \\ N(t) &= \tfrac{1}{2}\{\nu([a, t]) + \nu([a, t[)\}. \end{aligned} \tag{4}$$

† For real numbers a and b such that $a \leqq b$, we write $[a, b]$ to denote the closed interval $\{t: t \in R, a \leqq t \leqq b\}$, and $[a, b[$ to denote the half-closed interval $\{t: t \in R, a \leqq t < b\}$. The expressions $]-\infty, a]$ and $]-\infty, a[$ are defined similarly.

It is obvious that M and N are (weakly) monotone increasing, nonnegative functions defined on $[a, b]$. Since $\mu(\varnothing)=\nu(\varnothing)=0$, we see that $M(a)=\frac{1}{2}\mu(\{a\})$, $N(a)=\frac{1}{2}\nu(\{a\})$. We shall state the theorem on integration by parts in terms of the functions M and N.

THEOREM A. *Let μ and ν be measures on $\mathfrak{B}([a, b])$ and M and N the functions on $[a, b]$ defined in* (4). *Then**

$$\int_{[a,b]} M(t)d\nu(t) + \int_{[a,b]} N(t)d\mu(t) = \mu([a, b])\cdot\nu([a, b]). \tag{5}$$

Proof. Both integrals appearing in (5) exist. Since M and N are increasing real-valued functions on $[a, b]$, they are clearly Borel measurable and bounded; and bounded measurable functions are integrable. The proof of (5) is extremely simple. Let $[a, b]\times[a, b]$ denote the Cartesian product of $[a, b]$ with itself, that is, the closed square in the plane $\{(t, u): a\leqq t\leqq b,\ a\leqq u\leqq b\}$. Let Q be the set $\{(t, u): (t, u)\in[a, b]\times[a, b],\ t\geqq u\}$. One version of Fubini's theorem ([4, p. 85, Theorem (9.8)]) asserts that†

$$\int_{[a,b]} \mu\{t: t\in[a, b], (t, u)\in Q\}d\nu(u) = \int_{[a,b]} \nu\{u: u\in[a, b], (t, u)\in Q\}d\mu(t). \tag{6}$$

Equality (6) can be rewritten as

$$\int_{[a,b]} \mu([u, b])d\nu(u) = \int_{[a,b]} \nu([a, t])d\mu(t). \tag{7}$$

Since $[a, b]=[a, u[\cup[u, b]\ (a\leqq u\leqq b)$, the left side of (7) can be rewritten as

$$\begin{aligned}\int_{[a,b]} \mu([u, b])d\nu(u) &= \int_{[a,b]} \mu([a, b])d\nu(u) - \int_{[a,b]} \mu([a, u[)d\nu(u)\\ &= \mu([a, b])\cdot\nu([a, b]) - \int_{[a,b]} \mu([a, u[)d\nu(u).\end{aligned} \tag{8}$$

Combining (7) and (8), we have

$$\int_{[a,b]} \mu([a, u[)d\nu(u) + \int_{[a,b]} \nu([a, t])d\mu(t) = \mu([a, b])\cdot\nu([a, b]). \tag{9}$$

Interchanging the rôles of μ and ν, we can also write

* Here and below all integrals are of Lebesgue's type.

† Let $\phi=\phi(t, u)$ be the function defined on $[a, b]\times[a, b]$ such that ϕ is equal to 1 on the set Q and 0 off the set Q. Then (6) states that the iterated integrals $\int(\int\phi d\mu)d\nu$ and $\int(\int\phi d\nu)d\mu$ are equal. This very special case of Fubini's theorem can be proved by an elementary argument: see the Remark at the end of this paper.

$$(9') \qquad \int_{[a,b]} \mu([a,u])d\nu(u) + \int_{[a,b]} \nu([a,t[)d\mu(t) = \mu([a,b])\cdot\nu([a,b]).$$

Adding (9) and (9') and dividing by 2, we obtain (5).

Theorem A can be extended to measures defined for Borel subsets of the entire real line R. Let $\mathfrak{B}(R)$ be the smallest family of subsets of R containing all closed subsets of R and closed under the formation of countable unions and complements relative to R. By a measure on $\mathfrak{B}(R)$, we mean a nonnegative, finite, real-valued function, say μ, defined for all sets in $\mathfrak{B}(R)$, for which (3) holds for all pairwise disjoint sets $A_1, A_2, A_3, \cdots$ in $\mathfrak{B}(R)$. For $t\in R$, let us write

$$(10) \qquad \begin{aligned} M(t) &= \tfrac{1}{2}\{\mu(]-\infty,t]) + \mu(]-\infty,t[)\}, \\ N(t) &= \tfrac{1}{2}\{\nu(]-\infty,t]) + \nu(]-\infty,t[)\}, \end{aligned}$$

where μ and ν are any measures on $\mathfrak{B}(R)$. The analogue for Theorem A dealing with measures on $\mathfrak{B}(R)$ can now be stated.

THEOREM B. *Let μ and ν be measures on $\mathfrak{B}(R)$ and M and N functions on R as defined in* (10). *Then*

$$(11) \qquad \int_R M(t)d\nu(t) + \int_R N(t)d\mu(t) = \mu(R)\nu(R).$$

Proof. For every positive integer k and every real number t, let $M_k(t) = \frac{1}{2}\{\mu([-k,t])+\mu([-k,t[)\}$ if $|t|\leqq k$ and $M_k(t)=0$ if $|t|>k$. Let $N_k(t)$ be defined similarly, with μ replaced by ν. Theorem A shows that

$$(12) \qquad \int_R M_k(t)d\nu(t) + \int_R N_k(t)d\mu(t) = \mu([-k,k])\cdot\nu([-k,k]).$$

The sequence of functions $\{M_k\}_{k=1}^{\infty}$ is increasing and has pointwise limit M; similarly for $\{N_k\}_{k=1}^{\infty}$. B. Levi's theorem on integration of monotone increasing sequences of integrable functions shows that

$$(13) \qquad \begin{aligned} \lim_{k\to\infty} \int_R M_k(t)d\nu(t) &= \int_R M(t)d\nu(t), \\ \lim_{k\to\infty} \int_R N_k(t)d\mu(t) &= \int_R N(t)d\mu(t). \end{aligned}$$

The limit of the right side of (12) is clearly $\mu(R)\cdot\nu(R)$. Combining (12) and (13), we obtain (11).

Theorems A and B have obvious extensions to the case of *complex* measures defined on $\mathfrak{B}([a,b])$ and $\mathfrak{B}(R)$, respectively. A complex measure on $\mathfrak{B}(R)$, say, is a complex-valued function defined for all sets in $\mathfrak{B}(R)$ and satisfying (3). It can be proved that every complex measure μ on $\mathfrak{B}(R)$ has the form $\mu=\mu_1-\mu_2$

$+i\mu_3-i\mu_4$, where μ_1, μ_2, μ_3, and μ_4 are measures on $\mathfrak{B}(R)$ (see [2, pp. 120–123]). From this, Theorem B follows for complex μ and ν by an obvious computation. Theorem A has a similar obvious extension to complex measures.

Formula (1) is obtained from (5) for complex measures as follows. Let f and g be absolutely continuous complex-valued functions defined on $[a, b]$, that is, functions which are the integrals of their derivatives. For sets A in $\mathfrak{B}([a, b])$, let $\mu(A)=\int_A f'(t)dt$ and $\nu(A)=\int_A g'(t)dt$. Then, since f and g are absolutely continuous, we have

$$M(t) = \int_a^t f'(x)dx = f(t) - f(a) \quad \text{and} \quad N(t) = g(t) - g(a) \qquad (a \leqq t \leqq b).$$

Furthermore, for every bounded measurable function ψ on $[a, b]$, we have $\int_{[a,b]}\psi(t)d\mu(t)=\int_a^b\psi(t)f'(t)dt$ and $\int_{[a,b]}\psi(t)d\nu(t)=\int_a^b\psi(t)g'(t)dt$. Equality (5) now gives us

$$\int_a^b [f(t) - f(a)]g'(t)dt + \int_a^b [g(t) - g(a)]f'(t)dt = [f(b) - f(a)][g(b) - g(a)],$$

which is (1) in a slight disguise.

Remark. As pointed out in the footnote† (p. 421), the special case of Fubini's theorem needed to prove Theorem A is elementary. Let h be a continuous real-valued function on $[a, b]\times[a, b]$, and let μ and ν be measures on $\mathfrak{B}([a, b])$. The function h can be arbitrarily uniformly approximated on $[a, b]\times[a, b]$ by polynomials $p(t, u)=\sum a_{kl}t^k u^l$ (Weierstrass's polynomial approximation theorem for two variables). It is trivial that

$$\int_a^b \int_a^b p(t, u)d\mu(t)d\nu(u) = \int_a^b \int_a^b p(t, u)d\nu(u)d\mu(t). \tag{14}$$

It is then easy to see that (14) holds with p replaced by h. The function ϕ defined in the footnote† (p. 421) is the pointwise limit of a bounded decreasing sequence of continuous functions. Thus (14) holds with p replaced by ϕ. This is just the equality (6). Of course Fubini's theorem in its full generality is a much stronger assertion than (6), and requires a much more complicated proof. See [4, p. 87, Theorem (9.10)].

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A MINIATURE THEORY OF LEBESGUE AREA

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The problem of showing the existence of a surface minimizing the parametric two-dimensional integral considered by McShane, Danskin, Sigalov and Cesari [9, 6, 12, 4] is closely connected with the theory of Lebesgue area. When we reduce this problem to the one-dimensional case, the results, though not new, are interesting and illustrate the methods of that theory [1, 2, 3].

Let p and q be two points in Euclidean space E, and D be the collection of absolutely continuous functions x on an interval $[a, b]$ into E such that $x(a)=p$ and $x(b)=q$. We are interested in minimizing $I(f, x)=\int_a^b f(x(t), x'(t))dt$ for all $x\in D$, where f belongs to the class F to be described later. Tonelli observed that Hilbert's compactness theorem for curves of bounded length would be sufficient to provide the desired minimum if it were known that I_f were lower semicontinuous, and he showed [13] that I_f has that property under conditions weaker than we have imposed upon f. Later Menger introduced such metrics and distances which would enable him to interpret the integral as a generalized length and obtained another proof of Tonelli's result [10].

It is this theorem which we are interested in. We wish to present another proof that I_f is lower semicontinuous, provided $f\in F$, by interpreting $I(f, x)$ as a length (it might be more appropriate to consider the integral as the energy of a curve in a suitable field) where the length of a small line segment depends upon its position and direction, as well as its geometric length. To this end we introduce a "length" L_f in much the same way that Lebesgue defined his area. That L_f is lower semicontinuous is immediate, but we must now show that $L(f, x)=I(f, x)$ if x is absolutely continuous. The proof of this equality will be obtained by means of a process which is used in Lebesgue area theory. (For our present problem it would be sufficient to restrict ourselves to the functional P_f to be defined later.)

Let E^* be the surface of the unit sphere in E. Then F is the collection of real-valued functions f on $E\times E$ satisfying

(a) f is uniformly continuous on $E\times E^*$,

(b) there exist numbers $M\geqq m>0$ such that

$$m\|p\| \leqq f(a, p) \leqq M\|p\|, \qquad a\in E, p\in E,$$

(c) if $k>0$ then $f(a, kp)=kf(a, p)$,

(d) if $f^a(p)=f(a, p)$, then f^a is convex for each $a\in E$.

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Let $C(J)$ be the space of continuous functions x on a closed interval J into E. For absolutely continuous functions x let $I(f, x) = \int_J f(x(t), x'(t)dt$.

DEFINITION. *If $x \in C(J)$ then $L(f, x) = \lim \inf_{z \to x} I(f, z)$ where $z \in C(J)$ is quasilinear and the arrow indicates uniform convergence.*

If $z \in C(J)$ is quasilinear, and if we put $z_n = z$ for $n = 1, 2, \cdots$, then $z_n \to z$ and we obtain $L(f, z) \leqq \lim \inf I(f, z_n) = I(f, z)$.

Next we use a diagonalization process to show that L_f is lower semicontinuous. Let $x_n \to x$ in $C(J)$. Then there exist quasilinear functions z_{n_k} in $C(J)$ such that $z_{n_k} \to x_n$ and $I(f, z_{n_k}) \to L(f, x_n)$. Thus there is an integer k_n such that $|z_{n_{k_n}} - x_n| < 1/n$ and $I(f, z_{n_{k_n}}) < L(f, x_n) + 1/n$. Hence $z_{n_{k_n}} \to x$ and $L(f, x) \leqq \lim \inf I(f, z_{n_{k_n}}) = \lim \inf L(f, x_n)$.

Now suppose that S is a lower semicontinuous functional on $C(J)$ such that $S(z) = I(f, z)$ for quasilinear z. If $x \in C(J)$ then there exists $z_n \in C(J)$, z_n quasilinear, such that $z_n \to x$ and $I(f, z_n) \to L(f, x)$. Thus $S(x) \leqq \lim \inf I(f, z_n) = L(f, x)$.

It remains for us to show that $L(f, x) = I(f, x)$ if x is absolutely continuous. At this stage we do not even know that the equality holds for quasilinear functions, though we have just seen that $L_f \leqq I_f$ on such functions. This problem also arises in the theory of Lebesgue area. It is customary to answer it by introducing what Radó called a "lower" area and studying the relation between the lower and the Lebesgue areas. We shall define a "lower" length P_f, also lower semicontinuous, in analogy to a particular lower area, Peano area. It should be noticed that L_f was defined globally and, conceivably, is too small on quasilinear functions, while P_f will be defined locally and, at first glance, may appear to be too big on quasilinear functions. We shall use Theorem 5, analogous to a theorem of Tonelli concerning Jordan length, to obtain $P(f, x) \geqq I(f, x)$ for rectifiable x with the equality holding if and only if x is absolutely continuous. In particular, $P(f, z) = I(f, z)$ if z is quasilinear. From this last equality and the lower semicontinuity of P_f we have $P_f \leqq L_f$. In Theorem 6 we get $L_f \leqq P_f$. Thus we obtain our desired result: $L(f, x) = P(f, x) = I(f, x)$ if x is absolutely continuous. Important questions concerning Lebesgue area can also be solved by showing that a lower area agrees with the Lebesgue area, but the establishment of this equality, as distinguished from the one-dimensional case we are considering, is distinctly nontrivial, and is generally considered to be the principal theorem in the theory of Lebesgue area [5, 8].

We begin our discussion by making use of the fact that a convex function can be described by means of its supporting linear functionals.

If $p^* \in E^*$, then we interpret p^* as a linear functional on E by putting $p^*(p) = [p, p^*]$, the scalar product of p and p^*, for all p in E.

THEOREM 1. *Let f be a real-valued function on $E \times E$. A necessary and sufficient condition that $f \in F$ is that there exist points p_i^* dense in E^*, $i = \pm 1, \pm 2, \cdots$, $p_i^* + p_{-i}^* = 0$, and an equicontinuous family $\{f_i\}$ of real-valued functions on E such that range $f_i \subset [m, M]$ and $f(a, p) = \sup_i f_i(a)[p, p_i^*]$.*

Proof. The sufficiency is not hard and we will only show necessity. If $p^* \in E^*$ and $a \in E$ then f^a and p^* are continuous functions on E^* and $f^a(p) \geqq m > 0$ for all $p \in E^*$. Thus p^*/f^a is also continuous on the compact set E^* and we can define

$$\frac{1}{\psi(p^*, a)} = \max_{p \in E^*} \frac{[p, p^*]}{f(a, p)} .$$

Let us notice that there exists $T(p^*, a) \in E^*$ such that $f(a, T(p^*, a)) = \psi(p^*, a)[T(p^*, a), p^*]$. Furthermore, $f(a, p) \geqq \psi(p^*, a)[p, p^*]$ for all $p \in E^*$ and, by (c), for all $p \in E$. Since $[p, p^*] \leqq 1$ for all $p \in E^*$ we have $1/\psi(p^*, a) \leqq 1/m$. On the other hand,

$$\frac{1}{\psi(p^*, a)} \geqq \frac{[p^*, p^*]}{f(a, p^*)} \geqq \frac{1}{M} .$$

Thus $m \leqq \psi(p^*, a) \leqq M$ for all $p^* \in E^*$ and $a \in E$. It follows that $[T(p^*, a), p^*] \geqq m/M$. Now choose $\epsilon > 0$. By hypothesis there exists $\delta > 0$, $\delta < m\epsilon/(2M^2)$, such that $|f(a, p) - f(b, q)| < m\epsilon/(2M)$ whenever $a, b \in E$, $p, q \in E^*$, $\|a - b\| < \delta$ and $\|p - q\| < \delta$. Now suppose that $a, b \in E$, $p^*, q^* \in E^*$, $\|a - b\| < \delta$, and $\|p^* - q^*\| < \delta$. Then

$$\begin{aligned}\{\psi(p^*, a) - \psi(q^*, b)\}[T(q^*, b), q^*] &= \psi(p^*, a)[T(q^*, b), p^*] \\ &\quad + \{\psi(p^*, a)([T(q^*, b), q^*)] - [T(q^*, b), p^*])\} - \psi(q^*, b)[T(q^*, b), q^*] \\ &\leqq f(a, T(q^*, b)) + M\delta - f(b, T(q^*, b)) < \frac{m\epsilon}{2M} + \frac{m\epsilon}{2M} = \frac{m\epsilon}{M} .\end{aligned}$$

Thus $\psi(p^*, a) - \psi(q^*, b) < (m\epsilon/M) \cdot (M/m) = \epsilon$. Similarly $\psi(q^*, b) - \psi(p^*, a) < \epsilon$. Now let p_i^* be dense in E^*, $i = \pm 1, \pm 2, \cdots$, with $p_i^* + p_{-i}^* = 0$, and put $f_i(a) = \psi(p_i^*, a)$. We have just shown that $\{f_i\}$ is equicontinuous and range $f_i \subset [m, M]$. Now fix $a \in E$ and $p \in E^*$. By the Hahn-Banach Theorem there exists a linear functional ϕ^* on E such that $\phi^*(p) = f(a, p)$ and $\phi^*(q) \leqq f(a, q)$ for all $q \in E$. Since ϕ^* is a linear functional over E there exists $p^* \in E^*$ and a constant k such that $\phi^* = kp^*$. We compute

$$\frac{1}{\psi(p^*, a)} \geqq \frac{[p, p^*]}{f(a, p)} = \frac{[p, p^*]}{k[p, p^*]} = \frac{1}{k}$$

and conclude that $\psi(p^*, a) \leqq k$. On the other hand,

$$\frac{[q, p^*]}{f(a, q)} \leqq \frac{[q, p^*]}{\phi^*(q)} = \frac{[q, p^*]}{k[q, p^*]} = \frac{1}{k}$$

for all $q \in E^*$. Thus $1/\psi(p^*, a) = 1/k$. Hence $f(a, p) = \psi(p^*, a)[p, p^*]$. Next we compute $|\psi(p^*, a)[p, p^*] - \psi(p_i^*, a)[p, p_i^*]| \leqq |\psi(p^*, a) - \psi(p_i^*, a)| \|p\| + |\psi(p_i^*, a)| \|p^* - p_i^*\|$. Now let $\epsilon > 0$. Since $\{p_i^*\}$ is dense in E^*, and ψ, as we

have seen, is uniformly continuous on $E^* \times E$, there exists an integer j such that $|\psi(p^*, a)[p, p^*] - \psi(p_j^*, a)[p, p_j^*]| < \epsilon$. Thus $f(a, p) \leqq \psi(p_j^*, a)[p, p_j^*] + \epsilon$. Since we have already observed that $f(a, p) \geqq \psi(q^*, a)[p, q^*]$ for all $q^* \in E^*$ we conclude that $f(a, p) = \sup_i f_i(a)[p, p_i^*]$.

We now begin to duplicate some of the ideas used in studying Lebesgue area. Our immediate goal is to define the Peano length P_f, but to do this we require the notion of a flat map. A *flat map* is a continuous function on a closed interval J into E_1, where J and E_1 are both oriented. Let ϕ be such a function.

If U is an open interval in E_1, let $K(\phi, U)$ be the collection of components of $\{p \in J \mid \phi(p) \in U\}$. If $V \in K(\phi, U)$, define

$$D(\phi, U, V) = \begin{cases} 1 & \text{if } \phi \mid V \text{ maps } V \text{ onto } U \text{ preserving orientation,} \\ -1 & \text{if } \phi \mid V \text{ maps } V \text{ onto } U \text{ reversing orientation,} \\ 0 & \text{if } \phi \mid V \text{ does not map } V \text{ onto } U. \end{cases}$$

Then $D(\phi, U, V)$ is the *degree* of the mapping $\phi \mid V$, and can be defined in such a way as to make sense for higher-dimensional flat mappings [7]. It follows from the uniform continuity of ϕ that there are only finitely many $V \in K(\phi, U)$ for which $D(\phi, U, V) \neq 0$. Furthermore, if $U_1 \subset U_2$ and $D(\phi, U_2, W) \neq 0$, then there exists $V \in K(\phi, U_1)$ such that $V \subset W$ and $D(\phi, U_1, V) = D(\phi, U_2, W)$.

Let D^+ and D^- be the positive and negative parts of D, respectively, so that $D = D^+ - D^-$.

We introduce the following notation: $\lambda = +$ or $\lambda = -$; $\mu = +$, $\mu = -$, or μ is to be disregarded. Thus $A^\mu = B^\mu$ means that $A^+ = B^+$, $A^- = B^-$, and $A = B$. If U is a set then $c(U)$ is the characteristic function of U. We write $c(U, t)$ for $[c(U)](t)$. Finally, we put $f_i^\lambda = f_{\lambda i}$ (see Theorem 1).

Now let $x \in C(J)$ and define $x^i = [x, p_i^*]$. Evidently x^i is a flat map for each i.

We modify the degree of the mapping to take account of the fact that we wish our length to depend upon $f \in F$. In what follows, U is an open interval in E_1 and $V \in K(x^i, U)$. We define

$$O_i^\lambda(x, U, V) = \inf_{p \in V} f_i^\lambda(x(p)) D^\lambda(x^i, U, V),$$

$$O_i = \overset{+}{O}_i + \overset{-}{O}_i.$$

The function M_i^μ is defined on $C(J) \times E_1$ by

$$M_i^\mu(x, t) = \sup_U \sum_{V \in K(x^i, U)} O_i^\mu(x, U, V) c(U, t).$$

Let $N < M_i^\mu(x, t)$ and suppose that $t_n \to t$. Then there exists U such that $\sum_{V \in K(x^i, U)} O_i^\mu(x, U, V) c(U, t) > N$. Furthermore, if n is sufficiently large then $t_n \in U$, from which it follows that $\liminf M_i^\mu(x, t_n) > N$. Thus M_i^μ is lower semi-continuous in its second argument. Since f_i^λ is uniformly continuous, it requires only a slight modification of Federer's argument to show that M_i^μ is also lower

semicontinuous in its first argument.

Let $\bar{f}(a, p)=\sup_i [p, p_i^*]$. It is not hard to see that $\bar{f}(a, p)=\|p\|$ and that $\bar{f}_i \equiv 1$ for all i. The function M_i evaluated for $\bar{f}$ is a *multiplicity function* and, as defined here, follows very closely the definition of Federer, which is valid for flat maps of arbitrary finite dimension.

THEOREM 2. $M_i = M_i^+ + M_i^-$.

Proof. Let $N^\lambda < M_i^\lambda(x, t)$. Then there exists an open interval U^λ in E_1 such that

$$\sum_{V \in K(x^i, U^\lambda)} O_i^\lambda(x, U^\lambda, V)c(U^\lambda, t) > N^\lambda.$$

Now let $U = U^+ \cap U^-$. Then $U \subset U^\lambda$ and for each $V \in K(x^i, U^\lambda)$ there exists $V^\lambda \in K(x^i, U)$ such that $V^\lambda \subset V$ and $D(x^i, U, V^\lambda) = D(x^i, U^\lambda, V)$. Since $\inf_{p \in V^\lambda} f_i(x(p)) \geqq \inf_{p \in V} f_i(x(p))$, it follows that $\sum' O_i(x, U, V^\lambda)c(U, t) \geqq \sum'' O_i^\lambda(x, U^\lambda, V)c(U, t) > N^\lambda$ where $\sum''$ indicates summation over $V \in K(x^i, U^\lambda)$ and $\sum'$ indicates summation only over such $V^\lambda \in K(x^i, U)$ for which there exists $V \in K(x^i, U^\lambda)$ such that $V^\lambda \subset V$ and $D(x^i, U, V^\lambda) = D(x^i, U^\lambda, V)$. Thus the sum over all $W \in K(x^i, U)$ exceeds $N^+ + N^-$ and we get $M_i(x, t) \geqq N^+ + N^-$. Hence $M_i(x, t) \geqq M_i^+(x, t) + M_i^-(x, t)$. The opposite inequality is an immediate consequence of the definitions. The corresponding theorem for the two-dimensional problem asserts only that the equality holds except on a countable set.

DEFINITION. $W_i(x) = \int_{-\infty}^{\infty} M_i(x, t)dt$.

Where necessary we modify our previous notation to show that O, M, and W depend upon f. Thus we write $W_i(f, x)$ for $W_i(x)$.

THEOREM 3. *A necessary and sufficient condition that x^i be of bounded variation is that $W_i(f, x)$ be finite.*

Proof. Cesari, with definitions phrased slightly differently, and Federer showed that $W_i(\bar{f}, x) = L(x^i)$, the geometric length of the flat map x^i, [1, 7]. Since range $f_i \subset [m, M]$, it follows that

$$mO_i(\bar{f}, x, U, V) \leqq O_i(f, x, U, V) \leqq MO_i(\bar{f}, x, U, V),$$

where U is an open interval and $V \in K(x^i, U)$. Thus

$$mM_i(\bar{f}, x, t) \leqq M_i(f, x, t) \leqq MM_i(\bar{f}, x, t)$$

and, using the result just referred to, we get

$$mL(x^i) \leqq W_i(f, x) \leqq ML(x^i).$$

We can interpret $W_i(x)$ and $L(x^i)$ as interval functions by putting $[W_i(x)](K) = W_i(x|K)$ and $[L(x^i)](K) = L(x^i|K)$ for closed intervals $K \subset J$. We recall that

an additive function of bounded variation has a derivative and that $L(x^i)' = |x^{i\prime}|$ almost everywhere. It is easy to verify that $W_i(x)$ is additive.

THEOREM 4. *Let x^i be of bounded variation. Then*

$$W_i(x)' = \max\,(f_i^+(x)x', -f_i^-(x)x')$$

almost everywhere.

Proof. Since $W_i(x)$ and x^i are both of bounded variation, the indicated derivatives exist almost everywhere. Suppose they exist at p, and that $x^{i\prime}(p) \geqq 0$. Let $\epsilon > 0$. There is a closed interval $K \subset J$ centered at p such that $|f_i(x(p)) - f_i(x(q))| < \epsilon$ provided $q \in K$. Hence, as in the last theorem,

$$[f_i(x(p)) - \epsilon]L(x^i \mid K) < W_i(x \mid K) < [f_i(x(p)) + \epsilon]L(x^i \mid K).$$

Thus

$$[f_i(x(p)) - \epsilon]x^{i\prime}(p) \leqq W_i'(x, p) \leqq [f_i(x(p)) + \epsilon]x^{i\prime}(p).$$

The case $x^{i\prime}(p) < 0$ is handled similarly and the theorem follows.

DEFINITION. *If $x \in C(J)$ then $P(f, x) = \sup_\pi \sum_{I\in\pi} \sup_i W_i(x \mid I)$ where π is a finite set of nonoverlapping closed intervals contained in J.*

It follows from the lower semicontinuity of M that P_f is also lower semicontinuous.

LEMMA. *If the interval functions S_i, $i = 1, 2, \cdots, n$, all have derivatives at p and if $T = \max S_i$, then T has a derivative at p and $T'(p) = \max S_i'(p)$.*

Proof. Let $\underline{T}$ and $\overline{T}$ be the lower and upper derivates of T. Since $T(I) \geqq S_i(I)$ for all intervals I and each i, we obtain $\underline{T}(p) \geqq \max S_i'(p)$. Now choose $\{I_k\}$ so that $p \in I_k$, (meas I_k)$\to 0$, and $T(I_k)/(\text{meas } I_k) \to \overline{T}(p)$. Then for some $i = i_0$ there exists a subsequence $\{I_{k_m}\}$ with (meas I_{k_m})$\to 0$ and $T(I_{k_m}) = S_{i_0}(I_{k_m})$ for all m. Thus $\overline{T}(p) = S_{i_0}(p) \leqq \max S_i'(p) \leqq \underline{T}(p)$.

THEOREM 5. *If x is rectifiable then $P(f, x)' = f(x, x')$ almost everywhere. Thus $P(f, x) \geqq I(f, x)$ and the equality holds if and only if (each component of) x is absolutely continuous. In particular, $P(f, z) = I(f, z)$ if z is quasilinear.*

Proof. Let $T_n(I) = \max_{|i|\leq n} W_i(x \mid I)$, $T(I) = \sup_i W_i(x \mid I)$. Then $P(f, x)$ = Variation T and $\{\text{Variation } T_n\}$ is monotonically increasing with limit (Variation T). Thus [11, p. 116, 121], (Variation T)' = lim (Variation T_n)' = lim $|T_n'|$ = sup W_i' almost everywhere.

Let p_i^*, $i = 1, 2, \cdots, N$, be a basis for E consisting of points of E^* all of whose coordinates are zero except one. (It may be that the points have to be relabelled to satisfy the above condition.) Since f^a is convex for each $a \in E$, we can use the proof of Theorem 3 to get

$$mL(x^i) \leqq W_i(x) \leqq P(f, x) \leqq \sum W_i(x) \leqq M \sum L(x^i).$$

Thus $P(f, x)$ is absolutely continuous if and only if $L(x^i)$ is absolutely continuous for each i. Hence $P(f, x)$ is absolutely continuous if and only if x^i is absolutely continuous for each i. The remainder of the proof now follows from [11, p. 119].†

THEOREM 6. *If $x \in C(J)$ there exists a sequence $\{z_n\}$ of quasilinear functions inscribed in x such that $z_n \to x$ and $I(f, z_n) \to P(f, x)$. Thus $P_f = L_f$.*

Proof. Let π_n be a partition of J into intervals whose maximum diameter is less than $1/n$ and such that $|f(b, p^*) - f(a, p^*)| < m/n$ for all $p^* \in E^*$ whenever $\|b - a\| \leqq \max \|x(u) - x(t)\|$ for all u, t in an interval of π_n. Let $z_n \in C(J)$ be the quasilinear function inscribed in x whose intervals of linearity are the elements of π_n. For each i and each $K \in \pi_n$, $M_i(z_n | K, t) < (m + m/n)m^{-1} M_i(x | K, t)$. Hence $I(f, z_n) = \sum I(f, z_n | K) = \sum P(f, z_n | K) < (1 + 1/n) \sum P(f, x | K) = (1 + 1/n)P(f, x)$. Thus $L(f, x) \leqq \liminf I(f, z_n) \leqq P(f, x) \leqq L(f, x)$.

It follows that I_f is lower semicontinuous with respect to uniform convergence on the class of absolutely continuous functions. Thus we have another proof of the Tonelli theorem.

Our discussion would be incomplete without some mention of a "curve." One reason for the importance of a *Fréchet curve* is that it allows us to use arc-length to parametrize a rectifiable curve [2]. An easy argument shows that L_f has the same value on all Fréchet equivalent functions, and is lower semicontinuous with respect to Fréchet convergence.

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† The statements in [11] refer to additive interval functions, and we require the less restricted functions of Banach. The theorems hold for these functions too. See page 116.

ALMOST DIAGONAL MATRICES

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1. Introduction. In the following, capital English letters will denote $N \times N$ matrices with real elements, lower case English letters will denote vectors, and A^t, x^t denote the transposed matrix A and vector x, respectively. We use (x, y) for the inner product of x and y.

DEFINITION. *A matrix A is almost diagonal (a.d.) if there exist a diagonal matrix D and vectors x and y such that*

$$A = D + xy^t. \tag{1}$$

That is, A is a.d. if it differs from a diagonal matrix by a matrix of rank one. The class of a.d. matrices is of importance in many applications. The following result is well known [1]:

LEMMA 1. *If D is nonsingular and*

$$(y, D^{-1}x) \neq -1 \tag{2}$$

then A is nonsingular and

$$A^{-1} = D^{-1}\left[I - \frac{xy^tD^{-1}}{1 + (y, D^{-1}x)}\right]. \tag{3}$$

2. The diagonal form. Suppose A is a.d., and ξ is an eigenvector of A corresponding to the eigenvalue λ. Then

$$\lambda\xi_i = \sum A_{ij}\xi_j = D_{ii}\xi_i + (y, \xi)x_i. \tag{4}$$

If (y, ξ) does not vanish, we may take $(y, \xi) = 1$ for normalization, and find

$$\xi_i = \frac{x_i}{\lambda - D_{ii}}, \tag{5}$$

and substituting back into (4) we find

$$\phi(\lambda) \equiv \sum_{i=1}^{N} \frac{x_iy_i}{\lambda - D_{ii}} = 1. \tag{6}$$

Thus the eigenvalues of A corresponding to eigenvectors which are not orthogonal to y satisfy (6). Now (6) fails to have N roots if either an x_i vanishes, or a y_i vanishes, or two or more of the D_{ii} are repeated, in any of which cases it is easy to see that (6) reduces to a polynomial equation of degree less than N

after suitable clearing. Now if a y_i vanishes for some i, then y is orthogonal to the eigenvector of D for the eigenvalue D_{ii} and (4) shows that that same vector is an eigenvector of A with the same eigenvalue. Similarly if an x_i vanishes, we may argue with A^t to the same conclusion that D_{ii} is an eigenvalue of A. Finally, if D_{ii} is a repeated diagonal element of D, say p times, then there are at least $p-1$ independent eigenvectors of D which are orthogonal to y, and all of these are eigenvectors of A with the eigenvalue D_{ii}. If p eigenvectors of D are orthogonal to y, then p components of y vanish, and we are back to the previous case. Summarizing, we have

LEMMA 2. *The roots of* (6) *are the eigenvalues of* A. *If a term fails to appear in* (6) *because its numerator vanishes, then the corresponding element of* D *is an eigenvalue of* A. *If* p *terms of* (6) *all have the same denominator, so that one can collect terms, then the corresponding element of* D *is a* $(p-1)$*-fold eigenvalue of* A. *All of the eigenvalues of* A *are accounted for by the above.*

In the following we shall assume that no components of x or y vanish.

Now suppose the D_{ii} are distinct, and let π be the permutation of the integers $1, 2, \cdots, N$ such that

$$D_{\pi(1),\pi(1)} < D_{\pi(2),\pi(2)} < \cdots < D_{\pi(n),\pi(n)}. \tag{7}$$

Then for any $\alpha = 1, 2, \cdots, N$,

$$\phi(D_{\alpha\alpha} + \epsilon) = \frac{x_\alpha y_\alpha}{\epsilon} + O(1) \qquad (\epsilon \to 0), \tag{8}$$

$$\phi(O_{\alpha\alpha} - \epsilon) = -\frac{x_\alpha y_\alpha}{\epsilon} + O(1) \qquad (\epsilon \to 0), \tag{9}$$

and we have

LEMMA 3. *Suppose the* D_{ii} *are distinct, and that for some* k,

$$\operatorname{sgn}(x_{\pi(k)} y_{\pi(k)}) = \operatorname{sgn}(x_{\pi(k+1)} y_{\pi(k+1)}). \tag{10}$$

Then an eigenvalue of A *lies between* $D_{\pi(k),\pi(k)}$ and $D_{\pi(k+1),\pi(k+1)}$.

Clearly if A is symmetric, (10) always holds. In any case

LEMMA 4. *If*

$$\operatorname{sgn}(x_i y_i) = \textit{const.} \qquad (i = 1, 2, \cdots, N) \tag{11}$$

and the D_{ii} *are distinct, then all the eigenvalues of* A *are real, distinct, and separated by the* D_{ii}.

Letting $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the roots of (6), the matrix

$$P_{ij} = \frac{x_i}{\lambda_j - D_{ii}} \qquad (i, j = 1, 2, \cdots, N) \tag{12}$$

by (5) is the matrix whose columns are the eigenvectors of A, and if the λ_j are distinct, we have

$$A = P\Lambda P^{-1}, \tag{13}$$

with

$$\Lambda_{ij} = \lambda_i \delta_{ij}. \tag{14}$$

3. The matrix P^{-1}:

Our purpose now is to exhibit the elements of P^{-1} in closed form.

LEMMA 5. *Suppose the eigenvalues of A are distinct. Then we have*

$$(P^{-1})_{ij} = \frac{1}{(\lambda_i - D_{jj})\psi'(\lambda_i)}, \tag{15}$$

where

$$\psi(z) = 1 - \sum_{i=1}^{N} \frac{x_i y_i}{z - D_{ii}}. \tag{16}$$

To prove this, we make use of the Cauchy integral formula in matrix form [2], namely

THEOREM 2. *Let M be an $N \times N$ matrix with distinct eigenvalues. Let C be a simple, closed contour in the complex plane which encloses a domain Δ. Suppose that all the eigenvalues of M lie in Δ, and let $f(z)$ be a function regular in Δ. Then*

$$f(M) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(zI - M)}. \tag{17}$$

Now let A be a.d. with distinct eigenvalues, and take $f(z) = 1$ in the above theorem. Then

$$I = \frac{1}{2\pi i} \oint_C (zI - A)^{-1} dz. \tag{18}$$

Clearly $zI - A$ is also a.d., and by (3),

$$(zI - A)^{-1} = (zI - D)^{-1} + \frac{1}{\psi(z)} (zI - D)^{-1} x y^t (zI - D)^{-1}. \tag{19}$$

We see that the poles of the integrand are at the zeros of $\psi(Z)$, that is, at the λ_i, and that if C has been deformed to exclude the D_{ii}, the first term of (19) contributes nothing to (18).

Further,

$$\text{Res}\left\{(zI - A)^{-1}_{ij}\right\}_{z=\lambda_p} = \frac{x_i y_j}{(\lambda_p - D_{ii})(\lambda_p - D_{jj})\psi'(\lambda_p)},$$

and (18) gives

$$\delta_{ij} = \sum_{p=1}^{N} \frac{x_i y_j}{(\lambda_p - D_{ii})(\lambda_p - D_{jj})\psi'(\lambda_p)} = \sum_{p=1}^{N} P_{ip}(P^{-1})_{pj}. \tag{20}$$

Comparison with (12) shows that (15) and (16) hold, which completes the proof of Lemma 5.

4. Summary. We summarize the preceding developments with

THEOREM 2. *Let A be a.d. with distinct eigenvalues, and let $f(z)$ be regular in some region containing all the eigenvalues of A. Then*

$$f(A) = Pf(\Lambda)P^{-1}, \tag{21}$$

where P is given explicitly by (12), *P^{-1} by* (15), (16), *and Λ by* (14) *where the λ_i satisfy* (6).

Theorem 2 permits, for example, the explicit solution of a system of linear ordinary differential equations with an a.d. coefficient matrix.

We conclude with

THEOREM 3. *Let A be a.d. and symmetric, so that*

$$A = D + \sigma xx^t. \tag{22}$$

Suppose the elements of D are distinct and that no x_i vanishes. Then, necessary and sufficient conditions that A be nonnegative definite are

$$1^\circ \sum_{i=1}^{N} \frac{x_i^2}{D_{ii}} \leqq -\frac{1}{\sigma} \text{ and } 2^\circ\ D_{11} \geqq 0, \text{ if } \sigma \leqq 0;$$

$$1'\ D_{22} \geqq 0 \text{ and either } 1^\circ \text{ or } 2^\circ, \text{ if } \sigma > 0.$$

(In the above, the D_{ii} have been renumbered in ascending algebraic order.)

The proof follows by inspection of the characteristic equation.

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A LOOK AT MATHEMATICAL COMPETITIONS IN HUNGARY

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The readers of this MONTHLY may be interested in some up-to-date information on mathematical competitions in Hungary. Contrary to the implication in the recent article by R. C. Buck [1], the Eötvös prize competition did not terminate in 1928. It is, in fact, still held annually (except for the years 1919–1921, 1944–1946, 1956) so that in the autumn of 1960, the sixtieth competition will be held. Besides the winners mentioned in [1], one may also mention, T. Gallai, L. Kalmár, D. König, F. Lukács, G. Sándor, M. Schweitzer, T. Szele, E. Teller, L. Tisza, and E. Vázsonyi, so that the relationship between this contest and mathematical fertility in Hungary seems to continue. Since 1949, the mathematical contest has been called the Kürschák prize competition, and the Eötvös competition limited to physics—certainly a more suitable choice.

Information about the contest can be found in issues of the journal *Középiskolai Matematikai Lapok* [Mathematical Journal for High Schools], and in the book [2] by Kürschák, mentioned in [1], which was re-edited in 1955 and to which a second volume was added in 1957 [3].

It should be said that the contest is now open to younger students than high school graduates. Indeed, 12 such contestants have won prizes since 1947. It might be of interest to give here the problems on the 1958 contest [4]. There were a total of 417 contestants, of whom only 317 handed in their work.

1. Consider six points in the plane of which no three are collinear. Show that three may be chosen to form a triangle in which one angle is greater than or equal to 120°.

2. If u and v are integers such that u^2+uv+v^2 is a multiple of 9, show that u and v must both be multiples of 3.

3. In the convex hexagon $ABCDEF$, suppose that every pair of opposite sides are parallel. Prove that the triangles ACE and BDF have equal areas.

Contestants have five hours to work on three problems. They may use books or other references. Stress is laid on ability and insight, with bonus points given for generalization. Problems are proposed by research mathematicians. In an evaluating lecture, many possible solutions and generalizations are given. The contest is administered by the sections of the J. Bolyai Mathematical Society (analogous, I believe, to a combination of the MAA and the AMS, in the United States).

The reader might be interested in other mathematical contests at present held in Hungary. Describing these by the admission age of contestants, there are:

Age:	
15	Dániel Arany prize for beginners.
16	Dániel Arany prize for advanced.
17–18	National Students Competition.

(Dániel Arany was founder in 1894 and first editor of our high school mathematics journal.) These contests began in 1947. At the present time, all are two-stage contests, each round consisting of three questions. The first round is held in schools, and the second at central locations in the 19 divisions of the country. In 1958, there were respectively 2145, 1666, 1951 entrants to the first round and 189, 79, 323 contestants admitted to the second round, in the three contests. As an example, we give here the problems on the contest for beginners [5].

First round

1. Prove that any two numbers between 10 and 99 with the same tens digit can be multiplied thus: remove the units in the first factor and add to the second, then multiply the resulting numbers, and add to the result the product of the units in the original factors.

Example: (23) (28) = (20) (31) + (3) (8) = 644

2. In the right triangle ABC (with $A=90°$) construct a circle with AC as diameter. Let it meet BC in E. Draw a tangent to this circle at E, meeting the remaining side in D. Show that EBD is an isoceles triangle.

3. An arc AD of a circle is divided into three equal arcs by the points B and C. Is the chord AD also divided equally by the radii to B and C?

Second round

1. For what integer values of b is $97b^2+84b-55$ a multiple of b?

2. Find the missing numbers (indicated by X) in the following division problem.

$$\begin{array}{r|l} & XX8 \\ \hline XXX & XXXXXX \\ & \underline{XXX5} \\ & XXXX \\ & \underline{9XX} \\ & XXX \\ & \underline{XXX} \\ & 0 \end{array}$$

3. Let Q be a fixed point that is not on the perimeter of a given square $ABCD$. For any choice of a point P on the perimeter, choose a point R so that PQR is an equilateral triangle. As P moves along $ABCD$, what is the path traced out by R?

In addition to these contests, the high school journal also conducts several problem-solving contests. There are usually 72 elementary and 72 advanced problems proposed in each year. In 1957–58, 1691 students participated, and were rewarded by "points", with winners receiving various prizes, last but not least of which is to have their photographs appear in the journal.

There also are contests for university students. Let me mention here only the Schweitzer Memorial Competition sponsored by the Mathematical Society. This is held in all universities (students of age 19–24 in general) with problems proposed by research mathematicians. As a rule, there are about 10 problems in multiple-choice form which are to be worked by the contestants at home during a ten-day period. Here also evaluating lectures used to be held in all uni-

versity seats. As an example, we give here some of the problems for the 1957 competition which was organized by the Debrecen section and given to 19 contestants [6].

1. Let (C_{ij}) be a real orthogonal matrix. Show that the mapping $w=f(z)=\sum_{i,j=1}^{3} C_{ij}z_i\bar{z}_j$ sends the surface $\sum_{i=1}^{3} z_i\bar{z}_i=1$ into a triangle.

3. Let A be a set of points in n-space which contains one interior point, and is such that if P and Q are in A, so is their mid-point. Show that A consists of a convex set and certain of its boundary points.

4. For $0<\epsilon<1$, let $\mathfrak{F}_\epsilon$ be the class of non-negative continuous functions $f(x)$ defined for $0\leq x<\infty$ and such that for all x_1, x_2

$$f(x_1)f(x_2) \leqq \epsilon^{|x_1-x_2|}.$$

Evaluate the expression

$$C(\epsilon) = \sup_{f\in\mathfrak{F}_\epsilon} \int_0^\infty f(x)dx.$$

5. Determine the real continuous functions satisfying the equation

$$f(xyz) = f(x) + f(y) + f(z)$$

(a) for all x, y, z different from 0,

(b) for all x, y, z in the interval $[a, b]$, where $1<a^3<b$.

7. Show that every real number x in the open interval (0, 1) can be written in the form $x=\sum_1^\infty 1/n_j$ where the n_j are positive integers, and n_{j+1}/n_j is always one of the integers 2, 3, 4.

8. Find all integers $c>1$ such that, for every prime p, the smallest positive number integer n with $c^n\equiv 1 \pmod{p}$ is different from 6.

10. We call an abelian group splitting if the subgroup consisting of elements of finite order is a direct summand. Show that an abelian group G is splitting if, for some integer m, the group mG, *i.e.*, {all mg for $g\in G$} is splitting.

In 1952, a competition was held for engineering and technological students, but was discontinued because their backgrounds seemed too disparate [7]. My personal opinion is that common problems could be found, and these contests continued also.

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AN EXISTENCE THEOREM FOR CERTAIN AUTONOMOUS SYSTEMS

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1. The usual existence proofs for ordinary differential equations use either the Picard method of successive approximations or the Cauchy method of sequences of approximate solutions, [1]. Both methods are very wide in scope but each one, by itself, is of local character. That is, the solution is shown to exist for a limited range of the independent variable.

In this note we consider a very narrowly restricted class of equations. For these we present an existence theorem whose proof uses neither of the above-mentioned methods and, in fact, it calls for nothing more advanced than the Weierstrass M-test, [2, p. 3]. Moreover, it yields the existence of analytic solutions for all real values of the independent variable. Finally, the method can be used, either by itself or in conjunction with Picard's or Cauchy's, on certain integral, integrodifferential and other functional equations, [3], [4]. However, it must be added that one can easily prove by Picard's method a result considerably more general than Theorem 1 below.

2. All variables, functions, and constants in the sequel are real. Accordingly, a function is everywhere analytic if for any real s_0 it can be expanded in a power series convergent in some interval about s_0. A system of differential equations is called autonomous, [1], if the independent variable does not occur in it explicitly. A first integral of a system E of differential equations, [5], is any equation which contains an arbitrary constant, is satisfied by the unknowns of E, and in which at least one of them occurs to lower derivative order than in E. For example, the autonomous system

$$x'' = -xy^2, \qquad y'' = -yx^2, \tag{1}$$

where $x=x(s)$, $y=y(s)$ and accents denote differentiation, has the first integral

$$x'^2 + y'^2 + x^2y^2 = c^2, \tag{2}$$

obtained by multiplying the first equation by x', the second one by y', adding the results and integrating the sum. System (1) occurs in the isoperimetric problem of maximizing the volume of the convex hull of a closed rectifiable space curve whose length is kept fixed, [6].

We consider the following autonomous system of ordinary differential equations together with initial conditions:

$$\begin{aligned} x_j^{(K_j)} &= f_j(x_1, x_2, \cdots, x_J, x_1', \cdots, x_J', x_1^{(p_j)}, \cdots, x_J^{(p_j)}), \\ x_j^{(k_j)} &= \xi_{jk_j}, \quad k_j = 0, 1, \cdots, K_j - 1, \quad j = 1, 2, \cdots, J. \end{aligned} \tag{3}$$

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Here K_j is a positive integer, p_j a nonnegative one and $K_j > p_j$. The J unknowns $x_1, \cdots, x_J$ are functions of the independent variable s which is usually omitted, and upper indices denote differentiation with respect to s. The system (3) is called a polynomial one if each f_j is a polynomial in the indicated variables. It is then assumed that no polynomial f_j vanishes identically and that each f_j involves at least one term with a derivative of order p_j. Henceforth a system will mean a polynomial system. Let one of the monomial terms of f_j be

$$A[x_1^{(a_1)}]^{b_1}[x_1^{(a_2)}]^{b_2} \cdots [x_2^{(c_1)}]^{d_1}[x_2^{(c_2)}]^{d_2} \cdots, \tag{4}$$

where A is a constant and $0 \leqq a_1 < a_2 < \cdots$, $0 \leqq c_1 < c_2 < \cdots$; $b_1, b_2, \cdots$, $d_1, d_2, \cdots \geqq 1$. With this term we associate the weight

$$W = b_1(a_1 + 1) + b_2(a_2 + 1) + \cdots + d_1(c_1 + 1) + \cdots.$$

Let W_j be the maximum of the weights of the terms of f_j.

The system (3) is called bounded if it possesses a first integral, or a system of first integrals, implying the existence of majorizing functions ϕ_j, such that

$$|x_j^{(K_j-1)}(s)| \leqq \phi_j(|s|), \qquad j = 1, 2, \cdots, J, \tag{5}$$

where each $\phi_j(u)$ is a nonnegative nondecreasing continuous function for all $u \geqq 0$.

We shall prove

THEOREM 1. *A bounded polynomial system which satisfies the weight condition*

$$K_j \geqq W_j - 1, \qquad j = 1, 2, \cdots, J, \tag{6}$$

has a solution, necessarily unique, which is analytic for all s.

An example to which Theorem 1 applies is the system (1) with arbitrary initial conditions.

3. We assemble now some preliminary estimates and lemmas. If p and q are nonnegative integers we shall abbreviate $(p+q)!/p!$ to $[p]_q$. This will be used in the following context: if $x = \sum_0^\infty a_n s^n$ then $x^{(K)} = \sum_0^\infty a_{n+K}[n]_K s^n$.

LEMMA 1. *Let the constants $W, a_1, a_2, \cdots, b_1, \cdots, c_1, \cdots$ satisfy the conditions listed under* (4) *and let $K \geqq W - 1$. Then*

$$[N]_K \geqq S = \sum [n_1]_{a_1}[n_2[_{a_1} \cdots [n_{b_1}]_{a_1}[m_1]_{a_2}[m_2]_{a_2} \cdots [m_{b_2}]_{a_2}[p_1]_{c_1} \cdots, \tag{7}$$

where $N = 0, 1, \cdots$ and the sum S is taken over all nonnegative solutions of $n_1 + n_2 + \cdots + m_1 + m_2 + \cdots + p_1 + \cdots = N$, permutations counted distinct.

It is first verified that S is the coefficient of s^N in the power series of

$$F(s) = [f^{(a_1)}]^{b_1}[f^{(a_2)}]^{b_2} \cdots [f^{(c_1)}]^{d_1} \cdots,$$

where $f(s) = 1/(1-s)$. Since $f^{(a)} = a!/(1-s)^{a+1}$ and $[f^{(a)}]^b = (a!)^b/(1-s)^{b(a+1)}$,

$$S = (a_1!)^{b_1}(a_2!)^{b_2} \cdots [N]_{W-1}/(W-1)!.$$

By the hypothesis $K \geqq W-1$ and in order to prove (7) it suffices to show that $(a_1!)^{b_1}(a_2!)^{b_2} \cdots \leqq [b_1(a_1+1)+b_2(a_2+1)+ \cdots -1]!$. Since $b_1, b_2, \cdots \geqq 1$, the latter is clear and (7) is proved.

We define now the index I_j of the polynomial f_j to be the sum of the absolute values of the coefficients of f_j.

LEMMA 2. *In proving Theorem* 1 *it may be assumed without loss of generality that the indices* I_j *of the* f_j*'s in* (3) *satisfy the condition*

$$0 < I_j \leqq 1, \qquad j = 1, 2, \cdots, J. \tag{8}$$

This is obvious since a linear transformation $s \to \lambda t$ of the independent variable achieves the purpose. Condition (6) guarantees then that the new indices can be made arbitrarily small. The condition of boundedness still holds for the new system.

From the hypothesis of boundedness, expressed by (5), we get by an easy induction on K

$$|x_j^{(K_j-K)}(s)| \leqq \int_0^{|s|} \cdots \int_0^u \int_0^u \phi_j(u)dudu \cdots du + \sum_{q=2}^{K} |\xi_{jK_j-q}| \, |s|^{K-q}/(K-q)!,$$

where $K = 2, 3, \cdots, K_j$. Using the standard transformation of the above $(K-1)$-tuple integral we get

$$|x_j^{(K_j-K)}(s)| \leqq 1/(K-2)! \int_0^{|s|} (|s| - u)^{K-2} \phi_j(u)du + \sum_{q=2}^{K} |\xi_{j\,K-q}| \, |s|^{K-q}/(K-q)! = \phi_{jK}, \tag{9}$$

where we also define ϕ_{j1} to be ϕ_j. All the majorants $\phi_{jK}(u)$ are finite nonnegative nondecreasing functions for $u \geqq 0$.

4. Assume the following power series expansions

$$x_j(s) = \sum_{n=0}^{\infty} \alpha_{nj} s^n, \qquad j = 1, 2, \cdots, J. \tag{10}$$

Substituting these and their derivatives into (3) and comparing the coefficients of s^N we have

$$[N]_{K_j} \alpha_{N+K_j\,j} = \sum Q_r, \tag{11}$$

where on the right there is a finite number of summands, each one contributed

by a single term of f_j. Consider now the sample term (4) of f_j—the corresponding Q_{r_1} term is

$$Q = AS\alpha_{n_1+a_1\,1}\alpha_{n_2+a_1\,1}\cdots\alpha_{nb_1+a_1\,1}\alpha_{m_1+a_2\,1}\alpha_{m_2+a_2\,1}\cdots\alpha_{p_1+c_1\,2}\cdots, \tag{12}$$

where the numerical factor S is the sum in (7). Let

$$R_1 = \max\,(1,\ |\,\xi_{jk_j}\,|\,/k_j!), \tag{13}$$

the maximum being taken over $k_j=0,\ 1,\ \cdots,\ K_j-1,\ j=1,\ 2,\ \cdots,\ J$; (13) is equivalent to

$$|\,\alpha_{k_j\,j}\,| \leqq R_1 \leqq R_1^{k_j+1}. \tag{14}$$

Suppose now that (14) holds for $k_j \leqq N+K_j-1,\ j=1,\ 2,\ \cdots,\ J$. Then by (11)

$$[N]_{K_j}\,|\,\alpha_{N+K_j\,j}\,| \leqq \sum |\,Q_r\,|\,, \tag{15}$$

and by (12), Lemma 1, the induction assumption and the weight condition (6) we have

$$|\,Q_{r_1}\,| \leqq [N]_{K_j}\,|\,A\,|\,R_1^{N+W} \leqq [N]_{K_j}\,|\,A\,|\,R_1^{N+K_j+1}.$$

Therefore by (8) and (15)

$$|\,\alpha_{N+K_j\,j}\,| \leqq R_1^{N+K_j+1}\sum |\,A\,| = I_jR_1^{N+K_j+1} \leqq R_1^{N+K_j+1}.$$

This completes the induction and therefore

$$|\,\alpha_{kj}\,| \leqq R_1^{k+1},\quad k = 0,\ 1,\ \cdots;j = 1,\ 2,\ \cdots,\ J.$$

It follows that the power series (10) converge for $0 \leqq s < s_1 = 1/R_1$.

We choose now a small positive ϵ_1 and repeat the power series procedure at $s_1-\epsilon_1$ obtaining a continuation of the solutions $x_1(s),\ \cdots,\ x_J(s)$ from $0 \leqq s < s_1$ to $0 \leqq s < s_1+s_2-\epsilon_1$, where $s_2=1/R_2$ and R_2, like R_1, is given by

$$R_2 = \max\,(1,\ |\,x_j^{(k_j)}(s_1-\epsilon_1)\,|\,/k_j!),$$

the maximum being taken as in (13). Similarly, take a small $\epsilon_2>0$ and repeat the process at $s_1+s_2-\epsilon_1-\epsilon_2$, and so on. In n steps one obtains a continuation of the original power series (10) to the interval

$$0 \leqq s < \sum_{i=1}^{n} s_i - \sum_{i=1}^{n-1}\epsilon_i,$$

where $s_i=1/R_i$ and

$$R_i = \max\left(1,\left|x_j^{(k_j)}\left(\sum_{h=1}^{i-1}s_h - \sum_{h=1}^{i-1}\epsilon_h\right)\right| \Big/ k_j!\right).$$

Now by the estimates (9), following from the boundedness hypothesis,

$$R_i \leqq \max\left(1, \phi_{j\,K_j-k_j}\left(\sum_{h=1}^{i-1} s_h\right) \Big/ k_j!\right) \tag{16}$$

since the majorants ϕ_{jK} are nondecreasing. The sum $\sum_1^n \epsilon_i$ can be made uniformly small and by (16) the sum $\sum_1^n s_i$ must diverge with n since otherwise one of the majorants $\phi_{jK}(u)$ would have become infinite for some finite positive u, which is a contradiction. It follows that the solutions $x_j(s)$ of (3) are analytic for $s \geqq 0$. Their analyticity for $s \leqq 0$ is proved in exactly the same way and their uniqueness follows from the analyticity.

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MATHEMATICAL NOTES

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ON THE PROOF OF MERGELYAN'S APPROXIMATION THEOREM

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During the past year, the writer had the occasion to give an exposition in a seminar of the proof of the well-known theorem of Mergelyan [2], which states that a necessary and sufficient condition on a compactum K in the complex z-plane in order that every function f continuous on K and analytic in the interior of K should be uniformly approximable on K by polynomials in z, is that K should not disconnect the plane. Mergelyan's proof of this theorem, as given in [2] and Chapter 1 of [3], is quite elementary except at one point, where he makes use of a result for whose proof he refers to the general theory of capacity of planar sets. At first glance, this step does not appear to have the same elementary character as the rest of the proof. Upon closer examination, however, the writer found it possible to construct a direct and simple proof of the needed result. The method of proof has a superficial resemblance to the proof of the

"Flächensatz" in Bieberbach [1], pages 72–73, but argues on arc length rather than area.

PROPOSITION. *Let* $z=F(w)=aw+b+\sum_{j\geq 1} a_j w^{-j}$, *be a holomorphic function of* w *for* $|w|>1$ *with a simple pole at infinity. Suppose that* F *is continuous for* $|w|\geqq 1$, *that* F *maps the set* $|w|>1$ *into the complement of a compact subset* S *of the* z*-plane, and that* F *maps* $|w|=1$ *into the disk* $|z|\leqq 1$. *Suppose further that* S *has diameter* $\geqq k$, *for some positive number* k. *Then*

(α) $|a|\leqq 1$, $|b|\leqq 1$, $|a_j|\leqq 1$ *for all* $j\geqq 1$;

(β) $|a|\geqq g(k)>0$, *where* $[g(k)]^2=\max_{0<u<1}[k^2\pi^{-2}u-u^2(1+u)(1-u)^{-3}]$.

Proof. We begin with some elementary topological remarks.

(1) Since F maps the set $|w|=1$ into the disk $|z|\leqq 1$, and since F is continuous for $|w|\geqq 1$, given $\epsilon>0$, F must map the circle $C_r=\{w: |w|=r\}$ into the disk $|z|\leqq 1+\epsilon$ for $r>1$ and sufficiently close to 1.

(2) Let C_r' be the image of C_r under F. If z is a point in the complement of C_r' and C_r' is oriented in concordance with the usual positive direction on C_r, $0(z, r)$, the order of C_r' with respect to z, is well defined. For z in S, $0(z, r)$ is defined for all $r>1$ and hence independent of r. Since for r very large, $0(z, r)$ is different from zero, it must be nonzero for all $r>1$ for every z in S.

(3) Let z_0 and z_1 be two points of S with $|z_1-z_0|\geqq k$. For all real t, we define $z_t=z_0+t(z_1-z_0)$. We assert that there must exist t_1 and t_2, $t_1<0$ and $t_2>1$, such that z_{t_1} and z_{t_2} lie in C_r'. (Otherwise, $0(z_t, r)$ would be well defined and constant for $t\leqq 0$ and $t\geqq 1$. But $0(z_t, r)=0$ for $|t|$ sufficiently large, while $0(z_0, r)$ and $0(z_1, r)$ are different from zero and we would have a contradiction.) Obviously $|z_{t_1}-z_{t_2}|>k$, and the diameter of C_r' is greater than k. But C_r' is a rectifiable curve for $r>1$, and for a closed rectifiable curve, the arc length is at least twice the diameter. Thus, if $L(C_r')$ is the length of C_r', $L(C_r')>2k$ for all $r>1$.

Proof of (α). Let $r>1$, j an integer $\geqq -1$. Then,

$$\frac{1}{2\pi i}\int_{C_r} F(w)w^{j-1}dw=\begin{cases} a & j=-1,\\ b & j=0,\\ a_j & j\geqq 1.\end{cases}$$

Since $|F(re^{i\phi})|\leqq(1+\epsilon)$ for r sufficiently close to 1, it follows that $|a|\leqq(1+\epsilon)r^{-1}$, $|b|\leqq(1+\epsilon)$, $|a_j|\leqq(1+\epsilon)r^j$ for all r sufficiently close to 1, whence $|a|\leqq 1$, $|b|\leqq 1$, $|a_j|\leqq 1$.

Proof of (β). By the conclusion of Remark 3, above, we have

$$\text{(I)}\quad 4k^2r^{-2}\leqq r^{-2}[LC_r')]^2=r^{-2}\left(\int_0^{2\pi}|F'(re^{i\phi})|\,rd\phi)\right)^2\leqq 2\pi\int_0^{2\pi}|F'(re^{i\phi})|^2d\phi.$$

We note that

$$\text{(II)} \qquad F'(re^{i\phi}) = a - \sum_{j\geq 1} j a_j r^{-j-1} e^{-(j+1)\phi}.$$

Applying the Parseval equality to (II), we obtain

$$\text{(III)} \qquad \int_0^{2\pi} |F'(re^{i\phi})|^2 d\phi = 2\pi\left[|a|^2 + \sum_{j\geq 1} j^2 |a_j|^2 r^{-2(j+1)}\right].$$

Substituting (III) into (I) and applying the inequalities of (α) to the $|a_j|$, we obtain

$$|a|^2 \geq k^2\pi^{-2}r^{-2} - \sum_{j\geq 1} j^2 r^{-2(j+1)},$$

for all $r>1$. If we set $u=r^{-2}$, $0<u<1$, we obtain

$$\text{(IV)} \qquad |a|^2 \geq \max_{0<u<1} [k^2\pi^{-2}u - u^2(1+u)(1-u)^{-3}] = [g(k)]^2.$$

Indeed, a simple calculation shows that the maximum above is positive.

For Mergelyan's proof in [2], only the special case $k=\frac{1}{4}$ is necessary. We remark, finally, that the inequality of (β) can be strengthened under the assumption that the images under F of the sets $|w|>1$ and $|w|=1$ are disjoint (an assumption which holds if F is one-to-one for $|w|>1$. In that case, we let S_1 be the complement of $F(|w|>1)$. The diameter of S_1, which we denote by k_1, will be greater than k. By translating z, we may assume that 0 lies in S_1, with our only expense a change in b. Then S_1 lies in the set $|z|\leq k_1$, and the image under F of $|w|=1$ will lie in S_1. The inequalities (α) on $|a_j|$ become $|a_j|\leq k_1$, and proceeding as above, we obtain instead of (IV),

$$\text{(V)} \qquad |a|^2 \geq |k_1|^2[g(1)]^2 \geq k^2[g(1)]^2.$$

Hence $|a|/k$ is bounded from below.

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A NOTE ON THE EXTENSION OF LINEAR FUNCTIONALS

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Bourbaki [1] has suggested a method of deducing the Hahn-Banach extension theorem from its "geometrical form" (essentially Mazur's theorem on the separation of convex sets by hyperplanes). The purpose of this note is to show that the method can be made to yield an extension theorem, similar to one re-

cently announced by Nakano [2], which is somewhat sharper than the original Hahn-Banach theorem [3].

We define a *convex functional*, on a real vector space $\mathbf{X}$, to be a real-valued function g such that

$$g(\lambda\mathbf{x} + (1 - \lambda)\mathbf{x}') \leqq \lambda g(\mathbf{x}) + (1 - \lambda)g(\mathbf{x}'),$$

whenever $\mathbf{x}$, $\mathbf{x}'$ belong to $\mathbf{X}$ and $0 \leqq \lambda \leqq 1$. For this condition to be satisfied, it is evidently sufficient, but not necessary, that

$$g(\mathbf{x} + \mathbf{x}') \leqq g(\mathbf{x}) + g(\mathbf{x}') \quad \text{and} \quad g(\rho\mathbf{x}) = \rho g(\mathbf{x}),$$

for all $\mathbf{x}$, $\mathbf{x}'$ in $\mathbf{X}$ and all $\rho \geqq 0$. Thus the following theorem includes the Hahn-Banach theorem as a special case.

Let g be a convex functional on a real vector space $\mathbf{X}$, let $\mathbf{X}_0$ be a vector subspace of $\mathbf{X}$, and let f_0 be a linear functional on $\mathbf{X}_0$ such that $f_0(\mathbf{x}) \leqq g(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbf{X}_0$. Then there is a linear functional f on $\mathbf{X}$ such that $f(\mathbf{x}) = f_0(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbf{X}_0$, and $f(\mathbf{x}) \leqq g(\mathbf{x})$ for all $\mathbf{x}$ in $\mathbf{X}$.

To prove this theorem, let $\mathbf{X}^\#$ be the direct sum of $\mathbf{X}$ and the field of real numbers. Let C be the set of all points $(\mathbf{x}, \xi)$ of $\mathbf{X}^\#$ which lie above the graph of g, in the sense that $g(\mathbf{x}) < \xi$. The convexity of g ensures that C is a convex set. Let $(\mathbf{x}_0, \xi_0)$ be any point of C, and $(\mathbf{x}, \xi)$ any point of $\mathbf{X}^\#$. Then, if $0 \leqq \lambda \leqq 1$,

$$\begin{aligned} g(\mathbf{x}_0 + \lambda\mathbf{x}) - (\xi_0 + \lambda\xi) &= g(\lambda(\mathbf{x}_0 + \mathbf{x}) + (1 - \lambda)\mathbf{x}_0) - \xi_0 - \lambda\xi \\ &\leqq \lambda g(\mathbf{x}_0 + \mathbf{x}) + (1 - \lambda)g(\mathbf{x}_0) - \xi_0 - \lambda\xi \\ &= \lambda\{g(\mathbf{x}_0 + \mathbf{x}) - \xi_0 - \xi\} - (1 - \lambda)\{\xi_0 - g(\mathbf{x}_0)\}. \end{aligned}$$

Since $\xi_0 - g(\mathbf{x}_0) > 0$, it follows that, when λ is a sufficiently small positive number, $g(\mathbf{x}_0 + \lambda\mathbf{x}) - (\xi_0 + \lambda\xi) < 0$, so that the point $(\mathbf{x}_0, \xi_0) + \lambda(\mathbf{x}, \xi)$ belongs to C. This property, together with the fact that C is convex, ensures that C is an open set in the largest topology with which $\mathbf{X}^\#$ is a locally convex space.

Now let S be the graph of f_0, consisting of all points $(\mathbf{x}, f_0(\mathbf{x}))$ with $\mathbf{x}$ in $\mathbf{X}_0$. Then S does not meet C, and is a subspace of $\mathbf{X}^\#$: hence, by the separation theorem, there is a linear functional F on $\mathbf{X}^\#$ such that $F(\mathbf{x}, \xi) > 0$ when $(\mathbf{x}, \xi)$ is in C and $F(\mathbf{x}, \xi) = 0$ when $(\mathbf{x}, \xi)$ is in S. Since $(0, \xi)$ is in C when ξ is large, $F(0, 1) > 0$. Writing

$$f(\mathbf{x}) = \xi - F(\mathbf{x}, \xi)/F(0, 1),$$

we define on $\mathbf{X}$ a linear functional f, independent of ξ. When $\mathbf{x}$ is in $\mathbf{X}_0$, we can take $\xi = f_0(\mathbf{x})$, showing that $f(\mathbf{x}) = f_0(\mathbf{x})$. Also, for any $\mathbf{x}$ in $\mathbf{X}$, $f(\mathbf{x}) < \xi$ if $g(\mathbf{x}) < \xi$, so that f is dominated by g, as required.

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ON THE DERIVATIVE OF AN ANALYTIC FUNCTION

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Since the days of Riemann it has been the custom to define an analytic function $f(z)=u(x, y)+iv(x, y)$ by the requirement of differentiability, or, what amounts to the same thing, the requirement that the functions u and v possess total differentials and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x. \tag{1}$$

If these conditions hold in a neighborhood $|z-z_0|<\epsilon$ then, as shown in the textbooks on the subject, the four derivatives are necessarily continuous and $f'(z)=u_x+iv_x$ is again an analytic function in the neighborhood.

For people who like their mathematics neat it has always been a source of minor annoyance that the continuity and differentiability of $f'(z)$ could not be proved without recourse to complex integration. It apparently is felt that this procedure violates the time-honored division of the calculus into its differential and integral varieties.

In a recently published paper [1], R. L. Plunkett has succeeded in showing that the continuity of $f'(z)$ can be proved—with the help of a rather heavy dose of modern topological analysis—without even mentioning the notion of integration. The aim of the present note is much less ambitious. All the author wants to do is indicate a proof of the continuity and differentiability of $f'(z)$ which uses only ordinary Lebesgue integration and is formally very simple. The proof will be carried out under the simplifying assumption that $|f'(z)|<M<\infty$ in the given neighborhood. Purists may object that complex integration or, at any rate, Green's theorem is lurking in the background, barely visible but still there. Far from desiring to argue this point, the author is content to offer the following proof as a mere pedagogic device.

It is clearly sufficient to carry out the proof under the assumption that the conditions in question hold in the closed disk $|z|\leqq 1$. By a well-known theorem (see, e.g. [3], p. 368), the relation

$$\int_a^b g'(x)dx = g(b) - g(a)$$

holds if $g'(x)$ exists and is uniformly bounded in $[a, b]$. Hence

$$u(1, \theta) - u(0) = \int_0^1 u_r(r, \theta)dr, \qquad z = re^{i\theta}.$$

Both sides are continuous in θ, and thus

$$\int_0^{2\pi} u(1, \theta)d\theta - 2\pi u(0) = \int_0^{2\pi}\int_0^1 u_r(r, \theta)drd\theta. \tag{2}$$

We choose a small positive constant ϵ and remark that in the region $\epsilon\leqq r\leqq 1$,

$0 \leqq \theta \leqq 2\pi$, $u(r, \theta)$ is continuous and $u_r(r, \theta)$ exists. It follows from another well-known result (see, e.g. [2], p. 170) that u_r is measurable over this region. Since $|u_r| \leqq M$, it is also summable and Fubini's theorem may be applied. Hence

$$\int_0^{2\pi} \int_\epsilon^1 u_r(r, \theta) dr d\theta = \int_\epsilon^1 \int_0^{2\pi} u_r(r, \theta) d\theta dr.$$

By (1), $ru_r = v_\theta$, and we have

$$r \int_0^{2\pi} u_r(r, \theta) d\theta = \int_0^{2\pi} v_\theta(r, \theta) d\theta = v(r, 2\pi) - v(r, 0) = 0,$$

since v_θ exists and is uniformly bounded for $\theta \in [0, 2\pi]$. (2) thus leads to

$$\left| \int_0^{2\pi} u(1, \theta) d\theta - 2\pi u(0) \right| = \left| \int_0^{2\pi} \int_0^\epsilon u_r(r, \theta) dr d\theta \right| \leqq 2\pi\epsilon M,$$

which shows that $u(r, \theta)$ is subject to the mean-value theorem

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(1, \theta) d\theta.$$

The same procedure yields the corresponding result for $v(r, \theta)$, and we thus obtain

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta = \frac{1}{2\pi i} \int_{|z|=1} f(z) \frac{dz}{z}, \qquad z = e^{i\theta}. \tag{3}$$

The transformation $z = (\zeta + t)(1 + \bar{\zeta} t)^{-1} (|\zeta| < 1)$ maps $|z| \leqq 1$ onto $|t| \leqq 1$. Since the function

$$g(t) = f\left(\frac{\zeta + t}{1 + \bar{\zeta} t}\right)$$

is differentiable in $|t| \leqq 1$ if the same is true of $f(z)$ in $|z| \leqq 1$, we may apply (3). Hence,

$$f(\zeta) = g(0) = \frac{1}{2\pi i} \int_{|t|=1} g(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{|t|=1} f(z) \frac{dt}{t}.$$

But $t = (z - \zeta)(1 - \bar{\zeta} z)^{-1}$, i.e.,

$$\frac{dt}{t} = \left[\frac{1}{z - \zeta} + \frac{\bar{\zeta}}{1 - \bar{\zeta} z}\right] dz,$$

and we have

$$f(\zeta) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z - \zeta} dz + \frac{\bar{\zeta}}{2\pi i} \int_{|z|=1} \frac{f(z)}{1 - \bar{\zeta} z} dz. \tag{4}$$

If $|\zeta| = |\xi+i\eta| < 1$, differentiation with respect to ξ and η under the integral sign is permissible for trivial reasons. If we apply to (4) the operation $\frac{1}{2}(\partial/\partial\xi - i\partial/\partial\eta)$, the second integral drops out and we obtain the identity

$$f'(\zeta) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{(z-\zeta)^2}\,dz,$$

which contains the desired result.

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A CLASS OF PARTIALLY ORDERED SETS

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In this note we consider a class of partially ordered sets, which we call *fairly well-ordered*, and show that they have some simple, desirable properties. A linearly ordered set is fairly well-ordered if and only if it is well-ordered. Moreover, we show in Proposition 1 that any subset, finite cartesian product, or image under an order-preserving map of fairly well-ordered sets is fairly well-ordered.

Fairly well-ordered sets are characterized by any of the four properties which the following theorem proves equivalent. The most useful consequence of our results is probably the fact—no doubt already known to many—that the product of finitely many copies of the positive integers has property (d) below. This result is used by J. Ceder to prove a topological theorem in [1], and it was a conjecture by Ceder which provided the original motivation for this study.

The author is grateful for some very helpful conversations with J. Isbell, to whom the present form of this paper can largely be credited.

THEOREM 1. *The following properties of a partially ordered set X are equivalent.*

(a) *Every sequence $\{x_i\}_{i=1}^{\infty}$ in X has a subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ such that $x_{i_{n+1}} \geqq x_{i_n}$ for all n.*

(b) *Every infinite subset of X contains an infinite ascending chain (i.e., a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{n+1} > x_n$ for all n).*

(c) *X contains no infinite descending chain* (i.e. *no sequence $\{x_n\}_{n=1}^{\infty}$ with $x_{n+1} < x_n$ for all n), and no infinite, pairwise incomparable subset.*

(d) *Every $S \subset X$ has a finite subset $M(S)$ such that each $x \in S$ is $\geqq$ some $x' \in M(S)$.*

Proof. The implications $(a)\to(b)\to(c)\to(d)$ are trivial, and can be left to the reader to check. It remains to prove $(d)\to(a)$, for which we need the following lemma, known as König's infinity lemma:

LEMMA 1. *Let* $\{E_n\}_{n=1}^{\infty}$ *be a sequence of nonempty, pairwise disjoint, finite sets, and let* $f_n: E_{n+1} \to E_n$ *be a function for each* n. *Then we can select from each* E_n *an element* e_n *such that* $f_n(e_{n+1}) = e_n$ *for all* n.

Proof. Call $e \in E_n$ a successor of $e' \in E_m$ if $n > m$ and $[f_{n-1} \circ f_{n-2} \circ \cdots \circ f_m](e) = e'$. Since E_1 is finite and $\bigcup_{n=2}^{\infty} E_n$ infinite, there exists an $e_1 \in E_1$ with infinitely many successors. Since e_1 has finitely many successors in E_2, it follows that one of these, say e_2, has infinitely many successors. Continuing in this fashion, we inductively construct the desired sequence $\{e_n\}_{n=1}^{\infty}$, and that proves the lemma.

We now proceed with the proof that $(d) \to (a)$ in Theorem 1. By induction we shall define nonempty, finite sets I_n $(n = 1, 2, \cdots)$ of positive integers as follows: Pick I_1 such that $\{x_i\}_{i \in I_1} = M(\{x_i\}_{i=1}^{\infty})$. Having defined $I_1, \cdots, I_n$, let $a_n = \max(\bigcup_{i=1}^{n} I_i)$, and then pick I_{n+1} from among $\{k \mid k > a_n\}$ such that

$$\{x_i\}_{i \in I_{n+1}} = M(\{x_k \mid k > a_n\}).$$

It follows that for each n there is a function $f_n: I_{n+1} \to I_n$ such that $x_{f_n(i)} \leqq x_i$ for all $i \in I_{n+1}$.

By Lemma 1, we can pick from each I_n an element i_n such that $i_n = f(i_{n+1})$ for all n. But then $\{x_{i_n}\}_{n=1}^{\infty}$ is the desired subsequence of $\{x_i\}_{i=1}^{\infty}$, and the proof of the theorem is complete.

DEFINITION 1. *A partially ordered set is* fairly well-ordered *if it satisfies any of the four equivalent conditions in Theorem* 1 *above.*

PROPOSITION 1.

(1) *A linearly ordered set is fairly well-ordered if and only if it is well-ordered.*

(2) *A subset of a fairly well-ordered set is fairly well-ordered.*

(3) *The image of a fairly well-ordered set, under an order-preserving map* f (i.e. $x \leqq y$ *implies* $f(x) \leqq f(y)$), *is fairly well-ordered.*

(4) *The cartesian product of finitely many fairly well-ordered sets (with the natural order:* $(x_1, \cdots, x_n) \leqq (y_1, \cdots, y_n)$ *means* $x_1 \leqq y_1, \cdots, x_n \leqq y_n$), *is fairly well-ordered.*

Proof. Assertions (1) and (2) follow from any of the characterizations in Theorem 1; (3) follows from (a) or (d), while (4) follows from (a).

Only the last statement requires proof. First, observe that it suffices to prove (4) for the product of *two* spaces. So suppose that X and Y are fairly well-ordered, and let $\{(x_i, y_i)\}_{i=1}^{\infty}$ be a sequence in $X \times Y$. By (a), $\{x_i\}_{i=1}^{\infty}$ has a subsequence $\{x_{i_n}\}_{n=1}^{\infty}$ with $x_{i_{n+1}} \geqq x_{i_n}$ for all n. Again by (a), $\{y_{i_n}\}_{n=1}^{\infty}$ has a subsequence $\{y_{i_{n_p}}\}_{p=1}^{\infty}$ such that $\{y_{i_{n_{p+1}}}\} \geqq y_{i_{n_p}}$ for all p. It follows that $\{(x_{i_{n_p}}, y_{i_{n_p}})\}_{p=1}^{\infty}$ is the desired subsequence of $\{(x_i, y_i)\}_{i=1}^{\infty}$, and that completes the proof.

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ON THE SYMMETRY OF SEMISIMPLE RINGS

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By a semisimple ring we mean a ring containing no nonzero nilpotent left ideal and satisfying the descending chain condition for left ideals. It is known that for semisimple rings a complete symmetry prevails between concepts involving "right-hand" notions and between their "left-hand" analogues. Thus to any characterization of these rings there exists a "dual" characterization which arises if for the "right-hand" and "left-hand" concepts occurring in the original characterization we substitute the corresponding "left-hand" and "right-hand" ones. For example, by the well-known theorem of Noether an arbitrary ring R is semisimple if and only if it has a right unit element and can be decomposed into the direct sum of minimal left ideals (see [2], or [3], Sec. 123), and by the above-mentioned duality we obtain that the ring R is semisimple if and only if R has a left unit element and can be decomposed into the direct sum of minimal right ideals.* To the best of our knowledge the demonstration of the equivalence of the two dual characterizations of semisimple rings has been effected so far only in an indirect way, mostly with the aid of the self-dual Wedderburn-Artin theorem. It is the purpose of this note to give a simple and straightforward proof for the equivalence of the two dual noetherian characterizations.

We are going to prove the following theorem:

If a ring R has a right unit element and admits a decomposition into a direct sum of a finite number of minimal left ideals, then R has also a left unit element and can be decomposed into a direct sum of finitely many minimal right ideals, and conversely.

It will clearly be sufficient to prove the first assertion of the theorem.

Let e be a right unit element of the ring R, and let the direct decomposition

$$R = L_1 + \cdots + L_m \tag{1}$$

hold, where the L_i $(i=1, \cdots, m)$ are mininal left ideals of R. In view of (1) we have for e the decomposition

$$e = e_1 + \cdots + e_m \qquad (e_i \in L_i;\ i = 1, \cdots, m). \tag{2}$$

First we show that $L_i = Re_i$ $(i=1, \cdots, m)$ and $e_1, \cdots, e_m$ are orthogonal idempotents.† In view of (2) we have for each component of e, $e_i = e_ie = e_ie_1 + \cdots$

* In E. Noether's original formulation of the theorem there occurs a "two sided unit element." It was one of us who remarked that it is sufficient to postulate the existence of a one sided unit element [1].

† The proof of this assertion is essentially the same as that of Lemma 5 in §123 of [3]. We give here this proof only for completeness sake and in view of the circumstance that in our considerations we use only a right unit element instead of a two sided one.

$+e_i^2+\cdots+e_ie_m$, whence by the uniqueness of the decomposition the relations

$$e_ie_k=\begin{cases}e_i & \text{for } i=k,\\ 0 & \text{for } i\neq k,\end{cases} \tag{3}$$

follow. Since by $e_i^2=e_i(\neq 0)$ we have the relation $Re_i\neq 0$ and by $e_i\in L_i$ the relation $Re_i\subseteq L_i$, the minimality of L_i implies $Re_i=L_i$.

Let us now show that e is a left unit element in R. Consider an arbitrary element r of R, which, by (2), admits the decomposition

$$r=re=re_1+\cdots+re_m. \tag{4}$$

For a fixed i $(1\leqq i\leqq m)$ the set of all elements of the form

$$ere_i-re_i \tag{5}$$

is a left ideal $L(\subseteq Re_i)$ of R. By the minimality of Re_i we must have either $L=Re_i$ or $L=0$. The case $L=Re_i$ is, however, impossible since the product of two elements of (5) is always zero, and so L cannot contain the idempotent element e_i. Now $L=0$ implies

$$ere_i=re_i, \qquad (i=1,\cdots,m). \tag{6}$$

Hence, by (4), $er=ere_1+\cdots+ere_m=re_1+\cdots+re_m=re=r$ and this proves e to be a left unit element. Accordingly we have, by (2), for any element $r(\in R)$,

$$r=e_1r+\cdots+e_mr. \tag{7}$$

On the other hand, since from the relation $e_1r_1+\cdots+e_mr_m=0$ $(r_1,\cdots,r_m\in R)$ it follows by left multiplication with e_i $(i=1,\cdots,m)$ in view of (3) that $e_ir_i=0$, and hence any element of the ring R admits a unique representation as a sum of elements belonging to the right ideals $e_1R,\cdots,e_mR$, respectively. Consequently we have the direct decomposition

$$R=e_1R+\cdots+e_mR. \tag{8}$$

In order to prove the minimality of the right ideals e_iR $(i=1,\cdots,m)$, it will be sufficient to show that for an arbitrary fixed element $e_is(\neq 0;\ s\in R)$ there exists a $t(\in R)$ such that

$$e_is\cdot t=e_i. \tag{9}$$

First of all we remark that, by (4), the element $e_is(\neq 0)$ can be written in the form $e_is=e_ise_1+\cdots+e_ise_m$, where we may consider the component e_ise_k to be different from zero. Let us now consider the left ideal $Re_i\cdot e_ise_k$. By the idempotency of e_i and by the minimality of Re_k we have for this ideal,

$$Re_i\cdot e_ise_k=Re_k. \tag{10}$$

From (10) there follows $e_kRe_i\cdot e_ise_k=e_kRe_k$, and this assures the existence of an element $s^*(\in R)$ for which

$$e_k s^* e_i \cdot e_i s e_k = e_k \tag{11}$$

holds. Now we are going to prove that

$$e_i s e_k \cdot e_k s^* e_i = e_i. \tag{12}$$

In order to shorten our notations, we put $e_i s e_k \cdot e_k s^* e_i = z$. Multiplying equation (11) from the left by $e_i s e_k$ we obtain with the aid of $e_i s e_k \neq 0$ the relation $z \neq 0$, and again by (11) the equality

$$z^2 = z. \tag{13}$$

Finally, by the definition of z and by (3) we have

$$e_i z = z. \tag{14}$$

On the other hand, by $Rz \neq 0$, $Rz \subseteq Re_i$ and by the minimality of Re_i we have $Rz = Re_i$. From this $yz = e_i$ follows for a suitable element $y (\in R)$. However, by (13) this implies

$$e_i z = yz^2 = yz = e_i, \tag{15}$$

and (14) and (15) together prove (12). Since by (12) the equation (9) has the solution $t = e_k s^* e_i$, the right ideals $e_i R$ $(i = 1, \cdots, m)$ are minimal right ideals in R, and this completes the proof of the theorem.

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REMARKS ON THE EULER-MACLAURIN AND BOOLE SUMMATION FORMULAS

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This note contains a few simple observations which provide a point of view from which it is especially easy to see the structural similarity of the Euler-Maclaurin and Boole formulas and the analogy which each bears to Taylor's formula.

For any function f which is continuous on the closed interval $[x, x+1]$ let $If(x)$ denote the integral average $\int_0^1 f(x+t)dt$ and let $Af(x)$ denote the arithmetic average $\frac{1}{2}[f(x)+f(x+1)]$.

One sees immediately that for every polynomial ϕ there exists a unique polynomial ϕ^* such that $I\phi^* = \phi$ and a unique polynomial ϕ^{**} such that $A\phi^{**} = \phi$. (One sees also that ϕ^* and ϕ^{**} have the same degree as ϕ.)

We now define the Bernouilli polynomial B_n of degree n to be the unique polynomial solution of the equation $IB_n(x) = x^n$, and the Euler polynomial E_n of degree n to be the unique polynomial solution of the equation $AE_n(x) = x^n$.

(For the Euler polynomials the definition is classical. For the Bernouilli polynomials, for which the classical definition is stated in terms of the equation $\Delta B_n(x) = nx^{n-1}$, the present definition has the advantage that the troublesome arbitrary constant in the polynomial solution of $\Delta y = nx^{n-1}$ does not appear, even transiently. In conjunction with the present definition of B_n, the uniqueness of the polynomial solution of the equation $\mathrm{I}\phi^* = \phi$ can be used to simplify the derivations of the well-known properties of the Bernouilli polynomials. For example, the formula $B_n(x+h) = \sum_{s=0}^{n} \binom{n}{s} h^s B_{n-s}(x)$ follows at once from the fact that if $g_1(x)$ and $g_2(x)$ are the right and left members of the formula in question, then we see immediately that $\mathrm{I}g_1(x)$ and $\mathrm{I}g_2(x)$ are both equal to $(x+h)^n$, and therefore $g_1 = g_2$.)

The Euler-Maclaurin and Boole formulas can now be written

$$f(x) = \sum_{k=0}^{n} \frac{\mathrm{I}f^{(k)}(a)}{k!} B_k(x - a) + \text{remainder}, \tag{1}$$

$$f(x) = \sum_{k=0}^{n} \frac{\mathrm{A}f^{(k)}(a)}{k!} E_k(x - a) + \text{remainder}, \tag{2}$$

respectively, to be compared with each other and with Taylor's formula

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + \text{remainder}. \tag{3}$$

The summations appearing in the right members of (1), (2), and (3) are characterized as the uniquely determined polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$ of degree n such that $\mathrm{I}p_1^{(k)}(a) = \mathrm{I}f^{(k)}(a)$, $\mathrm{A}p_2^{(k)}(a) = \mathrm{A}f^{(k)}(a)$, and $p_3^{(k)}(a) = f^{(k)}(a)$, $(k = 0, 1, \cdots, n)$, respectively.

To find a formula for the remainder $R_n(x)$ in (1) we may proceed as follows: Let $h = x - a$, and assume that $0 \leqq h \leqq 1$. Then

$$R_0(x) = f(a + h) - \mathrm{I}f(a)B_0(h) = \int_0^1 [f(a + h) - f(a + t)]dt$$

$$= \int_0^1 \int_t^h f'(a + s)dsdt.$$

If the order of integration is reversed, the double integral can be rewritten as $\int_0^1 V(s, h)f'(a+s)ds$, where $V(s, h) = s$ $(0 < s < h)$, $s - 1$ $(h < s < 1)$. Hence

$$R_1(x) = \int_0^1 V(s, h)f'(a + s)ds - \int_0^1 f'(a + s)ds(h - \tfrac{1}{2}) = \int_0^1 W(s, h)f'(a + s)ds,$$

where $W(s, h) = s - h + \frac{1}{2}$ $(0 < s < h)$, $s - h - \frac{1}{2}$ $(h < s < 1)$. Thus $R_1(x) = -\int_0^1 \overline{B}_1(h - s)f'(a + s)ds$, where $\overline{B}_1(x) = B_1(x)$ $(0 \leqq x < 1)$ and $\overline{B}_1(x + 1) = \overline{B}_1(x)$ (all x). From here on the usual remainder formula

$$R_n(x) = -\int_0^1 \overline{B}_n(h-s)\frac{f^{(n)}(a+s)}{n!}\,ds$$

can be obtained by repeated integrations by parts, just as in the case of Taylor's formula with integral remainder. A similar discussion applies to the remainder term in (2).

In brief, the essential step in passing from Taylor's formula to the Euler-Maclaurin formula, or to the Boole formula, is the replacement of each derivative at a by the *integral average* of the derivative over $[a, a+1]$, or by the *arithmetic average* of the values of the derivative at the points a and $a+1$.

CLASSROOM NOTES

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THE EFFECTS OF SINGULAR COLLINEATIONS

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Generally in an elementary study of collineations, considerable attention is given to the classification of the nonsingular ones as to the nature of their fixed points, the conditions for their periodicity, etc. Also it is usually pointed out that when a collineation is singular there is a certain subspace of points which have no correspondent (since all their homogeneous coordinates become zero), while the remaining points of the space undergoing transformation are carried into another fixed subspace. For example, one often finds the statement that if a collineation in the plane ($\rho x_i' = \sum_{j=1}^{3} a_{ij}x_j$) has a matrix of rank 2, there is one point which has no correspondent, while the remaining points of the plane all have their correspondents on a fixed line. Seldom is anything said about a singular collineation having a matrix whose order exceeds its rank by more than one, and, though an exhaustive search has not been made, nowhere has any mention been found of the manner in which this "piling up" of points on a subspace is effected. The purpose of this note is to explain how this "piling up" is effected in the general case where an n-space, S_n, is subjected to a transformation whose matrix has rank $n-k$.

Before considering the most general case, perhaps we should restate what appears well known concerning a collineation in S_n whose matrix has rank n. Here it is easily shown that there is one singular point F, or an S_0, which has no correspondent, and that all other points of S_n are carried into a fixed S_{n-1}. The collineation then induces a collineation in the fixed S_{n-1}, and this collinea-

$$\rho x_i' = a_{i1}\left[\mu b_1 + \sum_{i=1}^{k+1} \lambda_i f_{i1}\right] + a_{i2}\left[\mu b_2 + \sum_{i=1}^{k+1} \lambda_i f_{i2}\right] + \cdots$$

$$+ a_{i,n+1}\left[\mu b_{n+1} + \sum_{i=1}^{k+1} \lambda_i f_{i,n+1}\right]$$

$$= \mu[a_{i1}b_1 + a_{i2}b_2 + \cdots + a_{i,n+1}b_{n+1}] + \lambda_1 \sum_{j=1}^{n+1} a_{ij}f_{ij} + \cdots + \lambda_{k+1} \sum_{j=1}^{n+1} a_{ij}f_{k+1,j}.$$

But since each of the points F_i is a point of the singular S_k, each of the summations above is zero, and we get

$$\rho x_i' = \mu \sum_{j=1}^{n+1} a_{ij}b_j;$$

thus we have shown that all points of the S_{k+1} which are outside the singular S_k are carried into one and the same point, the correspondent of the point B.

AN EXAMPLE IN DIFFERENTIAL EQUATIONS: THE N-BODY PROBLEM

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It is easy for a student in an elementary course in differential equations to acquire three inaccurate conceptions concerning the power series method: (i) that the method is suitable for a single equation but not for systems; (ii) that the method is suitable for linear equations only; (iii) that a solution by this method is not satisfactory unless the coefficients of the series are given explicitly as a function of n.

Misconception (iii) can be dispelled by pointing out that a recursive definition of the coefficients is more useful for computing than an explicit definition and can often be used directly in the ratio test for convergence of the series. Misconceptions (i) and (ii) can be dispelled by an example such as the one to follow, in which power series are obtained as (local) solutions of the n-body problem. The method follows [1] and [2] and has been programmed by Norman Levine for the IBM 704 computer.

Let there be given n particles distributed in space, the mass and position vector of the ith particle being denoted by M_i and $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3})$. In the sequel, the letters i, j, k, n, and p denote integers which satisfy (unless the contrary be stated) $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $j \neq i$, $1 \leqq h \leqq 3$, $0 \leqq k < \infty$, and $1 \leqq p \leqq 3$. Now define

$$R_{ij} = \sum_h (X_{ih} - X_{jh})^2, \qquad S_{ij} = (R_{ij})^{-3/2}. \tag{1}$$

By Newton's Law of Gravitation, the force exerted by M_j on M_i has magnitude GM_iM_j/R_{ij} and is directed along the unit vector $R_{ij}^{-1/2}(\mathbf{X}_j - \mathbf{X}_i)$. Thus the total force on M_i is

$$\sum_j GM_iM_j(\mathbf{X}_j - \mathbf{X}_i) \div R_{ij}R_{ij}^{1/2}.$$

By Newton's 2nd Law of Motion, then

$$\frac{d^2}{dt^2}\mathbf{X}_i = \sum_j GM_jS_{ij}(\mathbf{X}_j - \mathbf{X}_i). \tag{2}$$

By differentiation of the second of equations (1), we have

$$2R_{ij}\frac{d}{dt}S_{ij} + 3S_{ij}\frac{d}{dt}R_{ij} = 0. \tag{3}$$

Power series are next introduced:

$$X_{ih} = \sum_k X_{ihk}t^k, \qquad R_{ij} = \sum_k R_{ijk}t^k, \qquad S_{ij} = \sum_k S_{ijk}t^k. \tag{4}$$

Now let the series of (4) be substituted in equations (1), (3), and (2), the resulting products of series being formed by the Cauchy product formula. The identification of coefficients of like powers of t in the usual way leads to three recurrence relations:

$$R_{ijk} = \sum_h \sum_{v=0}^{k} [X_{ihv}X_{ih(k-v)} - 2X_{ihv}X_{jh(k-v)} + X_{jhv}X_{jh(k-v)}]. \tag{5}$$

$$S_{ijk} = -(2kR_{ij0})^{-1}\left\{\sum_{v=1}^{k-1} 2vR_{ij(k-v)}S_{ijv} + \sum_{v=1}^{k} 3vS_{ij(k-v)}R_{ijv}\right\}. \tag{6}$$

$$X_{ih(k+1)} = (k^2 + k)^{-1}\sum_{v=0}^{k-1}\sum_j GM_jS_{ij(k-v-1)}(X_{jhv} - X_{ihv}). \tag{7}$$

In (6) and (7), $0 < k < \infty$. In (6) the first sum is taken 0 when $k = 1$.

If initial conditions of position and velocity are known for all n bodies, then the numbers X_{ih0} and X_{ih1} are prescribed. From these one obtains R_{ij0} via (5); also $S_{ij0} = R_{ij0}^{-3/2}$. Thereafter, formulas (5), (6), and (7) may be employed cyclically to determine the remaining coefficients. A lower bound for the radius of convergence may be given by the Cauchy-Poincaré Theorem. This is done, for example, in [3], pp. 23–25. This lower bound need not be a good approximation to the radius of convergence, however. In one application of the above method, the radius of convergence turned out to be approximately 10^5 times the lower bound given by the Cauchy-Poincaré Theorem.

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TRIANGLES FORMED BY LINE SEGMENTS PARALLEL TO THE ALTITUDES AND ANGLE BISECTORS OF A TRIANGLE

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The theorem that three line segments respectively parallel to the medians of a triangle, and of proportional lengths, form a triangle, and its proof by vector methods, are well known. Corresponding theorems can be stated for the angle bisectors and the altitudes, and easily proved by vector methods, as follows:

THEOREM 1. *Three line segments respectively parallel to the angle bisectors of a triangle, and of lengths proportional to the cosines of half the bisected angles, form a triangle.*

THEOREM 2. *Three line segments respectively parallel to the altitudes of a triangle, and of lengths proportional to the sines of the originating angles, form a triangle.*

To prove these theorems, let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be the sides of the given triangle expressed as vectors, their directions taken counter-clockwise around the triangle; $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, unit vectors perpendicular to $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, respectively, their directions taken outward from the triangle; and α, β, γ, the angles opposite $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, respectively. The vectors

$$\frac{\mathbf{B}}{|\mathbf{B}|}-\frac{\mathbf{C}}{|\mathbf{C}|}, \qquad \frac{\mathbf{C}}{|\mathbf{C}|}-\frac{\mathbf{A}}{|\mathbf{A}|}, \qquad \frac{\mathbf{A}}{|\mathbf{A}|}-\frac{\mathbf{B}}{|\mathbf{B}|}$$

are respectively parallel to the angle bisectors and of lengths

$$2\cos\tfrac{1}{2}\alpha, \qquad 2\cos\tfrac{1}{2}\beta, \qquad 2\cos\tfrac{1}{2}\gamma.$$

Since their sum is zero, the first theorem follows. Also,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = k\mathbf{P}\sin\alpha, \qquad \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = k\mathbf{Q}\sin\beta,$$

and $\mathbf{C}\times(\mathbf{A}\times\mathbf{B})=k\mathbf{R}\sin\gamma$, where $k=|\mathbf{A}|\cdot|\mathbf{B}|\cdot|\mathbf{C}|$. Since $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})\equiv\mathbf{O}$, $\mathbf{P}\sin\alpha+\mathbf{Q}\sin\beta+\mathbf{R}\sin\gamma=\mathbf{O}$, and the second theorem follows.

A UNIFYING EQUATION IN MATHEMATICS OF FINANCE

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1. Introduction. The essential unity of many seemingly different topics discussed in textbooks on mathematics of finance has either been ignored or not been fully appreciated by textbook writers in this subject. It is the purpose of this paper to show the common origin of many of the seemingly unrelated formulas which appear in standard textbooks under a wide variety of notations which help to obscure their interrelationships. We first recall the following symbols:

$s_{\overline{n}|i}$: The accumulated value of n payments of 1, made at the end of each period for n periods at rate i per period, at the time of the last payment.

$a_{\overline{n}|i}$: The discounted value of the same series of payments one period before the time of the first payment.

$s^{-1}_{\overline{n}|i}$: The periodic payment made at the end of each period for n periods at rate i per period which will have an accumulated value of 1 at the time of the last payment.

$a^{-1}_{\overline{n}|i}$: The periodic payment made at the end of each period for n periods at rate i per period, which has a discounted value of 1.

$s^{(p)}_{1|i}$: The accumulated value at the end of one interest conversion period of p payments of $1/p$ made at the ends of each pth part of the interest conversion period.

The usual formulas developed in the study of (a) Sinking Funds, (b) Amortization, (c) Bonds, (d) Evaluation of Mining Property, (e) Capitalized Cost, (f) Reinvestment Rates and (g) Depreciation will be identified with a single formula,

$$R = Fi' + (F - S)s^{-1}_{\overline{n}|i} = Fi' + Ws^{-1}_{\overline{n}|i}, \tag{1}$$

where R, the annual return on an investment, F, to earn a rate i', must include a term $(F-S)s^{-1}_{\overline{n}|i}$ to restore the wearing value $W=F-S$ at the end of the useful lifetime, n, by creating a sinking fund earning a rate i.

Also a generalization of (1) will be given and applied to two examples.

2(a). Sinking funds. A debtor who desires to reduce a loan F to an amount S in n years by the sinking fund method will be required to make payments of $i'F$ to his creditor and $(F-S)s^{-1}_{\overline{n}|i}$ to a sinking fund growing at a rate i. If R represents the total annual outlay of the debtor then $R=Fi'+(F-S)s^{-1}_{\overline{n}|i}$ which is (1).

2(b). Amortization of a debt. It is customary to consider the complete amortization of a debt over a period of n years but it will serve our purpose better to consider the more general problem of the partial amortization of a debt from its initial value, F, to a final unpaid balance, S, where S may or may not be zero.

The necessary equation of value can be based on the fact that the accumulated value of the debt at the end of n years may be equated to the accumulated value of the payments made and the unpaid balance at that time.

$$F(1 + i')^n = Rs_{\overline{n}|i'} + S. \tag{2}$$

Solving this equation for R gives,

$$R = \frac{F(1 + i')^n - S}{s_{\overline{n}|i'}},$$

and, replacing $(1+i')^n$ by $1+i's_{\overline{n}|i'}$

$$R = \frac{F[1 + i's_{\overline{n}|i'}] - S}{s_{\overline{n}|i'}} = Fi' + (F - S)s_{\overline{n}|i'},$$

which is (1) for $i=i'$.

2(c). Bonds. Making use of the formula which states that the purchase price F of a bond is equal to the discounted value of the redemption price, S, for n years plus the discounted value of the coupons R, both at the investor's rate i', we may write

$$F = Sv^n + R \cdot a_{\overline{n}|i'} = [F - (F - S)]v^n + Ra_{\overline{n}|i'},$$

from which follows

$$R = \frac{F(1 - v^n) + (F - S)v^n}{a_{\overline{n}|i'}} = Fi' + (F - S)s^{-1}_{\overline{n}|i'},$$

which is (1) for $i=i'$.

2(d). Evaluation of mining property. The usual derivation of the formula for finding the value of mining property yields, in the notation of this article,

$$F = \frac{Rs_{\overline{n}|i}}{1 + i's_{\overline{n}|i}} = \frac{R}{s^{-1}_{\overline{n}|i} + i'}.$$

Solving this equation for R gives $R=Fi'+Fs^{-1}_{\overline{n}|i}$ which is (1) for $S=0$.

It may be pointed out in passing that many investments may lead to negative values of S which introduce no complication whatever because the sinking fund must, in such circumstances, not only restore the original fund F but also provide enough to care for the additional obligation, S.

2(e). Capitalized cost. The capitalized cost, K, of an article costing F dollars with a scrap value S at the end of n years, is usually defined as the initial cost F plus the cost of perpetual renewals or,

$$K = F + (F - S)s^{-1}_{\overline{n}|i}/i.$$

Multiplying this equation through by i gives $Ki=R=Fi+(F-S)s^{-1}_{\overline{n}|i}$, which is identical with (1) for $i'=i$.

2(f). Reinvestment rates. A published formula* for finding the value of an enterprise, P, with final value, L, to an investor who stipulates that during n years he will earn the rate i per period on his original investment assuming that it will produce dividends D per period and that the dividends can be reinvested at rate r per period is $P=(L+Ds_{\overline{n}|r})/(1+is_{\overline{n}|r})$.

* W. L. Hart, Mathematics of Investment (3rd ed.), Boston, 1946, (2), p. 181.

In the notation of this paper, this formula becomes

$$F = (S + rs_{\overline{n}|i})/(1 + i's_{\overline{n}|i}),$$

which on solving for R is identical with (1).

2(g). Depreciation. Among the traditional approaches to the study of depreciation there are only two which give recognition to the problem of replacement or the rate of return on a depreciating asset,—they are (i) Annuity Method and (ii) Compound Interest Method, which will be considered in this order.

(i) *Annuity method.* Following the usual development found in books on mathematics of finance, the formula for the periodic depreciation charge is derived as

$$R = [F(1 + i)^n - S]s_{\overline{n}|i}^{-1}.$$

Replacing $(1+i)^n$ by $1+is_{\overline{n}|i}$, this becomes

$$R = Fi + (F - S)s_{\overline{n}|i}^{-1},$$

an evident special case of equation (1) for $i'=i$.

(ii) *Compound interest method.* The formula* usually given in books may be written in the notation of this paper as

$$R_r = i'\left[F - \frac{W}{s_{\overline{n}|i}} s_{\overline{r-1}|i}\right] + \frac{W}{s_{\overline{n}|i}}(1 + i)^{r-1}, \tag{3}$$

where i' is defined as the rate of return desired on the investment, i is the sinking fund rate and R_r is the total charge for both interest on book value and for depreciation for the rth year.

If $(1+i)^{r-1}$ be replaced by $(1+is_{\overline{r-1}|i})$ in (3), the expression for R_r may be written in terms of the R of (1) as

$$R_r = \left(i'F + \frac{W}{s_{\overline{n}|i}}\right) + (i - i')s_{\overline{r-1}|i} = R + (i - i')s_{\overline{r-1}|i}.$$

In this form it is evident that unless $i=i'$ the property owner will not be earning the desired rate i' on the investment as stipulated. Furthermore the rate actually earned will be greater than or less than i' according as $i>i'$ or $i<i'$, respectively. It is also clear that when $i'=i$, $R_r=R$.

3. An extension of the basic equation (1). Up to this point in the consideration of (1) it has been assumed that the payment period and interest conversion periods are all equal. Seldom in actual practice will such ideal conditions be encountered, consequently several examples of some complexity will now be

* J. H. Moore, Handbook of Financial Mathematics, New York, 1945, (181), p. 1041.

given to illustrate the versatility of (1) when interpreted logically.

Multiplying (1) by $s_{\overline{n}|i}$ and rearranging terms gives

$$F + Fi's_{\overline{n}|i} = S + Rs_{\overline{n}|i},$$

which simply states the equivalence between

(4) [*the total investment+the accumulated value of the desired income*] and [*the terminal value S+the accumulated value of the net income received*];

both accumulated values are obtained at the available reinvestment rate.

Two examples will now be given illustrating the versatility of (4).

Example 1. How much should be paid for the use of a certain property for 25 years which will yield no return for a period of 5 years, \$10,000 per year for 5 years commencing at the end of six years, followed by \$15,000 per year for the succeeding 10 years and \$25,000 for each of the last 5 years, if the investor will be required to make an expenditure of \$10,000 at the end of 25 years to restore the property to its original condition and it is desired that a rate $i^{(2)}=.06$ be earned on the investment? Assume that all income can be reinvested at $i^{(4)}=.03$.

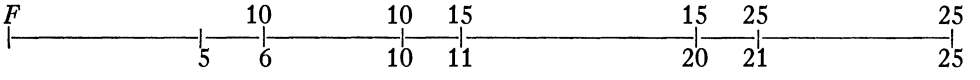


FIG. 1*

F+the accumulated value of the desired income is

$$F + (.03)Fs^{-1}_{\overline{2}|.0075}s_{\overline{100}|.0075}.$$

The accumulated value of the net income may be written

$$\$5{,}000\{2s^{-1}_{\overline{4}|.0075}s_{\overline{80}|.0075} + s^{-1}_{\overline{4}|.0075}s_{\overline{60}|.0075} + 2s^{-1}_{\overline{4}|.0075}s_{\overline{20}|.0075} - 2\}.$$

Equating and solving for F gives

$$F = \frac{\$5{,}000s^{-1}_{\overline{4}|.0075}\{2s_{\overline{80}|.0075} + s_{\overline{60}|.0075} + 2s_{\overline{20}|.0075}\} - \$10{,}000}{1 + (.03)s^{-1}_{\overline{2}|.0075}s_{\overline{100}|.0075}}$$

or $F=\$126{,}000$, to the nearest \$1,000.

Example 2. A man who pays \$100,000 for a piece of property which he keeps for 5 years and at the end of two years invests an additional \$50,000 desires to know what average interest rate he has earned per year on his investment if he sells it for \$20,000 at the end of 5 years and during the period has received a monthly income of \$3,000 per month for the first year, no income for the second

* The figures above the line are in thousands of dollars; those below the line, in years.

year and \$5,000 per month for the last three years, if it is assumed that money can be reinvested at $i^{(2)}=.025$ the first two years and at $i^{(2)}=.03$ during the last three years.

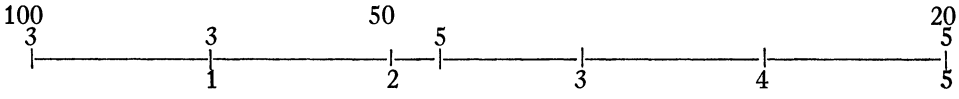


FIG. 2

$F+$the accumulated value of the desired income is

$$\$150{,}000 + \{i'\$100{,}000\, s^{-1}_{\overline{2}|.0125}s_{\overline{4}|.0125}(1.015)^6 + i'\$150{,}000\, s^{-1}_{\overline{2}|.015}s_{\overline{6}|.015}\}.$$

The accumulated value of the net income at the end of 5 years may be represented as

$$\$1{,}000\{20 + 18s^{(6)}_{\overline{1}|.0125}s_{\overline{2}|.0125}(1.0125)^2(1.015)^6 + 30s^{(6)}_{\overline{1}|.015}s_{\overline{6}|.015}\}.$$

Equating and solving for i' gives

$$i' = \frac{\$1{,}000\{18\, s^{(6)}_{\overline{1}|.0125}s_{\overline{2}|.0125}(1.0125)^2(1.015)^6 + 30\, s^{(6)}_{\overline{1}|.015}s_{\overline{6}|.015} - 130\}}{\$50{,}000\{2s^{-1}_{\overline{2}|.0125}s_{\overline{4}|.0125}(1.015)^6 + 3s^{-1}_{\overline{2}|.015}s_{\overline{6}|.015}\}}$$

or $i'=14.4\%$, to the nearest tenth percent.

It would be pointless to try to give general formulas to fit all the situations which can arise; however, the two examples given serve to illustrate two types of problems which can be solved by (4) and will suggest to ingenious teachers a limitless number of variations.

MATHEMATICAL EDUCATION NOTES

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PROGRAM OF VISITING LECTURERS OF THE M.A.A.

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The Visiting Lectureship Program of the Mathematical Association of America has now completed its fifth year under support of the National Science Foundation. During the five years it has provided lecturers to 231 different

institutions, involving a total of 798 days of visitation. The increasing size of the program is shown by the number of days of lecturing each year and number of institutions visited.

	Number Days of Visits	*Number of Institutions*	*Average Length of Visit*
1954–55	169	66	2.6
1955–56	104	57	1.8
1956–57	169	81	2.0
1957–58	122	66	1.8
1958–59	234	136	1.7

The lecturers have been:

1954–55: Professor R. H. Bing, University of Wisconsin
Professor W. L. Duren, Tulane University (now at the University of Virginia)
Professor Tomlinson Fort, University of South Carolina
Professor George Polya, Stanford University
Professor D. V. Widder, Harvard University

1955–56: Professor D. H. Lehmer, University of California, Berkeley
Professor George Polya, Stanford University
Professor G. Baley Price, University of Kansas

1956–57: Professor Edwin Hewitt, University of Washington
Professor Kurt Mahler, University of Manchester
Professor A. W. Tucker, Princeton University
Professor J. L. Walsh, Harvard University

1957–58: Professor H. S. Coxeter, University of Toronto
Professor Paul Halmos, University of Chicago
Professor John Kemeny, Dartmouth College
Professor Walter Prenowitz, Brooklyn College
Professor R. D. Schafer, University of Connecticut

1958–59: Professor Tom M. Apostol, California Institute of Technology
Dr. Robert E. Gaskell, Boeing Airplane Company
Professor R. E. Johnson, Smith College
Professor John L. Kelley, University of California, Berkeley
Professor Marguerite Lehr, Bryn Mawr College
Professor Lynn H. Loomis, Harvard University
Professor S. S. Wilks, Princeton University
Dr. Stanislaw Ulam, Los Alamos Scientific Laboratory

1959–60: Professor David Blackwell, University of California, Berkeley
Professor J. Sutherland Frame, Michigan State University
Professor Magnus Hestenes, University of California, Los Angeles
Professor S. A. Jennings, University of British Columbia
Professor Tibor Rado, Ohio State University
Professor Ernst Snapper, Indiana University

Mathematicians, the M.A.A., and the Committee are greatly indebted to these men for the time, energy, and talent which they have given to the program. It is their contribution which has made the program outstandingly successful, and they have our sincere appreciation.

In spite of the belief of the Committee on Visiting Lecturers that visits of three or more days would be optimum and in spite of the increase in lecturing time available, the growth of the demand for lecturers has continued to lower the average number of days spent at each institution, the average of 1.7 for 1958–59 being the minimum. The time requests of many institutions were reduced, and a good number of requests were not granted. A reasonable estimate of the total audience reached in 1958–59 is 25,000, counting duplications.

The program for 1959–60 was oversubscribed. Among the applications there were a large number from schools which had never before participated in the program. This means that further increases in the size of the project will occur. To cope with this increase the Committee has been enlarged, the current membership being: Burton W. Jones, University of Colorado; Paul B. Johnson, U.C.L.A.; R. C. Fisher, Ohio State University; R. E. Gaskell, Oregon State College; R. E. Johnson, Smith College; Robert Rosenbaum, Wesleyan University; C. L. Seebeck, Jr., University of Alabama; and, as chairman, Rothwell Stephens, Knox College.

In order to accommodate more institutions and to establish closer relations between the mathematics departments in the universities and the neighboring colleges, a Visiting Mathematicians Program was inaugurated this year. Interested universities have nominated members of their staffs to serve as Visiting Mathematicians to nearby colleges. The Visiting Mathematician functions in much the same way as the Visiting Lecturer.

The Program of Visiting Lecturers of the Mathematical Association was the first lecturing program undertaken on a nationwide basis in the sciences, and it would not have been possible without the generous support of the National Science Foundation. With the support of the Foundation, the Program is assured through 1960–61. The enthusiastic reports received from institutions and from lecturers indicate that it is a program which should be continued on a permanent basis.

A SUMMER INSTITUTE ON THE TEACHING OF ARITHMETIC FOR ELEMENTARY SCHOOL PERSONNEL

JOSEPH N. PAYNE, University of Michigan

Under the sponsorship of the National Science Foundation, the University of Michigan held a four-week institute, July 20 to August 15, 1959, on the teaching of arithmetic for personnel of the elementary school. Of the fifty-two participants in the institute, 20 were supervisors or consultants, 15 were principals or superintendents with supervisory functions, 13 were elementary school teachers, 2 were junior high school teachers, and 2 taught arithmetic method courses in college.

Purpose of the institute. An understanding of and appreciation for arithmetic is a first step in achieving mathematical and scientific literacy needed by

citizens of the modern world of today. That teachers may help children to learn arithmetic with meaning and understanding, teachers must: (1) know the fundamental mathematical ideas; (2) know the principles of psychology and their implications for teaching arithmetic; and (3) be able to translate these ideas into effective teaching in the classroom. The Institute was designed to improve instruction in arithmetic by helping teachers and supervisory staff improve their competency in the three areas listed above.

The program. The program designed to help achieve the above-named objectives consisted of the following:

1. A mathematics course in the foundations of arithmetic: This course presented the logical structure of the number system and the basic mathematical concepts of arithmetic. Special attention was given to the historical background of the development of these ideas.

2. An education course in the psychology and teaching of arithmetic: This course presented the basic concepts of psychology and human growth and development as they relate to the development of mathematical concepts by children. Guest lecturers and demonstration classes helped in bridging the gap between theory and practice.

3. A daily period for supervised reading, discussion, and writing: Each participant prepared some written materials—curriculum outlines, sample teaching units, suggestions for teaching aids and materials, or plans for in-service education of teachers. These were duplicated and given to each participant. A special library of texts, supplementary books, curriculum guides and teaching materials was available for use by participants.

4. Lectures and demonstration classes: Special lectures dealt with recent experimentation, revision of content, current trends, and curriculum organization in arithmetic education as well as psychology and human growth and development. The demonstration classes included one closed-circuit TV demonstration.

The staff. The staff members of the Institute consisted of: Professor E. Glenadine Gibb (Associate Director), Iowa State Teachers College; Professor Phillip S. Jones, The University of Michigan; Dr. Edwina Deans, Elementary Supervisor, Arlington County, Virginia, Public Schools; Miss Alice Hach, Mathematics and Science Consultant, Racine, Wisconsin, Public Schools; Professor Joseph N. Payne (Director), The University of Michigan.

THE CASE FOR BASIC RESEARCH ON THEORY OF INSTRUCTION

GERTRUDE HENDRIX, University of Illinois

Theory of instruction research now under way seeks an underlying explanation of the teaching approach most successful with courses developed by the UICSM Mathematics Project. The University of Illinois Nonverbal Awareness

Project, supported initially by small private grants and directed by the Teacher Coordinator of the UICSM Project, is now operating at a hypothesis-generating level only. Facilities have been relinquished and equipment stored for a two-year period in order to release research personnel for intensive work on a film project, the National Educational Television and Radio Center Mathematics Study.

The primary aim of the film project is to produce a series of films to replace the demonstration classes which are now an essential accompaniment to UICSM teacher-training programs. Incidentally, however, the films also are providing a rich store of recorded events from which data for the underlying theoretical study can proceed two years hence. Footage already obtained contains many vivid illustrations of the kind of learning the new courses seek to promote.

One phase of teaching looming large in things revealed to date is the enormous role played by nonverbal communication between teacher and students. Current research in paralinguistics—especially that involving kinesics—is revealing ways to identify and classify the nonverbal behavior which human beings learn to interpret in each other; it is thus that we produce the complicated stream of communication sometimes accompanying, sometimes independent of, words. Such an analysis might enable a teacher to cultivate desirable paralinguistic effects, and to avoid those which are destructive to his work.

Effective teacher-training films must help teachers learn to *observe* students. Sympathetic observation is prerequisite for clues as to what to say or do next in order to make learning "come clear." Since it was a recent re-interpretation of animal learning which led to identification of the nonverbal stage in creative abstract human learning, some have suggested that observations of animals in training might speed up acquiring ability to tell when someone has "caught on" to an idea. Two of the twenty films in the NETRC Math Study series will contain shots for testing this kind of observation power.

Those involved in the nonverbal awareness research at the University of Illinois feel that a re-examination of the entire learning process is in order. Even the simplest so-called conditioned reflex can be interpreted as evidence that the organism possesses a generalization of a type the logicians call 'universal conditional'.

Phenomena which indicate the advent of awareness are of basic importance in teaching for discovery, and all such phenomena are alike in that they consist of behavior that would have been impossible without awareness of the concept or generalization being taught. Usually these phenomena do *not* include a verbal formulation of the thing just discovered. In fact, in a creative sequence it is awareness of a new concept which impels one to seek or to invent a name for it.

Electroencephalographs of animals acquiring conditioned reflexes reveal decided changes in neutral activity. We wish that we could see brain-wave records of the seventh-graders who were becoming aware of the distributive principle for multiplication over addition in NETRC Math Study Film Number IX.

A REPORT ON THE BATON ROUGE CENTER FOR GRADES SEVEN AND EIGHT SCHOOL MATHEMATICS STUDY GROUP, 1958–59

HOUSTON T. KARNES, Louisiana State University

The Baton Rouge Center was one of twelve areas invited to engage in the 1958–59 experimental program for grades seven and eight. The other eleven centers were: University of Arizona; University of Chicago; University of Colorado; University of Delaware; University of Michigan; University of Minnesota; Brookline, Massachusetts; Pasadena, California; Princeton, New Jersey; Seattle, Washington; Westport, Connecticut.

Center reports indicate that the experience in Baton Rouge was typical of those in the other centers. A center was composed of seventh and eighth grade teachers, a chairman and a consultant. The chairman could be a person in the field of teacher training, a supervisor in mathematics, one from the field of mathematics education, or a college or university teacher of mathematics well versed in secondary school work. The consultant was a mathematician from the staff of a college or university. The Baton Rouge Center was composed of four seventh grade teachers, and two eighth grade teachers. The author doubled as the chairman and the consultant. The teachers were all duly certified to teach junior high school mathematics, which required a mininum of eighteen semester hours in mathematics. Only two had more than the minimum and none of the teachers had had recent courses in college mathematics. The college work of these teachers, in mathematics, was of the traditional type, and therefore did not include certain vocabulary, ideas and concepts these teachers would be using in the experimental material. Each teacher taught three classes of the SMSG seventh grade text and two or three classes using a traditional text. This gave a total of eighteen experimental classes involving some 640 pupils. The teachers were drawn from schools of various sections of the city so that there was a good socio-economic class distribution among the pupils. The pupils were not a select group; the I.Q. range was from well below average to superior. In only one school was there any attempt at homogeneous grouping.

Program of the center. Once per week, throughout the year, a two to three hour seminar was held. On several occasions it was found desirable to meet twice per week. Most of the time of these meetings was devoted to studying the text, introducing related mathematical ideas which would help the teachers, and in discussing the presentation of the text material. Time was allotted for the raising and answering of questions and also in discussing classroom experiences of the teachers. In addition, part of each seminar was devoted to the testing program.

At the conclusion of each unit of the text all teachers submitted unit reports. These reports consisted of a questionnaire, a page or two of additional comments by the teacher, and a copy of the unit test with the test results. These reports were of distinct aid to the writing team during the summer of 1959. The last part of the program dealt with the preparation of new materials and editing the

current materials. Each center was encouraged to either write new units or rewrite one of the existing units. In addition, each center was asked to edit one unit of the text as preparations were being made for a new printing.

Reaction to the program.

1. *The teachers.* The teachers were highly enthusiastic with respect to the materials used and to student reaction. In general the teachers found most all of the material used to be teachable. They found this approach to be a good motivating vehicle. Several teachers made comments to the effect that some below-average students who were taking no interest in the traditional material became alive and really took an interest in the activities of the class. This was even more noticeable among the above-average students.

Although no actual study was made, from time to time the teachers gave the experimental classes the same tests that were given to the traditional classes. In general it was found that the experimental classes did better on these tests than did the pupils of the traditional group. One teacher gave standardized achievement tests to all of his pupils in the spring of 1959. The result showed that the experimental classes were accelerated from one-half year to one and one-half years beyond the traditional classes.

2. *The pupils.* The new approach of the SMSG text was well received by the pupils. This seemed to be due to the following reasons:

1) Material which was new and different from the uninteresting traditional arithmetic of the first six years—arithmetic which has been a drudgery to many pupils and has caused many good pupils to be lost to the field of mathematics.

2) In the SMSG text the pupils found answers for previous perplexing questions and reasons for technique which were being developed.

3) The enthusiasm of the teachers.

4) The interest on the part of their parents.

Space does not permit quotations from the pupils; however, many made statements of approval with regard to the new material. Only seven pupils in the experimental classes requested to be changed to a traditional class and three of these were due to parental influence, probably because those parents found it difficult to help their children in this new approach.

3. *The parents.* In some sections of the city there were no parental comments at all. This, however, was to be expected. In those areas where professionally trained parents and parents who were business leaders lived the interest ran high. Many of these parents learned right along with their children. Some even requested permission to visit the classes. On parent visitation nights the teachers in the program would be swamped with visitors. A large number of parents, having heard of the program, made requests that their children be transferred to the experimental classes. These requests became so large and demanding last fall (1959), that it became necessary to add five new classes in one of our schools.

REACTION TO THE KANSAS TEPS CONFERENCE

W. V. PARKER, Auburn University

The Kansas Conference in June 1959 was organized in very much the same way as the 1958 Bowling Green Conference. (This MONTHLY, vol. 65, pp. 698–700 and vol. 66, pp. 59–60.) The approximately 1200 participants were divided into 34 groups designated as Set I and then were regrouped into 30 groups designated as Set II. The groups of Set I met on Wednesday to discuss the general problems of teacher training. In each group a report was given on the teacher training program of a selected institution.

The groups of Set II met on Thursday to discuss special training for the various areas. The ones relating to the training of elementary teachers were divided into several levels and those concerned with secondary teaching were divided according to subject matter areas. In some cases there were several groups concerned with the same area. There were two groups on mathematics. These groups were concerned with the training needed for their special areas.

The groups of Set I reassembled on Friday to attempt to bring together the ideas of the various groups of Set II and to formulate a pattern for teacher training at the various levels.

One was impressed by the wide-spread academic interest of the participants. Apparently there was a much larger representation from the nonprofessional areas than has been the case at previous meetings.

The need for strong subject matter training for teachers at all levels was emphasized in the discussions generally. Much of the controversy of academic versus professional work seemed to be giving way. There seemed to be general agreement that the training of teachers must be a college-wide problem and not just the problem of the schools of education.

The willingness of both sides to make concessions is encouraging. Other meetings of this kind on both regional and national levels should prove to be quite worthwhile.

MATHEMATICS COURSES FOR ELEMENTARY TEACHERS

One of the most important and difficult problems in mathematics teacher education is at the elementary school level. While there is agreement that mathematics should be required of all elementary teachers, there is no general agreement on what the content and nature of the course or courses should be. In the April issue Lenore John reported on the geometry content needed by the elementary teacher. The editors of this department have invited a number of departments of mathematics to contribute descriptions of courses in mathematics which they offer for elementary teachers. Below are reports received in response to this invitation from a liberal arts college, a teachers college, a privately endowed university, and four state universities.

Emory University. Emory currently has two mathematics courses for elementary school teachers. Both deal with the mathematical foundations of arithmetic and its relation to elementary school instruction; both have two instructors, one from Mathematics and one from Education. One course, sponsored by the National Science Foundation,

is for in-service teachers and is meeting once a week for 30 two-hour sessions. The second course, supported by the American Association for the Advancement of Science, was offered in the winter quarter to undergraduates preparing for teaching.

In both courses the mathematics instructor is responsible for the organization and presentation of the mathematical content and the education instructor is responsible for problems of application and interpretation in elementary school teaching and for relevant material on the psychology of learning and teaching. Actually it has turned out that both instructors have shared extensively in all tasks. It should be noted that a careful delineation for responsibility in such a "team-teaching" situation is nevertheless probably a sound precaution.

For the in-service course a graduate assistant, who has had school teaching experience and graduate courses in both Mathematics and Education, has been devoting 20 hours a week to visiting the teachers in their classrooms. This assistant has made a valuable contribution to the individual instruction and guidance for the teacher and has performed the important function of giving the teachers direct assistance in applying course material in their teaching.

Bevan K. Youse

Illinois State Normal University. The Mathematics Department at Illinois State Normal University offers a basic program in mathematics education for teachers of the elementary grades in two courses. The first course, usually taken in the freshman year, is primarily a content course, Basic Concepts of Arithmetic, which emphasizes those concepts growing out of counting and numbers as well as those growing out of measuring. This course gives credit of three semester hours. The second course, Arithmetic for the Elementary Grades, for two semester hours emphasizes the problems of learning and teaching elementary mathematics. This work is usually taken by juniors before they do their student teaching. Most of the students are doing junior participation in the elementary classrooms at this time.

Students preparing to teach at the junior high level are enrolled in a course called Mathematics for the Junior High School Grades. About half of these students are mathematics majors or minors and the remainder are elementary majors who have strong mathematics background from high school and have completed the five semester hours program for the elementary curriculum. Advanced work is offered in three graduate courses—Teaching of Arithmetic, Teaching of Junior High School Mathematics, and Teaching and Supervision of Mathematics. Throughout the program, mathematical content and significant historical developments are stressed.

Francis R. Brown

Knox College. Students majoring in elementary education are required to take six semester hours in the Mathematics Department and three semester hours in the teaching of arithmetic. It is usually recommended that students take a year course, Basic Mathematics, to meet the mathematics requirement. This course is designed for students who have had only two years of college preparatory mathematics. The course includes an axiomatic approach to the natural numbers, the integers, the rational numbers, the real numbers, and the complex numbers. Also discussed are prime numbers, induction, the decimal system, number scales, arithmetic and geometric progressions, linear equations, systems of linear equations, square roots, quadratic equations, mathematical systems, groups, isomorphism, miniature geometries, graphs and functions. The course concludes with brief introductions to trigonometry, analytic geometry, and calculus. The Teaching of Arithmetic is taught in the Department of Education, currently by the Director of Elementary Education in the local public school system.

Rothwell Stephens

Montana State College. At Montana State College two courses in mathematics are required of students majoring in Elementary Education. The two courses are: Intermediate Mathematics, 4 credits, and Teaching of Arithmetic, 3 credits.

In the first part of the course in Intermediate Mathematics, a review of arithmetic operations with ordinary and decimal fractions is given. Stress is placed on our number system, as well as numbers in some scale other than ten. Work is included on denominate numbers and the metric system. The second part of the course includes a review of elementary algebra. Emphasis is placed on exponents, evaluation of formulas, the fundamental operations with algebraic fractions, and the solution of linear equations.

In the Teaching of Arithmetic, which is a junior level course, emphasis is put on the understanding and mastery of the usual operations in arithmetic. Some attention is given to the methods of teaching the usual operations. The need for reading current periodicals on the teaching of arithmetic and the desirability of a mathematics library at the elementary level is pointed out.

Adrien L. Hess

University of California, Los Angeles. Fundamentals of Arithmetic. (3) I, II. Prerequisite: sophomore standing. Designed primarily for prospective teachers of arithmetic. The study of the fundamental operations on integers and fractions is stressed, together with suitable visual aids. Although efficiency in arithmetical skills is required, the emphasis is on the understanding of arithmetical procedures.

This course is required for all elementary credential candidates and furthermore, is highly recommended for general secondary candidates.

Clifford Bell

University of Maryland. In considering material for a mathematics course suited to the needs of students preparing to teach in the elementary schools, primary attention was given to those topics which would provide deeper understanding of the foundations of the arithmetic taught in the elementary grades. Accordingly, topics in the algebra of integers and of rational numbers, and in the elementary theory of numbers were selected. Included in the course are the following:

1. Inductive proof
2. The system of natural numbers, based on the Peano Axioms
3. Mathematical systems: Elements, operations, properties
4. The algebraic structures, group and field, modular arithmetic
5. The system of integers: Ordered pairs of natural numbers
6. The system of rational numbers: Ordered pairs of integers
7. Congruences and rules for divisibility
8. Euclidean algorithm for the greatest common factor
9. Number system on bases other than ten.

This is a four semester-hour course, the students meeting for one-hour sessions, four days a week.

Clearly, many of these topics receive a sophisticated treatment in advanced courses. At this level, while the rigor of the proofs is maintained, the presentation of the material is made from a point of view which requires only the skills acquired in a first course in high school algebra. It has been our experience that the students are sufficiently mature to grasp the concepts of the topics outlined above, if properly presented. Carefully planned developmental exercises help surmount the difficulty of limited manipulative algebraic skill. Upon completion of the course, the student is able to read with understanding, current literature in "new" mathematics for the pre-high school student.

Helen L. Garstens

The University of Oklahoma. The University of Oklahoma is offering a course in the fundamentals of arithmetic which is designed for the pre-service preparation of the elementary teacher. The primary aim of this course is to emphasize the structure of our numbers, beginning with the natural numbers and terminating with the real numbers as a field. The historical development of these numbers and the numerals used to denote them is studied with special consideration given to their contribution to our culture.

Elementary set theory is used to give a deeper understanding of the operations and the laws governing these operations. It is used as a unifying concept in the teaching of arithmetic at the elementary level. Accurate but beginning interpretations are given to many concepts which appear in more advanced courses, for example, the concept of variable. This is introduced through the study of various types of sentences. Some time is also given to the study of numeration in order that a better understanding of computational skills may be acquired.

Eunice Lewis

California Conference on Secondary School Mathematics

One of the most encouraging signs of the great progress which is being made in the improvement of mathematics instruction in the secondary schools and the important relationships of mathematicians with school administrators and educationists is given by the Conference on Secondary-School Mathematics recently held at the University of California, Santa Barbara, under the sponsorship of the California State Department of Education. When a state department of education arranges a conference to consider secondary-school mathematics curriculum and is able to bring together such outstanding mathematicians as E. G. Begle, Karl Menger, and R. L. Wilder, in addition to quite a number of leading mathematicians from California including John L. Kelley and Harold M. Bacon, then we in mathematics have cause for real optimism about present improvement of secondary instruction. Such a conference as this should remove, on the part of many, the fear that is sometimes heard that the mathematicians have not sufficiently sought the cooperation and assistance of secondary-school administrators and teachers, and hence that the product of the various mathematics curriculum studies, particularly the School Mathematics Study Group, will suffer therefrom in acceptance. The large numbers of people representing various interests that are involved in the School Mathematics Study Group in the try-out of materials in more than 100 schools throughout the country is largely responsible for this happy turn of events.

J. R. Mayor

Science Education News

The Conference Board of the Mathematical Sciences has named Dr. Leon W. Cohen, head of the department of mathematics of the University of Maryland, as Contributing Editor for *Science Education News.* This is a quarterly publication issued by the Science Teaching Improvement Program of the American Association for the Advancement of Science.

Science Education News reports science education activities of interest to scientists as members of scientific societies and in industrial and government positions. Interested persons may have their names added to the mailing list by writing to Mrs. Charlotte W. Colton, Editor, Science Education News, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

In the December 1959 issue of *Science Education News,* the editorial is written by Dr. Cohen.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1416. *Proposed by Leonard Cohen, City College of New York*

Prove that $x^2+y^2=z^n$, where n is a positive integer, always has nonzero integral solutions.

E 1417. *Proposed by Robert Hartop, Los Angeles, California*

Given a unit circle with point P on its circumference and a distance d_1, $0<d_1<2$. With center at P, construct the circle of radius d_1 to cut the given circle in two points. If d_2 is the distance between these points, construct a new circle of radius d_2 with center at P and so obtain $d_3, d_4, \cdots, d_n, \cdots$. Find $\lim_{n\to\infty} d_n$.

E 1418. *Proposed by R. C. Buck, Institute for Defense Analyses, Princeton, New Jersey*

In *Comptes Rend.* (1945) vol. 62, pp. 95–97, there appears the theorem: If $f(x, y)$ has partial derivatives of first and second orders, and if $f(a, b)+f(b, c)=f(a, c)$ for all a, b, c, then there is a function ϕ such that $f(x, y)=\phi(x)-\phi(y)$. Can these hypotheses be weakened?

E 1419. *Proposed by C. C. Yalavigi, Government College, Mercara, India*

In the *Collected Papers of Srinivasa Ramanujan*, edited by G. H. Hardy, Seshu Aiyar, and B. M. Wilson (1928), appears (p. 334, Question 1076 (XI, 199)) the following:

Show that

$$\text{(i)} \qquad [7(20)^{1/3} - 19]^{1/8} = (5/3)^{1/3} - (2/3)^{1/3},$$

$$\text{(ii)} \qquad [4(2/3)^{1/3} - 5(1/3)^{1/3}]^{1/6} = (4/9)^{1/3} - (2/9)^{1/3} + (1/9)^{1/3}.$$

Show that these relations are incorrect, and that to correct them we must interchange the exponents 1/8 and 1/6 appearing in the left-hand sides.

E 1420. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A', B', C' (A'', B'', C'') be the centers of squares described exteriorly (interiorly) on the sides BC, CA, AB of a triangle ABC. Show that the radical center of the circles $A(A')$, $B(B')$, $C(C')$, ($A(A'')$, $B(B'')$, $C(C'')$) coincides with the nine-point center of triangle ABC.

SOLUTIONS

Two Trains

E 1386 [1959, 812]. *Proposed by Monte Dernham, San Francisco, California*

In overtaking a freight, a passenger train which is x times as fast takes x times as long to pass it as it takes the two trains to pass when going in opposite directions. Find x.

Solution by Underwood Dudley, University of Michigan. If the combined length of the trains is d and the speed of the freight is v, then the conditions give $d/v(x-1)=xd/v(x+1)$. Keeping the positive root we find $x=1+\sqrt{2}$.

Also solved by A. N. Aheart, R. G. Albert, M. H. Alief, J. W. Baldwin, Leon Bankoff, Edward Barbeau, Robert Bart, J. F. Barth, Alan Beal, W. R. Becker, R. E. Blewster, Jr., Karl Bochert and Laurie Dodge (jointly), D. A. Breault, E. W. Brown, Charles Bush, Marcus Charles, I. Chorneyko and W. Zayachkowski (jointly), A. G. Clark, N. A. Court, F. T. Driscoll, J. M. Elkin, G. W. Erwin, Jr., Irma Esrig, Arthur Evans, Jr. and H. R. van Zoeren (jointly), W. D. Evans, Walter Feibes, Geoffrey Fox, Stuart Friedman, Susan L. Friedman, R. E. Glenn, Michael Goldberg, L. D. Goldstone, C. L. Goode, A. G. Grace, Jr., S. H. Greene, Newcomb Greenleaf, Spencer Hamilton, Ned Harrell, Dom Dunstan Hayden, E. E. Heimann, J. H. Hodges and D. B. Larson (jointly), Richard Holt, E. L. Hubbard, W. O. Hulser, A. R. Hyde, J. N. Issos, Gerald Janusz, Kim Jones, J. P. Jordan, A. F. Kaupe, Jr., R. W. Kilmoyer, J. M. Kingston, P. G. Kinney, Jr., Leonard Klafter, Donald Knuth, Frank Kocher, Sidney Kravitz, M. E. Lakser, W. E. Lawrence, Ruth Lemon, Joel Levy and H. J. Lieberman (jointly), Norton Levy, R. J. Lewyckyj, L. I. Lowell, J. R. Lux, Joseph McMeudie, Barry MacKichan, Wallace Manheimer, D. C. B. Marsh, Vincent Mead, J. W. Mettler, N. H. Mines, Morris Morduchow, C. S. Ogilvy, Bart Park, P. M. Pepper, J. L. Pietenpol, C. F. Pinzka, G. S. Plews, Thomas Porsching, D. K. Richards, L. A. Ringenberg, Mary J. Robertson, D. R. Rogers, M. T. Salhab, C. M. Sandwick, Sr., E. M. Scheuer, L. J. Schneider, R. T. Shannon, L. C. Shepley, Harold Shulman, J. K. Siberz, S.J., C. M. Sidlo, D. L. Silverman, Arnold Singer, Vencil Skarda, E. L. Spitznagel, Jr., C. L. Sterling, D. C. Stevens, P. C. Stewart, W. B. Stovall, Jr., Melvin Tainiter, Hoyt Taylor, C. W. Trigg, Chung-lie Wang, Larry Webster, W. E. Wiebenson, Ralph Wiggins, Donald Williams, C. Winston, Tim Winterer, G. L. Woodruff, C. C. Yalavigi, A. W. Yonda, I. Zucker, and the proposer. Late solutions by John Burslem, George Glauberman, J. B. Muskat, and D. J. Peterson.

A Line Dividing the Area Under a Sine Arch

E 1387 [1959, 812]. *Proposed by L. T. Van Tassel, San Diego, California*

Find p such that the line $y=p$ divides the area of an arch of the sine curve $y=\sin x$ into parts in a given ratio to one another.

Solution by A. R. Hyde, Bronxville, N. Y. The same ratio r may be applied to the half-arch of $y=\cos x$. Then

$$\int_0^p \arccos y\,dy = r/(r+1),$$

giving the equation

$$(1-p^2)^{1/2} - p \arccos p = 1/(r+1).$$

This may also be written $\sin t - t\cos t = 1/(r+1)$, where $\cos t = p$.

Also solved by A. N. Aheart, Robert Bart, D. A. Breault, E. W. Brown, Louise Grinstein,

E. L. Hubbard, Donald Knuth, Wallace Manheimer, D. C. B. Marsh, Vincent Mead, E. M. Scheuer, Arnold Singer, Larry Webster, and C. C. Yalavigi. Late solution by Kenneth Stolarsky.

Yalavigi showed that if we set $k=1/(r+1)$, then we have, approximately,

$$p^2 = [(1 + k^2)^2 - 4k^2]/2(3 + k^2).$$

A System of Inequalities

E 1388 [1959, 813]. *Proposed by H. W. Gould, West Virginia University*

Determine the largest integral value of n such that there exists a solution to the system of inequalities

$$k < x^k < k + 1, \qquad k = 1, \cdots, n.$$

Would it make any difference if $<$ were replaced by $\leqq$?

Solution by N. J. Fine, Institute for Advanced Study. The largest possible n is 4. If $n \geqq 5$, then the inequalities $3 \leqq x^3$ and $x^5 \leqq 6$ would imply $3^5 \leqq 6^3$, which if false. For $n=4$, any number x satisfying $3^{1/3} < x < 5^{1/4}$ will do. Changing $<$ to $\leqq$ makes no difference.

Also solved by R. G. Albert, J. W. Baldwin, Edward Barbeau, Robert Bart, D. A. Breault, E. W. Brown, Underwood Dudley, J. W. Ellis, Arthur Evans, Jr. and H. R. van Zoeren (jointly), Geoffrey Fox, S. P. Franklin and G. A. Hutchison (jointly), Leo Gafney, S.J., Michael Goldberg, S. H. Greene, J. H. Hodges, J. Hooley, A. R. Hyde, Gerald Janusz, J. P. Jordan, Donald Knuth, Sidney Kravitz, Joel Levy and H. J. Lieberman (jointly), R. J. Lewyckyj, Wallace Manheimer, D. C. B. Marsh, C. S. Ogilvy, Walter Penney, P. M. Pepper, J. L. Pietenpol, C. F. Pinzka, Thomas Porsching and R. H. Shaw (jointly), L. A. Ringenberg, L. C. Shepley, J. K. Siberz, S.J., D. L. Silverman, Vencil Skarda, C. D. Stevens, J. E. Vinson, W. E. Wiebenson, Donald Williams, and the proposer. Late solutions by George Glauberman and J. B. Muskat.

Franklin and Hutchison generalized by showing that if $\phi(m)$ is the maximum number of solvable inequalities

$$k < x^k < k + 1, \qquad k = m, m + 1, \cdots, n,$$

then for $m=1, 2, \cdots, 6$, $\phi(m)$ takes on the values 4, 3, 2, 2, 2, 1, and for $m \geqq 7$, $\phi(m)=1$.

Evaluation of a Determinant

E 1389 [1959, 813]. *Proposed by E. M. Scheuer, Space Technology Laboratories, Los Angeles, California*

Evaluate the $n \times n$ determinant A_n whose (i, j)th entry is $a^{|i-j|}$.

I. *Solution by E. E. Strock, Prudential Insurance Company, Newark, N. J.* Expand the determinant A_n by the elements of the first row. The minor of the first element is A_{n-1}; the minor of the second element is aA_{n-1}; the minors of the remaining elements of the first row are all zero. Hence $A_n=(1-a^2)A_{n-1}$. Since $A_1=1$, $A_n=(1-a^2)^{n-1}$.

II. *Solution by J. W. Ellis, Louisiana State University in New Orleans.* By subtracting a times the second row from the first row, we see that $A_n = (1-a^2)A_{n-1}$. Since $A_1=1$, it follows that $A_n=(1-a^2)^{n-1}$.

III. *Solution by T. A. Porsching, Fort Belvoir, Va.* By multiplying the

$(i-1)$st row by a, and subtracting it from the ith row, $i=n, n-1, \cdots, 2$, A_n is reduced to triangular form with $a_{11}=1$, $a_{ii}=1-a^2$ when $i\neq 1$. Therefore $A_n=(1-a^2)^{n-1}$.

Also solved by A. N. Aheart, R. G. Albert, Edward Barbeau, Robert Bart, Alan Beal, W. R. Becker, D. A. Breault, E. W. Brown, R. F. Brown, I. Chorneyko and W. Zayachkowski (jointly), Richard Cottle, Underwood Dudley, E. S. Eby, Arthur Evans, Jr., N. J. Fine, S. P. Franklin and G. A. Hutchison (jointly), Stuart Friedman, Michael Goldberg, Leonard Goldstone, H. W. Gould, S. H. Greene, Newcomb Greenleaf, J. H. Hodges, Vern Hoggatt, J. Hooley, A. R. Hyde, Gerald Janusz, J. P. Jordan, A. F. Kaupe, Jr., J. M. Kingston, Donald Knuth, Sidney Kravitz, W. E. Lawrence, Gerald Leibowitz, R. J. Lewyckyj, Wallace Manheimer, D. C. B. Marsh, P. T. Mielke, N. H. Mines, E. R. Mullins, Jr. and David Rosen (jointly), C. S. Ogilvy, F. D. Parker, J. L. Pietenpol, C. F. Pinzka, L. A. Ringenberg, R. T. Sandberg, C. M. Sandwick, Sr., L. J. Schneider, A. L. Schreiber, Donna Seaman, Harold Shulman, C. M. Sidlo, D. L. Silverman, Benjamin Sims, Arnold Singer, A. Sinkov, Vencil Skarda, J. R. Smart, E. L. Spitznagel, Jr., D. C. Stevens, J. R. Swenson, Chester Tsai, L. T. Van Tassel, Chung-lie Wang, W. E. Wiebenson, R. H. Wilson, Jr., Dale Woods, A. W. Yonda, S. M. Yusuf, David Zeitlin, the proposer, and an anonymous solver. Late solutions by J. W. Baldwin, H. F. Bechtell, C. H. Cunkle's *Modern Algebra* class, George Glauberman, Joseph Hammer, P. G. Kirmser, M. S. Klamkin, J. B. Muskat, D. J. Peterson, and B. E. Rhoades.

Hodges suggested, as an allied problem, the evaluation of the $n\times n$ determinant B_n whose (i, j)th entry is 0 for $i=j$ and $a^{|i-j|}$ for $i\neq j$. S. H. Eisman proposed the problem of finding the inverse of the $n\times n$ matrix whose (i, j)th entry is $a^{|i-j|}$ and where $|a|\neq 1$.

Primitive Decompositions

E 1390 [1959, 813]. *Proposed by J. Łoś and J. Mycielski, Wroclaw, Poland*

A *decomposition* of a rectangle (rectangular parallelepiped) R is a dissection of R into smaller rectangles (rectangular parallelepipeds) by lines (planes) parallel to the sides (faces) of R. If a decomposition of R is not a refinement of some other decomposition of R, then the decomposition is said to be *primitive*.

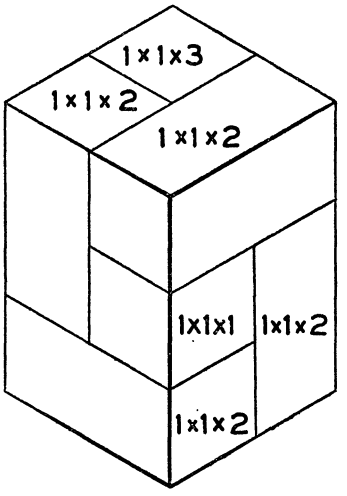
(1) Show that there exist primitive decompositions of a rectangle into any number $n>1$ of rectangles, except $n=3$, 4, 6.

(2) Show that there exist primitive decompositions of a rectangular parallelepiped into any number $n>1$ of rectangular parallelepipeds, except $n=3$, 4.

Composite solution by D. C. B. Marsh, Colorado School of Mines, and Michael Goldberg, Washington, D. C. Let us call any of the smaller rectangles into which the given rectangle is decomposed, a *part*; those parts with a vertex in common with the given rectangle will be called *corner parts*. Necessary conditions for a primitive decomposition with $n>2$ are (1) no two parts may have a common side, and (2) no corner part may have two vertices in common with the given rectangle. (The proof of (1) is that this common side, if removed, would result in a decomposition of $n-1$ parts, whence the original decomposition would be nonprimitive contrary to assumption; for (2), any decomposition with a corner part covering two vertices would be a refinement of a decomposition involving merely this part and the remaining portion of the given rectangle: i.e., a refinement of some $n=2$.) Thus we conclude there are at least four parts (each a distinct corner part) in any primitive decomposition having $n>2$. Thus $n\neq 3$.

For $n=4$, the interior vertices of the four corner parts would have to coincide, but such a decomposition obviously is not primitive. For $n>4$, the interior vertices of the four corner parts may be vertices of a rectangle, in which case we have a primitive decomposition with $n=5$. If the interior vertices of the four corner parts are not vertices of a rectangle, then the simplest remaining area is L-shaped and the least possible number of parts is 6. However, for the L-shaped configuration, the L may be broken into just two parts in only two ways, either of which results in a 6-part decomposition which is not primitive.

The following method may be used to construct primitive rectangular decompositions for $n=5, 7, 8, 9, \cdots$. Starting from a point (not a vertex) on one side of the given rectangle, draw an $(n-1)$-segmented rectilinear spiral with its segments parallel to sides of the rectangle and with the last "loop" closed; extend each segment (after the first) backward until it meets an orthogonal segment; there results a primitive decomposition with n parts.



For rectangular parallelepipeds, a strictly analogous argument involving the necessary condition that no corner part may have three (or more) vertices in common with the given parallelepiped and yield a primitive decomposition shows that $n=3, 4$ are not primitive. Each primitive decomposition for $n=5, 7, 8, 9, \cdots$ of a plane rectangle produces a primitive decomposition into the same number of parts of a rectangular parallelepiped by making the solid parts the same height in the third dimension. However, the primitive decomposition into 6 parts is also possible as is shown in the accompanying figure.

Also solved by Newcomb Greenleaf, Donald Knuth, and the proposers.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4905. *Proposed by Basil Gordon, Redstone Arsenal, Alabama*

Does the following limit exist:

$$\lim_{x\to 1^-} (1-x)^{1/2} \prod_{n=0}^{\infty} (1+x^{4^n})?$$

4906. *Proposed by P. L. Butzer, Technical University, Aachen, Germany*

If the function $f(x)$ is continuous on an interval (a, b) and, as $h\to 0$, $(1/h^3)\int_0^h [f(x+u)+f(x-u)-2f(x)]du\to 0$ for all x in (a, b), then $f(x)$ is a linear function.

4907. *Proposed by P. T. Bateman, University of Illinois*

Suppose $\lambda_1, \cdots, \lambda_r$ are among the elements of a finite abelian group G (written additively) and let H be the subgroup of G generated by $\lambda_1, \cdots, \lambda_r$. Show that there exist positive integers $k_1, \cdots, k_r$ such that every element of H is uniquely expressible in the form

$$n_1\lambda_1 + \cdots + n_r\lambda_r,$$

where n_i is a nonnegative integer less than k_i $(i=1, \cdots, r)$.

4908. *Proposed by John Rainwater, University of Washington, Seattle*

Consider a triangle *abc* divided into four smaller triangles, a central one *def* inscribed in *abc* and three others on the three sides of *def*. Show that *def* cannot have the smallest area of the four unless all four are equal with *d*, *e*, and *f* the midpoints of the sides of *abc*. (The problem seems to have originated with P. Erdös and been transmitted by N. D. Kazarinoff and J. R. Isbell.)

4909. *Proposed by D. J. Newman, Brown University*

Prove that the product of the zeros of $z^n \pm z^{n-1} \pm \cdots \pm z \pm 1$ which lie outside the unit circle is less than $\sqrt{(n+1)}$.

4910. *Proposed by J. S. Frame, Michigan State University*

Find all analytic solutions of the functional equation $f(2z) = 2f(z)f'(z)$.

SOLUTIONS

Derivative of Powers of a Matrix

4823 [1959, 66; 1959, 818]. *Proposed by G. Matthews, St. Dunstan's College, Catford, England*

Let X be a lower-semi-matrix whose elements x_{ij} $(i, j=1, 2, \cdots)$ are independent variables if $j \leqq i$ and zero if $j>i$, and let D be the differential operator matrix defined by $D_{ij}=\partial/\partial x_{ij}$ if $j<i$, $D_{ij}=0$ if $j>i$ and $D_{ii}=\sum_{k=1}^{i} \partial/\partial x_{kk}$. Prove that $D(X^2)=2X$, and hence by induction that $D(X^n)=nX^{n-1}$, where n is any positive integer greater than 2.

II. *Solution by R. D. Gordon, California State Polytechnic College.* There seems to be a gap in the former solution since the relation $DAB=A\cdot DB+DA\cdot B$ is not shown to hold for arbitrary matrices. In fact a counter-example can be cited.

Using the fact that, in the present instance, A and B are powers of the same semi-matrix, the proof can be completed as follows. Let $X^n=(x_{ij}^{(n)})$, then $x_{ij}^{(n)}=0$ if $j>i$. Suppose $DX^n=nX^{n-1}$. This is equivalent to $\sum_{(i\geq k\geq j)} D_{ik}x_{kj}^{(n)}=nx_{ij}^{(n-1)}$. Since

$$X^{n+1}=(x_{ij}^{(n+1)})=X^nX=\left(\sum_{(i\geq l\geq j)} x_{il}^{(n)}x_{lj}\right),$$

it follows that the elements of DX^{n+1} are given by

$$\sum_{(i\geq k\geq j)} D_{ik}x_{kj}^{(n+1)}=\sum_{(i\geq k\geq l\geq j)} D_{ik}x_{kl}^{(n)}x_{lj}=\sum_{(i\geq k\geq l\geq j)} (nx_{il}^{(n-1)}x_{lj}+x_{kl}^{(n)}D_{ik}x_{lj})$$

$$=nx_{ij}^{(n)}+x_{ij}^{(n)}=(n+1)x_{ij}^{(n)}.$$

This is so because the only term in $\sum x_{kl}^{(n)}D_{ik}x_{lj}$ which does not vanish corresponds to $k=i$, $l=j$ and is $x_{ij}^{(n)}(\partial/\partial x_{jj})x_{jj}=x_{ij}^{(n)}$. This gives the desired result $DX^{n+1}=(n+1)X^n$, and the proof is complete since the inductive hypothesis is easily verified for $n=1$.

Finite Ring

4852 [1959, 515]. *Proposed by P. T. Bateman, University of Illinois*

Give an example of a ring with a finite number of elements which cannot be isomorphic to a residue-class ring of the ring of integers of an algebraic number field (of finite degree over the rationals).

Solution by Irving Reiner and Shoji Sato, University of Illinois. Let K be an algebraic number field, I the ring of algebraic integers in K, A an ideal in I. If $A=P_1^{e_1}\cdots P_n^{e_n}$ is the factorization of A into prime ideals P_i in I, then the factor ring I/A is isomorphic with the direct sum S of the rings $I/P_i^{e_i}$, *viz.*,

$$I/A \cong \sum_{i=1}^{n} I/P_i^{e_i} = S.$$

Let $N = \sum_{i=1}^{n} P_i/P_i^{e_i}$. Then N is a subring of S and $N^e = 0$, $N^{e-1} \neq 0$, where $e = \max\{e_1, \cdots, e_n\}$. Also no element of S not in N is nilpotent, so that N is the radical of S. In fact,

$$S/N \cong \sum_{i=1}^{n} I/P_i,$$

where the I/P_i are finite fields. If g_i denotes the number of elements in I/P_i, then the ring $I/P_i^{e_i}$ has $g_i^{e_i}$ elements by the multiplicativity of the ideal-norm.

On the other hand let F be a finite field containing g elements and let $R = F\cdot 1 + Fu_1 + \cdots + Fu_m$ be the commutative algebra over F defined by the multiplication rules

$$1\cdot u_i = u_i\cdot 1 = u_i \ (i = 1, \cdots, m),\ 1^2 = 1,\ u_iu_j = 0 \ (i, j = 1, \cdots, m).$$

Then the radical $T = Fu_1 + \cdots + Fu_m$ of R satisfies $T^2 = 0$ and $R/T \cong F$. Also R contains g^{m+1} elements.

Now suppose the rings R and I/A of the two preceding paragraphs are isomorphic. Then S/N is isomorphic to a single field F, so that $n = 1$ and $A = P^e$. Since R and I/A must have the same number of elements, $e = m+1$. Moreover $N^{e-1} \neq 0$, so that $T^m \neq 0$. Thus m must be 0 or 1. Hence, if we take $m > 1$, the ring R provides an example of the type sought.

Dimension of a Banach Space

4853 [1959, 515]. *Proposed by D. J. Newman, Brown University*

The dimension of a Banach space can be defined, *e.g.* in the following two ways: (1) the smallest number (cardinal) of complex numbers necessary to specify a vector; (2) the smallest number of vectors such that linear combinations of them are dense. Are these two definitions equivalent?

Solution by Albert Wilansky, Lehigh University. The definitions are not equivalent. Two dimensional Euclidean space has dimension 2 by the second definition; but dimension 1 by the first, for, since the cardinality of the space is c, it can be put into one-to-one correspondence with the reals, hence any of its points can be specified by one number.

Also solved by Leopold Flatto, and the proposer.

Unitary Matrices Having Preassigned Eigenvalues

4854 [1959, 515]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let $\{\alpha_i\}$, $\{\beta_i\}$ $(1 \leqq i \leqq n)$ and γ be given complex numbers, each of absolute value 1. Prove: There exist two unitary matrices A, B of order n with the preassigned eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$, respectively, and such that γ is an eigenvalue

of AB, if and only if, in the complex plane, the convex hull of the n points $\{\alpha_i\}$ and the convex hull of the n points $\{\gamma\bar{\beta}_i\}$ have a point in common.

Solution by the proposer. Let A, B be two unitary matrices of order n with eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$, respectively and such that γ is an eigenvalue of AB. Then γ is also an eigenvalue of BA, so there exists a vector z in the unitary n-space such that $\|z\| = 1$ and $BAz = \gamma z$. Let $Ax_i = \alpha_i x_i$, $By_i = \beta_i y_i$ $(1 \leqq i \leqq n)$, where $\{x_i\}$ are n orthonormal eigenvectors of A, and $\{y_i\}$ are n orthonormal eigenvectors of B. Let $p_i = |(z, x_i)|^2$, $q_i = |(z, y_i)|^2$ $(1 \leqq i \leqq n)$. Then

$$\sum_{i=1}^{n} p_i\alpha_i = \sum_{i=1}^{n} \alpha_i |(z, x_i)|^2 = (Az, z), \qquad \sum_{i=1}^{n} q_i\gamma\bar{\beta}_i = \sum_{i=1}^{n} \gamma\bar{\beta}_i |(z, y_i)|^2 = (\gamma B^*z, z).$$

Since $BAz = \gamma z$, we have $Az = \gamma B^{-1}z = \gamma B^*z$. Hence

$$\sum_{i=1}^{n} p_i\alpha_i = \sum_{i=1}^{n} q_i\gamma\bar{\beta}_i. \tag{1}$$

As $p_i \geqq 0$, $q_i \geqq 0$, $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = \|z\|^2 = 1$, equation (1) means that the convex hull of $\{\alpha_i\}$ and the convex hull of $\{\gamma\bar{\beta}_i\}$ have a point in common.

To prove the converse, assume that (1) is satisfied for certain $p_i \geqq 0$, $q_i \geqq 0$ with $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. Let

$$x = \{\sqrt{p_1}, \sqrt{p_2}, \cdots, \sqrt{p_n}\}, \qquad y = \{\alpha_1\sqrt{p_1}, \alpha_2\sqrt{p_2}, \cdots, \alpha_n\sqrt{p_n}\},$$
$$u = \{\sqrt{q_1}, \sqrt{q_2}, \cdots, \sqrt{q_n}\}, \qquad v = \{\gamma\bar{\beta}_1\sqrt{q_1}, \gamma\bar{\beta}_2\sqrt{q_2}, \cdots, \gamma\bar{\beta}_n\sqrt{q_n}\}.$$

Then since $\sum p_i = \sum q_i$ and $|\alpha_i| = |\beta_i| = |\gamma| = 1$, we have $(x, x) = (u, u)$, $(y, y) = (v, v)$. From (1), we have also $(x, y) = (u, v)$. Hence there exists a unitary matrix U of order n such that $Ux = u$, $Uy = v$. Let $A = \text{diag}\,(\alpha_1, \alpha_2, \cdots, \alpha_n)$, $C = \text{diag}\,(\beta_1, \beta_2, \cdots, \beta_n)$ and $B = U^*CU$. Then A, B are unitary matrices with eigenvalues $\{\alpha_i\}$, $\{\beta_i\}$, respectively. We have $\gamma C^*u = v$, which can be written $\gamma C^*Ux = Uy$ or $\gamma x = U^*CUy = By$. But $Ax = y$, so $BAx = By = \gamma x$. Thus γ is an eigenvalue of BA, and therefore also an eigenvalue of AB.

Editorial Note. E. J. Taft and A. C. Mewborn call attention to the fact that the above is a special case of a theorem of H. Wielandt (*Die Einschliessung von Eigenwerten normaler Matrizen*, Mathematische Annalen, v. 121 (1949–1950), p. 234, Satz 1). Wielandt's theorem deals with normal matrices but if the normal matrix is taken as unitary and certain vectors are taken to be unit vectors the adaptation to the present problem imposes little difficulty.

The Growth Function

4856 [1959, 516]. *Proposed by L. A. Rubel, University of Illinois*

The growth function $h(\theta)$ of an entire function $f(z)$ of exponential type is defined by $h(\theta) = \limsup_{r\to\infty} r^{-1} \log |f(re^{i\theta})|$. It is well known that $f(z) \equiv f_1(z)f_2(z)$ need not imply that $h(\theta) \equiv h_1(\theta) + h_2(\theta)$. Suppose, however, that $h_1(\theta) \equiv h_2(\theta)$. Must $h(\theta) \equiv h_1(\theta) + h_2(\theta)$ in this case?

Solution by the proposer. A negative answer is indicated by the following

construction: Choose a large positive number R, and let $A=A_R=a_1, a_2, \cdots$ be the sequence of all positive integers a_n such that for some integer v, $R^{2v}\leqq a_n<R^{2v+1}$, and let $B=B_R=b_1, b_2, \cdots$ be the complementary sequence of positive integers. Put $f_1(z)=\prod(1-z^2/a_n^2)$, $f_2(z)=\prod(1-z^2/b_n^2)$.

Since more sophisticated counterexamples will soon be in print, the proof given here is considerably abbreviated. By symmetry, we need consider only $0\leqq\theta\leqq\frac{1}{2}\pi$, and by continuity of the h-functions, only $0<\theta\leqq\frac{1}{2}\pi$. We use the well-known formula $\log|f_1(re^{i\theta})|=r\int_0^\infty P(t,\theta)D_A(rt)dt$ and the corresponding formula for f_2 in terms of D_B. Here $D_A(x)=x^{-1}\sum_{a_n\leqq x}1$ and $P(t,\theta)=2(1-t^2\cos 2\theta)/(1-2t^2\cos 2\theta+t^4)$. From the fact that $D_B(t)=D_A(Rt)+o(1)$, it follows that $\log|f_2(re^{i\theta})|=R^{-1}\log|f_1(Rre^{i\theta})|+o(r)$ for each $\theta\in(0,\frac{1}{2}\pi]$ as $r\to\infty$, so that $h_1(\theta)\equiv h_2(\theta)$. Clearly, we may also choose R so large that $h_1(\frac{1}{2}\pi)>\frac{1}{2}\pi$. But since $f_1(z)\cdot f_2(z)=\prod(1-z^2/n^2)=\sin\pi z/\pi z$, $h(\theta)\equiv\pi|\sin\theta|$, and thus $h_1(\frac{1}{2}\pi)+h_2(\frac{1}{2}\pi)>\pi=h(\frac{1}{2}\pi)$.

Editorial Note. This proposal was submitted October 1957. J. P. Kahane and the proposer have shown that $h(\theta)=h_1(\theta)+h_2(\theta)$ is false under far more restrictive hypotheses on f_1 and f_2. (See *Sur les produits canoniques de type nul sur l'axe reel,* Comptes Rendus, Acad. Sci. Paris, v. 248, pp. 3102–3103, June 1959. A fuller version will appear in a forthcoming issue of the *Illinois Journal of Mathematics.*) Another example by C. Roumieu is in preparation for the *Annales de l'Ecole Normale Supérieure.*

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematics in Physics and Engineering. By J. Irving and N. Mullineux. Academic Press, New York, 1959. xvii+883 pp. $11.50.

This is Volume 6 of the series on Pure and Applied Physics for which H. S. W. Massey is consulting editor. Many topics in advanced calculus for engineers and physicists are found in this remarkable volume: differential equations, Laplace transforms, matrices, classical and quantum mechanics, calculus of variations, complex variable, residues, numerical methods, integral equations. There are 82 pages of appended material, many problems and their solutions. Because of its wide scope it is suitable for classes where shorter texts suitable to individual needs are not available. Teachers would naturally wish to examine the treatment of individual topics for themselves, but this reviewer was very well impressed by the judicious compromise between clear exposition and scholarly depth. Topics seem easy to locate, and the single volume is not too bulky.

ARTHUR BERNHART
University of Oklahoma

Abelian Varieties. (Interscience Tracts in Pure and Applied Mathematics, No. 7.) By Serge Lang. Interscience, New York, 1959, xii+256 pp. $7.25.

It has been over ten years since Weil's definitive *Variétés Abéliennes et Courbes Algébriques* first appeared. During this period a great deal of work has been done on algebraic groups and, in particular, on abelian varieties. Many facets of abelian varieties only mentioned in Weil's treatise have now reached a state of substantial development. This development has been the combined work of the French, Italian, Japanese and American schools. It has therefore become desirable to have a book available which incorporates the results of Weil's treatise with the newer developments in the field of abelian varieties.

Lang has attempted to do just this in the present book and, for the most part, succeeds admirably. Even though—as Lang points out—several special topics are not touched upon, an amazing amount of material is covered. The topics dealt with are the Jacobian, Picard, and Albanese varieties; the 1-adic representations; algebraic systems of abelian varieties. The emphasis is very definitely and very heavily on the "abstract." More examples drawn from "classical" algebraic geometry would help to illuminate and clarify the ideas developed—particularly for the beginner in algebraic geometry. This, however, is a minor flaw in an otherwise excellent book. The reader who is willing to give his time and close attention to this book will be amply rewarded by the insights he will receive into one of the most beautiful and fruitful branches of modern mathematics.

EDWARD H. BATHO
Harvard University

Differential Geometry. By A. V. Pogorelov (translated by Leo F. Boron). Noordhoff, Groningen, 1959. ix+171 pp. $3.90 (paper) or $4.50 (cloth).

This book is an introduction to local theory of curves and surfaces in three-dimensional Euclidean space. The author satisfactorily accomplishes his aim "to present a rigorous discussion of the fundamentals and of the methods of investigation without disturbing well-established tradition in the process." Regular curves and regular surfaces are defined after the manner of a differentiable manifold. The topics are well chosen to cover the essentials in 171 pages which include a bibliography and an index. A large amount of factual material is relegated to exercises and problems at the end of each chapter. Vector notation is used. This book differs from most books at the same level by its concise, rigorous and up-to-date treatment with emphasis on basic concepts. It merits serious consideration as a text or a reference book for beginners in the subject.

However, there are many obvious misprints in the book. In the definition of "order of contact" on page 32, for example, it seems necessary to indicate that n is the greatest integer. Similarly the reader should be aware that some terms like "onto" and "circle" are used in different senses at different places and that the word *respectively* is frequently put in unusual positions.

The chapters in Part One on *Theory of Curves* are: Concept of curve; Concepts for curves which are related to the concept of contact; Fundamental concepts for curves which are related to the concepts of curvature and torsion. Part Two on *Theory of Surfaces* consists of the following chapters: Concept of surface; Fundamental concepts for surfaces which are related to the concept of contact; First quadratic form of a surface and concepts related to it; Second quadratic form of a surface and questions about surface theory related to it; Fundamental equations of the theory of surfaces; Intrinsic geometry of surfaces.

T. K. PAN
The University of Oklahoma

Approximate Methods of Higher Analysis. By L. V. Kantorovich and V. I. Krylov (translated by Curtis D. Benster). Noordhoff, Groningen; Interscience, New York, 1958. 681 pp. $17.00.

This is the English translation of the third Russian edition (1950) of the book "Priblizhennye metody vysshego analiza" by the two famous authors. On the jacket, the book is praised as a "unique text and reference book of pure and applied mathematics." It can be considered as an almost unique text because of the wealth of information contained therein (for a table of content see the Mathematical Reviews, vol. 13, 1952, p. 77) and the great number of carefully worked out examples, many of which are not trivial. It certainly has to be considered as a "unique" reference book due to the complete absence of a subject index. This fact makes its use as a reference book very awkward indeed, especially as the section headings in the contents are often ambiguous and prevent an accurate guess as to their content. The translation is precise and clear, but rather stilted due to the translators obvious aim to stick to the Russian text as closely as possible. If the translator had supplied appropriate references to Anglo-American literature, the usefulness of this work would have been increased many fold. The printing is clear except for a surprisingly large number of space bar blurs. The formulas and tables are well arranged.

HANS SAGAN
University of Idaho

Combinatorial Topology, Vol. 1. By P. S. Aleksandrov. Graylock Press, Rochester, New York, 1956. xvi+225 pp. $4.95.

Combinatorial Topology, Vol. 2, *The Betti Groups*. By P. S. Aleksandrov. Graylock Press, Rochester, New York, 1957. xi+244 pp. $6.50.

These two volumes are a translation of Parts I, II and III of Aleksandrov's *Kombinatornaya Topologiya*. The translation into English has been made with a commendable lack of awkwardness, resulting in a masterful treatise, moderately paced. The book is designed to meet the needs of the student with maturity who has only modest familiarity with general topology and the concepts of algebra. Except for Chapter I, which is of an introductory nature, the arguments

are made in considerable detail. The material is beautifully organized, and there are numerous remarks and examples by way of clarification. There are very few exercises, but in all other respects the book constitutes an excellent textbook on classical homology theory.

Chapter II is devoted to proving the Jordan curve theorem. Chapter III covers the topological classification of closed two-dimensional surfaces. Euler's theorem (that two triangulations of a surface have the same Euler characteristic) is employed without proof to show that the connectivity of a surface is invariant under homeomorphism. (A proof is supplied later in Chapter X.) Chapter IV is a detailed study of complexes and Chapter V treats Sperner's lemma and its corollaries, principally the Pflastersatz, the invariance of interior points of subsets of R^n, and the fixed-point theorem for n-cells. Chapter VI is an introduction to dimension theory and treats only the covering dimension for compacta. These six chapters, with an appendix on n-dimensional geometry, complete Volume I.

Volume 2 (Part III of the original work) is devoted to the Betti groups. The notions of chain and boundary operator, and the resultant algebraic apparatus are developed in Chapter VII. The elementary theory of homology groups is developed in Chapter VIII and in Chapter IX the cohomology groups appear and are used to obtain relations among homology and cohomology groups with different coefficient domains. The invariance of the Betti groups is proved in Chapter X. In Chapters XI and XII the homology theory is extended from polyhedra to compacta. An appendix is devoted to basic information on Abelian groups.

Volume 3 has not yet appeared in English translation. Since Volume 2 is introductory to the remaining chapters and, in a sense, is motivated by what is to follow, it is to be hoped that the remainder of the book will be published in English at an early date. Among other topics, these chapters contain the duality theories of Poincaré and Alexander-Pontryagin, and the Lefschetz fixed-point formula.

L. E. WARD, JR.
University of Oregon

Industrial Mathematics. By Matthias Thies. Prentice Hall, Englewood Cliffs, N. J., 1959, 342 pp. $9.00.

A more appropriate title for this book would be "Industrial Arithmetic", for there is no real mathematics in it. The basic operations of arithmetic and the computational aspects of trigonometry are treated in very great detail. The author attains this simplicity in his presentation by paying the price of careless statements.

This book is definitely not suitable as either a high school or college text. This reviewer was also appalled at the high price of the book, considering the caliber of the contents.

LEONARD E. FULLER
Kansas State University

Elementary Decision Theory. By Herman Chernoff and L. E. Moses. Wiley, New York, 1959. xv+364 pp. $7.50.

The day of determinism is past, even in the field of engineering, and it is rather amusing to remark that the determinist has inevitably brought it on himself. By constructing more and more realistic models of the physical universe, of greater and greater complexity, he has slowly forced himself into a position where the only escape from stalemate lies in the introduction of new rules into the game. The result is that we must coat ignorance of true cause and effect and inability to handle the resultant equations, even if we know them, by the soothing, albeit sophisticated, salve of probability theory.

The modern engineer who is interested in feedback control processes, in automation, in processes involving adaptive control, and in any number of significant uses of digital and analogue computers, must acquaint himself with the mathematical theory of decision processes. He must understand the difficulties and advantages of uncertainty, and be familiar with the various tricks that are used to construct precise mathematical models of imprecise situations.

Unfortunately, many of the standard works in the field of decision-making start from a high mathematical level and continue upward. The result is that they are not suited to the needs of the beginner.

The book by Chernoff and Moses is a book designed specifically for the scientist interested in decision processes, but possessing only a rudimentary mathematical background. The authors have succeeded admirably in what they have set out to do. Carefully and slowly, they illustrate how mathematical models of decision-making under uncertainty are formulated. They discuss the choice of a criterion function, the choice of possible policies, the description by means of state variables, and so on.

All of this is done in a very pleasant, readable style, with numerous examples, and much important discussion. Most important, they are honest with the reader in constantly pointing out pitfalls and limitations. Their presentation illustrates very clearly that fundamental ideas can be transmitted without the pseudo-abstraction that befogs the Bourbaki and their devotees.

In the second half of the book, they give an introduction to classical statistics. Throughout, their aim is to follow the guiding idea of Wald, who conceived of statistics as decision-making under uncertainty. In their presentation they have followed the teachings of the late M. A. Girshick.

The book is recommended to mathematicians, physicists, engineers, and so forth, who wish to obtain a relatively painless introduction to this modern field, either for their own studies, or merely to keep abreast of current intellectual activity.

RICHARD BELLMAN
The RAND Corporation

La Localisation des Valeurs Caractéristiques des Matrices et ses Applications. By M. Parodi. Gauthier-Villars, Paris, 1959. x+172 pp. $7.76.

This book is an almost-complete survey of results on the determination of bounds for the eigenvalues of a square matrix which have been found by such workers as A. Ostrowski, A. Brauer, K. Fan, O. Taussky and the author, and which have as their prototype the well-known "Geršgorin circles" theorem. Some of the topics treated are: sufficient conditions for a matrix to be nonsingular, bounds (in the complex plane) for the eigenvalues of a matrix, sufficient conditions for the stability of an analogue computer, the location of the zeros of polynomials, the determination of the zeros of determinants with polynomial elements.

Perhaps the most interesting section is that concerned with polynomials, in which much unfamiliar and inaccessible material is given on bounds for the zeros of a polynomial in terms of its coefficients.

Anyone concerned with the numerical determination of eigenvalues will find this a useful and practical reference work.

MORRIS NEWMAN
National Bureau of Standards

A Philosopher Looks at Science. By John G. Kemeny. Van Nostrand, Princeton, N. J., 1959. 273 pp. $4.95.

Mathematical journals seldom find time to review books on philosophy or even the philosophy of science, but the author of this one, while having every right to call himself a philosopher, happens to be a mathematician known for his research in logic and contributions to undergraduate education. In the first part of the book, "What Science Presupposes," Kemeny speaks primarily as a mathematician with a philosophical bent, or perhaps one should say as a philosopher thoroughly imbued with mathematics and making full use of it to deflate pseudo-problems and to formulate ideas simply. In Part two, "Science," and in the third and final part, "Problems Raised by Science," the philosopher in Kemeny comes more to the fore, though even here there are constant reminders of his mathematical competence and point of view.

The book is essentially a popularization, directed to the interested layman or the undergraduate student. However, important ideas can be presented in a book at this level, and Kemeny writes in his preface that he hopes his book will be of some interest even to the expert. Possibly Kemeny could be accused of oversimplification and of sidestepping really difficult philosophical problems by redefinition. But this reviewer finds the discussion convincing and satisfying—it is difficult to be critical of an author whose ideas one finds congenial.

The mathematician should find the book enjoyable, since it presents a point of view that is consistent with mathematical knowledge and inclinations.

KENNETH O. MAY
Carleton College

General Homogeneous Coordinates in Space of Three Dimensions. (paper edition.) By E. A. Maxwell. New York: Cambridge University Press, 1959. xiv+165 pp. $2.75.

This is a sequel to the author's earlier work on the methods of plane projective geometry based on the use of general homogeneous coordinates (Cambridge University Press, 1946). A knowledge of the material in the earlier volume is assumed. The principal emphasis is on projective geometry of ordinary space. One chapter portrays applications to euclidean geometry.

It is curious that the author replaces the usual statement concerning linear dependence of elements by the remark that the elements are in *syzygy*. It is regrettable that use is not made of a notation which can be extended easily to the case of n-dimensions. The eighth and last chapter introduces matrices, and contains a clever and novel treatment of line coordinates. The belated mention of matrices deprives the reader of an opportunity to gain practice in the use of this tool. Many standard exercises appear under the heading of Theorem-examples. More difficult exercises are found at the end of each chapter. Following a development of quadric surfaces, there is a treatment of line geometry including the linear and tetrahedral complexes. The twisted cubic and systems of quadrics are studied.

There is much of interest in this well-written account of a selection of topics from classical algebraic geometry. It should be a welcomed addition to the literature because of its uniqueness of approach and its value to one who aspires to read more advanced treatises on geometry.

C. E. SPRINGER
University of Oklahoma

Introduction to Analysis. By N. B. Haaser, J. P. LaSalle, J. A. Sullivan. Ginn, Boston, 1959. 688 pp. $8.50.

This is a skillfully written rigorous presentation of introductory analysis, which should prove highly satisfactory for a student who has had thorough preparation in high school mathematics, and who is guided in this volume by a competent teacher. Even with the omission of about fifty pages which the preface indicates as optional, there is more here than can be covered in two five-hour semesters.

The language of sets is introduced at the beginning and is used throughout. Then the axioms for the real numbers (except the least-upper-bound axiom which is delayed appropriately until page 381) are stated, and the following treatment of algebra, geometry, trigonometry, and calculus is based upon this set of axioms. Vectors as ordered pairs of real numbers are employed extensively in the development of analytic geometry and trigonometry. A function is defined as a "set of ordered pairs of elements such that no two distinct pairs have the same first element." A transformation of the Euclidean plane R^2 is a function T with domain R^2 and range R^2, so that transformations constitute a special

class of functions. Following an intuitive discussion of arc length, the real number line R is mapped on the unit circle. To a number u on R there corresponds a point on the circle. A function $F(u)$ is thus defined, the ordered pair for which are cos u and sin u. The law of sines and the law of cosines are developed by vector methods. Mathematical induction is treated rigorously and a careful foundation for the calculus is laid. Even the Heine-Borel theorem, and uniform continuity appear, although these are in sections described as optional.

It is curious to find a discussion of the solution of a quadratic equation following chapters involving real variable theory, problems on maxima and minima, and the graphing of functions more complicated than the quadratic function. Of course, this is to open the way for complex numbers, which are defined as ordered pairs of real numbers with certain operations of addition and multiplication. Complex numbers are shown to be an instance of a field. The logarithm, exponential, and inverse functions are introduced near the end of the volume. Such topics as infinite series and partial differentiation are not treated. These will presumably appear in the projected Volume II. The Taylor theorem is introduced and is used in a discussion of numerical integration.

The authors and publishers are to be commended for producing this excellent volume which is in the spirit of contemporary mathematics. All teachers of college mathematics should be familiar with its contents.

C. E. SPRINGER
The University of Oklahoma

Introductory Calculus. By Donald E. Richmond. Addison-Wesley, Reading, Mass., 1959. xv+207 pp. $6.75.

Richmond's *Introductory Calculus* is refreshingly different, and it merits special attention for its faithfulness to the spirit of modern mathematics. Recognizing that "Mathematics is concerned with logical consequences of different sets of assumptions," the author states assumptions and definitions with precision, and he leaves no doubt about the logical status of statements which are made. Theorems are proved with unusual care, but there is no quixotic tendency to make a fetish of rigor. The stage for an important idea or proof generally is first set by an intuitive approach which is fortified by the sound conviction that "it is important to lay bare the essential idea as clearly as possible."

This text, consisting of the first five chapters of Richmond's *Calculus* (appearing also in 1959), is intended to serve as the basis for a semester course, and as such it should meet with favor. An instructor who does not adopt it as the textbook for the course may yet wish to use it as supplementary material, for it affords delightfully different approaches and unconventional proofs. The power rule for the differentiation of positive integral powers of functions, for example, is derived through mathematical induction from the product rule; and the derivative of the logarithmic function is obtained as the inverse of an exponential function. More strikingly novel are the definitions of the sine and

cosine functions as the imaginary and real parts of a number representing a point on a unit circle in the complex plane, from which the derivatives and other properties are obtained.

Richmond's *Introductory Calculus* resembles, in size, scope, and purpose, the first six chapters of his *Fundamentals of the Calculus* (1950), but it is clearly a new and more sophisticated volume. There is increased confidence in novelty of treatment, and there is greatly heightened emphasis upon depth of understanding.

A few peccadillos may catch the eyes of an attentive reader—the ubiquitous use of "may" for "can," an inadequacy in the definition of point of inflection (p. 68), and the reversal of the inequality signs in Archimedes' bounds for π (p. 159). These will in no way jeopardize the final judgment on the book as a sound work which exhibits major expository and mathematical virtues.

CARL B. BOYER
Brooklyn College

Contributions to the Theory of Games, Vol. IV. Edited by A. W. Tucker and R. D. Luce. Annals of Mathematics Study No. 40, Princeton University Press, 1959. vii+453 pp. $6.00.

This is the fourth (and last) volume of the series *Contributions to the Theory of Games*. It carries a dedication to the late John von Neumann and is devoted entirely to an area in which he took a special interest, the theory of non-strictly competitive games. These games, which are roughly all games except zero-sum two-person games, bear the greatest promise for fruitful application to the social sciences. At the same time, this is the branch of game theory that is least developed mathematically and which is replete with problems unformulated or unsolved. All of the current directions of research are well represented in this collection of nineteen papers by sixteen authors.

A detailed review of the contents is rendered superfluous by the editors' introduction, which provides a synopsis of each paper and arranges them in a natural order. The contributions fall into two broad classes: those which investigate the von Neumann-Morgenstern concept of a solution and those which attempt to modify or improve it. Unfortunately, the question of the existence of a von Neumann-Morgenstern solution for every finite game is still open.

With its wide coverage of research, this is an indispensable book for anyone who wishes to keep abreast of progress in the theory of games. It also includes two bonus features, which make it even more valuable: a translation of von Neumann's classic 1928 paper, which initiated the modern theory of games, and a bibliography on game theory with more than 1,000 entries.

HAROLD W. KUHN
Princeton University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, *New York. Items must be submitted at least two months before publication can take place.*

SUMMER SESSIONS

University of California, Los Angeles, June 20 to July 29: Professor Horn, functions of a complex variable; Visiting Professor Grünbaum, linear programming.

University of Illinois, June 20 to August 13: Professor Day, combinatorial topology; Professor Ketchum, measure and integration; Professor Ribenboim, theory of rings; Professor Brahana, linear transformations and matrices; Professor Schubert, introduction to higher algebra; Professor Reiner, number, length, area. In addition, the following courses will be given: fundamental concepts of geometry; advanced algebra; introduction to higher geometry; complex variables and applications; introduction to higher analysis; advanced statistics; mathematical methods in engineering and science.

State University of Iowa, June 13 to August 10: Professor Dye, introduction to analysis, introduction to algebraic topology; Professor Hogg, topics in mathematical statistics, statistical hypotheses, seminar in mathematical statistics; Dr. Jakobsen, differential equations, matrices and determinants; Professor Muhly, elementary theory of numbers, topics in commutative algebra, seminar in algebra; Professor Price, construction of technical aids in the teaching of mathematics, supervision of mathematics; Professor Reid, integral equations, seminar in analysis.

University of Michigan, June 20 to August 13: Dr. Chu, vector analysis; Professor Craig, statistical analysis, mathematical theory of probability; Professor Dushnik, operational mathematics, topics in modern mathematics for teachers; Professor Dwyer, calculus of finite difference; Professor Griffin, advanced calculus; Dr. Hedstrom, differential equations; Dr. Hicks, fourier series and applications; Professor D. Jones, mathematical theory of probability, theory of statistics; Professor P. Jones, history of mathematics; Dr. Kincaid, introduction to functions of a complex variable with applications; Dr. Kister, set-theoretic topology; Professor Livingstone, theory of equations and determinants, topics in geometry for teachers; Professor McLaughlin, introduction to matrices, introduction to theory of numbers; Mr. Morrill, mathematics of life insurance; Dr. Mrowka, introduction to differential equations, introduction to the foundations of mathematics; Professor Murdoch, introduction to matrices, algebra; Dr. Ramanujan, advanced mathematics for engineers; Professor Reade, introduction to functions of a complex variable with applications, differential geometry; Dr. Rosen, theory of dimensions; Dr. Schaetz, operational mathematics, intermediate course in differential equations; Dr. Shimrat, advanced mathematics for engineers; Professor Ullman, fourier series and applications, complex analysis; no instructor, introduction to differential equations; no instructor, differential equations.

PERSONAL ITEMS

Dr. R. W. Bass, RIAS, Baltimore, Maryland, has been appointed Chief Scientist at Aeronca Manufacturing Corporation, Aerospace Division, Baltimore, Maryland.

Mr. L. W. Chinnery, Grinnell College, has accepted a position as Computer Programmer Trainee with the System Development Corporation, Santa Monica, California.

Mr. R. H. Christ, RCA, Camden, New Jersey, has been appointed Data Reduction Mathematician with the RCA Missile Test Project, Patrick Air Force Base, Florida.

Mr. E. A. Covey, Abilene Christian College, has been appointed Mathematician with the U. S. Civil Service, White Sands Missile Range, New Mexico.

Mr. Charles Drescher, Ramo-Wooldridge Corporation, Los Angeles, California, has accepted a position as Electrical Systems Engineer with the System Development Corporation, Lodi, New Jersey.

Mr. W. G. Fair, University of Missouri, has accepted a position as Junior Engineer with the Bendix Aviation Corporation, Kansas City, Missouri.

Mr. S. I. Gass, Operations Research Branch, Council for Economic and Industrial Research, Arlington, Virginia, has accepted a position as Senior Mathematician with the Federal Systems Division of the I.B.M. Corporation, Washington, D. C.

Mr. R. Q. Jennett, The Martin Company, Florida, has accepted a position as Senior Aerophysics Engineer for Convair, Fort Worth, Texas.

Dr. A. T. Lauria, Purdue University, has accepted a position as Mathematician with the Research Planning Group of the Linde Company, Tonawanda, New York.

Mr. C. D. Robbins, Douglas Aircraft, Santa Monica, California, has accepted a position as Research Engineer with the Temco Aircraft Company Experimental Laboratories, Dallas, Texas.

Professor J. W. Tukey, Princeton University, has been appointed to membership on President Eisenhower's Science Advisory Committee for the four-year term beginning January 1, 1960.

Mr. W. S. Wenger, University of Kansas, has accepted a position as Mathematician in the Ballistic Research Laboratory, Army Proving Ground, Aberdeen, Maryland.

Dr. Albert Wolinsky, General Precision Laboratory Inc., Pleasantville, New York, has been promoted to Senior Staff Member.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. L. Alder, Secretary, announces that the following 345 persons have been elected to membership by the Board of Governors on applications duly certified.

CESAR A. ABUAUAD, Licenciado (Chile) Prof., University of Chile

ALFRED AEPPLI, Ph.D. (Swiss Federal Inst. Tech.) Asst. Prof., Cornell University

JOSIAH P. ALFORD, M.A. (George Washington) Grad. Asst., Pennsylvania State University

FARID R. AHMED, Ph.D. (Leeds U.) Asso. Research Officer, National Research Council, Canada

ARNOLD O. ALLEN, M.A. (California, Los Angeles) Fellow, University of California, Los Angeles

CLYDE E. ALLEN, JR., B.S. (Gustavus Adolphus Coll.) Mathematician, Remington Rand Univac

BINYAMIN A. AMIRA, D.Sc. (Geneva) Asso. Prof., Hebrew University, Israel

EDWARD S. K. ANSAH, A.B. (Wilkes Coll.) Grad. Asst., Michigan State University

MRS. LUCEIL K. ARNETT, M.A. (West Virginia) Instr., American University

CLAYTON V. AUCOIN, Ph.D. (Auburn University) Asst. Prof., Southwestern Louisiana Institute

JASON M. AUSTIN, Colonel, U. S. Army, Retired

JOHN D. AYDLOTT, A.B. (Wayne S.U.) Instr., Highland Park Junior College

LOTFALI BADII, Student, San Jose State College

IRVING F. BARDITCH, B.S.E.E. (Johns Hopkins) Senior Engr., Westinghouse Electric Corporation

LEE W. BARIC, B.S. (Dickinson Coll.) Instr., Lafayette College

LARRY BARNES, Student, Classen High School, Oklahoma City

BRUCE R. BARRETT, Student, University of Kansas

JAMES E. BARRY, B.S. (Carnegie Inst. Tech.) Engr., E. I. duPont de Nemours and Company

ANTHONY R. BATTISTA, B.S. (St. Francis Coll.) Jr. Mathematician, Teleregister Corporation

GEORGE F. BEATTIE, M.A. (Claremont Coll.) Instr., San Bernardino Valley College

THEODORE S. BECK, Student, San Diego State College; Sr. Research Engr., Ryan Aeronautical Co.

DALE R. BEDGOOD, M.A. (Arkansas) Grad. Asst., University of Oklahoma

LELAND K. BELDEN, B.A. (Iowa S.C.) Iowa City

CLARENCE W. BENNETT, M.S. (Massachusetts) State Supervisor of Mathematics, Maine

GORDON M. BENNETT, M.A. (Columbia T.C.) Asst. Prof., Ohio Northern University

MORRIS M. BENNETT, Student, Long Beach State College

JAMES A. BJUSTROM, M.A. (Columbia) Mathematician, Remington Rand Univac

LESTER M. BLUM, B.E.E. (C.C.N.Y.) Member, Technical Staff, Hughes Aircraft Company

LESLIE E. BLUMENSON, M.S. (N.Y.U.) Staff Mathematician, Columbia University

RAJ C. BOSE, D.Litt. (Calcutta) Prof., Statistics, University of North Carolina

ROBERT N. BOYD, B.A.S. (Toronto) Supervising Engr., DuPont of Canada

RICHARD B. BRIAN, M.A. (Maryland) Asso. Mathematician, Applied Physics Lab., Johns Hopkins University

RAYMOND F. BRINKER, Student, North Central College

BROTHER ALBAN ALBRIGHT, Ph.D. (Pennsylvania) Asst. Prof., La-Salle College

BROTHER PATRICK MATTHEW, B.A.

(Marian Coll.) Head of Dept., St. Joseph's Academy, Laredo, Texas

JOHN ALAN BROWN, M.D. (Minnesota) Instr., Montana State College

RICHARD H. BROWN, Ph.D. (Columbia) Asst. Prof., Washington College

STEFAN A. BURR, Student, University of California, Berkeley

FAY B. BURRAS, Student, Lebanon Valley College

SYLVIA BURTON, B.A. (St. Joseph's Coll.) Grad. Student, New York University

JOHN E. BRUHA, M.A. (Wisconsin) Instr., Iowa State Teachers College

DAVID M. BURTON, A.M. (Rochester) Asst. Prof., University of New Hampshire

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ROGER I. CANFIELD, M.E. (Stevens Inst. Tech.) Engr., Curtiss-Wright

OTTO M. CAROTHERS, JR., M.A. (Ball S.C.) Grad. Student, Vanderbilt University

BENJAMIN P. CARTER, Student, Stanford University

CHARLES CASTONGUAY, Student, University of Ottawa

ERNEST F. CHACE, Student, Oklahoma State University

GULBANK D. CHAKERIAN, M.A. (California) Teaching Asso., University of California

JIN TZE CHEN, M.A. (Oklahoma S.U.) Instr., Central State College

V. GENE CHILD, B.S. (Kansas) Grad. Student, University of Kansas

MILFORD L. CLABAUGH, B.A. (Ohio S.U.) Instr., Arizona State University

WILLIAM W. COLLIER, Student, Harvard College

WILLIAM J. CONGDON, Ph.D. (Cornell) Manager of Estimating, Allied Chemical Corporation

CHARLES T. CONWAY, M.S. (St. Louis) Mathematics Chairman, St. Louis University High School

LT. CDR. FRANK B. CORREIA, M.S. (New Mexico) Instr., NROTC Unit, University of Colorado

DENNIS I. COUZIN, Student, California Institute of Technology

GARY D. CROWN, Student, University of Wichita

CHARLES A. CUMMINS, Student, Southwest Missouri State College

THOMAS R. CURRY, Student, State University College on Long Island

EDWARD C. DAHL, Student, American University

AUBERT DAIGNEAULT, Ph.D. (Princeton) Asst. Prof., University of Ottawa

DONALD A. DARIA, Student, Manhattan College

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JOHN N. DAVIS, M.A. (California) Instr., Willow Glen High School, San Jose, California

ROBERT F. DAVIS, M.A. (Syracuse) Instr., Shepherd College

JAMES T. DAY, M.S. (Oklahoma) Instr., University of Tulsa

COMMODORE C. DEARMAN, JR., M.A. (Mississippi) Mathematician, Redstone Arsenal

RALPH W. DECKER, Student, San Jose State College

NICHOLAS J. DELILLO, Student, Manhattan College

MATTHEW DEMARIA, M.A. (Purdue) Head of Dept., Holton-Arms School, Washington, D. C.

SPENCER E. DICKSON, Student, University of Kansas

DOM DIPOLLINA, Owner, Dom's Jewelers, Utica, New York

ROBERT D. DOLAN, B.S. (California S.T.C.) Instr., California Community High School, Pennsylvania

ELBERT G. DONSBACH, Computer Programmer, American Oil Company

EDWIN C. DOUGLAS, M.A. (Harvard) Asst. Headmaster, The Taft School, Watertown, Connecticut

ELINOR V. DOUGLAS, M.A. (Iowa S.U.) Teacher, Woodrow Wilson High School, Washington, D. C.

JAMES DOW, B.S. (M.I.T.) Engr., Sylvania Electric Company

DALE H. DUEL, B.A. (Wartburg) Engr., American Machine and Foundry Company

WILLIAM V. DURANTE, M.E. (Boston S.T.C.) Teacher, Boston Technical High School, Massachusetts

MRS. VERENA H. DYSON, Ph.D. (Zurich) Asst. Prof., San Jose State College

JAMES B. ECKSTEIN, M.A. (Detroit) Asst. Prof., University of Detroit

MARTIN ENGERT, Student, Carleton College

IRVING J. EPSTEIN, Ph.D. (N.Y.U.) Mathematician, Fort Monmouth, New Jersey

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BOBBY R. FARRIS, Student, University of Kentucky

RICHARD F. FAWCETT, A.M. (Missouri) Asst. Prof., Central College, Morrison Observatory

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DONALD H. FLANDERS, B.A. (Reed) Grad. Asst., University of Maryland

THOMAS J. FLYNN, M.S. (Purdue) Inst., California State Polytechnic College

JON H. FOLKMAN, Student, University of California

JOE G. FOREMAN, Student, University of Oklahoma

MRS. TEMPIE R. FRANKLIN, M.A. (Yale) Supervisor, Secondary Mathematics, Arlington County Public Schools, Virginia

ELLIOTT D. FREDLAND, Student, Washington and Jefferson College

RAYMOND W. FREESE, A.M. (Missouri) Instr., University of Missouri

JOSEPH M. GANGLER, M.A. (Columbia) Asst. Prof., Southern Connecticut State College

REGINA H. GARB, Ed.D. (Columbia) Asso. Prof., Newark State College

JACKIE B. GARNER, M.A. (Auburn) Grad. Asst., Auburn University

EDWARD D. GAUGHAN, M.S. (Kansas S.C.) Asst. Instr., University of Kansas

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HERMAN S. GRAF, B.A. (Alfred) Mathematician, Teleregister Corporation

JAMES W. GRAHAM, M.A. (Toronto) Asst. Prof., University of Waterloo

ANDREW R. GRANT, B.A. (Washington) Lt. Col., Army Air Defense Command, Colorado Springs, Colorado

DONALD S. GRANT, B.S. (Manitoba) Winnipeg, Manitoba

DOUGLAS M. GRANT, Student, Amherst College

CHARLES A. GREEN, M.S. (Ohio) Teaching Asst., University of Wisconsin

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THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twenty-second annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, January 16, 1960. Professor G. C. Preston, Chairman of the Section, presided. There were 153 persons in attendance, including 131 members of the Association.

At the business meeting a report of the Committee to Study the Activities was presented; an amendment to the Section by-laws was adopted; and, under the new by-laws, the following officers were elected: Chairman and Program Chairman elect, Professor S. P. Hughart, Sacramento State College; Vice-Chairman and Chairman elect, Professor David Blakeslee, San Francisco State College; Program Chairman, Professor G. C. Preston, San Jose State College; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College. The term of the Secretary-Treasurer will be for three years and the others for one year. In the future, under the new by-laws, the only election will be that for Vice-Chairman except in the years when the term of the Secretary-Treasurer expires.

By invitation of the section, two addresses were given. Professor A. N. Milgram, University of Minnesota and University of California, Berkeley, spoke on *The Heat Equation* and traced the influence of this equation on the development of modern points of view on analysis. Professor G. B. Price, University of Kansas and California Institute of Technology, spoke on *Generalized Area and Certain Formulas in Calculus*. An abstract of this address follows:

A ribbon, a special type of surface determined by two space curves, has an invariant called generalized area. Under suitable hypotheses, generalized area can be expressed as a definite integral. Conditions are determined under which generalized area reduces to ordinary area. It is shown that the integral formula for generalized area reduces in special cases to the integral formulas for the following: area of a surface of a cylinder, a cone, and a frustum of a cone; area under a curve in the plane; area in polar coordinates; and the area enclosed by a curve in the plane.

The following papers were presented:

1. *The centroid in absolute geometry*, by Professor C. M. Fulton, University of California, Davis.

The following theorem is proved: The lines joining the midpoints of opposite edges of a tetrahedron intersect in the centroid. For the proof only Hilbert's axioms of incidence, order, and

congruence are used. No assumption about continuity and parallelism is required. The main tool is a known theorem of absolute geometry, namely: The line joining the midpoints of two sides of a triangle is orthogonal to the perpendicular bisector of the third side.

2. *Remarks on the S.M.S.G. Program in Northern California*, by Professor Bernard Friedman, University of California, Berkeley, and Mr. Robert Starkey, Cubberley Senior High School, Palo Alto.

3. *An integral inequality derived from convexity*, by Dr. Albert Novikoff, Stanford Research Institute, Menlo Park.

A well-known inequality (due to Young) between two functions, each the inverse of the other, has been generalized recently by Fenchel. The resulting inequality between functions of n variables is geometrically interpretable and seems more natural than earlier attempts at generalization (cf. Hardy, Littlewood, and Polya's *Inequalities*). An interesting by-product is the ability to find the potential function for certain conservative fields without the necessity of integration.

4. *Satellite orbits*, by Dr. J. L. Brenner, Stanford Research Institute.

Finite (closed-form, approximate) equations for the motion of a near satellite of the earth have been derived. The analysis parallels and refines that of King-Hele, Proc. Roy. Soc. (A) 247, 49–72 (1958). The equations have been checked by numerical integration of the differential equations. In view of the form of the equations, the "pear-shaped" theory of the earth's gravitational field does not seem to be established. The motion of satellite 1958 Beta Two can be explained on the basis of an earth axially symmetric, and with northern and southern hemispheres congruent.

ROY DUBISCH, *Secretary*

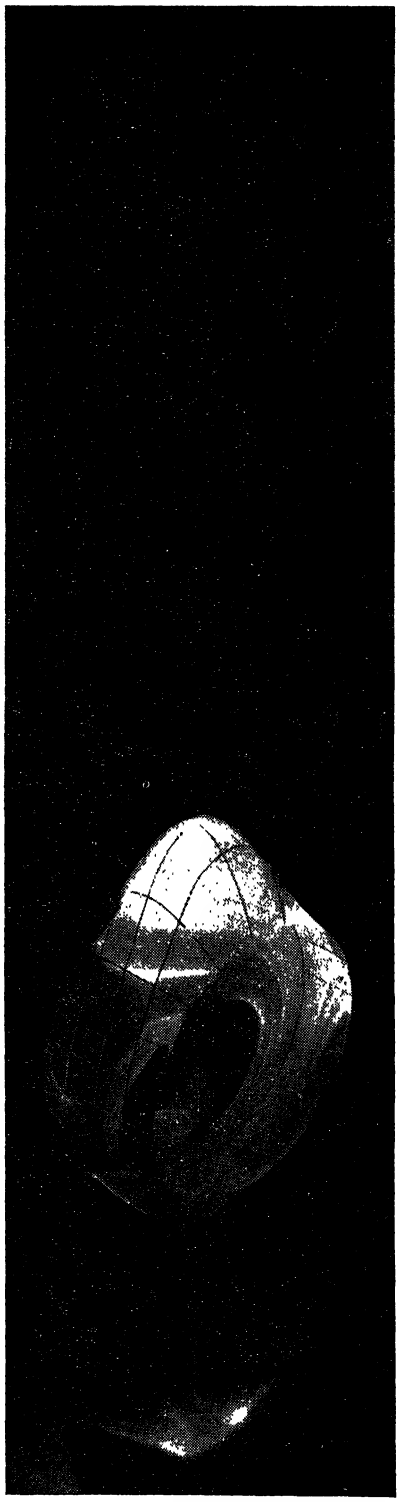
CALENDAR OF FUTURE MEETINGS

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.
NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.
OHIO
OKLAHOMA
PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.
PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
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Model of oil pressure distribution for a non-rotating journal with a reciprocating load.

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Besides being a valuable contribution to the theory of lubrication, this work has its practical side: namely, accurate, serviceable design curves for engineers. At GM Research, we believe delving into both the theoretical and applied sides of a problem is important to progress.
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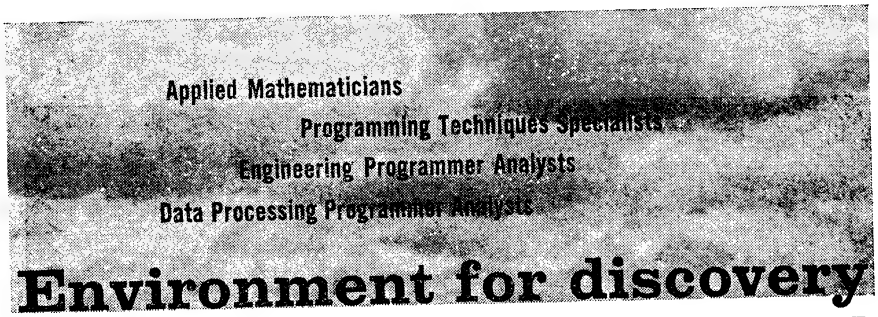
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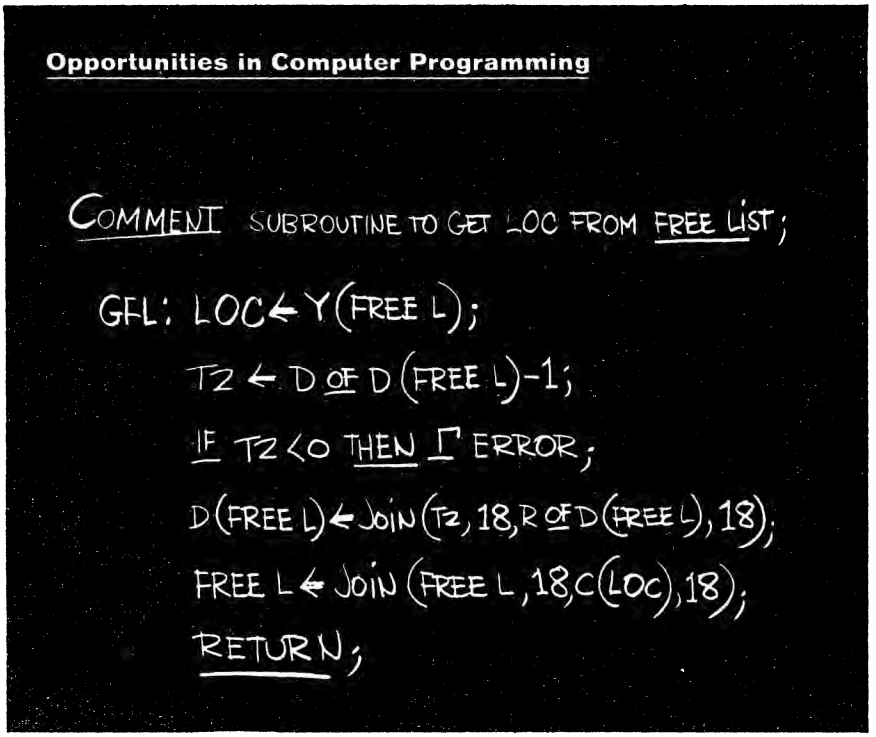
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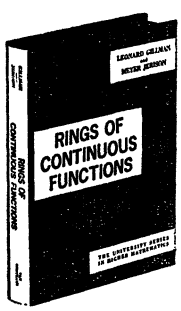
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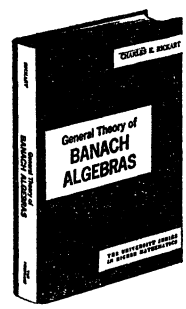
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THE RÔLE OF COMPACTNESS IN ANALYSIS

EDWIN HEWITT, University of Washington*

Introduction. Analysis is an immense field of mathematics, and compactness concepts and arguments enter in a great many different branches of analysis. To give a really adequate picture of the rôle of compactness in analysis would require in fact a survey of much of analysis: a task obviously impossible of accomplishment within the confines of a short essay, to say nothing of the limitations of the writer. After some remarks about the concept of compactness as such, therefore, we shall give examples from various parts of analysis in which compactness enters as a necessary hypothesis. These examples range from highly elementary to fairly sophisticated. They have been chosen partly to illuminate the concept and partly as important theorems of analysis. No attempt has been made to be exhaustive.

Compactness also plays a vital rôle in many existence theorems as a technique of proof. For reasons of space, we must limit ourselves to a brief description of how this is done.

Compact subsets of the line. Compactness for subsets of the real line R has been explicitly recognized, studied, and used for many years. Let A be a subset of the line, and consider the following three properties, which A may or may not have.

(1) Let $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ be a sequence of points such that every t_n belongs to A. Then there is a subsequence $\{t_{n_1}, t_{n_2}, t_{n_3}, \cdots, t_{n_k}, \cdots\}$ of the original sequence that converges to a limit t, where t is a point of the set A.

(2) Every infinite subset of A has at least one limit point lying in A.

(3) Let $\{J_\iota\}_{\iota\in I}$ be a family of open intervals such that every point of A lies in at least one of the open intervals J_ι, that is, $\bigcup_{\iota\in I} J_\iota \supset A$. Then there is a finite subfamily $\{J_{\iota_1}, J_{\iota_2}, \cdots, J_{\iota_m}\}$ of $\{J_\iota\}_{\iota\in I}$ such that $J_{\iota_1}\cup J_{\iota_2}\cup\cdots\cup J_{\iota_m}\supset A$.†

It is easy to show, and well known, that properties (1), (2), and (3) are equivalent to each other. A subset of R possessing one, and hence all, of them is called *compact*. It is also elementary and well known that a subset of R is compact if and only if it is closed and bounded.

It is obvious that every finite subset F of R is compact. If F is void, properties (1), (2), and (3) hold trivially. Hence suppose that F is nonvoid. If $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ is a sequence such that every t_n lies in F, there must be some point t of F such that $t_n = t$ for an infinite set of positive integers n. (If every $x\in F$ appears as a t_n only a finite number of times, and F is finite, then the sequence $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ can have only a finite number of terms.) Let

* Written with financial support from the National Science Foundation. This essay is based on a lecture given several times by the writer while a visiting lecturer of the Mathematical Association of America (March–May 1957).

† The following paraphrase of (3) is attributed to Hermann Weyl (1885–1955). "If a city is compact, it can be guarded by a finite number of arbitrarily near-sighted policemen."

n_1 be the least positive integer n such that $t_n = t$, n_2 the least positive integer n such that $n > n_1$ and $t_n = t$, and so on. The sequence $\{t_{n_1}, t_{n_2}, t_{n_3}, \cdots, t_{n_k}, \cdots\}$ is constant and so has the limit $t \in F$. Property (2) holds vacuously for every finite set $F \subset R$, since F contains no infinite subsets. Property (3) is almost trivial. Let $F = \{x_1, x_2, \cdots, x_p\}$. Then each x_k is in some open interval J_{ι_k} $(k = 1, 2, \cdots, p)$, and the finite subfamily $\{J_{\iota_1}, J_{\iota_2}, \cdots, J_{\iota_p}\}$ of $\{J_\iota\}_{\iota \in I}$ obviously has the property that $J_{\iota_1} \cup J_{\iota_2} \cup \cdots \cup J_{\iota_p} \supset F$.

Of course there are many *infinite* compact subsets of R. The set $\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ is such a set. The closed interval $[a, b]$,* for any real numbers a and b such that $a < b$, is also. A curious and instructive example of an infinite compact subset of R is Cantor's ternary set, defined as the set of all numbers of the form $2(\sum_{n=1}^{\infty} x_n 3^{-n})$, where $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ is an arbitrary sequence consisting only of 0's and 1's.†

It thus appears that compactness for subsets of R is a generalization of finiteness. The thesis of this essay is that *a great many propositions of analysis are:*

(A) *trivial for finite sets;*
(B) *true and reasonably simple for infinite compact sets;*
(C) *either false or extremely difficult to prove for noncompact sets.*

As we shall see, some care is usually needed in moving from the finite situation to the corresponding compact infinite situation. A given assertion true for finite sets often needs some qualification to be provable for all compact sets.

Example I. As the first illustration of our thesis, consider the following proposition.

I_f. A real-valued function f defined on a (nonvoid) finite set is bounded and attains its least upper bound and greatest lower bound.

This proposition is perfectly obvious. Let f be defined on $\{x_1, x_2, \cdots, x_p\}$ (a subset of R or not). Then the numbers $\max\{f(x_1), f(x_2), \cdots, f(x_p)\}$ and $\min\{f(x_1), f(x_2), \cdots, f(x_p)\}$ are the least upper bound and greatest lower bound, respectively, of the values of f, and are attained by f at points of its domain of definition.

For arbitrary compact subsets of R, we have the following paraphrase of Proposition I_f.

I_c. A continuous real-valued function f defined on a (nonvoid) compact subset of R is bounded and attains its least upper bound and greatest lower bound.

Propositions I_f and I_c illustrate the typical change required in going from the finite to the infinite compact situation. Proposition I_f is true of *all* real-valued

* We use the expression $[a, b]$ to denote the *closed* interval $\{x: x \in R, a \leq x \leq b\}$, and $]a, b[$ to denote the *open* interval $\{x: x \in R, a < x < b\}$. The expressions $[a, b[$ and $]a, b]$ have similar definitions.

† We make no effort here to classify all compact subsets of R.

functions, while Proposition I_c requires that f be continuous. Similar changes are almost always needed in going from the finite to the infinite compact situation.*

Proposition I_c is proved in every carefully written treatment of the elementary theory of functions of a real variable.† Nevertheless we sketch a proof. Let A be a compact nonvoid subset of R and f a continuous real-valued function defined on A. Let x be any point of A. By continuity, there is a number $\delta_x > 0$ such that $|f(x)-f(t)| < 1$ for all $t \in A$ such that $|t-x| < \delta_x$. The family of open intervals $\{]x-\delta_x, x+\delta_x[\}_{x\in A}$ thus satisfies the conditions of property (3) above. Hence there are finitely many of these intervals, say $]x_1 - \delta_{x_1}, x_1 + \delta_{x_1}[$, $]x_2-\delta_{x_2}, x_2+\delta_{x_2}[, \cdots,]x_m-\delta_{x_m}, x_m+\delta_{x_m}[$, whose union contains A. It then follows that $|f(t)| < 1+\max\{|f(x_1)|, |f(x_2)|, \cdots, |f(x_m)|\}$ for all $t \in A$: thus f is bounded. To prove that the least upper bound d of the set $\{f(x): x \in A\}$ is a value of f, it is convenient to use property (1) of compact subsets of R. By definition of least upper bound, we can find, for every positive integer n, a point $t_n \in A$ such that $f(t_n) > d - 1/n$. By property (1), there is a subsequence $\{t_{n_1}, t_{n_2}, t_{n_3}, \cdots, t_{n_k}, \cdots\}$ of the sequence $\{t_1, t_2, t_3, \cdots, t_n, \cdots\}$ that converges to a limit t, where $t \in A$. The continuity of f then implies that $f(t) = d$. Similarly one shows that the greatest lower bound of the set $\{f(x): x \in A\}$ is a value of f.

Parts (A) and (B) of our thesis are thus demonstrated for the simple propositions I_f and I_c. Part (C) is also true for Proposition I, as the following statement shows.

I_n. Let B be a noncompact subset of R. Then there is an unbounded continuous real-valued function on B, and there is a bounded continuous real-valued function on B that assumes neither its least upper nor greatest lower bound.

If B is noncompact and contained in R, it is either unbounded or nonclosed. If B is unbounded, let $f_1(x) = x$ for $x \in B$ and let $f_2(x) = |x|/(1+|x|)$ for $x \in B$. Clearly f_1 and f_2 are continuous, f_1 is unbounded, and f_2 is bounded and does not attain its least upper bound 1. (It is easy, although a little more complicated, to produce a bounded continuous real-valued function on B assuming neither its least upper nor greatest lower bound: we omit the details.) If B is nonclosed, let u be a limit point of B that does not lie in B. Then let $f_1(x) = 1/(x-u)$ for $x \in B$ and $f_2(x) = e^{-|x-u|}$ for $x \in B$. Clearly f_1 and f_2 are continuous, f_1 is unbounded, and f_2 is bounded and does not attain its least upper bound 1.

Example II. We now take up the classical Weierstrass approximation theorem and its neoclassical generalization due to M. H. Stone.‡ For functions on

* Since every real-valued function defined on a finite subset of R is continuous, we can insert "continuous" before "function" in Proposition I_f without altering its force, and thus make the hypotheses on f identical in I_f and I_c. This however seems to be a little artificial.

† See for example [1], p. 73.

‡ Stone's general approximation theorem appeared in [19] as Theorem 82. An extended account of the theorem and its applications appears in [20]. For a somewhat different treatment, see [7].

finite sets, we have as before a simple assertion.

II_f. *Let F be any finite set (a subset of R or not). Let $\mathfrak{S}$ be a set of real-valued functions on F such that for every pair x and y of distinct points in F, there is some function f belonging to the set $\mathfrak{S}$ such that $f(x) \neq f(y)$.* Then every real-valued function on F is equal to a polynomial in functions chosen from the set $\mathfrak{S}$.*

Proposition II_f may appear somewhat artificial, and perhaps it is. Its generalization II_c to the compact situation, however, is an extremely powerful result, which is rightly regarded as one of the landmarks of modern analysis.

We proceed to a proof of II_f. Write the points of F as a finite sequence with no repetitions: $F = \{x_1, x_2, \cdots, x_p\}$. By hypothesis, there is a function f_{1j} in $\mathfrak{S}$ such that $f_{1j}(x_1) \neq f_{1j}(x_j)$, for $j = 2, 3, \cdots, p$. Let φ_1 be the function on F defined by

$$\varphi_1 = \prod_{j=2}^{p} [f_{1j} - f_{1j}(x_j)][f_{1j}(x_1) - f_{1j}(x_j)]^{-1}.$$

Plainly φ_1 is a polynomial in functions from $\mathfrak{S}$. It is also plain that $\varphi_1(x_1) = 1$ and $\varphi_1(x_j) = 0$ for $j = 2, 3, \cdots, p$. In like manner, we construct functions $\varphi_2, \varphi_3, \cdots, \varphi_p$, each a polynomial in functions from $\mathfrak{S}$, such that $\varphi_j(x_j) = 1$ and $\varphi_j(x_k) = 0$ for $k \neq j$ ($j = 1, 2, \cdots, p$; $k = 1, 2, \cdots, j-1, j+1, \cdots, p$). Let g be any real-valued function on F. Then $g = g(x_1)\varphi_1 + g(x_2)\varphi_2 + \cdots + g(x_p)\varphi_p$; this proves Proposition II_f.†

The infinite compact analogue for II_f follows.

II_c. *Let A be an arbitrary compact subset of R, and let $\mathfrak{S}$ be a separating set of continuous real-valued functions on A. Let g be any continuous real-valued function on A and ϵ any positive real number. Then there is a polynomial $p = p(f_1, f_2, \cdots, f_r)$ in functions $f_1, f_2, \cdots, f_r$ taken from $\mathfrak{S}$ such that $|g(x) - p(x)| < \epsilon$ for all $x \in A$. In other words, every continuous real-valued function on A is the uniform limit (on A) of a sequence of polynomials formed from functions of the set $\mathfrak{S}$.*

There are many proofs of Proposition II_c. A basic point in several of these is the fact that the absolute value of any bounded real-valued function ψ on A is the uniform limit of polynomials with real coefficients in the function ψ^2.‡

It follows at once from Raabe's test for the convergence of series of positive terms ([13], p. 285) that the series

$$\sum_{k=0}^{\infty} \left| \binom{\frac{1}{2}}{k} \right|$$

converges. We also have $|t| = (1 + (t^2 - 1))^{1/2}$ for every real number t. Let α be

* Such a set of functions is called a *separating* set of functions, for an obvious reason.

† Proposition II_f is true for functions with values in any division ring; hence for complex- or quaternion-valued functions defined on F.

‡ This is the fundamental idea in Lebesgue's proof [15], which deals with a special case. It is also pointed out explicitly by Stone ([19], p. 467) and is used by Hewitt [7].

the least upper bound of the set $\{|\psi(x)| : x \in A\}$. Then, for every $x \in A$, we have

$$\left| |\psi(x)| - \sum_{k=0}^{n} \alpha \binom{\frac{1}{2}}{k} (\alpha^{-2}\psi^2(x) - 1)^k \right|$$

$$= \alpha \left| |\alpha^{-1}\psi(x)| - \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} (\alpha^{-2}\psi^2(x) - 1)^k \right|$$

$$= \alpha \left| \sum_{k=n+1}^{\infty} \binom{\frac{1}{2}}{k} (\alpha^{-2}\psi^2(x) - 1)^k \right| \leqq \alpha \sum_{k=n+1}^{\infty} \left| \binom{\frac{1}{2}}{k} \right|.$$

Since the last expression is arbitrarily small if n is sufficiently large, the first observation is proved.

We next notice that $\max(a, b) = \frac{1}{2}(|a - b| + (a + b))$ and $\min(a, b) = \frac{1}{2}(-|a-b| + (a+b))$ for real numbers a and b. Hence, if ψ_1 and ψ_2 are any bounded real-valued functions on A, the functions $\max(\psi_1, \psi_2)$ and $\min(\psi_1, \psi_2)$ are uniform limits of sequences of polynomials in the functions ψ_1 and ψ_2. The same is also true, clearly, for $\max(\psi_1, \psi_2, \cdots, \psi_k)$ and $\min(\psi_1, \psi_2, \cdots, \psi_k)$, if $\psi_1, \psi_2, \cdots, \psi_k$ are bounded real-valued functions on A $(k = 2, 3, 4, \cdots)$.*

Now let f be any continuous real-valued function on A. Let

$$c = \min\{f(x) : x \in A\} \quad \text{and} \quad d = \max\{f(x) : x \in A\}.$$

If $c = d$, f is a constant and is trivially a polynomial in functions from $\mathfrak{S}$. If $c < d$, let $g = (f - c)/(d - c)$. Then $\min\{g(x) : x \in A\} = 0$, $\max\{g(x) : x \in A\} = 1$, and if g can be arbitrarily uniformly approximated by polynomials in functions from $\mathfrak{S}$, then f can also be so approximated.

Let $E = \{x : x \in A,\ g(x) \leqq \frac{1}{3}\}$ and $F = \{x : x \in A,\ g(x) \geqq \frac{2}{3}\}$. Let x and y be any points in E and F respectively. Let g_{xy} be a function from the set $\mathfrak{S}$ such that $g_{xy}(x) \neq g_{xy}(y)$. Let h_{xy} be the function

$$h_{xy} = \frac{1}{4} + \frac{1}{2(g_{xy}(y) - g_{xy}(x))} (g_{xy} - g_{xy}(x)).$$

Then $h_{xy}(x) = \frac{1}{4}$ and $h_{xy}(y) = \frac{3}{4}$ and h_{xy} is obviously a polynomial in functions from $\mathfrak{S}$. Now regard y as fixed for a moment and let x vary in E. Since h_{xy} is continuous, there is an open interval I_x containing x such that $h_{xy}(t) < \frac{1}{3}$ for all $t \in I_x \cap A$. It is easy to see that E, as a closed subset of A, is compact. Hence there are a finite number of open intervals $I_{x_1}, I_{x_2}, \cdots, I_{x_m}$ whose union contains E. Let φ_y be the function defined by

$$\varphi_y = \min\{h_{x_1 y}, h_{x_2 y}, \cdots, h_{x_m y}\}.$$

Then φ_y is the uniform limit of polynomials in functions from $\mathfrak{S}$ and has the

* Obviously the last two observations hold for bounded real-valued functions defined on *any* set.

property that $\varphi_y(t) < \frac{1}{3}$ for $t \in E$ and $\varphi_y(y) = \frac{2}{3}$. Now letting y vary over F, using the compactness of F, and forming a maximum, we find a function ψ that is the uniform limit of polynomials in functions from $\mathfrak{S}$ with the properties that $\psi(t) < \frac{1}{3}$ for $t \in E$ and $\psi(t) > \frac{2}{3}$ for $t \in F$. The function

$$\omega_1 = \min\{\max[\tfrac{1}{3}, \psi], \tfrac{2}{3}\}$$

is also the uniform limit of polynomials in functions from $\mathfrak{S}$. It is clear that $\omega_1(t) = \frac{1}{3}$ for $t \in E$, $\omega_1(t) = \frac{2}{3}$ for $t \in F$, and $\frac{1}{3} \leqq \omega_1(t) \leqq \frac{2}{3}$ for all $t \in A$. The function $g_1 = g - \omega_1$ is continuous and has minimum and maximum values $-\frac{1}{3}$ and $\frac{1}{3}$ on A. Applying the foregoing argument to the function $\frac{3}{2}g_1 + \frac{1}{2}$ we find a function ω_2 that is the uniform limit of polynomials in functions from $\mathfrak{S}$ such that $\max\{|\frac{3}{2}g_1(t) + \frac{1}{2} - \omega_2(t)| : t \in A\} = \frac{1}{3}$. Thus we have

$$\max\{|\, g(t) + \tfrac{1}{3} - \omega_1(t) - \tfrac{2}{3}\omega_2(t)\,| : t \in A\} = 2 \cdot 3^{-2}.$$

It is easy to see that this process can be repeated as many times as desired to obtain as close an approximation as one wishes to g, by polynomials in functions from $\mathfrak{S}$. This completes the proof.*

The reader may find it instructive to try to carry over the proof of Proposition II_f to the compact infinite case, and to see just where it breaks down.

Part (C) of our thesis is amply supported by the Stone-Weierstrass theorem.

II_n. *Let B be any noncompact subset of R. Then there is a separating set $\mathfrak{S}$ of bounded continuous real-valued functions on B such that not every bounded continuous real-valued function on B is the uniform limit of polynomials in functions from $\mathfrak{S}$.*

Proof. The function arctan (x) (principal value) with domain restricted to B by itself separates the points of B and will be taken as constituting the set $\mathfrak{S}$. Suppose that B is unbounded above. Then there is a subset of B of the form $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ such that $x_1 < x_2 < x_3 < \cdots < x_n < \cdots$ and $\lim_{n\to\infty} x_n = \infty$. Let $f(x_n) = (-1)^n$ $(n = 1, 2, 3, \cdots)$, let f be linear in the intervals

* Proposition II_c may fail for complex-valued continuous functions defined on a compact subset A of R. This is a fairly recondite fact. W. Rudin has shown [17] that if A is a compact subset of R that contains a "replica" of Cantor's ternary set, then there is a separating set of continuous complex-valued functions $\mathfrak{S}$ on A such that not every complex-valued continuous function on A is the uniform limit of polynomials (with complex coefficients) formed from the functions of $\mathfrak{S}$. It is not hard to see that if one allows the formation of complex conjugates as well as polynomials, then Proposition II_c holds for complex-valued continuous functions and polynomials with complex coefficients. Curiously enough, Proposition II_c holds without any exception for *quaternion*-valued continuous functions, as has been shown by J. C. Holladay [10]. The theorem follows immediately from the identity $q - iqi - jqj - kqk = 4a$, where q is the quaternion $a + bi + cj + dk$, i, j, and k are the usual quaternion units, and a, b, c, and d are real numbers. Holladay's observation has recently been generalized by Alvin Hausner [A generalized Stone-Weierstrass theorem, Arch. der Math., vol. 10, 1959, pp. 85–87] to the case of continuous functions defined on compact sets and having values in a Cayley-Dickson algebra of dimension 2^n $(n = 2, 3, \cdots)$ or a Clifford algebra of dimension 2^{2l} $(l = 1, 2, 3, \cdots)$. Hausner's proof generalizes Holladay's.

$]x_n, x_{n+1}[$ $(n=1, 2, 3, \cdots)$, and $f(x)=-1$ for $x<x_1$. Then f is a bounded continuous real-valued function on R and hence, with its domain restricted to B, is continuous on B. Plainly $\lim_{n\to\infty} f(x_n)$ does not exist. However, if φ is a function on B that is the uniform limit of polynomials in arctan (x), then $\lim_{n\to\infty} \varphi(x_n)$ must exist. Therefore f cannot be the uniform limit of polynomials in arctan (x).

The cases in which B is bounded above but unbounded below, or bounded but nonclosed, are treated very similarly. We omit the details.

Example III. The third illustration of our thesis is drawn from functional analysis. It deals with integral representations for certain linear functionals.

III_f. *Let F be a finite set (a subset of R or not) and L a real-valued functional defined for all real-valued functions on F such that: $L(\alpha f+\beta g)=\alpha L(f)+\beta L(g)$ for all functions f and g on F and all real numbers α and β (that is, L is linear); $L(f)\geqq 0$ if $f(x)\geqq 0$ for all $x\in F$ (that is, L is a positive linear functional). Then there is a nonnegative real-valued function λ defined on F such that $L(f)=\sum_{x\in F} f(x)\lambda(x)$ for all real-valued functions f on F.*

Like Proposition II_f, Proposition III_f has a somewhat artificial air. Nevertheless, it has great importance in linear algebra (see for example the discussion in [6], pp. 20–24).

Proposition III_f is grotesquely easy to prove. Write F as $\{x_1, x_2, \cdots, x_p\}$, where the points $x_1, x_2, \cdots, x_p$ are all distinct. Let φ_j be the function on F such that $\varphi_j(x_j)=1$ and

$$\varphi_j(x_k) = 0 \text{ for } j \neq k \ (j = 1, 2, \cdots, p; k = 1, 2, \cdots, j-1, j+1, \cdots, p).$$

Then any real-valued function f on F can be written in just one way in the form $f=\sum_{j=1}^{p} f(x_j)\varphi_j$. If L is a linear functional defined for all real-valued functions on F, then we have $L(f)=L(\sum_{j=1}^{p} f(x_j)\varphi_j)=\sum_{j=1}^{p} f(x_j)L(\varphi_j)$. Since φ_j is a nonnegative function, we have $L(\varphi_j)\geqq 0$ if L is a positive linear functional. Hence, writing $\lambda_j=L(\varphi_j)$ $(j=1, 2, \cdots, p)$, we obtain Proposition III_f.

The reader may well have noticed that positivity of L is really unnecessary in Proposition III_f: if L is *any* real-valued linear functional on all real-valued functions on F, then $L(f)=\sum_{x\in F} f(x)\lambda(x)$, where λ is now a real-valued function on F that may assume values of arbitrary sign. Here, as in comparing the proofs of Propositions II_f and II_c, we encounter a genuine difference between the finite and the infinite compact cases. In order to obtain an analogue of Proposition III_f valid for all compact subsets of R, we are forced to restrict L sharply. As usual, we replace the set of *all* real-valued functions by the set of *continuous* real-valued functions. The following theorem is known as F. Riesz's representation theorem.

III_c. *Let A be any compact subset of R. Let L be any real-valued functional defined for all continuous real-valued functions on A such that: $L(\alpha f+\beta g)=\alpha L(f)+\beta L(g)$ for all real numbers α and β and continuous real-valued functions f and g on A (L is linear); $L(f)\geqq 0$ if f is a nonnegative real-valued continuous function*

on A. Then there is a measure λ on the family $\mathfrak{B}(A)$ of Borel subsets of A such that $L(f)=\int_A f(x)d\lambda(x)$ for all continuous real-valued functions f on A.*†

A few words of explanation of Proposition III_c may be appropriate. A subset E of A is closed if it is a closed subset of R, *i.e.*, if it contains all of its own limit points. Let $\mathfrak{F}(A)$ denote the family of all closed subsets of A. There are families of subsets $\mathfrak{G}$ of A that contain $\mathfrak{F}(A)$ and are also closed under the formation of countable unions and complements relative to A: that is, if $Y_1, Y_2, Y_3, \cdots, Y_n, \cdots$ are in $\mathfrak{G}$, then $\bigcup_{n=1}^{\infty} Y_n$ is in $\mathfrak{G}$, and if Y is in $\mathfrak{G}$, then $Y' \cap A$ is in $\mathfrak{G}$. The family of all subsets of A is an example. From this it is easy to see that there is a smallest family, $\mathfrak{B}(A)$, of subsets of A, that contains $\mathfrak{F}(A)$ and is closed under the formation of countable unions and of complements relative to A. The family $\mathfrak{B}(A)$ is called the family of Borel subsets of A. A real-valued function λ defined for every set in $\mathfrak{B}(A)$ is called a measure (in this essay) if $0 \leqq \lambda(E) < \infty$ for all E in $\mathfrak{B}(A)$ and $\lambda(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$ for every countably infinite subfamily $\{E_1, E_2, E_3, \cdots, E_n, \cdots\}$ of $\mathfrak{B}(A)$ for which $E_n \cap E_m = \varnothing$ if $n \neq m$ ($\varnothing$ denotes the void set). This property of λ is called countable additivity.

The most remarkable feature of Proposition III_c is the fact that the linear functional L has a representation as an integral with respect to a *countably* additive measure. Countable additivity is the property of measures that makes the most powerful theorems of integration theory valid (*e.g.*, Lebesgue's theorem on term-by-term integration of convergent sequences of functions, and Fubini's theorem); and it is a quite delicate property. As we shall see below, countable additivity of the measure that gives the integral representation for L depends entirely upon the compactness of A.

We proceed to the proof of Proposition III_c. There are a number of measure-theoretic technicalities in it, a complete explanation of which would require several pages. We shall therefore limit ourselves to a sketch of the construction, giving references to the literature where needed.

We build up the measure λ from the functional L in steps. First, let F be any closed subset of A. Define $\lambda(F)$ as the greatest lower bound of the set $\{L(\varphi)$: φ is a continuous real-valued function on A, $\varphi(x) \geqq 1$ for $x \in F$ and $\varphi(x) \geqq 0$ for all $x \in A\}$. It is clear that $0 \leqq \lambda(F) \leqq L(1) < \infty$, that $\lambda(A) = L(1)$, and that $\lambda(\varnothing) = 0$. Next, suppose that G is an open subset of A (*i.e.*, that $G' \cap A$ is closed).‡ Then define $\lambda'(G)$ as the least upper bound of the set of numbers $\{\lambda(F)$: F is closed in A and $F \subset G\}$. Again it is clear that $0 \leqq \lambda'(G) \leqq L(1) < \infty$. If a subset

* Here and below all integrals are of Lebesgue's type, unless the contrary is explicitly stated.

†The theorem was proved by F. Riesz [16] for the case in which $A = [0, 1]$, and with the integral $\int_A f(x)d\lambda(x)$ replaced by a Riemann-Stieltjes integral $\int_0^1 f(x)dl(x)$, where l is a function of finite variation on $[0, 1]$. For a general compact subset A of R, it is most convenient to present the theorem as we have done, in terms of measures and integrals of Lebesgue's type.

‡ Open subsets of A need not be open when considered as subsets of R. For example, if $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$, then any finite set excluding 0 is open in A. An open subset of A, however, is always the intersection with A of a countable union of intervals $]a, b[$. Closed subsets of A, on the other hand, are always closed in R, since A itself is closed in R.

G of A is both open and closed, then $\lambda(G)=\lambda'(G)$, since $\lambda(F_1)\leqq\lambda(F_2)$ whenever F_1 and F_2 are closed in A and $F_1\subset F_2$. Finally, for an arbitrary subset X of A, let $\lambda''(X)$ be the greatest lower bound of the set of numbers $\{\lambda'(G): G \text{ is open in } A \text{ and } G\supset X\}$. It is obvious that $\lambda'(G)=\lambda''(G)$ if G is open in A and that $\lambda''(F)\geqq\lambda(F)$ if F is closed in A. To prove that $\lambda''(F)\leqq\lambda(F)$ for F closed in A, let ϵ and α be any positive real numbers, and select a continuous real-valued function φ on A such that $\varphi(x)\geqq 1$ for $x\in F$, $\varphi(x)\geqq 0$ for $x\in A$, and $\lambda(F)>L(\varphi)-\epsilon$. Let $\psi=(1+\alpha)\varphi$, $F_1=\{x: x\in A,\ \psi(x)\geqq 1\}$, and $G_1=\{x: x\in A,\ \psi(x)>1\}$. The definitions of λ and λ' show that $\lambda(F_1)\geqq\lambda'(G_1)$. Now we have $L(\varphi)=(1/(1+\alpha))L(\psi)$, and $L(\psi)\geqq\lambda(F_1)\geqq\lambda'(G_1)$. Since $G_1\supset F$, we also have $\lambda'(G_1)\geqq\lambda''(F)$. Combining these estimates, we have

$$L(F) > L(\varphi) - \epsilon = \frac{1}{1+\alpha}L(\psi) - \epsilon \geqq \frac{1}{1+\alpha}\lambda'(G_1) - \epsilon \geqq \frac{1}{1+\alpha}\lambda''(F) - \epsilon.$$

Since α and ϵ can be arbitrarily small, it follows that $\lambda(F)\geqq\lambda''(F)$. Hence $\lambda=\lambda''$ whenever λ is defined, and we may regard λ'' as an extension of λ. For typographical convenience, we will now drop the "$''$" and write $\lambda(X)$ instead of $\lambda''(X)$, for all subsets X of A.

We next show that λ is a Carathéodory outer measure (see for example [18], pp. 43–47). That is,

$$\lambda(X) \leqq \lambda(Y) \qquad \text{for } X \subset Y \subset A, \tag{1}$$

$$\lambda\Big(\bigcup_{n=1}^{\infty} X_n\Big) \leqq \sum_{n=1}^{\infty} \lambda(X_n) \qquad \text{if } X_n \subset A \qquad (n = 1, 2, 3, \cdots). \tag{2}$$

The inequality (1) is obviously valid. Inequality (2) is at the root of countable additivity of λ for Borel sets and, not surprisingly, depends upon the compactness of A.

The weaker inequality

$$\lambda(X_1 \cup X_2 \cup \cdots \cup X_p) \leqq \sum_{n=1}^{p} \lambda(X_n), \qquad \bigcup_{n=1}^{p} X_n \subset A, \tag{2'}$$

is established by a routine argument from the foregoing definitions: we will not reproduce it. (See for example [11], p. 1009, footnote 12.) Inequality (2′) does *not* depend upon compactness of A. We prove (2) from (2′). If the series on the right side of (2) diverges, there is nothing to prove. Suppose that this series converges, and let ϵ be any positive real number. For $n=1, 2, 3, \cdots$, let G_n be an open subset of A such that $G_n\supset X_n$ and $\lambda(G_n)-\epsilon 2^{-n}<\lambda(X_n)$. Let F be a closed subset of A such that $F\subset\bigcup_{n=1}^{\infty} G_n$ and $\lambda(F)+\frac{1}{2}\epsilon>\lambda(\bigcup_{n=1}^{\infty} G_n)$. As a closed subset of A, F is compact, and each G_n is the intersection with A of a countable union of open intervals: $G_n=(\bigcup_{k=1}^{\infty} I_{n,k})\cap A$, where each $I_{n,k}$ is an open interval. Thus we have $F\subset\bigcup_{n=1}^{\infty}(\bigcup_{k=1}^{\infty} I_{n,k})$. By property (3) of compact sets, we have $F\subset I_{n_1,k_1}\cup\cdots\cup I_{n_l,k_l}$, and hence $F\subset(\bigcup_{k=1}^{\infty} I_{n_1,k})\cup\cdots\cup(\bigcup_{k=1}^{\infty} I_{n_l,k})$. This

implies that $F \subset G_{n_1} \cup \cdots \cup G_{n_l}$. Properties (1) and (2′) of λ imply that $\lambda(F) \leqq \lambda(G_{n_1} \cup \cdots \cup G_{n_l}) \leqq \lambda(G_{n_1}) + \cdots + \lambda(G_{n_l})$. We can now put together our estimates to find

$$\lambda\left(\bigcup_{n=1}^{\infty} X_n\right) \leqq \lambda\left(\bigcup_{n=1}^{\infty} G_n\right) < \lambda(F) + \tfrac{1}{2}\epsilon \leqq \lambda(G_{n_1}) + \cdots + \lambda(G_{n_l}) + \tfrac{1}{2}\epsilon$$

$$\leqq \sum_{n=1}^{\infty} \lambda(G_n) + \tfrac{1}{2}\epsilon < \sum_{n=1}^{\infty} (\lambda(X_n) + \epsilon 2^{-n}) + \tfrac{1}{2}\epsilon = \sum_{n=1}^{\infty} \lambda(X_n) + \epsilon.$$

Since ϵ can be chosen arbitrarily small, inequality (2) is demonstrated.

The argument now follows a familiar course. A set $X \subset A$ is (Carathéodory) measurable if $\lambda(Y) = \lambda(Y \cap X) + \lambda(Y \cap X')$ for all $Y \subset A$. The family $\mathfrak{M}$ of measurable sets is proved to be closed under the formation of countable unions and complements relative to A, and λ is proved countably additive on $\mathfrak{M}$. This can be taken *verbatim* from [18], pp. 44–46. One next shows that closed subsets of A are measurable. This again is done by a routine argument, which we omit.

To finish the proof, we must show that

$$L(f) = \int_A f(x)\, d\lambda(x) \tag{3}$$

for all continuous real-valued functions f on A. The equality (3) is obvious if f is a constant function. If f is nonconstant, let $c = \min \{f(x) : x \in A\}$, $d = \max \{f(x) : x \in A\}$, and let n be any positive integer. Let $\alpha_k = [(n-k)c + kd]/n$ for $k = 0, 1, 2, \cdots, n$. That is, the numbers $c = \alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_n = d$ cut up the interval $[c, d]$ into n subintervals of equal length. We now subdivide A as is often done in constructing Lebesgue integrals: let $A = D_0 = \{x : x \in A, f(x) \geqq \alpha_0\}$, $D_1 = \{x : x \in A, f(x) \geqq \alpha_1\}$, $\cdots$, $D_k = \{x : x \in A, f(x) \geqq \alpha_k\}$, for $k = 0, 1, 2, 3, \cdots, n$. We also write E_k for the set $\{x : x \in A, \alpha_k \leqq f(x) < \alpha_{k+1}\}$ for $k = 0, 1, 2, \cdots, n-1$, and $E_n = \{x : x \in A, f(x) = \alpha_n\} = D_n$. Then plainly enough $E_k = D_k \cap D_{k+1}'$ for $k = 0, 1, \cdots, n-1$.

We now write the function f as a linear combination of other continuous functions, in terms of which we can estimate the numbers $\lambda(E_0), \lambda(E_1), \cdots, \lambda(E_n)$. For $k = 1, 2, \cdots, n$, let f_k be the function on A such that

$$f_k(x) = 0 \qquad \text{if } f(x) \leqq \alpha_{k-1},$$

$$f_k(x) = \frac{1}{\alpha_k - \alpha_{k-1}} (f(x) - \alpha_{k-1}) \qquad \text{if } \alpha_{k-1} < f(x) < \alpha_k,$$

$$f_k(x) = 1 \qquad \text{if } f(x) \geqq \alpha_k.$$

It is clear that f_k is a continuous real-valued function on A, and it follows from the definition of λ that $L(f_k) \geqq \lambda(D_k)$ $(k = 1, 2, \cdots, n)$. Finally, it is easy to check that

$$f = \alpha_0 + (\alpha_1 - \alpha_0) f_1 + (\alpha_2 - \alpha_1) f_2 + \cdots + (\alpha_n - \alpha_{n-1}) f_n.$$

Now apply the functional L to both sides of this identity, and use the inequalities $L(f_k) \geqq \lambda(D_k)$. This yields

$$\begin{aligned} L(f) &= \alpha_0 L(1) + (\alpha_1 - \alpha_0)L(f_1) + (\alpha_2 - \alpha_1)L(f_2) + \cdots + (\alpha_n - \alpha_{n-1})L(f_n) \\ &\geqq \alpha_0\lambda(A) + (\alpha_1 - \alpha_0)\lambda(D_1) + (\alpha_2 - \alpha_1)\lambda(D_2) + \cdots + (\alpha_n - \alpha_{n-1})\lambda(D_n) \\ &= \alpha_0(\lambda(D_0) - \lambda(D_1)) + \alpha_1(\lambda(D_1) - \lambda(D_2)) + \alpha_2(\lambda(D_2) - \lambda(D_3)) + \cdots \\ &\quad + \alpha_{n-1}(\lambda(D_{n-1}) - \lambda(D_n)) + \alpha_n\lambda(D_n). \end{aligned}$$

We also have $\lambda(D_k) = \lambda(D_{k+1}) + \lambda(E_k)$ for $k = 0, 1, \cdots, n-1$, since λ is a (countably) additive measure. Thus

$$L(f) \geqq \alpha_0\lambda(E_0) + \alpha_1\lambda(E_1) + \cdots + \alpha_{n-1}\lambda(E_{n-1}) + \alpha_n\lambda(E_n).$$

The right side of the last inequality is the integral $\int_A h_n(x)d\lambda(x)$, where $h_n(x) = \alpha_k$ for $x \in E_k$ $(k = 0, 1, \cdots, n)$. Since $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in A$, a standard theorem on convergence of Lebesgue integrals implies that $\lim_{n\to\infty} \int_A h_n(x)d\lambda(x) = \int_A f(x)d\lambda(x)$. (See for example [18], p. 29, Theorem (12.11).) It follows that $L(f) \geqq \int_A f(x)d\lambda(x)$. Replacing f by $-f$, we have $-L(f) = L(-f) \geqq \int_A -f(x)d\lambda(x) = -\int_A f(x)d\lambda(x)$, which shows that $L(f) \leqq \int_A f(x)d\lambda(x)$. Thus $L(f) = \int_A f(x)d\lambda(x)$ for every continuous real-valued function on A. This completes the proof.*

It should be remarked that Proposition III_c admits an immediate generalization. Suppose that L is a linear functional defined for all continuous real-valued functions on A that is not positive but satisfies the weaker condition $|L(f)| \leqq C \cdot \max\{|f(x)| : x \in A\}$ for all f, where C is a constant independent of f. Then one can prove that $L = L_1 - L_2$, where L_1 and L_2 are linear and positive (see for example [3], p. 245, Theorem 8). Thus $L(f) = \int_A f(x)d\lambda(x)$, where $\lambda = \lambda_1 - \lambda_2$ and λ_1 and λ_2 are measures on $\mathfrak{B}(A)$.

Part (C) of our thesis is fully supported by the Riesz representation theorem.

III_n. *Let B be any noncompact subset of R. Then there is a positive linear functional L defined for all bounded continuous real-valued functions on B that is representable by no countably additive measure defined on the Borel sets of B.*

Proof. Suppose that B is unbounded above and that $\{x_n\}_{n=1}^{\infty}$ is a subset of B such that $x_1 < x_2 < x_3 < \cdots < x_n < \cdots$ and $\lim_{n\to\infty} x_n = \infty$. For every bounded continuous real-valued function f on B, let $p(f) = \overline{\lim}_{n\to\infty} f(x_n)$. We then have $p(f+g) \leqq p(f) + p(g)$ and $p(\alpha f) = \alpha p(f)$ if f and g are bounded continuous real-valued functions on B and α is a nonnegative real number. The Hahn-Banach theorem implies (see [2], p. 29, Corollaire) that there is a linear functional L defined for all bounded continuous real-valued functions f on B such that $L(f) \leqq p(f)$ for all f.† Thus we have $-p(-f) \leqq L(f) \leqq p(f)$. Since $-p(-f) = \underline{\lim}_{n\to\infty} f(x_n)$, we may write

* The proof just given is based on a proof given by Kakutani ([11], Theorem 9). The history of representation theorems like Proposition III_c is long and involved, and we cannot go into it here.

† This is a sophisticated result, which depends upon the well-ordering theorem.

$$\underline{\lim}_{n\to\infty} f(x_n) \leqq L(f) \leqq \overline{\lim}_{n\to\infty} f(x_n).$$

Thus, if $f(x)\geqq 0$ for all $x\in B$, we have $\underline{\lim}_{n\to\infty} f(x_n)\geqq 0$ and thus $L(f)\geqq 0$. Also, if $\lim_{n\to\infty} f(x_n)=0$, it follows that $L(f)=0$. Now suppose that L can be represented by an integral with respect to a measure λ defined at least for the closed subsets E of B. Let f be any continuous function on B such that $0\leqq f(x)\leqq 1$ for all $x\in B$ and $f(x)\geqq 1$ for $x\in E$. A moment's reflection shows that $L(f)\geqq\lambda(E)$. However, if E is bounded, f can be chosen so that $f(x_n)=0$ for all sufficiently large n. That is, we must have $\lambda(E)=0$ for all closed bounded subsets E of B. The set B is the union of an increasing sequence of such sets: $B=\bigcup_{k=1}^{\infty} E_k$. If λ were countably additive, we would have $\lambda(B)=\lim_{k\to\infty}\lambda(E_k)=0$. But $\lambda(B)=L(1)=1$. Thus there is no countably additive measure defined for closed subsets of B that produces an integral representation for L. A virtually identical argument is used if B is bounded and nonclosed.

Remark. Something can be salvaged from Riesz's representation theorem for positive linear functionals on the bounded continuous real-valued functions on noncompact sets. Such functionals have integral representations, but with respect to *finitely* and not always *countably* additive measures. The main theorems of integration theory (term-by-term integration of sequences, and Fubini's theorem) are in general false for finitely additive measures, so that the analytic utility of such representations is small. The properties of, and representations using, finitely additive measures are discussed at length in [22] and [8].

Example IV. At the risk of tediousness, we cite still another example in support of our thesis. Consider the set of *all* real-valued continuous functions on a subset A of R, denoted by the symbol $\mathfrak{C}(A)$. The set $\mathfrak{C}(A)$ is a ring under the operations of pointwise addition and multiplication. As such, $\mathfrak{C}(A)$ admits the concept of an *ideal*: an ideal $\mathfrak{I}$ in $\mathfrak{C}(A)$ is a nonvoid subset of $\mathfrak{C}(A)$ such that if f, g are in $\mathfrak{I}$ and φ is in $\mathfrak{C}(A)$, then $f-g$ is in $\mathfrak{I}$ and φf is in $\mathfrak{I}$. To avoid uninteresting special cases in our proofs, we suppose that all ideals considered are distinct from $\mathfrak{C}(A)$ itself.

It is simple to find all ideals in $\mathfrak{C}(F)$ if F is a finite subset of R.

IV$_f$. *Let F be a finite subset of R, so that $\mathfrak{C}(F)$ is the ring of all real-valued functions on F. Let $\mathfrak{I}$ be an ideal in $\mathfrak{C}(F)$. Then there is a nonvoid subset H of F such that $\mathfrak{I}$ consists exactly of the functions in $\mathfrak{C}(F)$ that vanish on H.*

Proof. If $\mathfrak{I}$ consists of the function 0 and nothing else, the result asserted holds with $H=F$. Suppose then that $\mathfrak{I}\neq\{0\}$, that $f\in\mathfrak{I}$, and $f(x_j)\neq 0$, where as before we write $F=\{x_1, x_2, \cdots, x_p\}$. Defining φ_j as in the proof of Proposition III$_f$, we see that $(1/f(x_j))\varphi_j f=\varphi_j\in\mathfrak{I}$. Let $J=\{x_{j_1}, x_{j_2}, \cdots, x_{j_l}\}$ be the set of all $x_j\in F$ for which there is an $f\in\mathfrak{I}$ such that $f(x_j)\neq 0$. Then every real-valued function g on F that vanishes on $H=F\cap J'$ has the form $g=g(x_{j_1})\varphi_{j_1}+\cdots+g(x_{j_l})\varphi_{j_l}$ and is hence in $\mathfrak{I}$. Clearly every function in $\mathfrak{I}$ must vanish on H, and thus IV$_f$ is proved.

The infinite compact analogue of Proposition IV_f, if stated directly, is rather weak: we cannot classify *all* ideals in $\mathfrak{C}(A)$ for any infinite subset A of R. However, all ideals in $\mathfrak{C}(A)$ have a very special property if A is compact.

IV_c. *Let A be a compact subset of R, and let $\mathfrak{I}$ be an ideal in $\mathfrak{C}(A)$. Then there is at least one point $p \in A$ such that $f(p) = 0$ for all $f \in \mathfrak{I}$.*

Proof. Assume that $\mathfrak{I}$ is an ideal in $\mathfrak{C}(A)$ and that for every $p \in A$, there is some $f_p \in \mathfrak{I}$ such that $f_p(p) \neq 0$. Then $f_p^2 \in \mathfrak{I}$ and $f_p^2(p) > 0$. Since f_p^2 is continuous, there is an open interval I_p containing p such that $f_p^2(t) > 0$ for all $t \in A \cap I_p$. Since A is compact, a finite number of the intervals I_p, say $I_{p_1}, I_{p_2}, \cdots, I_{p_m}$, cover A: A is contained in $I_{p_1} \cup I_{p_2} \cup \cdots \cup I_{p_m}$. If we examine the function $g = f_{p_1}^2 + \cdots + f_{p_m}^2$, we see that $g \in \mathfrak{I}$ and $g(x) > 0$ for all $x \in A$. Then, for every $\varphi \in \mathfrak{C}(A)$, we have $\varphi g^{-1} g \in \mathfrak{I}$, so that $\mathfrak{I}$ is not an ideal in our present sense.

Proposition IV_c has the following immediate corollary. *Let $\mathfrak{M}$ be a maximal ideal in $\mathfrak{C}(A)$, where A is any compact subset of R. (An ideal $\mathfrak{M}$ in $\mathfrak{C}(A)$ is said to be maximal if there is no ideal $\mathfrak{N}$ in $\mathfrak{C}(A)$ such that $\mathfrak{M} \subsetneq \mathfrak{N} \subsetneq \mathfrak{C}(A)$.) Then there is a point $p \in A$ such that $\mathfrak{M}$ consists of all functions in $\mathfrak{C}(A)$ that vanish at p.* Thus the maximal ideals of $\mathfrak{C}(A)$ are readily classified.

To see the complications that arise if one attempts to find all of the ideals in $\mathfrak{C}(A)$, consider $\mathfrak{C}([0, 1])$ and the set of all functions $f \in \mathfrak{C}([0, 1])$ that vanish in some interval $[0, \delta]$, where $0 < \delta \leqq 1$ (δ depends upon f). Clearly this set is an ideal in $\mathfrak{C}([0, 1])$, and it is contained in only one maximal ideal. A whole class of ideals is found by the following construction. Choose a sequence $\{t_n\}_{n=1}^{\infty}$ of numbers such that $0 < t_n \leqq 1$ for all n and $\lim_{n \to \infty} t_n = 0$. Consider the set of all $f \in \mathfrak{C}([0, 1])$ such that $f(t_n) = 0$ for $n \geqq N$ (N depends upon f). Finally, consider all $f \in \mathfrak{C}([0, 1])$ for which $|x^{-1} f(x)|$ is bounded for $0 < x \leqq 1$. Here again we have an ideal contained in only one maximal ideal. These examples give a hint of the complexity of the class of all ideals in $\mathfrak{C}([0, 1])$. Similar constructions can be given in $\mathfrak{C}(A)$ for any infinite compact subset A of R.

Nevertheless, Proposition IV_c can be greatly improved on. The reader will see that the following assertion bears much the same relation to IV_f as II_c bears to II_f.

IV_c'. *Let A be a compact subset of R and $\mathfrak{I}$ an ideal in $\mathfrak{C}(A)$ that is closed under the formation of uniform limits. That is, if $f_1, f_2, f_3, \cdots, f_n, \cdots$ are in $\mathfrak{I}$, and $\lim_{n \to \infty} f_n = \varphi$, the limit being uniform on A, then φ is also in $\mathfrak{I}$. In this case, there is a nonvoid closed subset E of A such that $\mathfrak{I}$ consists exactly of the functions in $\mathfrak{C}(A)$ that vanish on E. Conversely, if E is any nonvoid closed subset of A, the set of all functions in $\mathfrak{C}(A)$ that vanish on E is an ideal in $\mathfrak{C}(A)$ closed under the formation of uniform limits.*

Proof. The second assertion is very easy to prove: we shall not go into its details. To prove the first assertion, let E be the set of all $x \in A$ such that $f(x) = 0$ for all $f \in \mathfrak{I}$. Proposition IV_c shows that E is nonvoid. It is quite obvious that

every function in $\mathfrak{I}$ vanishes everywhere on the set E. We must prove that every function φ in $\mathfrak{C}(A)$ that vanishes on E is the uniform limit (on A) of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $\mathfrak{I}$: this will show that φ is in the ideal $\mathfrak{I}$. If $E=A$, there is nothing to prove: in this case, $\mathfrak{I}=\{0\}$. Suppose that $E\subsetneq A$ and that D is any closed subset of A disjoint from E. Let p be an arbitrary point of D. Since p is not in E, there is a function $f_p\in\mathfrak{I}$ such that $f_p(p)\neq 0$. The functions f_p^2 are nonnegative and lie in $\mathfrak{I}$. We argue as in the proof of Proposition IV$_c$ to infer that there are a finite number of points $p_1, p_2, \cdots, p_m$ in D such that the function $f=f_{p_1}^2+f_{p_2}^2+\cdots+f_{p_m}^2$ is positive throughout D. Since D is compact, f has a positive lower bound, say α, on D. Since f is in $\mathfrak{I}$, f vanishes throughout E. Define the function g by the relations

$$g(x)=\frac{\alpha}{f(x)} \text{ if } f(x)\geqq\alpha, \qquad g(x)=1 \text{ if } f(x)<\alpha.$$

Plainly g is continuous on A, and the function $h=(1/\alpha)gf$ is in $\mathfrak{I}$. Also we have $h(x)=1$ for $x\in D$, $h(x)=0$ for $x\in E$, and $0\leqq h(x)\leqq 1$ for all $x\in A$.

Now let φ be any function in $\mathfrak{C}(A)$ such that $\varphi(x)=0$ for all $x\in E$. Let $D_n=\{x: x\in A, |\varphi(x)|\geqq 1/n\}$ $(n=1, 2, 3, \cdots)$. If $\varphi\neq 0$, then D_n is nonvoid for all sufficiently large integers n. For all such n, let h_n be a function in $\mathfrak{I}$ such that $h_n(x)=1$ for $x\in D_n$ and $0\leqq h_n(x)\leqq 1$ for all $x\in A$. Such an h_n exists because D_n is a closed subset of A disjoint from E. Since h_n is in $\mathfrak{I}$, the function $h_n\varphi$ is in $\mathfrak{I}$. The construction of h_n shows that $|h_n(x)\varphi(x)-\varphi(x)|\leqq 1/n$ for all $x\in A$. Thus $\lim_{n\to\infty} h_n\varphi=\varphi$, the limit being uniform on A. This completes the proof.

Proposition IV fails in a most satisfactory way for noncompact subsets of R.

IV$_n$. *Let B be any noncompact subset of R. Then there is an ideal $\mathfrak{I}$ in $\mathfrak{C}(B)$ closed under the formation of uniform limits such that for every $x\in B$, there is an $f\in\mathfrak{I}$ such that $f(x)\neq 0$.*

Proof. Suppose that B is unbounded above. For every positive integer n, let f_n be the function on R such that $f_n(x)=0$ for $x\geqq n$, $f_n(x)=1$ for $x\leqq n-1$, and f_n is linear in the interval $[n-1, n]$. With its domain restricted to B, f_n is clearly in $\mathfrak{C}(B)$. Let $\mathfrak{I}_0$ be the set of all functions of the form $\varphi=\varphi_1 f_{n_1}+\varphi_2 f_{n_2}+\cdots+\varphi_l f_{n_l}$, where $\varphi_1, \varphi_2, \cdots, \varphi_l$ are in $\mathfrak{C}(B)$. Obviously $\mathfrak{I}_0$ is an ideal in $\mathfrak{C}(B)$ (in fact the smallest ideal containing $f_1, f_2, f_3, \cdots$). It is clear that $\mathfrak{I}_0$ is not all of $\mathfrak{C}(B)$, since $\varphi(x)=0$ for $x\geqq\max\{n_1, n_2, \cdots, n_l\}$ and B is unbounded above. Since all f_n are in $\mathfrak{I}_0$ and $f_n(x)>0$ for $n>[x]+1$, it is clear that $\mathfrak{I}_0$ has the property asserted. Let $\mathfrak{I}$ be the set of all uniform limits of sequences of functions in $\mathfrak{I}_0$. Then $\mathfrak{I}$ is an ideal, as can be easily verified, and $\mathfrak{I}$ is not equal to $\mathfrak{C}(B)$ since $\lim_{x\to\infty}\varphi(x)=0$ for all $\varphi\in\mathfrak{I}$. Very similar constructions are used if B is unbounded below or nonclosed.

Compactness in more general spaces. Suppose now that we have any metric space, say X. That is, for every pair of points (x, y) from X, a distance $\rho(x, y)$

is defined, satisfying the following axioms: $\rho(x, x)=0$, and $\rho(x, y)$ is a positive real number if $x \neq y$; $\rho(x, y)=\rho(y, x)$ for all x and y in X; $\rho(x, z) \leqq \rho(x, y)+\rho(y, z)$ for all x, y, and z in X. A sequence $\{t_n\}_{n=1}^{\infty}$ of elements of X converges to a point t of X if $\lim_{n\to\infty} \rho(t_n, t)=0$. A limit point of a subset Y of X is a point t such that for every positive real number ϵ, there is some y in Y such that $0<\rho(t, y)<\epsilon$. A subset G of X is open if for every $x \in G$, there is a positive real number α such that G contains all points $y \in X$ such that $\rho(x, y)<\alpha$.* With these concepts, compactness of the metric space X can be defined by repeating properties (1) and (2) of compactness *verbatim*. Property (3) of compactness is changed slightly, to the following.

(3′) Let $\{G_\iota\}_{\iota \in I}$ be a family of open subsets of X such that $\bigcup_{\iota \in I} G_\iota = X$. Then there is a finite subfamily $\{G_{\iota_1}, G_{\iota_2}, \cdots, G_{\iota_m}\}$ of $\{G_\iota\}_{\iota \in I}$ such that $G_{\iota_1} \cup G_{\iota_2} \cup \cdots \cup G_{\iota_m} = X$.

Properties (1), (2), and (3′) are equivalent for a metric space, and are collectively called compactness. Furthermore, all parts of all of the propositions I, II, III, and IV remain true for metric spaces. (A real-valued function f on a metric space X is continuous if $x_n \to x$ in X implies $f(x_n) \to f(x)$ in R: similarly with complex- or quaternion-valued functions on X.) The proofs of Propositions I_f, II_f, III_f, and IV_f are trivially unchanged, since a function on a finite set F is indifferent to the imbedding of F in the line, or in any other metric space for that matter. The proofs of Propositions I_c, II_c, III_c, and IV_c are repeated with the set A replaced by an arbitrary compact metric space. A little care is needed with Propositions I_n, II_n, III_n, and IV_n, where B is replaced by an arbitrary noncompact metric space. However, examples showing the falsity of the four propositions can be constructed. We shall not go into details here (see [9], p. 69, Theorem 30; [7], p. 423, Theorem 3; [9], p. 53, Theorem 7). Generalizations of I_n, II_n, and IV_n, respectively, are given in these three references. The writer knows of no explicit generalization in print of III_n for noncompact metric spaces. Such a generalization is easily obtained, however, along lines very like those of the construction in the proof of Proposition III_n.

Compactness for the most general topological spaces exhibits peculiarities not found in metric spaces. A set X is called a topological space if there is prescribed in it a family $\mathfrak{O}$ of subsets, called *open sets*, satisfying the following axioms: $\varnothing$ is in $\mathfrak{O}$; if $G_1, G_2, \cdots, G_n$ are in $\mathfrak{O}$, then $G_1 \cap G_2 \cap \cdots \cap G_n$ is in $\mathfrak{O}$; if $\{G_\iota\}_{\iota \in I}$ is contained in $\mathfrak{O}$, then $\bigcup_{\iota \in I} G_\iota$ is in $\mathfrak{O}$. A subset E of X is closed if its complement E' (relative to X) is open. To avoid technicalities of no present interest, we suppose that all topological spaces X discussed are completely regular. That is, given any two points x and y in X distinct from each other, there is an open set G in X containing x but not y; and given a closed set E and a point $x \in X \cap E'$, there is a continuous real-valued function f defined on X such that $f(x)=0$ and $f(y)=1$ for all $y \in E$. (A real-valued function f on X is continuous if $f^{-1}(]a, b[)$

* There is a very extensive theory of metric spaces. Perhaps the most complete treatise on the subject is [14].

is an open subset of X for all open intervals $]a, b[$ of R.)* For general topological spaces, we take (3′) as the definition of compactness: every open covering of X admits a finite subcovering. In this very general situation, (1) and (2) are no longer equivalent to (3′). There is an analogue of (1) dealing with generalizations of sequences, which is equivalent to (3′) (see [12], p. 136, Theorem 2). There is also a reformulation of (2) which is equivalent to (3′) (see [12], p. 163, Problem I).

Our propositions I_c, II_c, III_c, and IV_c are true for all compact topological spaces. In fact, all of the proofs carry over without change if A is replaced by an arbitrary compact space X, except for the proof in I_c that every real-valued continuous function on X attains its least upper and greatest lower bounds. For this, a nearly trivial substitute proof using open coverings can be supplied.

Some changes appear in the propositions about noncompact spaces. Proposition I_n is false for general topological spaces: there are noncompact completely regular spaces on which every continuous real-valued function is bounded. These spaces seem first to have been studied by the writer ([9], pp. 67–71). Propositions II_n and IV_n hold for all noncompact completely regular spaces. (See [9], p. 53, Theorem 7 and [7], p. 423, Theorem 3, respectively.) Proposition III_n seems not to have been studied explicitly for noncompact spaces. There are some curious measure-theoretic and topological technicalities here; suffice it to say that there are noncompact completely regular spaces X such that every positive linear functional on the space of all bounded continuous real-valued functions can be represented by the integral with respect to a countably additive measure.†

Our final illustration of the power of compactness as an hypothesis is one which we can only cite without proof. Consider a group G. A unitary representation of G is a homomorphism $x \to U(x)$ mapping G into the group of unitary operators on some Hilbert space $\mathfrak{H}$. If there is no proper closed subspace of $\mathfrak{H}$ carried into itself by all of the operators $U(x)$, the representation U is called irreducible. It is a classical algebraic fact (which we state in somewhat disguised form) that if G is finite, then all of its irreducible representations are finite-dimensional. Also, every complex-valued function on G is a linear combination (with complex coefficients) of functions $(U(x)\xi, \eta)$, where U is an irreducible unitary representation of G and ξ, η are in the finite-dimensional complex Euclidean space $\mathfrak{H}$ on which U acts. This is merely a restatement of the classical fact that the group algebra of G over the complex numbers is semisimple.‡

For infinite compact groups,§ the Peter-Weyl theorem replaces the foregoing

* There is a highly developed theory of topological spaces. See for example the treatise [12].

† The space of ordinal numbers less than Ω with the interval topology is an example. The $\{0, 1\}$ measure that represents the functional $\lim_{\alpha \to \Omega} f(\alpha)$ is, however, not a *regular* measure.

‡ For a discussion of this theorem, see for example [4], p. 62, Satz 4.3.

§ A topological group is a group and a topological space in which the mapping $(x, y) \to xy^{-1}$ is continuous. A (locally) compact group is a topological group which as a topological space is (locally) compact.

result. Every irreducible continuous unitary representation of a compact group G is finite-dimensional. Every complex-valued continuous function on G can be arbitrarily uniformly approximated by linear combinations of functions $(U(x)\xi, \eta)$ where U is a continuous irreducible unitary representation of G and ξ, η are elements of the finite-dimensional complex Euclidean space $\mathfrak{H}$ on which U acts.*

If G is a noncompact, non-Abelian, locally compact group, then it may have no nontrivial finite-dimensional unitary representations. The group of all complex matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad-bc=1$ is an example.†

Other applications of compactness. The preceding sections of this essay treat compactness as an essential hypothesis in theorems about various entities met with in analysis. These theorems clearly show the vital part played by compactness in obtaining usable theorems. Perhaps as important as this, however, is the rôle played by compactness in existence theorems. The general idea used is the following. Suppose that one has a collection S of objects of a certain sort—they may be functions, measures, linear functionals, *etc.*—and suppose that one wishes to show the existence of one of the objects, s_0, having a certain property P. The property P is often numerical in character. Suppose further that S is topologized so as to be compact, and for simplicity let us say that S is a metric space. Suppose that one can find a sequence $\{s_1, s_2, s_3, \cdots, s_n, \cdots\}$ of objects in S that come closer and closer to having the property P. Then, extracting from this sequence a subsequence $\{s_{n_1}, s_{n_2}, s_{n_3}, \cdots, s_{n_k}, \cdots\}$ with limit s_0 in S, one can often show that s_0 actually has the property P. The argument is a little more sophisticated if S is compact but not metric, but the basic idea is identical.

There is a formidable list of existence theorems in analysis that employ this compactness technique. However, as Kipling said, that is another story.

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THE GENERALIZED COCONUT PROBLEM*

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1. Introduction. The coconut problem has long interested students of mathematics. Accordingly, many methods have been applied to its solution, including congruences, properties of number bases, and elaborate sieves. In this paper, the calculus of finite differences is used to solve the following generalization.

Suppose that n sailors plan to divide a pile of coconuts in the morning. During the night, one of them decides to take his share. After discarding V_1 coconuts to make the division come out even, he takes one nth of the pile. The process is repeated by $m-2$ more sailors, the xth sailor discarding V_x coconuts.

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In the morning, after discarding V_m coconuts, the remaining pile is divided equally. What is the minimum number of coconuts that could have been in the pile originally?

The traditional problem is the special case where $n=5$, $m=6$, and $V_i=1$, $i=1, 2, \cdots, 6$. A formula for the solution of the generalized problem will now be derived.

2. The difference equation and its solution. Abstracting from sailors and coconuts, we wish to find the smallest integer A such that for m nonnegative integers $V_1, \cdots, V_m$, there exist m positive integers $U_1, \cdots, U_m$, that satisfy

$$U_1 = \frac{A - V_1}{n}, \tag{1}$$

$$U_x = \frac{(n-1)U_{x-1} - V_x}{n}. \tag{2}$$

Since the V_i, $i>m$ do not affect the solution, we shall choose these such that $\Delta^k V_x=0$, $k\geqq m$. This can always be done, and it will simplify the following equations.

The difference equation (2) with initial condition (1) is seen to be linear with constant coefficients, and defined for all positive x. Its solution is

$$U_x = \frac{(A - V_1 - nP_1)}{(n-1)}\left(\frac{n-1}{n}\right)^x + P_x, \tag{3}$$

where P_x is any particular solution to the nonhomogeneous equation. To determine P_x, we write the difference equation in the following equivalent form: $n\Delta U_x + U_x = -V_{x+1}$. A particular solution may be verified to be

$$P_x = -\sum_{k=0}^{m-1} (-n)^k \Delta^k V_{x+1}. \tag{4}$$

Using Newton's formula, V_{x+1} can be expressed in terms of the differences $\Delta^k V_1$, $k=0, 1, \cdots$, as

$$V_{x+1} = \sum_{j=0}^{m-1} \Delta^j V_1 C_j^x, \tag{5}$$

where

$$C_j^x = \begin{cases} \dfrac{x(x-1)\cdots(x-j+1)}{j!}, & j \geqq 0; \\ 0, & j < 0. \end{cases}$$

Noting that $\Delta^k C_j^x = C_{j-k}^x$, and that $C_j^x=0$, $j<0$, we have upon substituting (5) into (4) and interchanging the order of summation:

$$P_x = -\sum_{j=0}^{m-1}\sum_{k=0}^{j}(-n)^k\Delta^j V_1 C_{j-k}^x. \tag{6}$$

We note that both V_x and P_x are integral when x is integral.

3. The determination of A. In order for A to be a solution, necessary conditions are that (i) A is integral, and (ii) U_k, $k=1, \cdots, m$, are positive integers. Therefore we must have

$$U_m = \frac{(A - V_1 - nP_1)}{(n-1)}\left(\frac{n-1}{n}\right)^m + P_m = r, \tag{7}$$

where r is an integer. Solving this equation for A in terms of r,

$$A = \frac{(r - P_m)n^m}{(n-1)^{m-1}} + V_1 + nP_1. \tag{8}$$

Since $n-1$ is relatively prime to n, and V_x and P_x are integers when x is an integer, a necessary and sufficient condition that A be integral is that

$$r - P_m = s(n-1)^{m-1} \tag{9}$$

for some integer s. From (2) we note that $U_i < U_{i-1}$ since $V_k \geqq 0$, $k=1, \cdots, m$. Thus if $U_m = r$ is positive then so will A and U_k, $k=1, \cdots, m$, be positive. To make A as small as possible, we choose the smallest integer s for which r is positive. This value of s is given by

$$s = \left[\frac{-P_m}{(n-1)^{m-1}}\right] + 1. \tag{10}$$

It is an easy matter to show that all the U_i are integral. From (7) and (9), we have $U_m = s(n-1)^{m-1} + P_m$. Using this and the original difference equation (2), solved for U_{x-1}, it follows by induction that

$$U_{m-k} = s(n-1)^{m-1-k}(n^k) + P_{m-k}, \qquad 0 \leqq k < m.$$

Therefore, for the choice of s in (10), conditions (i) and (ii) above hold. Furthermore, the A thus determined is the smallest compatible with the conditions on the U_i. Hence it is the solution. To obtain A explicitly, we substitute (9) and (10) into (8). Evaluating (6) at $x=1$ and $x=m$, and substituting these values into (8), we obtain

$$A = \left(\left[\frac{\sum_{j=0}^{m-1}\sum_{k=0}^{j}(-n)^k\Delta^j V_1 C_{j-k}^m}{(n-1)^{m-1}}\right] + 1\right)n^m + (1-n)\sum_{j=0}^{m-1}(-n)^j\Delta^j V_1. \tag{11}$$

If n and m are very large, the square bracket term will in general be difficult to calculate. However, this calculation may sometimes be avoided by noting that any two "solutions" will differ by a multiple of n^m. We can thus write

$$A = cn^m + (1 - n)\sum_{j=0}^{m-1} (-n)^i \Delta^i V_1, \tag{12}$$

where c is the smallest integer such that $A > 0$, and $U_k > 0$, $k = 1, \cdots, m$. Fortunately, if n and m are larger than three or four, and the V_i are not too large, this value is usually the one between 0 and n^m. That is, A equals the right hand side of (12) modulo n^m.

It may be remarked that we have used the nonnegative character of the V_i only to establish that the U_i are positive. Therefore, with c defined as in (12), (12) is valid even if some of the V_i are negative.

To summarize, if a pile of coconuts is divided m times among n sailors in the manner described in the introduction, such that before the ith division, V_i coconuts are discarded, the minimum number of coconuts that could have been in the pile originally is given by formula (11) or (12).

4. Examples.

1. Consider the "generalization" where n sailors divide the pile m times and discard p coconuts before each division. Since $V_x \equiv p$, all the differences are zero. Formula (11) yields the solution

$$A = \left(\left[\frac{p}{(n-1)^{(m-1)}}\right] + 1\right) n^m + (1 - n)p.$$

If $0 \leqq p < (n-1)^{m-1}$, this simplifies to

$$A = n^m + (1 - n)p.$$

The classic coconut problem is the special case where $n = 5$, $m = 6$, and $p = 1$. Substituting these values in, we find $A = 15{,}621$.

2. This example illustrates the use of (12). Suppose $n = 5$, $m = 4$, $V_1 = 4$, $V_2 = 3$, $V_3 = 1$, $V_4 = 1$. The differences can be obtained from a difference table:

x	V_x	ΔV_x	$\Delta^2 V_x$	$\Delta^3 V_x$
1	4	−1	−1	3
2	3	−2	2	
3	1	0		
4	1			

We have $A = (1-n)\sum_{j=0}^{m-1} (-n)^i \Delta^i V_1 (\text{mod } n^m) = 314$. This may be verified to be correct.

3. Finally, we use (12) in a case where the V_i are negative. This means that coconuts are being added to the pile (say by a monkey). Let $n = 3$, $m = 2$, $V_1 = V_2 = -10$. Then

$$A = 3^2 c + (1 - 3)(-10) = 9c + 20, \quad U_1 = 3c + 10, \quad U_2 = 2c + 10.$$

Thus $c = -2$, and $A = 2$.

ON THE BROCARD POINTS OF A TRIANGLE

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1. Preliminaries. Let (T) be any triangle $A_1A_2A_3$, with vertices numbered in counterclockwise order. The interior angle at A_i is denoted by α_i $(i=1, 2, 3)$, and the length of the opposite side is a_i.

The Brocard angle of (T) is called ω, for which one formula is $\cot\omega=\cot\alpha_1+\cot\alpha_2+\cot\alpha_3$. The inequality

$$\cot\omega \geqq \sqrt{3} \tag{1}$$

is well known, and the relationship

$$\cot\omega \leqq \frac{9}{8}\csc\alpha_1\csc\alpha_2\csc\alpha_3 \tag{2}$$

may be easily verified. In each case equality occurs only when (T) is equilateral.

Johnson [2] shows that, if ω is given, the shape of (T) is restricted by $\delta' \leqq \alpha_i \leqq \delta$ $(i=1, 2, 3)$, in which

$$\cot\delta = \tfrac{1}{3}(\cot\omega - 2\sqrt{\cot^2\omega - 3}), \tag{3}$$

$$\cot\delta' = \tfrac{1}{3}(\cot\omega + 2\sqrt{\cot^2\omega - 3}). \tag{4}$$

The value of δ or δ' is realized only by the vertex angle of an isosceles triangle.

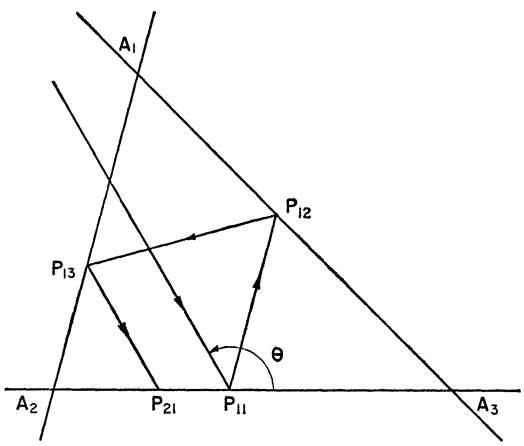


FIG. 1

2. A convergent path. From any point in the plane of (T) a line is drawn (Fig. 1) to meet A_2A_3 (extended if necessary) at a given angle $\theta(0<\theta<\pi)$, θ being measured counterclockwise from A_2A_3 to the line. From P_{11}, the point at which the line meets A_2A_3, a line is drawn to meet A_3A_1 at P_{12}, forming the angle θ in the

same sense as before. This path is continued indefinitely in the same manner, proceeding from one side of (T) to the next in counterclockwise order. It meets A_2A_3 for the nth time at P_{n1} $(n=1, 2, 3, \cdots)$, and the directed segment $P_{n1}A_3$ is denoted by u_n.

By trigonometry it is seen that $u_2=au_1+b$, in which

$$a = \frac{\sin(\alpha_1-\theta)\sin(\alpha_2-\theta)\sin(\alpha_3-\theta)}{\sin^3\theta} \quad \text{and} \quad b = \frac{a_1\sin^2\alpha_2}{\sin^2\theta}.$$

It follows that $u_n=a^{n-1}u_1+b(1+a+a^2+\cdots+a^{n-2})$. When $|a|<1$, $\lim_{n\to\infty} u_n$ exists and is equal to $b/(1-a)$. Clearly then the path converges to meet each side of (T) at a unique point.

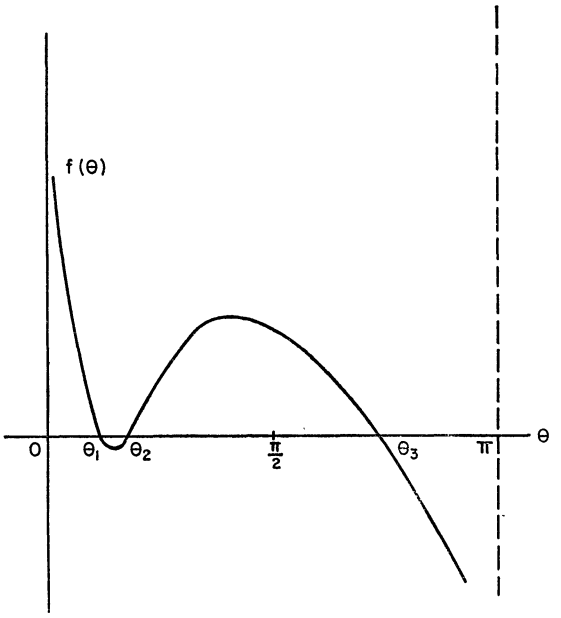


FIG. 2

The inequality $|a|<1$ may be reduced to $0<\csc^2\theta(\cot\omega-\cot\theta)<2\csc\alpha_1\csc\alpha_2\csc\alpha_3$, in which the left-hand inequality yields the result $\theta>\omega$. The other inequality may be written $f(\theta)>0$, if $f(\theta)=\cot^3\theta-\cot\omega\cot^2\theta+\cot\theta-\cot\omega+2\csc\alpha_1\csc\alpha_2\csc\alpha_3$. The essential properties of $f(\theta)$ are shown in Figure 2.

There are three values of θ for which $f(\theta)=0$. They are called θ_1, θ_2, θ_3 (in order of increasing magnitude when all are real). The root θ_3 is always real and lies within certain bounds which will now be determined.

When θ is near π, $f(\theta)<0$. But

$$f\left(\frac{2\pi}{3}\right) = -\frac{4}{3\sqrt{3}} - \frac{4}{3}\cot\omega + 2\csc\alpha_1\csc\alpha_2\csc\alpha_3$$

$$\geqq -\frac{4}{3\sqrt{3}} - \frac{4}{3}\cot\omega + \frac{16}{9}\cot\omega \quad \text{(from (2))}$$

$$= \frac{4}{9}(\cot\omega - \sqrt{3})$$

$$\geqq 0 \quad \text{(from (1))}.$$

Therefore $\theta_3 \geqq 2\pi/3$.

An upper bound may be obtained as follows. Given ω, $\csc\alpha_1\csc\alpha_2\csc\alpha_3$ becomes a function of one variable, and its extremes may be computed to be $(2/27)[\cot^3\omega + 9\cot\omega \pm (\cot^2\omega - 3)^{3/2}]$. Now

$$f(\pi - 2\omega) = -\frac{1}{8\cot^3\omega}\Big[3\cot^6\omega + 5\cot^4\omega + \cot^2\omega - 1 - 16\cot^3\omega\csc\alpha_1\csc\alpha_2\csc\alpha_3\Big]$$

$$\leqq -\frac{1}{8\cot^3\omega}\Big[3\cot^6\omega + 5\cot^4\omega + \cot^2\omega - 1 - \frac{32}{27}\cot^3\omega\{\cot^3\omega + 9\cot\omega + (\cot^2\omega - 3)^{3/2}\}\Big]$$

$$= -\frac{\cot^2\omega - 3}{216\cot^3\omega}[49\cot^4\omega - 6\cot^2\omega + 9 - 32\cot^3\omega(\cot^2\omega - 3)^{1/2}]$$

$$\leqq -\frac{\cot^2\omega - 3}{216\cot^3\omega}(17\cot^4\omega - 6\cot^2\omega + 9)$$

$$\leqq 0.$$

Therefore $\theta_3 \leqq \pi - 2\omega$.

In case θ_1 and θ_2 are imaginary, the condition for convergence of the path is stated as $\omega < \theta < \theta_3$. However, they may be real. Several facts which may be derived will now be stated without proof.

The relative minimum and maximum of $f(\theta)$ are at $\theta = \frac{1}{2}(\pi - \delta)$ and $\theta = \frac{1}{2}(\pi - \delta')$ respectively. The relative minimum has the same sign as $\theta_3 - \delta$; hence θ_1 and θ_2 are real when $\theta_3 \leqq \delta$. If θ_1 and θ_2 are real, then $\theta_1 + \theta_2 + \theta_3 = \pi$ and $\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = \cot\omega$; this means that $\theta_1, \theta_2, \theta_3$ are the angles of a triangle with the same Brocard angle as that of (T).

Also it is easily shown that $\theta_1 > \omega$. Therefore, when θ_1 and θ_2 are real, the domain of convergence of the path is separated into two disjoint intervals: $\omega < \theta < \theta_1$ and $\theta_2 < \theta < \theta_3$.

In particular, the path converges when $\theta = \alpha_i$ $(i = 1, 2, 3)$. If we assume

$\alpha_1 \leqq \alpha_2 \leqq \alpha_3$, then $\alpha_1 \leqq \frac{1}{2}(\pi - \delta) \leqq \alpha_2$. Hence, in the case for which θ_1 and θ_2 are real, α_1 lies in the first subset of the domain of convergence, while α_2 and α_3 lie in the second subset.

3. A convergent sequence of triangles. Whether or not the path converges, there always exists, with one exception, a triangle inscribed in (T) each of whose sides makes the angle θ with the corresponding side of (T). This triangle is also similar to (T), the ratio of similitude being $\sin\omega/\sin(\theta-\omega)$. Thus the exceptional case is that for which $\theta=\omega$.

These inscribed triangles are known to be the Miquel triangles of Ω_2, the second Brocard point of (T), which is also the center of similitude of (T) and each inscribed triangle [2, p. 269].

Let (T_1) be the inscribed triangle for a given value of θ. Then inscribe (T_2) in (T_1) in the same way that (T_1) was inscribed in (T). Continue this process to obtain an infinite sequence of triangles. This sequence converges to a point if $|\sin\omega/\sin(\theta-\omega)| < 1$, from which it follows that $2\omega < \theta < \pi$. The limit point is Ω_2, since it is the center of similitude of every pair of successive triangles and is always an interior point.

If, from the beginning, θ had been measured clockwise and the path had been defined as going in clockwise order from each side of (T) to the next, the limit point of the resulting sequence of triangles would have been Ω_1, the first Brocard point of (T).

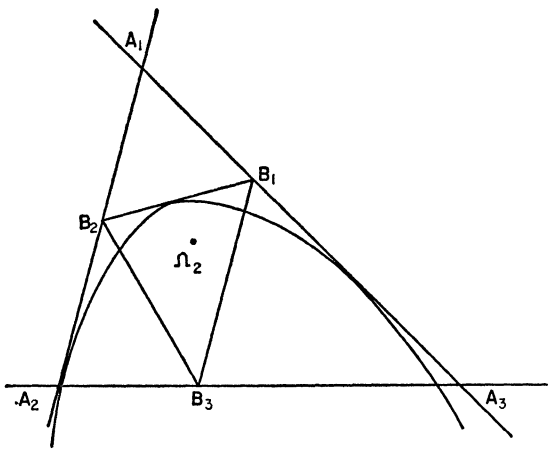


FIG. 3

4. Three confocal parabolas. Now let (T) be the triangle of reference in a trilinear homogeneous coordinate system; and let triangle $B_1B_2B_3$ be the inscribed triangle corresponding to θ, as described above (Fig. 3). Then the coordinates of its vertices are as follows:

$$B_1(a_2 \sin(\theta - \alpha_3),\ 0,\ a_3 \sin\theta),$$
$$B_2(a_1 \sin\theta,\ a_3 \sin(\theta - \alpha_1),\ 0),$$
$$B_3(0,\ a_2 \sin\theta,\ a_1 \sin(\theta - \alpha_2)).$$

The equation of the line B_1B_2 may now be derived. When it is differentiated with respect to θ, and θ is eliminated, the equation of the envelope is obtained:

$$\left(x_1 - \frac{a_2 \sin(\alpha_1 - \alpha_3)}{a_3 \sin\alpha_1} x_3\right)^2 - \frac{4a_2}{a_3} x_2x_3 = 0. \tag{5}$$

Thus the envelope of B_1B_2, as θ varies, is a conic. Solving its equation simultaneously with that of the line at infinity, $a_1x_1+a_2x_2+a_3x_3=0$, we learn that it is tangent to this line and is therefore a parabola. This parabola is tangent to A_1A_2 at A_2 and is also tangent to A_1A_3.

The parabolic envelopes of B_1B_3 and B_2B_3 are obtained by cyclic permutation of the subscripts in (5). It will now be shown that the three parabolas have a common focus.

The envelope of B_1B_2 is tangent to each side of triangle $A_1B_2B_1$. Therefore its focus is known to be on the circumcircle of this triangle for every value of θ. The trilinear coordinates of Ω_2 with respect to triangles $A_1A_2A_3$ and $B_1B_2B_3$ are $(a_2/a_3,\ a_3/a_1,\ a_1/a_2)$. With respect to triangle $A_1B_2B_1$, the second and third coordinates are still in the same ratio, since this triangle has two sides in common with triangles $A_1A_2A_3$. Now the distance of Ω_2 from B_1B_2 equals its distance from A_1A_2 multiplied by the ratio of similitude $\sin\omega/\sin(\theta-\omega)$. This distance is also negative because Ω_2 is exterior to triangle $A_1B_2B_1$. Therefore its coordinates are $(-a_1 \sin\omega/[a_2 \sin(\theta-\omega)],\ a_3/a_1,\ a_1/a_2)$. Since the angles of triangle $A_1B_2B_1$ are α_1, $\pi-\theta$, and $\theta-\alpha_1$, the equation of its circumcircle is $y_2y_3 \sin\alpha_1+y_3y_1 \sin\theta+y_1y_2 \sin(\theta-\alpha_1)=0$ [3, p. 357]. The coordinates of Ω_2 satisfy this equation identically in θ, and thus it is proved that Ω_2 is on the circumcircle of every triangle $A_1B_2B_1$. Therefore it must be the focus of the parabola enveloping the lines B_1B_2. Similarly it may be shown that Ω_2 is on the circumcircles of triangles $A_2B_3B_2$ and $A_3B_1B_3$. Thus it is the common focus of the three parabolic envelopes.

At this time it may be noted that the usual construction of a Brocard point involves three concurrent circles, each one passing through two vertices of the triangle and tangent to one side [1, p. 274]. For example, one such circle through Ω_2 passes through A_2 and is tangent to A_1A_3 at A_1. But the foregoing discussion has shown that this circle is the limiting case, as θ approaches zero, of the coaxal system through A_1 and Ω_2.

Finally, the relative sizes of the three parabolas may be determined. Triangle $B_1B_2B_3$ is smallest when $|\sin\omega/\sin(\theta-\omega)|$ is minimum, that is, when $\theta=\omega+\frac{1}{2}\pi$. This is the pedal triangle of Ω_2. For this value of θ each side of $B_1B_2B_3$ is nearest the focus of its parabola; in other words, each side is tangent at the vertex. It follows that the distance from focus to vertex in each parabola is

proportional to the corresponding trilinear coordinate of Ω_2.

Similar results may be obtained for Ω_1, the first Brocard point.

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FIBONACCI SERIES MODULO m

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This inquiry is concerned with determining the length of the period of the recurring series obtained by reducing a Fibonacci series by a modulus m. The problem arose in connection with a method for generating random numbers, but it turned out to be unexpectedly intricate, and so quickly became of interest in its own right. The function studied (length of period as a function of starting values and modulus) exhibits quite a few apparent properties which are established here. At least two questions remain unanswered: see remarks after Theorems 5 and 7.

Let f_n denote the nth member of the Fibonacci series $f_0=a$, $f_1=b$, $f_{n+1}=f_n+f_{n-1}$. We reduce f_n modulo m, taking least nonnegative residues, and let h denote the length of the period of the repeating series that results. The letter p is reserved to designate a prime, but a, b, and m may be arbitrary integers, except that we assume, without loss of generality, that a, b, and m are relatively prime: $(a, b, m)=1$. We will also refer to the two special Fibonacci series u_n and v_n defined by $u_0=0$, $u_1=1$, and $v_0=2$, $v_1=1$, and will make use of many of the known properties of these series. It will be convenient to let $k=k(m)$ denote the length of the period of u_n (mod m), in distinction from h which depends on a and b as well as m.

Example: The values of u_n (mod 7) are 0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, and then repeat; so $k(7)=16$. We are curious as to the relation between the number 16 and the number 7. Note that $u_8\equiv 0 \pmod 7$ so the 16 terms in the period form two sets of 8 terms each, the terms of the second half being 6, or -1, times the corresponding terms of the first half. Theorem 7 generalizes this property and explains the relation $k(7)=16$.

THEOREM 1. *f_n (mod m) forms a simply periodic series. That is, the series is periodic and repeats by returning to its starting values.*

Proof. The series repeats because there are only a finite number m^2 of pairs of terms possible, and the recurrence of a pair results in recurrence of all following terms. From the defining relation we have $f_{n-1}=f_{n+1}-f_n$, so if $f_{t+1}\equiv f_{s+1}$ and $f_t\equiv f_s$, (mod m), then $f_{t-1}\equiv f_{s-1}$, $\cdots$, $f_{t-s+1}\equiv f_1$, and $f_{t-s}\equiv f_0$, so that the series is simply periodic.

COROLLARY. $u_k\equiv 0 \pmod m$. (A direct consequence of Theorem 1.)

THEOREM 2. *If m has the prime factorization $m=\prod p_i^{e_i}$ and if h_i denotes the length of the period of $f_n \pmod{p_i^{e_i}}$, then $h=\operatorname{lcm}[h_i]$, the least common multiple of the h_i.*

Proof. The statement, "h_i is the length of the period of $f_n \pmod{p_i^{e_i}}$," implies that the series $f_n \pmod{p_i^{e_i}}$ repeats only after blocks of length ch_i; and the statement, "h is the length of the period of $f_n \pmod m$," implies that $f_n \pmod{p_i^{e_i}}$ repeats after h terms for all values of i. Therefore h is of the form ch_i for all values of i, and since any such number gives a period of $f_n \pmod m$, we conclude that $h=\operatorname{lcm}[h_i]$.

In view of Theorem 2, we may henceforth assume that m is of the form $m=p^e$.

The next five theorems establish properties of the special series u_n with $u_0=0$, $u_1=1$; k is the length of the period of this series, mod m. A close relationship between the special series u_n and the more general series f_n is contained in the formula $f_n=bu_n+au_{n-1}$, which shows that f_n repeats after k terms, so that h is a divisor of k: $h \mid k$.

THEOREM 3. *The terms for which $u_n\equiv 0 \pmod m$ have subscripts that form a simple arithmetic progression. That is, $n=xd$ for $x=0, 1, 2, \cdots$, and some positive integer $d=d(m)$, gives all n with $u_n\equiv 0 \pmod m$.*

Proof. From the known relations $(u_n, u_{n+1})=1$ and $u_{n+t}=u_{n+1}u_t+u_nu_{t-1}$, we see that $u_i\equiv 0 \pmod m$ and $u_j\equiv 0 \pmod m$ imply $u_{i+j}\equiv 0 \pmod m$ and (with $i\geqq j$) $u_{i-j}\equiv 0 \pmod m$. The first follows by setting $n=i$, $t=j$; and setting $n+t=i$, $n=j$ gives $u_{n+1}u_t\equiv 0 \pmod m$, which with $(u_n, u_{n+1})=1$ along with $u_n\equiv 0 \pmod m$ gives the second congruence $u_t=u_{i-j}\equiv 0 \pmod m$. Therefore, the subscripts n that we are concerned with comprise the nonnegative terms of a module, and so are of the form $n=xd$. The corollary to Theorem 1 shows that u_0 is not the only $u_n\equiv 0 \pmod m$, so $d>0$, and this completes the proof of the theorem.

Remark. We note that $d \mid k$. Empirically we find many m with $d=k$, but also many with $d<k$.

THEOREM 4. *If $m>2$ then k is an even number.*

Proof. Suppose that k is odd: $k=2x+1$. Then by working both ends to the middle with the defining relation, all congruences being mod m, we find:

$$-u_k \equiv 0 = u_0, \quad u_{k-1} \equiv 1 = u_1, \quad -u_{k-2} = -u_k + u_{k-1} \equiv u_0 + u_1 \equiv u_2,$$

.

$$(-1)^{t-1}u_{k-t} = (-1)^{t-1}u_{k-t+2} + (-1)^t u_{k-t+1} \equiv u_{t-2} + u_{t-1} = u_t,$$

.

$$(-1)^{x-2}u_{x+2} \equiv u_{x-1}, \qquad (-1)^{x-1}u_{x+1} \equiv u_x.$$

From this last congruence and the defining relation, it follows that $u_{x-1} \equiv 0 \pmod{m}$ if x is odd, and $u_{x+2} \equiv 0 \pmod{m}$ if x is even; but then the congruence just prior to the last one in the sequence shows that again $u_{x-1} \equiv 0 \pmod{m}$.

Now from Theorem 3, $d \mid (x-1)$ so $d \mid (2x-2)$, also $d \mid k$ so $d \mid (2x+1)$; therefore, $d \mid (2x+1-2x+2)$ so $d=3$. Finally $u_d = u_3 = 2 \equiv 0 \pmod{m}$ shows $m=2$ is implied by the hypothesis that k is odd. Therefore for $m>2$, k must be even.

THEOREM 5. *If $k(p^2) \neq k(p)$, then $k(p^e) = p^{e-1}k(p)$. Also, if t is the largest integer with $k(p^t) = k(p)$, then $k(p^e) = p^{e-t}k(p)$ for $e>t$.*

Proof. By solving the standard formulas $u_n = (r^n - s^n)/\sqrt{5}$ and $v_n = r^n + s^n$ for r^n and s^n in terms of u_n and v_n (where r and s satisfy $x^2 = x+1$), expanding $(r^n)^a$ and $(s^n)^a$ by the binomial theorem, and recombining, we obtain the relations

$$u_{an} = (r^{an} - s^{an})/\sqrt{5} = [2^{-a}(\sqrt{5}u_n + v_n)^a - 2^{-a}(-\sqrt{5}u_n + v_n)^a]/\sqrt{5}$$
$$= 2^{1-a} \sum_{j \text{ odd}} \binom{a}{j} 5^{(j-1)/2} u_n^j v_n^{a-j} = 2^{1-a} u_n (K u_n^2 + a v_n^{a-1}),$$

where K is an integer, and similarly

$$u_{an+1} = 5^{-1/2} 2^{-a} \sum_0^a \binom{a}{j} 5^{j/2} u_n^j v_n^{a-j} \left[\frac{1+\sqrt{5}}{2} - (-1)^j \frac{1-\sqrt{5}}{2}\right]$$
$$= 2^{-a}(K u_n^2 + a u_n v_n^{a-1} + v_n^a),$$

where K is an integer. The theorem follows from these relations by induction on e, except for the case $p=2$. We will outline the induction step, observing that the theorem is trivially true for $e=1$ ($e=t$ in the second statement), which enables the induction to begin.

On noting that $v_n = u_{n-1} + u_{n+1}$ has a g.c.d. with u_n of either 1 or 2, and applying Theorem 3, the first formula above gives the "law of repetition of primes": *if u_n is the first term $\equiv 0 \pmod{p^e}$ but $\not\equiv 0 \pmod{p^{e+1}}$, then u_{pn} is the first term $\equiv 0 \pmod{p^{e+1}}$, also $u_{pn} \not\equiv 0 \pmod{p^{e+2}}$.* For this value of n, the terms u_{nx} for $x = 0, 1, 2, \cdots$ are the terms $\equiv 0 \pmod{p^e}$, and u_{pnx} gives all terms $\equiv 0 \pmod{p^{e+1}}$. On setting $u_{pnx+1} \equiv 1 \pmod{p^{e+1}}$, to obtain $k(p^{e+1}) = pnx$, the second formula above gives

$$(v_{nx}/2)^p \equiv 1 \pmod{p^{e+1}},$$
$$v_{nx} \equiv 2 \pmod{p^e}, \qquad u_{nx+1} \equiv 1 \pmod{p^e},$$

so $k(p^e)=nx$ and $k(p^{e+1})=pk(p^e)$. For the exceptional case $p=2$ we use the formulas $u_{2n}=u_n(u_{n-1}+u_{n+1})$ and $u_{2n+1}=u_{n+1}^2+u_n^2$ to establish by induction that $k(2^e)=2^{e-1}k(2)$; the details are omitted here.

Remark. The most perplexing problem we have met in this study concerns the hypothesis $k(p^2)\neq k(p)$. We have run a test on a digital computer which shows that $k(p^2)\neq k(p)$ for all p up to 10,000; however, we cannot yet prove that $k(p^2)=k(p)$ is impossible. The question is closely related to another one, "can a number x have the same order mod p and mod p^2?", for which rare cases give an affirmative answer (*e.g.*, $x=3$, $p=11$; $x=2$, $p=1093$); hence, one might conjecture that equality may hold for some exceptional p.

THEOREM 6. *If* $m=p=10x\pm1$, *then* $k(p)\mid(p-1)$.

LEMMA. *The congruence* $x^2\equiv x+1 \pmod p$ *has a double root only for* $p=5$.

Proof of Lemma. $x^2-x-1\equiv(x-r)^2$ implies $2r\equiv1$ and $r^2\equiv-1$; these give $4r^2\equiv1$ and $4r^2\equiv-4$, so $5\equiv0$ which implies a modulus of 5.

Proof of Theorem 6. The number 5 is a quadratic residue for primes of the form $p=10x\pm1$, so the congruence $x^2\equiv x+1 \pmod p$, which is equivalent to $(2x-1)^2\equiv5 \pmod p$, has distinct roots r and s; therefore $u_n\equiv(r^n-s^n)/(r-s) \pmod p$.

Let g denote the lcm of the order of $r \pmod p$ and the order of $s \pmod p$. Since $rs\equiv-1 \pmod p$, we see that g is equal to the order of r if this number is even, and otherwise is equal to twice the order of r. In either case, $u_n \pmod p$ repeats after g terms, so $k(p)\mid g$, and Fermat's theorem shows $g\mid(p-1)$, so the theorem is proved.

THEOREM 7. *If* $m=p=10x\pm3$, *then* $k(p)\mid(2p+2)$.

Proof. The number 5 is a quadratic nonresidue for primes of this form, so $5^{(p-1)/2}\equiv-1 \pmod p$. We let $n=p$ and then let $n=p+1$ in the known formula

$$u_n = 2^{1-n}\left[\binom{n}{1}+5\binom{n}{3}+5^2\binom{n}{5}+\cdots\right].$$

The first substitution gives

$$u_p \equiv 5^{(p-1)/2}\binom{p}{p} \equiv -1 \pmod p,$$

and the second gives

$$u_{p+1} \equiv 2^{-1}\left[\binom{p+1}{1}-\binom{p+1}{p}\right] \equiv 0 \pmod p.$$

These congruences and the defining relation show that $u_{p+2}\equiv-u_1$, $u_{p+3}\equiv-u_2$, $\cdots$, $u_{2p+1}\equiv-u_p\equiv1$, $u_{2p+2}\equiv-u_{p+1}\equiv0 \pmod p$. Therefore, $u_n \pmod p$ repeats beginning with $u_{2p+2}\equiv u_0$, so that $k(p)\mid(2p+2)$.

COROLLARY. *If* $p=10x\pm 3$ *then* $k(p)\equiv 0 \pmod 4$.

Proof. Otherwise $k(p)$ would divide $(p+1)$ which would imply $u_p\equiv +1$.

Remarks. Theorems 6 and 7 furnish upper bounds for the function $k(p)$, and we easily find cases where $k(p)$ has these maximum values. On the other hand, we cannot find a nontrivial lower bound for $k(p)$, and a table of values of $k(p)$ shows many entries with $k(p)$ smaller than the upper limits of Theorems 6 and 7. Examples: $k(491)=490$, $k(521)=26$, $k(4993)=9988$, $k(9349)=38$.

We now return to the more general series f_n. One of the most interesting properties of h, the length of the period of $f_n \pmod m$, is that it is often independent of the starting values a and b. Theorems 8–12 describe this property. In such cases we will write $h=h(m)$, and relate h to $k(m)$, the length of the period of $u_n \pmod m$.

THEOREM 8. *If* $p=10x\pm 3$, *then* $h(p^e)=k(p^e)$.

Proof. The congruences which indicate that $f_n \pmod m$ repeats with period h may be written in the following form:

$$f_h - a = bu_h + a(u_{h-1} - 1) \equiv 0 \pmod m,$$
$$f_{h+1} - b = (b + a)u_h + b(u_{h-1} - 1) \equiv 0 \pmod m.$$

Considering a and b as given coefficients, the determinant of this system is $D=b^2-ab-a^2$. With $m=p^e$, if $D\equiv 0 \pmod p$ then $4a^2+4ab+b^2=(2a+b)^2 \equiv 5b^2 \pmod p$; and $b\not\equiv 0 \pmod p$ simultaneously with $D\equiv 0 \pmod p$ since this would imply $a\equiv 0 \pmod p$, contradicting $(a, b, m)=1$. Therefore $D\equiv 0 \pmod p$ implies that 5 is a quadratic residue of p. But 5 is not a quadratic residue of primes $p=10x\pm 3$, so for these p and $m=p^e$ we have $(D, m)=1$ and the only solution to the system of congruences is $u_h\equiv 0$, $u_{h-1}\equiv 1 \pmod{p^e}$, which shows that $k \mid h$. Since also $h \mid k$, we therefore have $h=k$, and the theorem is proved.

COROLLARY. *Whenever* $D=b^2-ab-a^2$ *satisfies* $(D, m)=1$, *then* $h=k$. *In particular*, v_n *has* $D=-5$ *so if* $(5, m)=1$, *then the length of the period of* $v_n \pmod m$ *is* $k(m)$.

THEOREM 9. *If* $m=2^e$, *then* $h=k$, *and if* $m=5^e$, *then either* $h=k$ *or else* $h=(1/5)k$, *according as* $D=b^2-ab-a^2$ *is not or is divisible by* 5.

Proof. As in Theorem 8, we examine the determinant D. By observing the three cases with $(a, b, 2)=1$, we see that $D\not\equiv 0 \pmod 2$, so $(D, 2^e)=1$ and $h=k$ for $m=2^e$. If a and b give $D\not\equiv 0 \pmod 5$, then $h=k$ for $m=5^e$. Although it is possible to have $D\equiv 0 \pmod 5$, the congruence $(2a+b)^2\equiv 5b^2 \pmod m$, which is equivalent to $D\equiv 0 \pmod m$, shows that $D\not\equiv 0 \pmod{5^2}$. In this case the congruences $f_h-a\equiv 0 \pmod{5^e}$ and $f_{h+1}-b\equiv 0 \pmod{5^e}$, from Theorem 8, give $u_h\equiv 0 \pmod{5^{e-1}}$ and $u_{h+1}\equiv 1 \pmod{5^{e-1}}$, so $k(5^{e-1}) \mid h(a, b, 5^e)$ and $h(a, b, 5^e)$ $=$either $k(5^e)$ or $(1/5)k(5^e)$. But the second value always holds, for $D\equiv 0 \pmod 5$ implies $b=-2a+5t$, and the formula

$$u_n = 2^{1-n}\left[\binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + \cdots\right]$$

shows that $u_{(1/5)k(5^e)} = u_{4\cdot 5^{e-1}} \equiv 3\cdot 5^{e-1} \pmod{5^e}$ and $u_{4\cdot 5^{e-1}+1} \equiv 1+4\cdot 5^{e-1} \pmod{5^e}$, since the terms $5\binom{n}{3}$, $5^2\binom{n}{5}$, $\cdots$, are all divisible by 5^e. Therefore $f_{(1/5)k(5^e)} \equiv (-2a + 5t)\cdot 3\cdot 5^{e-1} + a(1 + 5^{e-1}) \equiv a \pmod{5^e}$, and $f_{(1/5)k(5^e)+1} \equiv (-2a + 5t)\cdot(1+4\cdot 5^{e-1}) + a(3\cdot 5^{e-1}) \equiv b \pmod{5^e}$. These formulas require $e>1$, but for, $e=1$, an enumeration of cases shows $k=20$ and $h=20$ for $D \not\equiv 0 \pmod 5$ and $h=4$ when $D \equiv 0 \pmod 5$, so the theorem is proved.

COROLLARY. *If $m=p^e$ and h is odd, then $p=10x\pm 1$ or $m=2$.*

(A direct consequence of Theorems 4, 8, and 9.)

THEOREM 10. *If $m=p^e$, $p>2$, and if (a, b) give $h=2t+1$, then $k=4t+2$.*

Proof. The congruences which indicate that $f_n \pmod m$ repeats with period h may also be written as follows:

$$bu_h + a(u_{h-1} - 1) \equiv 0 \pmod m,$$
$$b(u_{h+1} - 1) + au_h \equiv 0 \pmod m.$$

The condition $(a, b, m)=1$ implies that $u_h^2-(u_{h+1}-1)(u_{h-1}-1)\equiv 0 \pmod m$, and the known formula $u_h^2-u_{h+1}u_{h-1}=(-1)^{h-1}$ permits this congruence to be simplified to $u_{h+1}+u_{h-1}=v_h=u_{2h}/u_h\equiv 1+(-1)^h \pmod m$. Therefore, h odd implies $u_{2h}\equiv 0 \pmod m$. Since $f_n \pmod m$ also repeats with period $2h$, the analogous congruences stating this fact lead to the analogous condition $u_{2h+1}+u_{2h-1} \equiv 1+(-1)^{2h}=2 \pmod m$, so $u_{2h}=0 \pmod m$ further implies $2u_{2h+1}\equiv 2$, $u_{2h+1} \equiv 1 \pmod m$, so $k \mid 2h$. Finally, k is even since $m>2$, so h odd requires $k=2h$, and the theorem is proved.

Examples:

$m = 11$, $a = 1$, $b = 4$ gives $h = 5$, while $k(11) = 10$;

$m = 29$, $a = 1$, $b = 24$ gives $h = 7$, while $k(29) = 14$;

$m = 121$, $a = 1$, $b = 37$ gives $h = 55$, while $k(121) = 110$.

THEOREM 11. (Converse to Theorem 10.) *If $m=p^e$, $p>2$, and if $k=4t+2$, then $h=2t+1$ for some (a, b).*

LEMMA. *If $k(p^e)=4t+2$, then $u_{2t+2}\equiv -u_{2t} \pmod{p^e}$.*

Proof of Lemma. This condition follows directly from the chain of congruences developed in Theorem 4, k now being even.

Proof of Theorem 11. Let

$$a = f_0 \equiv -u_{2t+1} - u_0 \pmod{p^e}, \qquad b = f_1 \equiv u_{2t} - u_1.$$

Then $f_n \equiv (-1)^{n-1}u_{2t+1-n} - u_n$, so $f_{2t+1} \equiv f_0$ and $f_{2t+2} \equiv -u_{-1} - u_{2t+2} \equiv -u_1 + u_{2t} = f_1$. Therefore $h \mid (2t+1)$; but then Theorem 10 shows $k = 2h$. This completes the proof except it must be verified that $(a, b, p^e) = 1$, which follows from Theorem 4 since $a \equiv -u_{2t+1} - u_0 \equiv b \equiv u_{2t} - u_1 \equiv 0 \pmod{p}$ would give $k(p) = 2t+1$, which is impossible. Incidentally, the case $p^e = 4$ is actually an exceptional case, since $k(4) = 6 \equiv 2 \pmod 4$, but no series $f_n \pmod 4$ has $h = 3$.

THEOREM 12. *If $m = p^e$, $p > 2$, $p \neq 5$, and h is even, then $h = k$.*

Proof. We use the condition $v_h \equiv 1 + (-1)^h \pmod m$, from Theorem 10, and the relation $v_n = r^n + s^n$ where r and s are the real roots of the equation $x^2 = x + 1$. Then, since h is even, and since $rs = -1$,

$$r^h + s^h - 2 \equiv 0 \pmod m,$$

$$r^{2h} + s^{2h} + 4 + 2(rs)^h - 4r^h - 4s^h \equiv 0 \pmod{m^2},$$

$$r^{2h} - 2 + s^{2h} \equiv (r^h - s^h)^2 \equiv 0 \pmod{m^2}.$$

Now $r^h - s^h$ is not an integer, but is of the form $x\sqrt{5}$; and since $p \neq 5$ assures $5 \not\equiv 0 \pmod m$, we may divide by $5 = (r-s)^2$ to obtain

$$[(r^h - s^h)/(r - s)]^2 = u_h^2 \equiv 0 \pmod{m^2}, \qquad u_h \equiv 0 \pmod m.$$

Finally $u_h \equiv 0$ and $v_h \equiv 2$ imply $u_{h-1} \equiv u_{h+1} \equiv 1$, which in turn implies $h = k$.

COROLLARY 1. *If $p \neq 5$ and $k(p) \equiv 0 \pmod 4$, then $h(p^e) = k(p^e)$.* (A direct consequence of Theorems 10 and 12, except for $p = 2$ which is covered by Theorem 9.)

COROLLARY 2. *If $h(a, b, p) = k(p)$ and $k(p^2) \neq k(p)$, then $h(a, b, p^e) = k(p^e)$.*

Proof. $h(a, b, p^e)$ must be a multiple of $h(a, b, p) = k(p)$, and must be a divisor of $k(p^e) = p^{e-1}k(p)$, hence must be of the form $p^{-c}k(p^e)$ for $c \geqq 0$. But Theorems 10 and 12 show that $h = k$ or $h = k/2$, so $h(a, b, p^e) = k(p^e)$. The cases $p = 2$ and $p = 5$ are covered by Theorem 9; for $p = 5$ note that $h = k$ or $h = (1/5)k$ independently of e.

Remark. The converse of Corollary 2 is false. For example, $h(3, 1, 11) < k(11)$, but $h(3, 1, 11^2) = k(11^2)$.

Example. Find h and k for $m = 10^{10}$.

As per Theorem 2, we consider $m_1 = 2^{10}$ and $m_2 = 5^{10}$. To apply Theorem 5, we check that $k(2) = 3 \neq k(2^2)$ and $k(5) = 20 \neq k(5^2)$. Therefore $k_1 = 3.2^9$, $k_2 = 20.5^9$, $k = \text{lcm}\,[k_1, k_2] = 15.10^9$. Finally, Theorem 9 shows that $h = 15.10^9$ or $h = 3 \cdot 10^9$ according as $D = b^2 - ab - a^2$ is not or is divisible by 5.

Remark. We note in passing that the number of ordered pairs $(a, b) \pmod m$ with $(a, b, m) = 1$ is given by the formula $m^2 \prod_{p \mid m} (1 - 1/p^2)$.

Since the results of this study direct interest to the function $k(p)$, we append a table of $k(p)$, listing all cases with $5<p<2000$ for which $k(p)$ is smaller than the maximum value permitted by Theorems 6 and 7.

p	$k(p)$	p	$k(p)$	p	$k(p)$
29	14	743	496	1279	426
47	32	761	380	1289	322
89	44	769	192	1291	430
101	50	797	228	1307	872
107	72	809	202	1361	680
113	76	811	270	1381	460
139	46	829	276	1409	704
151	50	859	78	1427	168
181	90	881	176	1471	490
199	22	911	70	1483	424
211	42	919	102	1511	302
229	114	941	470	1523	1016
233	52	953	212	1549	774
263	176	967	176	1553	1036
281	56	977	652	1579	526
307	88	991	198	1597	68
331	110	1009	126	1601	160
347	232	1021	510	1621	810
349	174	1031	206	1669	834
353	236	1049	262	1699	566
401	200	1061	530	1709	854
421	84	1069	356	1721	430
461	46	1087	128	1733	1156
509	254	1097	732	1741	870
521	26	1103	96	1789	894
541	90	1109	554	1823	1216
557	124	1151	230	1861	930
563	376	1217	812	1871	374
619	206	1223	816	1877	1252
661	220	1229	614	1913	1276
677	452	1231	410	1951	390
691	138	1249	624	1973	1316
709	118	1277	852	1999	666

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MATHEMATICAL INDUCTION IN COMMUTATIVE SEMIGROUPS†

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1. The principle of mathematical induction in commutative semigroups. One formulation of the First Principle of Mathematical Induction is that if a subset S of the positive integers N^* contains 1 and also contains $n+1$ whenever it contains n, then $S=N^*$. The set N^*, with addition, is an example of a semigroup, that is, a set furnished with a composition $+$ satisfying $(x+y)+z=x+(y+z)$ for all members x, y, z of the set. It is further a commutative semigroup, that is, one satisfying $x+y=y+x$ for all elements x and y. We shall say that a semigroup D with composition $+$ satisfies (PMI) if there exists $\beta\in D$ such that for any subset S of D, if $\beta\in S$ and if $x+\beta\in S$ whenever $x\in S$, then $S=D$. (We shall see that any such semigroup is necessarily commutative.) The natural numbers (nonnegative integers) N form a commutative semigroup which does not satisfy (PMI), the proper inductive statement for N being, of course, that if subset S of N contains 0 and contains $n+1$ whenever it contains n, then $S=N$. Analogously, we shall say that a commutative semigroup D with composition $+$ satisfies (PMI)′ if there exist members α and β of D such that for any subset S of D, if $\alpha\in S$ and if $x+\beta\in S$ whenever $x\in S$, then $S=D$. It is natural to ask which commutative semigroups, in addition to N^*, satisfy (PMI), and which ones, in addition to N^* and N, satisfy (PMI)′. We shall see that it is a simple matter to give a complete description of all such semigroups.

2. Dipper semigroups. For any natural number n and any positive integer m let H_n be the set of integers k satisfying $0\leqq k\leqq n$, H_n^* the set of integers k satisfying $1\leqq k\leqq n$ (H_0^* is thus the empty set), and let C_m be the cyclic group of the integers modulo m under addition. For any integer k, we let k_m be the integer modulo m determined by k. Let $D(n, m)$ be the set $H_n\cup C_m$. We define a composition $+_{n,m}$ on $D(n, m)$ as follows: If x, $y\in H_n$ and $x+y\leqq n$, then $x+_{n,m}y=x+y$, but if $x+y>n$, then $x+_{n,m}y=(x+y)_m\in C_m$; if $x\in H_n$ and $y\in C_m$, then $x+_{n,m}y=x_m+y\in C_m$, and if $x\in C_m$ and $y\in H_n$, then $x+_{n,m}y=x+y_m\in C_m$; finally, if x, $y\in C_m$, then $x+_{n,m}y=x+y$. It is easy to verify that $+_{n,m}$ is a commutative associative composition on $D(n, m)$. Let $D^*(n, m)=H_n^*\cup C_m$. $D^*(n, m)$ is clearly a subsemigroup of $D(n, m)$, obtained by deleting the identity element 0 from $D(n, m)$. Semigroups $D(n-1, m)$ and $D^*(n, m)$ both have $n+m$ elements but are not isomorphic, for the latter has only one element x satisfying $x+x=x$, namely 0_m, whereas the former has two, namely 0 and 0_m. We shall say a semigroup D is a *dipper semigroup* if it is isomorphic either with $D(n, m)$ or $D^*(n, m)$ for some $n\geqq 0$, $m>0$. Dipper semigroups have been discussed in [2] and [3]. The terminology is suggested by the fact that the Big Dipper is a natural model for both $D^*(3, 4)$ and $D(2, 4)$. We imagine the cup of the Big Dipper in an upright position and the handle pointing to the left. Starting from its end, we label the three stars of the handle 1, 2, 3, and proceed-

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ing clockwise around the cup, we identify its four stars with 4_4, 5_4, 6_4, 7_4, 4_4 being the star joining the cup to the handle. If we consider the sequence of elements 1, 1+1, 1+1+1, *etc.*, we first trace out the handle and then encircle the cup infinitely many times in a clockwise fashion. Similarly the Big Dipper is a model for $D(2, 4)$ where we identify the stars of its handle with 0, 1, 2, and the stars of its cup with 3_4, 4_4, 5_4, and 6_4. In an analogous fashion, we may picture a dipper constellation with n stars in the handle and m in the cup as the set of elements for both semigroups $D^*(n, m)$ and $D(n-1, m)$. We note that $D^*(0, m)$ (a dipper with no handle) is simply the cyclic group of integers modulo m, whereas $D^*(n, 1)$ [respectively, $D(n-1, 1)$] (a dipper with no cup to speak of) is isomorphic with the semigroup of integers k satisfying $1 \leqq k \leqq n+1$ [respectively, $0 \leqq k \leqq n$], where composition $\circ$ is defined by $x \circ y = \min\{x+y, n+1\}$ [respectively, $\min\{x+y, n\}$]. $D^*(n, m)$ clearly satisfies (PMI), for we may choose $\beta = 1$ if $n \geqq 1$ and $\beta = 1_m$ if $n = 0$. $D(n, m)$ does not satisfy (PMI) but does satisfy (PMI)′, for we may choose $\alpha = 0$ and $\beta = 1$ if $n \geqq 1$, $\beta = 1_m$ if $n = 0$.

THEOREM 1. *Let D be a semigroup satisfying (PMI). Then D is isomorphic either with the positive integers N^* or with a dipper semigroup $D^*(n, m)$. In particular, D is commutative.*

Proof. We define inductively $n \cdot \beta$ for each positive integer n by $1 \cdot \beta = \beta$, $(k+1) \cdot \beta = k \cdot \beta + \beta$. Clearly the function $n \to n \cdot \beta$ is a homomorphism from N^* into D, and it is onto D by (PMI). Consequently D is commutative. If the homomorphism is an isomorphism, then N^* and D are isomorphic. In the contrary case, there is a smallest natural number n such that $(n+1) \cdot \beta = s \cdot \beta$ for some $s > n+1$; let m be the smallest positive integer such that $(n+1) \cdot \beta = (n+m+1) \cdot \beta$. For any $k \geqq n+1$ and any natural number s it is easy to see by induction that $(k+sm) \cdot \beta = k \cdot \beta$; indeed, if $k = n+1+r$ and $(k+(s-1)m) \cdot \beta = k \cdot \beta$, then $(k + sm) \cdot \beta = (n + 1 + r + m + (s - 1)m) \cdot \beta = (n + 1 + m) \cdot \beta + (r + (s - 1)m) \cdot \beta = (n + 1) \cdot \beta + (r + (s - 1)m) \cdot \beta = (n + r + 1 + (s - 1)m) \cdot \beta = (k+(s-1)m) \cdot \beta = k \cdot \beta$. Further, if $n+1 \leqq r < s \leqq n+m$, then $r \cdot \beta \neq s \cdot \beta$, for if $r \cdot \beta = s \cdot \beta$, then $(n+r+m-s+1) \cdot \beta = r \cdot \beta + (n+m+1-s) \cdot \beta = s \cdot \beta + (n+m+1-s) \cdot \beta = (n+m+1) \cdot \beta = (n+1) \cdot \beta$ and $0 < r+m-s < m$, contradicting the definition of m. Let $f(k \cdot \beta) = k$ if $1 \leqq k \leqq n$, and $f(k \cdot \beta) = k_m$ if $n+1 \leqq k \leqq n+m$. From the preceding, one easily establishes that f is a well-defined isomorphism from D onto $D^*(n, m)$.

A problem equivalent to finding all semigroups satisfying (PMI) is that of finding all systems $(D, ')$ where D is a set, $'$ a function from D into D satisfying the property (Ind): *There exists $\beta \in D$ such that for any subset S of D, if $\beta \in S$ and if $x' \in S$ whenever $x \in S$, then $S = D$.* (The author is indebted to the referee for this observation.) Indeed, if D satisfies (PMI), the function $'$ defined by $x' = x + \beta$ satisfies (Ind) and also the property $x + y' = (x + y)'$ for all $x, y \in D$. On the other hand, if $'$ is a function satisfying (Ind), there is a unique composition $+$ on D such that (i) $x + \beta = x'$ for all $x \in D$, and (ii) $x + y' = (x + y)'$ for all $x, y \in D$, and further D with this composition is a semigroup satisfying (PMI);

this is proved in [1, pp. 4–6] by an argument (depending only on (Ind) and not on the other Peano postulates) ascribed to Kalmár. Consequently by Theorem 1, every system $(D, ')$ satisfying (Ind) is isomorphic to N^* or some $D^*(n, m)$ in the sense that either there exists a one-one function f from D onto N^* or $D^*(n, m)$ for some $n>0$, $m>0$ satisfying $f(x')=f(x)+1$ for all $x\in D$, or else there exists a one-one function f from D onto $D^*(0, m)=C_m$ for some $m>0$ satisfying $f(x')=f(x)+1_m$ for all $x\in D$. Alternatively, one might first prove that a system $(D, ')$ satisfying (Ind) is isomorphic with N^* or $D^*(n, m)$ for some $n\geqq 0$, $m>0$; from this follows Theorem 1 by the uniqueness of the composition $+$ satisfying (i) and (ii).

3. Almost cyclic semigroups. A particularly simple type of semigroup is one which contains in addition to a subgroup just one element. We first inquire into the possibilities that may occur (in the following theorem we violate the convention that a composition denoted by "$+$" is always commutative).

THEOREM 2. *Let D be a (not necessarily commutative) semigroup with composition $+$ which contains a group G and an element α not in G such that $D=G\cup\{\alpha\}$. Then one and only one of the following holds*: (1) $\alpha+x=x=x+\alpha$ *for all* $x\in D$; (2) $\alpha+x=\alpha=x+\alpha$ *for all* $x\in D$; (3) *there exists* $b\in G$ *such that* $\alpha+x=b+x$ *and* $x+\alpha=x+b$ *for all* $x\in G$ *and* $\alpha+\alpha=b+b$; (4) *G has only one element β, and either* (4a) $\alpha+\beta=\beta+\alpha=\alpha$, $\alpha+\alpha=\beta$, *i.e., D is isomorphic with the additive group of integers modulo* 2, *or* (4b) $\alpha+\alpha=\alpha+\beta=\alpha$, $\beta+\alpha=\beta$, *or* (4c) $\alpha+\alpha=\beta+\alpha=\alpha$, $\alpha+\beta=\beta$. *Conversely if G is a group and if $D=G\cup\{\alpha\}$ where α is not in G, then the composition $+$ of G may be extended to an associative composition on D by defining $\alpha+x$ and $x+\alpha$ to satisfy the identities of* (1), (2), (3), *or, if G has only one element β,* (4a), (4b), *or* (4c).

Proof. We first assert that either $\alpha+x=\alpha$ for all $x\in G$, or else $\alpha+x\in G$ for all $x\in G$. For suppose $\alpha+c\in G$ for some $c\in G$ but $\alpha+d=\alpha$ for some $d\in G$. Then $\alpha+0=\alpha+(c-c)=(\alpha+c)-c\in G$, but $(\alpha+0)+d=\alpha+(0+d)=\alpha+d=\alpha$, whence $\alpha+0$ cannot belong to G, a contradiction. Similarly, either $x+\alpha=\alpha$ for all $x\in G$, or else $x+\alpha\in G$ for all $x\in G$. We next assert that if G has more than one element, either $\alpha+x=\alpha=x+\alpha$ for all $x\in G$, or else $\alpha+x$, $x+\alpha\in G$ for all $x\in G$. Indeed, let $z\neq 0$ be in G. If $\alpha+x=\alpha$ for all $x\in G$ but $x+\alpha\in G$ for all $x\in G$, then $(z+\alpha)+z=z+(\alpha+z)=z+\alpha\in G$ whence $z=0$, a contradiction; similarly it is not possible that $\alpha+x\in G$ for all $x\in G$ but $x+\alpha=\alpha$ for all $x\in G$. Hence by the preceding, the assertion is established. We now consider four cases. (i) $\alpha+\alpha=\alpha$ and either $\alpha+0\in G$ or $0+\alpha\in G$. If $\alpha+0\in G$, then for any $x\in G$, $\alpha+x=y\in G$, whence $y=\alpha+x=(\alpha+\alpha)+x=\alpha+(\alpha+x)=\alpha+y=\alpha+(0+y)=\alpha+((x-x)+y)=(\alpha+x)+(-x+y)=y-x+y$, and therefore $0=-x+y$, *i.e.*, $x=y=\alpha+x$. Similarly if $0+\alpha\in G$, then $x+\alpha=x$ for all $x\in G$. Hence by our previous assertion, if G has more than one element or if D is commutative, (1) holds. In the contrary case, clearly either (4b) or (4c) holds. (ii) $\alpha+\alpha=\alpha$ and $\alpha+0=\alpha=0+\alpha$. Then by a preceding assertion, as $0\in G$, $\alpha+x=\alpha=x+\alpha$ for all $x\in D$, *i.e.*, (2) holds. (iii) $\alpha+\alpha\in G$ and either $\alpha+0\in G$ or $0+\alpha\in G$. If $\alpha+0\in G$,

let $b=\alpha+0$. Then for all $x\in G$, $\alpha+x=\alpha+(0+x)=(\alpha+0)+x=b+x$, and $\alpha+\alpha=(\alpha+\alpha)+0=\alpha+(\alpha+0)=\alpha+b=\alpha+(0+b)=(\alpha+0)+b=b+b$. Similarly, if $0+\alpha\in G$ and if $c=0+\alpha$, then $x+\alpha=x+c$ for all $x\in G$ and $\alpha+\alpha=c+c$. Suppose both $\alpha+0$, $0+\alpha\in G$. Then $b=b+(b+(-b))=b+(\alpha+(-b))=(b+\alpha)+(-b)=b+c+(-b)$, whence $0=c-b$, *i.e.*, $b=c$. Thus if G has more than one element or if D is commutative, (3) holds. But if G has only one element β, then $\alpha+\alpha=\beta$ so $\alpha+\beta=\alpha+(\alpha+\alpha)=(\alpha+\alpha)+\alpha=\beta+\alpha$, so D is indeed commutative. Hence (3) holds. (iv) $\alpha+\alpha\in G$ and $\alpha+0=\alpha=0+\alpha$. Let $c=\alpha+\alpha$. By the preceding, $\alpha+x=\alpha=x+\alpha$ for all $x\in G$. Hence for any $x\in G$, $c=\alpha+\alpha=\alpha+(\alpha+x)=(\alpha+\alpha)+x=c+x$, whence $x=0$. Thus G has only one element 0, and so $\alpha+\alpha=0$, $\alpha+0=\alpha=0+\alpha$, *i.e.*, (4a) holds. Clearly (1), (2), (3), (4a), (4b), (4c) are mutually exclusive. The verification of the remaining assertion of the theorem is easy.

For each positive integer m and each integer n satisfying $0\leqq n\leqq m-1$, let $G(n, m)$ be the semigroup $C_m\cup\{\alpha\}$ where C_m is the additive group of integers modulo m, α is not a member of C_m, and where $\alpha+x=x+\alpha=n_m+x$ for all $x\in C_m$, and $\alpha+\alpha=n_m+n_m$. By Theorem 2, $G(n, m)$ is a commutative semigroup. $G(n, m)$ satisfies (PMI)′, for we may choose α to be the element adjoined to C_m, β to be 1_m. We shall call a semigroup D *almost cyclic* if it is isomorphic to some $G(n, m)$. The function taking α onto 1 and each member of C_m onto itself is an isomorphism of $G(1, m)$ with $D^*(1, m)$. It is easy to see that $G(0, 1)$, which is isomorphic with $D^*(1, 1)$, is the only other almost cyclic semigroup which is a dipper.

THEOREM 3. *Let D be a commutative semigroup satisfying (PMI)′. Then either D is isomorphic with the positive integers N^* or with the natural numbers N, or is a dipper semigroup, or is an almost cyclic semigroup.*

Proof. Let α and β be such that the only subset of D which contains α and contains $x+\beta$ whenever it contains x is D itself. Let E be the subset of all elements of the form $n\cdot\beta$ for positive integers n. We assert that $D=E\cup\{\alpha\}$; by (PMI)′ it suffices to show $\alpha+\beta\in E$. At any rate, by (PMI)′ D is the set of all elements of the form $\alpha+n\cdot\beta$ for $n\geqq 0$. In particular, there exist nonnegative integers k and n such that $\alpha+\alpha=\alpha+k\cdot\beta$ and $\beta=\alpha+n\cdot\beta$. Hence $\alpha+\beta=\alpha+(\alpha+n\cdot\beta)=(\alpha+\alpha)+n\cdot\beta=(\alpha+k\cdot\beta)+n\cdot\beta=(\alpha+n\cdot\beta)+k\cdot\beta=\beta+k\cdot\beta=(k+1)\cdot\beta\in E$. Hence $D=E\cup\{\alpha\}$. Clearly E is a semigroup satisfying (PMI) and so by Theorem 1 is isomorphic either with N^* or with $D^*(n, m)$ for some $n\geqq 0$, $m>0$. Hence we may assume α is not a member of E, for otherwise the proof is complete. If $\alpha+\beta=\alpha$, then $\{\alpha\}$ contains α and contains $x+\beta$ whenever it contains x, so $D=\{\alpha\}$ and thus $\alpha=\beta$, contradicting our assumption α is not in E. Hence $\alpha+\beta\neq\alpha$. Let p be the smallest positive integer such that $\alpha+\beta=p\cdot\beta$. We consider two cases. (1) $p=1$ and $\alpha+\alpha=\alpha$. Then by induction $\alpha+k\cdot\beta=k\cdot\beta$ for all $k\geqq 1$. Therefore D is isomorphic with N if E is isomorphic with N^*, and D is isomorphic with $D(n, m)$ if E is isomorphic with $D^*(n, m)$. (2) Either $\alpha+\alpha=r\cdot\beta$ for some $r\geqq 1$ and $p=1$, or else $p>1$. In the first case,

$(r+1)\cdot\beta=r\cdot\beta+\beta=(\alpha+\alpha)+\beta=\alpha+(\alpha+\beta)=\alpha+\beta=\beta$. In the second case, by (PMI)′ D consists of the elements α and $k\cdot\beta$ for $k\geqq p$, so in particular there exists $k\geqq p$ such that $\beta=k\cdot\beta$. Thus in either case there exists $k>1$ such that $\beta=k\cdot\beta$. Let m be the smallest positive integer such that $\beta=(m+1)\cdot\beta$. Then as in Theorem 1, we may conclude that the function f defined by $f(k\cdot\beta)=k_m$, $1\leqq k\leqq m$, is a well-defined isomorphism from E onto the additive group of integers modulo m. By Theorem 2, one of the following possibilities occurs: (1) $\alpha+x=x$ for all $x\in D$; then D is isomorphic with $D(0, m)$. (2) $\alpha+x=\alpha$ for all $x\in D$; but this contradicts $\alpha+\beta\neq\alpha$. (3) There exists $b\in E$ such that $\alpha+x=b+x$ for all $x\in E$ and $\alpha+\alpha=b+b$. Then if the isomorphism f of E onto C_m carries b into n_m, D is clearly isomorphic with $G(n, m)$. (4) D is isomorphic with the integers modulo 2 under addition, the dipper semigroup $D^*(1, 1)$. This completes the proof.

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A TOPOLOGY FOR SEQUENCES OF INTEGERS II

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1. Introduction. Let S denote the set of all increasing sequences of positive integers which have asymptotic density and define $\rho(A, B)=0$ if $A=B$ and $\rho(A, B)=k^{-1}+|\delta(A)-\delta(B)|$ otherwise, where $\delta(A)$ denotes the asymptotic density of A and where A and B differ in the kth term but do not differ in any previous term. Then S is a totally disconnected nowhere locally compact separable metric space [1]. This metric for S seems to be a very natural one, although most of the simple topological properties of S do not depend on the fact that $\delta(A)$ represents the asymptotic density of A. The most characteristic properties of $\delta(A)$ in this respect appear to be that $\delta(A)=\delta(B)$ if A and B differ in only a finite number of terms and that the set of values of $\delta(A)$ is an interval.

It would of course be of some interest to determine properties which completely characterize $\delta(A)$, *i.e.*, to provide an axiomatic approach to density. We believe that a certain multiplicative property due to Niven [1, 2, 3] may be of considerable importance in this connection. In the present note we continue the study of the properties of S begun in Part I.

2. Decomposition of S. It has been shown previously [1] that if $S=E_1\cup E_2$ then $\rho(E_1, E_2)=0$ or n^{-1} where n is a positive integer, and that the second possibility actually occurs for every n. We shall derive a necessary condition for

$\rho(E_1, E_2)>0$. The following notation will be used for this purpose. If $E\subset S$, let E^* denote the set of all real numbers α such that $\alpha=\delta(A)$ for some $A\in E$. Also, let I denote the open interval (0, 1).

THEOREM 1. *If $E\subset S$, then either $\rho(E, S-E)=0$ or $E^*=\bar{I}$.*

Proof. Assume that $\rho(E, S-E)>0$. Then $S-E$ is closed. If E^* contains a limit point of $\bar{I}-E^*$, then E contains a limit point of $S-E$. Hence $\bar{I}-E^*$ is closed and $E^*\cap I$ is open. By the component interval theorem [4] $E^*\cap I$ is a countable union of mutually disjoint open intervals. If $E^*\cap I\neq I$, then there exists a $t\in I$ which is an end point of one of these intervals. Now let $T=\{A\in S\mid \delta(A)=t\}$. Then there exists $\{A_n\}\subset E$ and $\{B_n\}\subset T\subset(S-E)$ such that $\rho(A_n, B_n)\to 0$. But this implies that $\rho(E, S-E)=0$ and so $E^*\cap I=I$. Thus $I\subset E^*$ and the same argument applied to $t=0$ and $t=1$ shows that $E^*=\bar{I}$.

COROLLARY. *If $S=E_1\cup E_2$ and if $\rho(E_1, E_2)>0$, then $E_1^*=E_2^*=\bar{I}$.*

3. Category of S. We recall that a set is of the first category (in itself) if it is a countable union of nowhere-dense sets.

THEOREM 2. *S is a set of the first category.*

Proof. Let $A=\{a_n\}$ and let R denote the set of all real numbers of the form $a=\sum_{\nu=1}^{\infty}2^{-a_\nu}$ with $A\in S$. Then $\pi: A\to a$ is a continuous one-one mapping of S on R. It follows from results due to Hill [5] that R is a set of the first category in (0, 1]. Thus $S=\bigcup_{n=1}^{\infty}E_n$ where $\pi(E_n)$ is nowhere-dense in (0, 1]. Suppose that $\overline{E}_n$ contains a sphere $S(A, r)=\{B\in S\mid \rho(A, B)<r\}$. Now $S(A, r)=\bigcup_{k>r^{-1}}\{S_k(A)\cap S_k(A, r)\}$ where $S_k(A)$ is the set of all $B\in S$ which agree with A in the first $k-1$ or more terms and $S_k(A, r)=\{B\in S\mid \delta(A)-\delta(B)\mid <r-k^{-1}\}$. For convenience we may take $r=m^{-1}$ for some sufficiently large positive integer m. Then it is clear that $\pi\{S(A, r)\}$ is everywhere-dense in the interval $(x, x+2^{-a_{m+1}})$, where $x=\sum_{\nu=1}^{m+1}2^{-a_\nu}$. Since $\pi(\overline{E}_n)\subset\overline{\pi(E_n)}=\overline{\overline{\pi(E_n)}}$, it follows that $\overline{\pi(E_n)}$ contains an interval. But this is impossible and so E_n is nowhere-dense in S and S is a set of the first category.

4. A related space. Let T denote the set of all increasing sequences (finite or infinite) of positive integers with the metric $\rho(A, B)=0$ if $A=B$ and $\rho(A, B)=k^{-1}$ otherwise, where k has the same meaning as before. This type of metric has been discussed by Ostmann [6].

THEOREM 3. *T is homeomorphic to the set of positive elements of the Cantor set.*

Proof. Let $A=\{a_n\}$ and $x=2\sum_{a_\nu\in A}3^{-a_\nu}$ and consider the mapping $\tau: A\to x$. Since none of the digits in the ternary expansion of x are one's, the mapping is one-one. Also, several simple calculations show that $3^{-a_{k_n}}\leqq|x_n-x|\leqq 3\cdot 3^{-a_{k_n}}$ if $x_n\neq x$ and where $\rho(A_n, A)=k_n^{-1}$. Thus $x_n\to x$ if and only if $k_n\to\infty$, or if all but a finite number of the A_n's are equal to A, *i.e.*, if $A_n\to A$, and τ is a homeomorphism.

Let T^* denote the space of all nonempty subsets of the set of all positive integers where the limit of a sequence of sets $\{E_n\}$ is defined by $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} (\bigcap_{k=n}^{\infty} E_k)$. It is known [7, p. 145] that T^* is homeomorphic to the set of positive elements of the Cantor set under the mapping τ. We may therefore conclude that T and T^* are homeomorphic under the identity mapping.

It can be shown by means of results such as Theorem 3 that S is homeomorphic to a certain subset of the plane. Let C denote the set of elements $x = \sum_{n=1}^{\infty} \epsilon_n(x)3^{-n}$ of the Cantor set for which $\phi(x) = \lim_{n\to\infty} (1/2n) \sum_{\nu=1}^{n} \epsilon_\nu(x)$ exists and where $\epsilon_n(x) = 0$ or 2 with the 2 occurring an infinite number of times. Then $\phi(x)$ is a real-valued function defined on C and, since the ternary representation of any $x \in C$ is unique, $\phi(x)$ is also single-valued. The following result is now obvious.

THEOREM 4. *S is homeomorphic to the graph of the function $\phi(x)$.*

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SOME REMARKS ON ARITHMETICAL IDENTITIES*

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1. Introduction. A single-valued function $f(n, r)$, where n is a nonnegative integer and r is a positive integer, having values in the field of complex numbers, will be said to belong to the class E_k if for all n and r, $f(n, r) = f((n, r^k)_k, r)$; here $(a, b)_k$ is the largest integral kth power which divides both a and b. This class of functions was studied in [9] (it was defined in a slightly different manner there) and it was shown there that if $f(n, r)$ belongs to the class E_k then

$$f(n, r) = \sum_{d|r} \alpha(d, r)c_k(n, d),$$

where $c_k(n, r)$ is the extension of Ramanujan's sum $c(n, r)$ defined by Cohen in [2], and the "Fourier coefficients" $\alpha(d, r)$ are uniquely determined by $f(n, r)$.

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This extended a result of Cohen, who had previously studied the class E_1 [4]. Using his results for the class E_1 Cohen developed in [6] and [7] a systematic procedure for the study of arithmetical identities involving the classical and some new arithmetic functions. In a previous paper [9] we have shown that Cohen's methods, when applied to the class E_k for arbitrary positive integral k, yield much more general identities, including many of the classical identities in their full generality. This paper is a continuation of [9] in which we shall apply the methods used by Cohen in [7] to the class E_k.

In Sections 2, 3, 4 and 5 we make use of an orthogonality property of the functions $c_k(n, r)$ under the Cauchy product to study the Cauchy products of functions of the class E_k. This leads to generalizations of all the identities of [7], but we shall give only a few examples to illustrate how these generalizations may be obtained. In particular, in Section 4, we obtain a generalization of an identity of Brauer and Rademacher [1]. In Section 6 we study the number of solutions of a certain congruence, and in Section 7 we discuss generalizations of certain Busche-Ramanujan identities.

Two functions which we shall have occasion to use often are defined as follows: for an arbitrary complex number s,

$$\phi_s(r) = \sum_{de=r} d^s\mu(e), \qquad \sigma_s(r) = \sum_{d|r} d^s.$$

When s is a positive integer, $\phi_s(r) = J_s(r)$, Jordan's extension of Euler's totient function. We shall also make use of the fact that [2, Theorem 2]

$$c_k(n, r) = \sum_{d^k|(n,r^k)_k} d^k\mu(r/d).$$

Then $c_k(0, r) = \phi_k(r)$. For a discussion of the function $\phi_k(r)$ (k a positive integer) the reader is referred to [5].

2. The Cauchy product. Let $f(n, r)$ and $g(n, r)$ belong to the class E_k. Then [9, Theorem 1]

$$f(n, r) = \sum_{d|r} \alpha(d, r)c_k(n, d), \tag{1}$$

$$g(n, r) = \sum_{d|r} \beta(d, r)c_k(n, d), \tag{2}$$

where $\alpha(d, r)$ and $\beta(d, r)$ are uniquely determined by $f(n, r)$ and $g(n, r)$, respectively. The function $c_k(n, r)$ has the following orthogonality property [3, Theorem 1]: if $d|r$ and $e|r$ then

$$\sum_{n\equiv a+b \pmod{r^k}} c_k(a, d)c_k(b, e) = \begin{cases} r^k c_k(n, d) & \text{if } d = e, \\ 0 & \text{if } d \neq e. \end{cases}$$

Using this it is easily seen that

THEOREM 1. $\sum_{n\equiv a+b \pmod{r^k}} f(a, r)g(b, r) = r^k \sum_{d|r} \alpha(d, r)\beta(d, r)c_k(n, d)$.

Let $g(r)$ be a function of a single positive integral variable, and set

$$G_s(r) = \sum_{d|r} d^s g(r/d), \qquad \Gamma_k^{(s)}(n, r) = \sum_{d^k|(n,r^k)_k} d^s g(r/d),$$

where s is a complex number. We have shown in [9] that

$$\Gamma_k^{(ks)}(n, r) = \sum_{de=r} d^{k(s-1)} G_{k(s-1)}(e) c_k(n, d).$$

Starting with a second function $h(r)$ we define $H_s(r)$ and $\Delta_k^{(s)}(n, r)$ in the same way as we defined $G_s(r)$ and $\Gamma_k^{(s)}(n, r)$, respectively. If we apply Theorem 1 to $\Gamma_k^{(ks)}(n, r)$ and $\Delta_k^{(kt)}(n, r)$ we obtain

$$(3) \qquad \sum_{n\equiv a+b \pmod{r^k}} \Gamma_k^{(ks)}(a, r)\Delta_k^{(kt)}(b, r) = r^k \sum_{de=r} d^{k(s+t-2)} G_{k(s-1)}(e) H_{k(t-1)}(e) c_k(n, d).$$

Example 1. Let $g(r) = 1$ for all r. Then $G_s(r) = \sigma_s(r)$ and $\Gamma_k^{(s)}(n, r) = \sigma_s((n, r^k)_k^{1/k})$, which we denote by $\sigma_{s,k}(n, r)$. If we also let $h(r) = 1$ for all r we have from (3), replacing ks by s and kt by t,

$$\sum_{n\equiv a+b \pmod{r^k}} \sigma_{s,k}(a, r)\sigma_{t,k}(b, r) = r^k \sum_{de=r} d^{s+t-2k} \sigma_{s-k}(e)\sigma_{t-k}(e) c_k(n, d)$$

(see [7, Corollary 1]).

Example 2. Let $Q_k(n, r, s)$ be the number of solutions $x_i \pmod r$, $y_i \pmod{r^k}$, $i = 1, 2, \cdots, s$, of the congruence

$$n \equiv a_1 x_1^k y_1 + \cdots + a_s x_s^k y_s \pmod{r^k},$$

where $(a_i, r) = 1$ for all i. Cohen has shown [3, Theorem 5] that

$$Q_k(n, r, s) = r^{k(s-1)+s} \sum_{d|r} \frac{c_k(n, d)}{d^s}.$$

If we set $g(r) = r^{k(s-1)}\phi_s(r)$ then [9, Example 3] $G_{k(s-1)}(r) = r^{k(s-1)+s}$ and $\Gamma_k^{(ks)}(n, r) = Q_k(n, r, s)$. Hence from (3), with kt replaced by t,

$$\sum_{n\equiv a+b \pmod{r^k}} Q_k(a, r, s)\sigma_{t,k}(b, r) = r^{k(s-1)} \sum_{de=r} d^{t-s-k} \sigma_{t-k}(e) c_k(n, d).$$

Example 3. Let $\phi_k^{(s)}(n, r)$ be the number of solutions of the linear congruence

$$n \equiv x_1 + \cdots + x_s \pmod{r^k},$$

where $(x_i, r^k)_k = 1$ for $i = 1, 2, \cdots, s$. Then [9, Example 6],

$$\phi_k^{(s)}(n, r) = \frac{\phi_k^s(r)}{r^k} \sum_{d|r} \frac{\mu^s(d)}{\phi_k^s(d)} c_k(n, d).$$

We then have

$$(4)\qquad \sum_{n\equiv a+b\,(\mathrm{mod}\ r^k)} \sigma_{s,k}(a, r)\phi_k^{(t)}(b, r) = \phi_k^t(r) \sum_{de=r} \frac{d^{s-k}\sigma_{s-k}(e)\mu^t(d)}{\phi_k^t(d)}\, c_k(n, d)$$

(see [7, Corollary 4]). Let $\tau_k(r)$ be the number of divisors of r whose kth power also divides r, and set $\tau_k(n, r)=\tau_k((n, r^k)_k)$. Then $\tau_k(n, r)=\sigma_{0,k}(n, r)$. If we set $s=0$ and $t=2$ in (4) we obtain

$$\sum_{n\equiv a+b\,(\mathrm{mod}\ r^k)} \tau_k(a, r)\theta_k(b, r) = \frac{\phi_k^2(r)}{r^k} \sum_{de=r} \frac{\sigma_k(e)\,|\mu(d)|}{\phi_k^2(d)}\, c_k(n, d),$$

where $\theta_k(n, r)$ generalizes Nagell's totient function [9, Example 6]. As an identity connecting the number of solutions of the linear and semilinear congruences we have

$$\sum_{n\equiv a+b\,(\mathrm{mod}\ r^k)} Q_k(a, r, s)\phi_k^{(t)}(b, r) = r^{k(s-1)+s} \sum_{d|r} \frac{\mu^t(d)}{d^s\phi_k^t(d)}\, c_k(n, d).$$

3. The case $n=0$. If $f(n, r)$ and $g(n, r)$ are in the class E_k and are given by (1) and (2), respectively, then

THEOREM 2. $\sum_{de=r} f(e^k, r)g(e^k, r)\phi_k(d) = r^k \sum_{d|r} \alpha(d, r)\beta(d, r)c_k(n, d)$.

The proof is similar to that of [7, Theorem 5]. It follows immediately from this theorem that

$$\sum_{de=r} \Gamma_k^{(s)}(e^k, r)\Delta_k^{(t)}(e^k, r)\phi_k(d) = r^k \sum_{de=r} d^{s+t-2k} G_{s-k}(e)H_{t-k}(e)\phi_k(d).$$

Example 4. If we set $g(r)=1$ for all r then $G_s(r)=\sigma_s(r)$ and if $e|r$, $\Gamma_k^{(s)}(e^k, r)=\sigma_s(e)$. Hence

$$\sum_{de=r} \sigma_s(e)\sigma_t(e)\phi_k(d) = r^k \sum_{de=r} d^{s+t-2k}\sigma_{s-k}(e)\sigma_{t-k}(e)\phi_k(d)$$

(see [7, Corollary 29]). When $s=t=k$ this yields

$$\sum_{de=r} \sigma_k^2(e)\phi_k(d) = r^k \sum_{de=r} \tau^2(e)\phi_k(d).$$

Example 5. When $g(r)=\mu(r)$ for all r, $G_s(r)=\phi_s(r)$ and if $e|r$, $\Gamma_k^{(s)}(e^k, r)=\phi_s(r)\mu(r/e)/\phi_s(r/e)$ [9, Example 5]. Hence

$$\phi_s(r)\phi_t(r) \sum_{d|r} \frac{|\mu(d)|\,\phi_k(d)}{\phi_s(d)\phi_t(d)} = r^k \sum_{de=r} d^{s+t-2k}\phi_{s-k}(e)\phi_{t-k}(e)\phi_k(d)$$

(compare this, when $s=t$, with [7, Corollary 26]). If we combine the functions used in this example and in the last we obtain

$$\phi_s(r)\sum_{de=r}\frac{\mu(d)\phi_k(d)\sigma_t(e)}{\phi_s(d)} = r^k\sum_{de=r} d^{s+t-2k}\phi_{s-k}(e)\sigma_{t-k}(e)\phi_k(d).$$

4. Generalization of the Brauer-Rademacher identity. In [7, Section 5] Cohen gave a proof of an interesting identity due to Brauer and Rademacher [1]. If we apply the method of his proof to functions of the class E_k we can generalize this identity. We have

$$\begin{aligned}\sum_{(b,r^k)_k=1} c_k(n-b,r) &= \sum_{(b,r^k)_k=1}\ \sum_{d^k\mid(n-b,r^k)_k} d^k\mu(r/d)\\ &= \sum_{(b,r^k)_k=1}\ \sum_{\substack{d\mid r\\ n\equiv b\,(\mathrm{mod}\ d^k)}} d^k\mu(r/d)\\ &= \sum_{d\mid r} d^k\mu(r/d)\sum_{(b,r^k)_k=1} 1\\ &= \phi_k(r)\sum_{\substack{d\mid r\\ (n,d^k)_k=1}}\frac{d^k}{\phi_k(d)}\mu(r/d),\end{aligned}$$

where we have used the fact that if $(n, d^k)_k=1$, where $d\mid r$, then there are exactly $\phi_k(r)/\phi_k(d)$ k-reduced residues b (mod r^k) congruent to n (mod r^k).

Now choose a function $g(r)$, define $G_s(r)$ and $\Gamma_k^{(s)}(n, r)$ as in Section 2, and set $G_s^*(n, r)=m^sG_s(r/m)$ where $m^k=r^k/(n, r^k)_k$. Then we have shown [9, Section 4] that

$$G_{ks}^*(n, r) = r^{-k}\sum_{de=r}\Gamma_k^{(k(s+1))}(e^k, r)c_k(n, d).$$

Hence, if $f(n, r)$ is any function of the class E_k and is given by (1), we have by Theorem 1,

$$\sum_{n\equiv a+b\,(\mathrm{mod}\ r^k)} f(a, r)G_{ks}^*(b, r) = \sum_{de=r}\alpha(d, r)\Gamma_k^{(k(s+1))}(e^k, r)c_k(n, d). \tag{5}$$

We are interested in the case when $s=0$ and $g(r)=\mu(r)$ for all r. Then $G_0(r)=1$ or 0 according as $r=1$ or $r\neq 1$ and so $G_0^*(n, r)=1$ or 0 according as $(n, r^k)_k=1$ or $\neq 1$. Furthermore, $\Gamma_k^{(k)}(e^k, r)=\phi_k(r)\mu(r/e)/\phi_k(r/e)$ if $e\mid r$. Hence (5) becomes

$$\sum_{(b,r^k)_k=1} f(n-b, r) = \phi_k(r)\sum_{d\mid r}\frac{\alpha(d, r)\mu(d)}{\phi_k(d)}c_k(n, d) \tag{6}$$

(see [7, (5.4)]). In particular, when $f(n, r)=c_k(n, r)$ we have

$$\phi_k(r)\sum_{\substack{d\mid r\\ (n,d^k)_k=1}}\frac{d^k}{\phi_k(d)}\mu(r/d) = \mu(r)c_k(n, r)$$

(see [7, Corollary 34]), and if we use Cohen's evaluation of $c_k(n, r)$ we obtain the desired generalization of the Brauer-Rademacher identity:

$$\phi_k(r) \sum_{\substack{d|r \\ (n,d^k)_k=1}} \frac{d^k}{\phi_k(d)} \mu(r/d) = \mu(r) \sum_{d^k|(n,r^k)_k} d^k \mu(r/d).$$

5. Additional identities. If $f(n, r)$ belongs to the class E_k and has Fourier coefficients $\alpha(d, r)$, we can evaluate $\phi_k^{-1}(r) \sum_{(b,r^k)_k=1} f(n-b, r)$ in two ways (one of which is given by (6)) to obtain the following extension of [7, Theorem 6]:

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^k}{\phi_k(d)} \left(\sum_{e|(r/d)} \alpha(ed, r)\mu(e) \right) = \sum_{d|r} \frac{\alpha(d, r)\mu(d)}{\phi_k(d)} c_k(n, d).$$

Then, if we choose a function $h(r)$ of a single positive integral variable r, set $g(r) = \sum_{d|r} h(d)$, and consider the function $f(n, r) = \sum_{d|r} g(r/d)c_k(n, d)$, we obtain

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^k}{\phi_k(d)} h(r/d) = \sum_{de=r} \frac{g(e)\mu(d)}{\phi_k(d)} c_k(n, d)$$

(see [7, Theorem 7]).

Example 6. Let $h(r) = r^{k-s}$. Then $g(r) = \sigma_{s-k}(r)/r^{s-k}$ and so we have

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^s}{\phi_k(d)} = \sum_{de=r} \frac{d^{s-k}\sigma_{s-k}(e)\mu(d)}{\phi_k(d)} c_k(n, d)$$

(see [7, Corollary 36]).

Example 7. If we let $h(r) = \mu(r)r^{k-s}$ then $g(r) = r^{k-s}\phi_{s-k}(r)$, and so

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^s}{\phi_k(d)} \mu(r/d) = \sum_{de=r} \frac{d^{s-k}\phi_{s-k}(e)\mu(d)}{\phi_k(d)} c_k(n, d)$$

(see [7, Corollary 37]).

Example 8. Let $h(r) = \phi_s(r)$. Then $g(r) = r^s$ and we have

$$\sum_{\substack{d|r \\ (d,n^k)_k=1}} \frac{d^k}{\phi_k(d)} \phi_s(r/d) = r^s \sum_{d|r} \frac{\mu(d)}{d^s\phi_k(d)} c_k(n, d).$$

When $n = 1$ this yields

$$\sum_{de=r} d^k \frac{\phi_s(e)}{\phi_k(d)} = r^s \sum_{d|r} \frac{|\mu(d)|}{d^s\phi_k(d)}.$$

6. A congruence. In this section we obtain a formula for the number of solutions $N_k^{(s)}(n, r)$ of the congruence $n \equiv x_1 + \cdots + x_s \pmod{r^k}$, where $(x_i, r^k)_k$

is a $2k$th power for $i=1, 2, \cdots, s$. Our discussion follows that of [7, Section 3], where Cohen considered the case $k=1$. Let $\psi_k(r)$ be the number of integers $a \pmod{r^k}$ such that $(a, r^k)_k$ is a $2k$th power. Also, let $\nu(r)=1$ or 0 according as r is or is not a square, and $\nu_k(n, r)=\nu((n, r^k)_k^{1/k})$. Finally, let $\lambda(r)=(-1)^{\Omega(r)}$, where $\Omega(r)$ is the number of primes dividing r with repetitions counted.

If we set

$$\beta_s(r) = \sum_{d|r} d^s\lambda(r/d), \qquad L_k^{(s)}(n, r) = \sum_{d^k|(n,r^k)_k} d^s\lambda(r/d),$$

then by the results described in Section 2,

$$L_k^{(s)}(n, r) = \sum_{de=r} d^{s-k}\beta_{s-k}(e)c_k(n, d)$$

(see [6, Corollary 3]), which yields when $s=k$,

$$\sum_{d^k|(n,r^k)_k} d^k\lambda(r/d) = \sum_{de=r} \nu(e)c_k(n, d) = \sum_{de^2=r} c_k(n, d)$$

(see [6, Corollary 4]). When $n=0$ we obtain

$$\beta_k(r) = \sum_{de^2=r} \phi_k(d).$$

This implies that $\psi_k(r)=\beta_k(r)$ (see [6, Corollary 4.2]). Now applying the identity involving $G_s^*(n, r)$ in Section 4 to the case under consideration here, we obtain

$$\frac{\beta_s((n, r^k)_k^{1/k})}{(n, r^k)_k^{s/k}} = r^{-(s+k)} \sum_{de=r} \lambda(d)\beta_{s+k}(e)c_k(n, d)$$

(see [6, Corollary 10]) which yields, when we set $s=0$,

$$\nu_k(n, r) = r^{-k} \sum_{de=r} \lambda(d)\beta_k(e)c_k(n, r)$$

(see [6, Corollary 10.4]). We are now able to prove

THEOREM 3. $N_k^{(s)}(n, r)=r^{-k}\sum_{de=r}(\lambda(d)\beta_k(e))^s c_k(n, d)$.

Proof. We have just shown that $N_k^{(1)}(n, r)=\nu_k(n, r)$, *i.e.*, that the theorem is true when $s=1$. Now assume that the theorem has been proved for all $t<s$. Then, using Theorem 1,

$$N_k^{(s)}(n, r) = \sum_{n\equiv a+b \pmod{r^k}} N_k^{(s-1)}(a, r)N_k^{(1)}(b, r) = r^{-k} \sum_{de=r} (\lambda(d)\beta_k(e))^s c_k(n, d).$$

7. Some Busche-Ramanujan identities. In his development of the theory of multiplicative functions [10], Vaidyanathaswamy considered, for a multiplicative function $f(r)$, identities of the following type:

$$f(MN) = \sum_{d|(M,N)} f(M/d)f(N/d)F(d). \tag{7}$$

He called such an identity a Busche-Ramanujan identity because of the classical identity involving the function $\sigma_k(r)$,

$$\sigma_k(MN) = \sum_{d|(M,N)} d^k \sigma_k(M/d)\sigma_k(N/d)\mu(d). \tag{8}$$

Vaidyanathaswamy obtained a necessary and sufficient condition for a multiplicative function to admit a Busche-Ramanujan identity [10, Theorem XXXV]. The identity (8) has been generalized by Cohen [7, Theorem 10] and he obtained as special cases of this generalization several interesting arithmetical identities [7, Section 6]. However, there is a still more general identity that includes (8) and Cohen's generalization of (8).

THEOREM 4.

$$\sum_{d|(M,N)} d^s \sigma_s(M/d)\sigma_s(N/d)c_k(n, d) = \sum_{\substack{d|(M,N) \\ d^k|n}} d^{s+k}\sigma_s(MN/d^2). \tag{9}$$

Proof. The left-hand side of (9) is equal to

$$\begin{aligned} &\sum_{d|(M,N)} d^s \sigma_s(M/d)\sigma_s(N/d) \sum_{e^k|(n,d^k)_k} e^k \mu(d/e) \\ &= \sum_{\substack{e|(M,N) \\ e^k|n}} e^k \sum_{\substack{d|(M,N) \\ d=\delta e}} d^s \sigma_s(M/d)\sigma_s(N/d)\mu(d/e) \\ &= \sum_{\substack{e|(M,N) \\ e^k|n}} e^{k+s} \sum_{\delta|(M/e,N/e)} \delta^s \sigma_s((M/e)/\delta)\sigma_s((N/e)/\delta)\mu(\delta), \end{aligned}$$

and so, using (8), we obtain the right-hand side of (9).

Note that in (9), k need not be a positive integer. If we set $n=0$ in (9) we obtain

$$\sum_{d|(M,N)} d^s \sigma_s(M/d)\sigma_s(N/d)\phi_k(d) = \sum_{d|(M,N)} d^{s+k}\sigma_s(MN/d^2). \tag{10}$$

If we set $M=N=r$ in (10) we obtain

$$\sum_{de=r} d^s \sigma_s^2(e)\phi_k(d) = \sum_{de=r} d^{s+k}\sigma_s(e^2)$$

which for $s=0$ gives

$$\sum_{de=r} \tau^2(e)\phi_k(d) = \sum_{de=r} d^k \tau(e^2) \tag{11}$$

(see [7, Corollaries 45.3 and 45.4]). (11) is an identity due originally to Gegenbauer [8, page 298].

A divisor d of an integer r is called a direct divisor of r if $(d, r/d)=1$. We say that a multiplicative function $f(r)$ admits a restricted Busche-Ramanujan identity if (7) holds whenever M and N have no direct divisors in common, *i.e.*, whenever M and N share no prime divisor to the same power. This termi-

nology is also due to Vaidyanathaswamy, and he obtained a necessary and sufficient condition for $f(r)$ to admit such a restricted identity [10, Theorem XXXVIII]. He attributes the fact that Euler's function $\phi(r)$ admits the restricted identity

$$\phi(MN) = \sum_{d|(M,N)} d\phi(M/d)\phi(N/d) \tag{12}$$

to Pillai. We shall now generalize (12). According to Vaidyanathaswamy's characterization of the functions which admit a restricted identity, the function $\phi_s(r)$ admits such an identity since it is a totient function in the sense of [10]. The results of [10] actually give the correct identity for this function, but we shall obtain this identity in such a way that gives us an idea of the "error" in the exceptional cases when the identity does not hold. Note that s need not be a positive integer.

THEOREM 5. *If M and N have no common direct divisors, then*

$$\phi_s(MN) = \sum_{d|(M,N)} d^s\phi_s(M/d)\phi_s(N/d). \tag{13}$$

Proof. We set $f(M, N)$ equal to the right-hand side of (13). If $(M_1N_1, M_2N_2) = 1$ then $((M_1, N_1), (M_2, N_2)) = 1$ and $(M_1M_2, N_1N_2) = (M_1, N_1)(M_2, N_2)$, and therefore $f(M_1M_2, N_1N_2) = f(M_1, N_1)f(M_2, N_2)$. Consequently, in order to evaluate $f(M, N)$ we need only evaluate $f(p^a, p^b)$ where p is a prime. We have, assuming without loss of generality that $a \leqq b$,

$$f(p^a, p^b) = \sum_{i=0}^{a} p^{si}\phi_s(p^{a-i})\phi_s(p^{b-i}),$$

and after some manipulation we obtain

$$f(p^a, p^b) = \begin{cases} \phi_s(p^{a+b}) & \text{if } a \neq b, \\ \phi_s(p^{2a}) + p^{s(a-1)} & \text{if } a = b. \end{cases}$$

This completes the proof of the theorem.

We now set $\sigma_k(n, r) = \sigma_k((n, r))$ and obtain the following extension of (13).

THEOREM 6. *If M and N have no common direct divisors, then*

$$\sum_{d|(M,N)} d^s\phi_s(M/d)\phi_s(N/d)(n, d)^{-k}\sigma_k(n, d) = \sum_{\substack{d|(M,N) \\ d|n}} d^{s-k}\phi_s(MN/d^2). \tag{14}$$

Proof. The left-hand side of (14) is equal to

$$\sum_{d|(M,N)} d^s\phi_s(M/d)\phi_s(N/d) \sum_{e|(n,d)} e^{-k} = \sum_{\substack{e|(M,N) \\ e|n}} e^{-k} \sum_{\substack{d|(M,N) \\ d=\delta e}} d^s\phi_s(M/d)\phi_s(N/d)$$

$$= \sum_{\substack{e|(M,N) \\ e|n}} e^{s-k} \sum_{\delta|(M/e,N/e)} \delta^s\phi_s((M/e)/\delta)\phi_s((N/e)/\delta).$$

Now if M and N have no common direct divisors, then the same is true of M/e and N/e where $e \mid (M, N)$. Hence, using (13), we obtain the right-hand side of (14).

If we set $n=0$ in (14) we obtain

$$\sum_{d \mid (M,N)} d^{s-k}\phi_s(M/d)\phi_s(N/d)\sigma_k(d) = \sum_{d \mid (M,N)} d^{s-k}\phi_s(MN/d^2), \tag{15}$$

which holds whenever M and N have no common direct divisors. If we set $M=r$ and $N=r^2$ then M and N have no common direct divisors and so we have from (15)

$$\sum_{de=r} d^{s-k}\phi_s(e)\phi_s(re)\sigma_k(d) = \sum_{de=r} d^{s-k}\phi_s(re^2). \tag{16}$$

If we do the same thing in (13) we obtain

$$\sum_{de=r} d^s\phi_s(e)\phi_s(re) = \phi_s(r^3).$$

We also have the following remarkable pair of identities: for all integers r and all s,

$$\sum_{de=r} d^s\phi_s(e)\phi_s(re)\sigma_{-s}(d) = \sum_{d \mid r} \phi_s(rd^2)$$

and

$$\sum_{de=r} d^s\sigma_s(e)\sigma_s(re)\phi_{-s}(d) = \sum_{d \mid r} \sigma_s(rd^2).$$

The first of these identities is obtained from (16) by setting $s=k$ and using the fact that $\sigma_s(r)=r^s\sigma_{-s}(r)$. The second is obtained from (10) by setting $M=r$ and $N=r^2$ and replacing k by $-s$.

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AXIOMS FOR THE AFFINE LINE*

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1. Introduction. The subject of this paper is the structure of a *line*, by which we mean a one-dimensional affine (or Euclidean) space. This structure is less rich than that of the *real line*, defined (somewhat redundantly) as a complete ordered field, but richer than that of the *ordered line*, defined as the real line with all structure other than its order ignored. A precise definition of "line" as mathematical system can be given in terms of the real line: *it is the set of real numbers with just such structure as is invariant under the group of affine functions*: $f(x)=ax+b$, *with* $a\neq 0$. However, in view of the intuitively geometric (rather than arithmetic) nature of the line, it would be more natural to proceed in the opposite direction, namely give an axiomatic definition of *line* and then show its relationship to the real line. Various axiomatic descriptions are certainly possible, based on different primitive concepts such as congruence or translations. The one we are proposing begins with order and prescribes further *how much* structure the line is to have, rather than specifically *what* structure. The quantity of structure is measured by the degree of transitivity of its automorphism group, as will be explained below.

It would be even better if, having given an axiomatic description of the line, we could define from it a model for the real field in a natural way. This is still undone; all we achieve is a proof that the axiomatic description does give the system we intended and no others.

The idea of Klein's Erlanger Program applies not only to geometry but everywhere in mathematics. A mathematical system consists of a set S and certain *constants* associated with S. These constants may be of various logical types: elements of S, subsets of S, families of subsets (as a topology), functions, operations, *etc.* Any one to one transformation of S onto itself gives rise in obvious ways to induced transformations of all these logical types, so maps each constant into an object of the same type. Those which take each given constant into itself are called *automorphisms* (or *symmetries*) of the system, and form a group G. The *structure* of the system is then defined as the aggregate of all objects, of whatever logical type, which are left fixed by G. There is a one to one correspondence between groups and structures, the richer the structure the smaller the group.

A transformation group G is *transitive* if given a and b in S there exists f in G such that $f(a)=b$. G is *simply transitive* if this f is unique for each a and b. In a sense transitivity keeps G from being very small, while *simple* transitivity keeps G from being very large. Among groups which are transitive but not simply transitive are doubly transitive groups. A transformation group G is *doubly transitive* if given $a_1\neq a_2$ and $b_1\neq b_2$ in S, there exists f in G such that

* A somewhat different proof of the theorem of this paper was submitted by the second author to Wesleyan University Honors College as an undergraduate Honors Thesis, 1959.

$f(a_1)=b_1$ and $f(a_2)=b_2$. We shall call such a group *just doubly transitive* if the f is unique for each a_1, a_2, b_1, b_2. (The restriction $a_1 \neq a_2$, $b_1 \neq b_2$ is unavoidable, because f must be one to one.) Similarly *n-transitivity* and *just n-transitivity* can be defined for each nonnegative integer n. (All groups are 0-transitive, while only the trivial group is just 0-transitive.) It is unusual, however, for an automorphism group to be just n-transitive for some n. Examples: The automorphism group of the real line is just 0-transitive. The group of translations in the real line (or, for that matter, in any abstract group) is simply transitive, *i.e.*, just 1-transitive. The group of affine functions $f(x)=ax+b$ $(a \neq 0)$ of the real line (or for that matter any field) is just doubly transitive. The group of projectivities of a projective line is just triply transitive. The group of homeomorphisms of a plane (but not of a line) is n-transitive for every n.

Our axiomatic description of the line can be stated thus: *the line is a set S having enough structure so that its automorphism group G is just doubly transitive, and having as part of that structure a complete (in the Dedekind sense) betweenness order.* We shall prove (see more explicitly the theorem in §2) that all such systems are isomorphic.

Axioms for the *ordered line* are well known [4]. Stated briefly they are that the order be complete, unbounded, and dense, and that S have a countable dense subset. Our line is this ordered line so far as its order alone is concerned, but as remarked earlier, it has more structure not definable in terms of order. It is interesting that the existence of such structure, expressed by G, is enough to make redundant most of the above order axioms.

2. Formulation of the problem. We begin by establishing notation and stating the theorem to be proved in detail. We are given a set S, with elements $a, b, \cdots$, ordered by a betweenness relation. We can dispense with restating the axioms for betweenness (see [3]) because betweenness gives rise to order, $<$, of the ordinary familiar kind as soon as an arbitrary choice of direction in S is made, for example by stipulating $a_0<a_1$ for two arbitrarily selected points a_0 and a_1. Only two such orders arise (the other by stipulating instead $a_1<a_0$); we adopt one of them arbitrarily as a notational device and state the order axioms in terms of $<$, as follows:

2.1 (Transitive law) *If $a<b$ and $b<c$ then $a<c$.*

2.2 (Law of Trichotomy) *For any a and b one and only one of the following is true: $a<b$, $a=b$, $b<a$.*

2.3 (Completeness in the sense of Dedekind) *Any nonempty subset A of S having an upper bound has a least upper bound, denoted by* sup A.

It is well known that 2.3 implies its dual, so we have greatest lower bounds, denoted inf A, as well.

The assumption of just double homogeneity of S is embodied by a given family G of functions of S onto S governed by the following axioms:

2.4 *G is a transformation group.*

2.5 *The functions in G preserve betweenness, that is, if* $a<b<c$, *then either* $f(a)<f(b)<f(c)$ or $f(c)<f(b)<f(a)$.

2.6 *Given* $a\neq a'$ *and* $b\neq b'$, *there exists in G a unique function f such that* $f(a)=b$ *and* $f(a')=b'$.

Finally, to give Axiom 2.6 force we need a nontriviality axiom:

2.7 *S has at least three elements.*

The most obvious example, hereinafter referred to as the *prototype*, of a system satisfying these axioms is the real line, namely the set of all real numbers with the usual order, and with the family of all affine functions $ax+b$ $(a\neq 0)$ for G. Our conclusion will be that any system satisfying the axioms is isomorphic to the prototype. More explicitly we shall prove:

THEOREM. *Given* $a_0<a_1$ *in S, there exists a unique one to one correspondence* ϕ *of the real line onto S such that*

1) $\phi(0)=a_0$, $\phi(1)=a_1$ *and*

2) *the correspondence* $f\to\phi f\phi^{-1}$ *of real functions of a real variable to functions of S to S maps the family of affine functions onto G.*

Moreover this correspondence ϕ *is order preserving.*

The group operation in Axiom 2.4 is of course composition, and the functions of which G is made up must have inverses in G, so must be one to one. The correspondence used in 2) of the theorem is not capricious, but standard; it is designed so that if $\phi(x)=a$ and $\phi(y)=b$, and if f takes x into y, then the corresponding function takes a into b. This correspondence, if it satisfies 2), is automatically an isomorphism of the two groups.

Some easy consequences of the axioms can be drawn immediately. By the uniqueness in 2.6, any function in G which leaves two points fixed must be the identity, because the identity leaves those points (in fact all points) fixed.

2.8 *The order in S is unbounded, that is, no c in S is maximum or minimum.*

Proof. Given c, there are two other points a and b by 2.7. If $a<c<b$ there is nothing to prove, so suppose for example that $a<b<c$. Let f be the function in G such that $f(a)=b$ and $f(b)=c$. Then by 2.5, $f(a)<f(b)<f(c)$, *i.e.*, $c<f(c)$. Thus c is not minimum or maximum. The remaining case $c<a<b$ is similar.

2.9 *The order in S is dense.*

Proof. Similar to 2.8.

2.10 *Every function in G is either order preserving or order reversing.*

Proof. Clear from 2.5.

The uniqueness of the correspondence ϕ asserted in the theorem can be

proved quite easily, and we might as well dispose of this now. Suppose that ϕ and ψ are one to one correspondences of the real line onto S, both satisfying 1) and 2). Then the composite function $g=\psi^{-1}\phi$ has the following properties:

(a) g is one-to-one and maps the real line onto itself.
(b) $g(0)=0$, $g(1)=1$.
(c) If f is any affine function, then $gfg^{-1}=\psi^{-1}(\phi f\phi^{-1})\psi$ is an affine function.

We want to prove that g is the identity: $g(x)=x$ for all x. We begin by applying (c) to the particular affine function $f(x)=ax$, $a\neq 0$, obtaining $g(ag^{-1}(x))=bx+c$, identically in x for some b and c. Setting successively $x=0$ and $x=1$ and using (b) we find $c=0$ and $b=g(a)$. Then putting u for $g^{-1}(x)$, we get the functional equation $g(au)=g(a)\cdot g(u)$, which holds for all a and u (trivially for $a=0$). In particular this gives $g(-1)^2=g(1)=1$, whence using (a) and (b) $g(-1)=-1$. Now apply (c) again, this time to the affine function $f(x)=x+1$. A similar evaluation of the constants leads to the functional equation $g(u+1)=(g(2)-1)g(u)+1$. In particular, $u=-1$ leads to the evaluation $g(2)=2$, whence $g(u+1)=g(u)+1$. Now for $c\neq 0$, $g(u+c)=g((uc^{-1}+1)c)=g(uc^{-1}+1)g(c)=[g(uc^{-1})+1]g(c)=g(uc^{-1})g(c)+g(c)=g(u)+g(c)$. Since this functional equation is also trivially true for $c=0$, g turns out to be a homomorphism of the real number field, and not identically 0. There is only one such homomorphism, and that is the identity function [1, p. 47].

3. Inversions and fixed points. Denote the identity function on S by i: $i(a)=a$ for $a\in S$. A function in G other than the identity which is its own inverse, *i.e.*, such that $f^2=i$, is called an *inversion*. Given any pair of distinct points, the function in G which interchanges them is an inversion, since its square leaves both points fixed. Clearly inversions are order reversing.

3.1 *If $f\in G$ and $a<f(a)$, $f(b)<b$ then f has a fixed point between a and b.*

Proof. Case I: f is order preserving. We can suppose $a<b$, for the other case reduces to this by interchanging a and b and replacing f by f^{-1}. Since f is order preserving we have $a<f(a)<f(b)<b$. Let A be the set of those x such that $a\leqq x<b$ and $x<f(x)$. Then $a\in A$ and A is bounded, so $c=\sup A$ exists. Suppose $c<f(c)$. Then $c<b$ and there exists x such that $c<x<b$ and $x<f(c)$. Since f is order preserving $f(c)<f(x)$, so that $x\in A$, contradicting $c<x$. On the other hand suppose $f(c)<c$. Then there exists x in A such that $f(c)<x\leqq c$. Since f is order preserving, $f(x)\leqq f(c)<x$, contradicting $x\in A$. Thus $f(c)=c$ and c is a fixed point.

Case II: f is order reversing. In this case certainly $a<b$, for $b<a$ would give $f(b)<f(a)$, making f order preserving. Let A be the set of all x such that $x<f(x)$. Then $a\in A$ and b is an upper bound for A. Indeed $b<x$ implies $f(x)<f(b)<b<x$, whence $x\notin A$. Thus $c=\sup A$ exists. Note that since f and hence also f^{-1} are order reversing, for any x other than a fixed point (we can assume there are none) either x or $f^{-1}(x)$ belongs to A. Indeed, $f(x)<x$ implies $f^{-1}(x)<f^{-1}(f(x))$

$=x=f(f^{-1}(x))$. Now suppose $c<f(c)$ and find x such that $c<x<f(c)$. Then applying f^{-1}, $c<f^{-1}(x)$. But either x or $f^{-1}(x)$ belongs to A, and $c=\sup A$, a contradiction. On the other hand suppose $f(c)<c$ and find $x\in A$ so that $f(c)<x<f(x)$. Applying f^{-1} get $x<f^{-1}(x)<c$. Now find y in A so that $f^{-1}(x)<y$. Then $y<f(y)<x<f^{-1}(x)<y$, a contradiction. Thus finally $c=f(c)$, and c is a fixed point.

3.2. COROLLARY. *Every inversion has a fixed point (unique by* 2.6).

3.3. COROLLARY. *If* $f\in G$, $a\leqq f(a)$, $b<f(b)$ *and* $a<c<b$, *and* $c\leqq f(c)$.

Proof. Otherwise f would have to have two fixed points.

We shall need the following sharpening of 3.3.

3.4 *If* $f\in G$, $a\leqq f(a)$, $b<f(b)$ *(or* $a<f(a)$, $b\leqq f(b)$*) and if* $a<c<b$, *then* $c<f(c)$.

Proof. Suppose first that $f(b)<f(a)$. Then f is order reversing, and $c<b$ leads to $f(b)<f(c)$, whence $c<f(c)$. Assume then that f is order preserving. By 3.3 we need only rule out the eventuality $f(c)=c$, which is certainly impossible if $a=f(a)$. Assume, then, that $a<f(a)$ and $f(c)=c$. Let g be the function in G such that $g(f(a))=a<f(a)$ and $g(f(b))=f(b)$. Then $c<g(c)$ would require g to have another fixed point between $f(a)$ and c (where $f(b)$ isn't), and $c=g(c)$ likewise would give g two fixed points. Hence $g(c)<c$. Now the function gf leaves a fixed and also (since $gf(b)=f(b)>b$ and $gf(c)=g(c)<c$) has another fixed point between b and c, an impossibility.

3.5 *Each point* c *of* S *is the fixed point of a unique inversion in* G.

Proof. Let $a<c$. For each $x\neq c$ there is a unique function f_x in G such that $f_x(c)=c$ and $f_x(a)=x$. In fact $x\to f_x$ is a one to one correspondence of $S-\{c\}$ to the subgroup of G leaving c fixed. Evidently any solution of the quadratic equation $f_x(x)=a$, other than the trivial one $x=a$, corresponds to an inversion with c as fixed point, and conversely.

We shall consider only f_x for $x>c$; each such f_x is order reversing. Let $x'=f_x^{-1}(a)$. Since $f_{x'}$ and f_x^{-1} agree at a and c, they are equal. It follows that if $a<f_x(x)$, then $x=f_{x'}f_x(x)<f_{x'}(a)=x'$ and also conversely. Thus $a<f_x(x)$, $x<x'$ and $f_{x'}(x')<a$ are equivalent. Evidently $x=x'$ gives $f_x(x)=a$ and solves the problem.

Suppose $c<x<y$ and $c<u$. We shall prove that $f_y(u)<f_x(u)$. Assume on the contrary that $f_x(u)\leqq f_y(u)$ and consider the function $g=f_yf_{x'}$. Evidently $g(x)=y>x$ and $g(f_x(u))=f_y(u)\geqq f_x(u)$. Since $f_x(u)<c<u$ and $g(c)=c$, we have a contradiction of 3.4. We conclude that $f_y(u)<f_x(u)$.

Two applications of this result: First, taking $u=x'$, we get $f_y(x')<a$, whence $y'<x'$. Thus $x<y$ implies $y'<x'$. Secondly, take $u=x$ to get $f_y(x)<f_x(x)$. Since $f_y(y)<f_y(x)$ we have that $x<y$ implies $f_y(y)<f_x(x)$. This, incidentally, insures the uniqueness of the solution.

Now let A be the set of all $x>c$ such that $f_x(x)>a$. We have shown that for

any $x>c$ either x or x', not both, belongs to A, and therefore A is not empty. Also if $x\in A$, then x' is an upper bound for A, by the second result of the last paragraph. Hence $b=\sup A$ exists. If b and b' are not equal, then there exists x between them, and by the first result of the last paragraph, x' also lies between them. Then one of x and x' belongs to A, while the other is an upper bound for A, which is impossible because $b=\sup A$. Hence $b=b'$ is a solution.

3.6 *Let $f\in G$, let $f(a_1)=a_2$ and $f(b_1)=b_2$. Let c_i be the fixed point of the inversion in G which interchanges a_i and b_i $(i=1, 2)$. Then $f(c_1)=c_2$.*

Proof. Call the two inversions in question f_1 and f_2 and let $g=f_2ff_1$. Then $g(a_1)=a_2=f(a_1)$ and $g(b_1)=b_2=f(b_1)$ so that $g=f$, *i.e.*, $f_2f=ff_1$. Then $f_2(f(c_1))=f(f_1(c_1))=f(c_1)$, so that $f(c_1)$ is a fixed point of f_2. Since f_2 has only one fixed point, $f(c_1)=c_2$.

3.7 *Let $f\in G$, let $f(a_1)=a_2$, $f(c_1)=c_2$. Let $b_i=f_i(a_i)$, where f_i is the inversion in G with c_i for fixed point $(i=1, 2)$. Then $f(b_1)=b_2$.*

Proof. Let g be the function in G such that $g(a_1)=a_2$ and $g(b_1)=b_2$. Then by 3.6 $g(c_1)=c_2$. Since g agrees with f at a_1 and c_1, $g=f$, so $f(b_1)=g(b_1)=b_2$.

4. Ladders and ramps. Our information about inversions can be summarized in an algebraic way by defining two operations in S, inverse to each other, as follows:

Interpolation: *define $(a*b)$ to be the fixed point of the inversion in G which interchanges a and b.*

Extrapolation: *define $(ac*)$ to be the image of a under the inversion in G which has c for its fixed point.*

Then we have evidently the following algebraic rules:

4.1 $(a*a)=(aa*)=a$.

4.2 $(a*b)=(b*a)$; $(ab*)=c$ *if and only if* $(cb*)=a$.

4.3 $(a*b)=c$ *if and only if* $(ac*)=b$.

Using this terminology, the content of 3.6 and 3.7 is that functions in G preserve interpolation and extrapolation. Applying this in particular to an inversion we have:

4.4 *If* $(a*a')=(b*b')=c$, *then also*
(1) $(a*b)*(a'*b')=c$ *and*
(2) $(ab*)*(a'b'*)=c$.

We shall use this property of functions in G to propagate information about them from a set consisting of only two points to a much larger set, in fact a set large enough to be decisive, as will come out in the next section.

We shall call a subset L of S a *ladder* when L is closed under extrapolation: $a\in L$ and $b\in L$ imply $(ab*)\in L$. Given any two points a_0, a_1 (more generally any subset) there exist ladders containing a_0 and a_1, for example S; the intersection of all such ladders is a ladder, called the ladder *generated* by a_0, a_1.

In our prototype, the real line, the ladder generated by two numbers is the both-ways-infinite arithmetic progression with these two numbers as adjacent terms, for example the integers. In the integers regarded as a ladder the operation is $(mn*) = 2n - m$.

4.5 *Given $a_0 \neq a_1$ in S, there exists a unique isomorphism $n \to a_n$ of the real ladder of integers onto the ladder in S generated by a_0, a_1 such that $0 \to a_0$ and $1 \to a_1$.*

Proof. For $n \to a_n$ to be an isomorphism means that $a_{2n-m} = (a_m a_n *)$ must be satisfied for all integers m, n. Writing p for $n - m$ and using 4.3 this can be put in the more symmetric form $(a_{n-p} * a_{n+p}) = a_n$. In particular this requirement dictates the following recursion formulae

$$a_{n+1} = (a_{n-1} a_n *) \qquad n = 1, 2, \cdots$$
$$a_{n-1} = (a_{n+1} a_n *) \qquad n = 0, -1, \cdots$$

which together with the given values of a_0 and a_1 determine the correspondence uniquely.

To show that the correspondence thus recursively defined is an isomorphism, it is sufficient to verify $(a_{n-p} * a_{n+p}) = a_n$ for all n and p. Do this by induction on p. For $p = 0$ it is trivial, and for $p = 1$ it is the recursive condition. Suppose $(a_{n-p} * a_{n+p}) = a_n$ for $p = 0, 1, \cdots, k$, in particular for $p = k - 1$ and $p = k$. Since $a_{n-k-1} = (a_{n-k+1} a_{n-k} *)$ and $a_{n+k+1} = (a_{n+k-1} a_{n+k} *)$ by definition, 4.4 (2) gives $(a_{n-k-1} * a_{n+k+1}) = a_n$, completing the induction. Finally, the correspondence being an isomorphism, its range must be a ladder, clearly the ladder generated by a_0, a_1.

Obviously 4.5 could have been stated more generally for any arithmetic progression in place of the integers, at some cost in notation. The same applies to the next lemma, which is in preparation for what follows:

4.6 *Given an isomorphism $2m \to a_{2m}$ of the real ladder of the even integers onto a ladder in S, it can be uniquely extended to an isomorphism of the ladder of all integers.*

Proof. As in 4.5, the condition which must be satisfied by the extended correspondence is $(a_{n-p} * a_{n+p}) = a_n$ for all n, p. In particular this uniquely dictates the definition $a_{2m+1} = (a_{2m} * a_{2m+2})$ for the odd integers $2m+1$, and the uniqueness is proved. To see that this does give an isomorphism we can use the proof of 4.5 once we verify $(a_{n-1} * a_{n+1}) = a_n$. For n odd this was the definition. For $n = 2m$ even, it reads $(a_{2m-1} * a_{2m+1}) = a_{2m}$. This is an application of 4.4 (1), since $(a_{2m} * a_{2m}) = a_{2m}$ trivially and $(a_{2m-2} * a_{2m+2}) = a_{2m}$ is given by hypothesis.

When the rungs of a ladder are very close together, it is a ramp. Explicitly we shall call a subset R of S a *ramp* when R is closed under extrapolation and interpolation: $a \in R$ and $b \in R$ imply $(ab*) \in R$ and $(a * b) \in R$. Given any two points a_0, a_1 there exist ramps containing them, for example S; the intersection of all such ramps is a ramp, called the ramp *generated* by a_0, a_1. This ramp is also a ladder, so contains the ladder generated by a_0, a_1.

In our prototype, the real line, the ramp generated by 0, 1 consists of all the dyadic rationals $m2^{-n}$. In the dyadic rationals regarded as a ramp, the operations are of course $(rs\ *) = 2s - r$ and $(r * s) = \frac{1}{2}(r+s)$.

4.7 *Given $a_0 \neq a_1$ in S, there exists a unique isomorphism $r \to a_r$ of the real ramp of the dyadic rationals onto the ramp in S generated by a_0, a_1 such that $0 \to a_0$ and $1 \to a_1$.*

Proof. The isomorphism is uniquely determined for integers r by 4.5. It is then uniquely determined for dyadic rationals with higher and higher denominators by repeated use of 4.6. The induction hinted at here proves the uniqueness. This also defines recursively a correspondence $r \to a_r$ whose domain in the limit includes all dyadic rationals. We have seen in the proof of 4.5 that in proving this correspondence is an isomorphism, both operations are disposed of at once by verifying the one formula $(a_{r-s} * a_{r+s}) = a_r$. But any instance of this formula involves only dyadic rationals with a single denominator 2^{-n}, the common denominator of r and s. From 4.6 applied n times, the correspondence is an isomorphism of the ladder of multiples of 2^{-n}, so this instance of the formula is true. Finally, the correspondence being an isomorphism, its range is a ramp, clearly the ramp generated by a_0, a_1.

5. Archimedean properties. As a result of Section 4, given two points a_0 and a_1 we know completely the structure of the ramp R they generate, and according to 3.6 and 3.7 any function in G is determined on R by its values at a_0 and a_1. To complete the picture we need to know the relationship of R to the rest of S, and this is a topological relationship (based on order) rather than an algebraic one.

The obvious properties of extrapolation and interpolation in relation to order are:

5.1 *If $a < b$, then $a < (a * b) < b$ and $b < (ab\ *)$.*

5.2 (a) *If $a \leqq a'$ and $b < b'$ then $(a * b) < (a' * b')$.*
(b) *If $b < b'$ then $(ab\ *) < (ab'\ *)$.*

Proof. To prove (a), let f be the function in G such that $f(a) = a'$ and $f(b) = b'$. Then by 3.6 $(a' * b') = f(a * b)$, and using 5.1 and 3.4, $(a * b) < (a' * b')$. Now (b) follows easily from (a).

5.3 *If $a_0 < a_1$, then the ladder and ramp isomorphisms of* 4.5 *and* 4.7 *are order-preserving.*

Proof. An inequality $a_{n-1} < a_n$ and the recursion relation $a_{n+1} = (a_{n-1}a_n\ *)$ together with 5.1 imply $a_n < a_{n+1}$. Hence by induction $a_n < a_{n+1}$ for $n = 0, 1, \cdots$, and similarly for the negative integers. Thus the isomorphism of 4.5 is order preserving. It is clear that each extension of the isomorphism by an application of 4.6 is still order-preserving. Therefore the ramp isomorphism of 4.7 is also.

5.4 *Every ladder containing at least two points is unbounded above and below.*

Proof. Let L be a ladder containing two points, say a and b, and suppose that $c=\sup L$ exists. Now $c>a$ because either $a<b$ or else $a<(ba*)\in L$. Hence $a<(a*c)<c$. Thus there exists $x\in L$ such that $(a*c)<x$. But then by 5.2 (b) $c=(a(a*c)*)<(ax*)$, while $(ax*)\in L$, a contradiction. Existence of inf L is ruled out similarly.

5.5 *Every ramp R containing at least two points is dense; that is, given $b_0<b_1$ in S, there exists a in R such that $b_0<a<b_1$.*

Proof. Let L be the ladder generated by b_0, b_1 and let $n\to b_n$ be the isomorphism of the integers onto L such that $0\to b_0$ and $1\to b_1$. This correspondence is order preserving by 5.3. Now R and L being both unbounded (5.4) there are elements of R greater than b_0, and then elements of L greater than these. Hence there is a largest integer n such that no r in R satisfies $b_0<r<b_n$. Similarly find the smallest (negative) integer m such that no r in R satisfies $b_m<r<b_0$. We want to prove $n=0$. If $n>0$ then we have $b_{m-1}<r\leqq b_m<b_n\leqq s<b_{n+1}$ for some r and s in R, and no point of R lies between b_m and b_n. But on the contrary we shall show that $(r*s)$ is such a point. Indeed, $(r*s)\in R$ because R is a ramp. By two applications of 5.2 (a) $(r*s)<(b_m*b_{n+1})\leqq(b_{n-1}*b_{n+1})=b_n$, and similarly $b_m<(r*s)$. This contradiction proves that R is dense.

6. Completion of the theorem. We can now finish the proof of the theorem of Section 2. We start with $a_0<a_1$ in S and define $\phi(x)$ for dyadic rationals x to be the ramp isomorphism determined by 4.7 of the dyadic rationals onto the ramp generated by a_0, a_1. According to 5.3 and 5.5 ϕ is an order-preserving function mapping a dense subset of the real line onto a dense subset of S. Since S is complete, it is well known that ϕ has a unique one-to-one order-preserving extension (still denoted ϕ) mapping all of the real line onto all of S. (For x irrational $\phi(x)$ is defined as the least upper bound of $\phi(r)$ for r dyadic rational, $r<x$. See [4] p. 49.) We must prove that this ϕ satisfies 2) of the theorem.

Let f be an affine function, and let $b_0=\phi f(0)$, $b_1=\phi f(1)$. Let g be the function in G such that $g(a_0)=b_0$ and $g(a_1)=b_1$. Then g and $\phi f\phi^{-1}$ are functions of S onto S, and they agree at a_0 and a_1. Moreover both these functions preserve extrapolation and interpolation on the ramp R generated by a_0 and a_1, g because it belongs to G, and $\phi f\phi^{-1}$ because ϕ is an isomorphism of this ramp and f is affine. Hence g and $\phi f\phi^{-1}$ agree on R. Finally g and $\phi f\phi^{-1}$ both preserve betweenness in S, g because it belongs to G (Axiom 2.5), and $\phi f\phi^{-1}$ because ϕ is order-preserving and f is affine. Since R is dense, g and $\phi f\phi^{-1}$ agree everywhere, that is $\phi f\phi^{-1}\in G$. The same argument with the roles of S and the real line interchanged shows that every function in G corresponds to an affine function, and the proof is complete.

7. Final remarks. The theorem of Section 2 has a counterpart for the projective line. The prototype is in this case the *real projective line*, whose points are the real numbers and ∞, and whose automorphisms are the real projective functions

$$f(x) = \frac{ax + b}{cx + d} \qquad ad - bc \neq 0.$$

The order involved is a circular order [2] defined by a quaternary relation "a and b *separate* c and d".

THEOREM. *Let S be a set with a complete circular order, G a just triply transitive transformation group on S preserving circular order. Then given any three distinct points a_0, a_1, a_∞ of S there is a unique one-to-one correspondence ϕ of the projective real line onto S such that*

1) $\phi(0) = a_0$, $\phi(1) = a_1$, $\phi(\infty) = a_\infty$ *and*

2) $f \to \phi f \phi^{-1}$ *maps the family of real projective functions onto G.*

Moreover ϕ is circular order-preserving.

We shall sketch a proof based on the affine theorem. The group G_0 which leaves a_∞ fixed acts just doubly transitively on the set $S_0 = S - \{a_\infty\}$. The circular order on S gives a betweenness order on S_0 which is preserved by G_0. Hence applying the affine theorem to S_0 with G_0 gives a unique betweenness preserving correspondence ϕ of the real line onto S_0 such that $\phi(0) = a_0$, $\phi(1) = a_1$ and the affine functions are mapped onto G_0. Extend ϕ by defining $\phi(\infty) = a_\infty$.

It remains to prove that the rest of the real projective functions correspond to the rest of G. This follows easily as soon as we show that the reciprocal function $1/x$ goes into a function in G. To prove this, note that there is a function g in G such that $g(a_0) = a_\infty$, $g(a_\infty) = a_0$ and $g(a_1) = a_1$. Let $f = \phi^{-1} g \phi$. Then f is a real function satisfying $f(0) = \infty$, $f(\infty) = 0$ and $f(1) = 1$, and it suffices to prove that $f(x) = 1/x$ identically. Now for any $a \neq 0$ the function $h(x) = f(ax)/f(a)$ is the composition of f with two affine functions, whence $\phi h \phi^{-1}$ belongs to G. But by computation $\phi h \phi^{-1}$ has the same effect on a_0, a_1 and a_∞ as g has, whence $\phi h \phi^{-1} = g$, $h = f$. This gives the functional equation $f(ax) = f(a)f(x)$. The only circular order preserving solutions of this equation are $f(x) = x^n$ for $n \neq 0$. Now since g^2 has a_0, a_1 and a_∞ as fixed points, g, and therefore also f, are involutions. We have $(x^n)^n = x$, and $n = \pm 1$. Since f is clearly not the identity $n = -1$ and $f(x) = 1/x$.

These characterizations appear to be special to one dimension. The obstacle in more dimensions is not only the problem of finding a substitute for order, but worse, the fact that the group of nonsingular affine transformations in n dimensions is not quite $(n+1)$-transitive: $a_0, a_1, \cdots, a_n$ can be mapped onto arbitrary $b_0, b_1, \cdots, b_n$ only when the a's and the b's satisfy the same coincidences of collinearity, for example when both sets are in general position. Thus the group cannot be adequately described without already having a structure of straight lines. But the straight line structure of the space is enough to determine the group, so mention of the group would be superfluous. Indeed we can say, rather, only in one dimension is a characterization like the one given needed.

In fact, in the light of the last remark one would say that the group G re-

garded as structure of the affine line is inherited from the more-dimensional geometry in which the line can be embedded, its just double transitivity being a consequence of the incidence axioms of that geometry. The presence of order should then be regarded as a further axiomatic restriction on the geometry (or alternatively on the group) which selects the real affine geometry from among all abstract affine geometries. This raises the interesting question: what is the relation between just doubly transitive groups and affine geometries in general? On this question see R. D. Carmichael, *Groups of Finite Order*, Chapters XI–XIII.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twentieth William Lowell Putnam Mathematical Competition held on November 21, 1959, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of the Polytechnic Institute of Brooklyn, Brooklyn, New York. The members of the team were Martin Isaacs, Donald Passman, and Gerald Stoller; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of the California Institute of Technology, Pasadena, California. The members of the team were Donald W. Anderson, Alfred W. Hales, and Harold M. Stark; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were Edward J. Barbeau, Andrew Korsak, and Joseph Lipman; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Stephen L. Adler, Stephen Lichtenbaum, and Robert Wells; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of Case Institute of Technology, Cleveland, Ohio. The members

of the team were Daniel J. Gans, Donald E. Knuth, and David F. Monck; to each of these a prize of ten dollars is awarded.

The eight persons ranking highest in the examination (the ranks five to eight being tied), named in alphabetical order, are Stephen L. Adler, Harvard University; Donald Gorman, Harvard University; Alfred W. Hales, California Institute of Technology; Martin Isaacs, Polytechnic Institute of Brooklyn; Samuel Klein, College of the City of New York; Stephen Lichtenbaum, Harvard University; Donald Passman, Polytechnic Institute of Brooklyn; and Daniel G. Quillen, Harvard University. Each of these will receive a prize of seventy-five dollars, and the three-thousand-dollar William Lowell Putnam Prize Scholarship at Harvard will be awarded to one of this number.

The two succeeding persons ranking highest in the examination, named in alphabetical order, are Alan Landman, Harvard University; and Harold M. Stark, California Institute of Technology. Each of these will receive a prize of thirty-five dollars.

The following teams, named in alphabetical order, won honorable mention: College of the City of New York, New York, New York, the members of the team being Stephen Berger, Samuel Klein, and Eugene Luks; Dartmouth College, Hanover, New Hampshire, the members of the team being William A. Veech, Albert R. Whitcomb, and Seth I. Zimmerman; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being Richard L. Faber, Steven A. Orszag, and Stephen Scheinberg; Oberlin College, Oberlin, Ohio, the members of the team being Stephen A. Andrea, Christopher Billings, and Peter J. Kahn; Yale University, New Haven, Connecticut, the members of the team being Richard William Beals, Peter Diamond, and Theodore William Gamelin; and Yeshiva University, New York, New York, the members of the team being Stanley Boylan, Jonathan Ginsberg, and Benjamin Weiss.

Eighteen individuals were given honorable mention. The names, alphabetically arranged, are: Donald W. Anderson, California Institute of Technology; Edward J. Barbeau, University of Toronto; Richard William Beals, Yale University; Stefan Burr, University of California, at Berkeley; B. P. Carter, Stanford University; William Easton, Cornell University; Jon Folkman, University of California, at Berkeley; Donald E. Knuth, Case Institute of Technology; Andrew Korsak, University of Toronto; Joseph Lipman, University of Toronto; John Lovin, Reed College; Peter T. Saunders, University of Toronto; Stanley A. Sawyer, California Institute of Technology; Jack Silver, Montana State University; Gerald Stoller, Polytechnic Institute of Brooklyn; Floris Tsang, University of California, at Berkeley; James Wirth, University of Notre Dame; and Seth I. Zimmerman, Dartmouth College.

A total of eight hundred and twenty-nine contestants from one hundred and forty-one colleges and universities (one hundred and twelve of these having teams) entered the Competition. Six hundred and thirty-three contestants from one hundred and thirty-five colleges and universities (ninety-five having teams)

participated in the examination on November 21, 1959.

The following is a list, in alphabetical order, of all colleges and universities which entered teams in the competition:

Agricultural and Mechanical College of Texas, Alabama College, American University, Arizona State College (Flagstaff), Arizona State University (Tempe), Austin College, Bethel College, Brooklyn College, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Case Institute of Technology, Central Michigan University, College of Saint Catherine, College of Saint Thomas, College of the City of New York, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Eastern Illinois University, Eastern New Mexico University, Fairleigh-Dickinson University, Fordham College, George Pepperdine College, Georgia Institute of Technology, Glenville State College, Grambling College, Harvard University, Iowa State University, Iowa Wesleyan College, Ithaca College, Kenyon College, Kent State University, Lafayette College, Lebanon Valley College, Long Beach State College, Mankato State College, Massachusetts Institute of Technology, McGill University, McMaster University, Memphis State University, Michigan State University, Mississippi State University, Northeastern University, Oberlin College, Ohio State University, Oregon State College, Pennsylvania State University, Polytechnic Institute of Brooklyn, Pomona College, Purdue University, Queen's University (Ontario), Reed College, Rice Institute, Rutgers University, Sacramento State College, Saint Francis Xavier University, Saint Joseph's College (West Hartford, Connecticut), Saint Martin's College, San Jose State College, Seattle University, Siena College, Shaw University, South Dakota School of Mines, Southern University, Southwestern at Memphis, Southwestern Louisiana Institute, Stanford University, State University of Iowa, Swarthmore College, Texas Technological Institute, Valparaiso University, Vanderbilt University, United States Naval Academy, University of Alberta, University of Arizona, University of British Columbia, University of Buffalo, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Colorado, University of Dayton, University of Idaho, University of Illinois, University of Kansas, University of Kansas City, University of Manitoba, University of Michigan, University of Minnesota, University of Nebraska, University of Notre Dame, University of Ottawa, University of Pennsylvania, University of Rochester, University of Santa Clara, University of Tampa, University of Toronto, University of Washington, University of Western Ontario, University of Wichita, University of Wisconsin, Ursinus College, Washington State University (Pullman), Washington University (Saint Louis), Wayne State University, Wesleyan University, William Jewell College, Wofford College, Yale University, and Yeshiva University.

The following colleges and universities, in alphabetical order, entered individual contestants only:

Anderson College, Assumption University (Ontario), Bennington College, Brandeis University, Caldwell College, Colorado College, Colorado State University, Drew University, Eastern Baptist College, Fresno State College, Grinnell College, Lake Forest College, Lindenwood College, Manhattan College, Montana State University, Newark College of Engineering, New Mexico State University, North Carolina State College, Parsons College, Saint Joseph's College (Philadelphia), Saint Mary's College, Saint Olaf College, Stevens Institute of Technology, University of Chicago, University of Dallas, University of Maine, University of the South, University of Vermont, and Wake Forest College.

The individual rankings of contestants (except for the relative ranks of the first eight) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions, will appear in a later issue of the MONTHLY.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for the department should be sent to Roy Dubisch, Department of Mathematics, University of California, Berkeley 4, California.

REMARKS ON UNIFORM CONTINUITY IN METRIC SPACES

NORMAN LEVINE, University of Pittsburgh

Every continuous transformation from one metric space into another is in a certain sense uniformly continuous. The sense in which this is so is implied in the following

THEOREM 1. *Let $f: S_1 \to S_2$ be a continuous transformation where S_1 and S_2 are metric spaces with metrics d_1 and d_2 respectively. Then there exists a metric d_1^* on S_1 which is equivalent to d_1 such that $f: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2.*

Proof. Let $d_1^*(a, b) = d_1(a, b) + d_2(f(a), f(b))$. We will show that (1) $d_1^*(a, b)$ is single valued and nonnegative; (2) $d_1^*(a, b) = d_1^*(b, a)$; (3) $d_1^*(a, b) = 0$ if and only if $a = b$; (4) $d_1^*(a, b) + d_1^*(b, c) \geqq d_1^*(a, c)$; (5) $\lim d_1^*(y_n, y) = 0$ if and only if $\lim d_1(y_n, y) = 0$ for every sequence $\{y_n\}$ and every point y in S_1; (6) $f: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2. (1) and (2) are clear. To show (3) let $d_1^*(a, b) = 0$. Then $d_1(a, b)$ is 0 and hence $a = b$. Conversely let $a = b$. Then $d_1(a, b)$ and $d_2(f(a), f(b))$ are both 0 and hence $d_1^*(a, b) = 0$. Now $d_1^*(a, b) + d_1^*(b, c) = d_1(a,b) + d_2(f(a),f(b)) + d_1(b,c) + d_2(f(b),f(c)) \geqq d_1(a,c) + d_2(f(a),f(c)) = d_1^*(a, c)$. This proves (4). We now prove (5). Let $\{y_n\}$ be a sequence in S_1 and y a point in S_1. Suppose $\lim d_1^*(y_n, y) = 0$. Then $\lim d_1(y_n, y) = 0$ since $d_1(y_n, y) \leqq d_1^*(y_n, y)$. Conversely let $\lim d_1(y_n, y) = 0$. Then since $f: S_1 \to S_2$ is continuous, $\lim d_2(f(y_n), f(y)) = 0$. Thus $d_1^*(y_n, y)$ has a limit 0. Lastly $d_2(f(a), f(b)) \leqq d_1^*(a, b)$ and choosing $\delta = \epsilon$, it is clear that $f: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2.

Let $f: S_1 \to S_2$ be continuous and d_1 and d_2 be metrics for S_1 and S_2 respectively. In general it is not possible to replace d_2 with an equivalent metric d_2^* such that $f: S_1 \to S_2$ is uniformly continuous relative to d_1 and d_2^*. Consider the following example. Let $S_1 \equiv \{x \mid x \text{ is real and } 0 \leqq x < 1 \text{ or } 1 < x \leqq 2\}$. Let $d_1(x, y) = |x - y|$ on S_1. Let $S_2 = \{a, b\}$ with $a \neq b$ and $d_2(a, b) = 1$. Let $f: S_1 \to S_2$ be defined by $f(x) = a$ if $0 \leqq x < 1$ and $= b$ if $1 < x \leqq 2$.

Clearly no equivalent metric d_2^* for d_2 will make $f: S_1 \to S$ uniformly continuous relative to d_1 and d_2^*.

Theorem 1 has an obvious extension to

THEOREM 2. *Let $f_i: S_1 \to S_2$ be continuous ($i = 1, \cdots, n$) where S_1 and S_2 are metric spaces with metrics d_1 and d_2 respectively. Then there exists a metric d_1^* for S_1 which is equivalent to d_1 such that $f_i: S_1 \to S_2$ is uniformly continuous relative to d_1^* and d_2 for $i = 1, \cdots, n$.*

Proof. Let $d_1^*(a, b)=d_1(a, b)+d_2(f_1(a), f_1(b))+\cdots+d_2(f_n(a), f_n(b))$. The remainder of the proof is almost identical to the proof for Theorem 1 and will be omitted.

We can get a stronger theorem by replacing d_1 and d_2 by equivalent metrics.

THEOREM 3. *Let f_i: $S_1\to S_2$ be continuous for $i=1, 2, \cdots$ where S_1 and S_2 are metric spaces with metrics d_1 and d_2 respectively. Then there exist metrics d_1^* and d_2^* for S_1 and S_2 respectively which are equivalent to d_1 and d_2 such that every f_i: $S_1\to S_2$ is uniformly continuous relative to d_1^* and d_2^*.*

Proof. Let d_2^* be any equivalent metric for S_2 which is bounded by 1. It is well known that such exist. Let $d_1^*(a, b)=d_1(a, b)+\sum_{i=1}^{\infty}2^{-i}d_2^*(f_i(a), f_i(b))$. It is clear that $d_1^*(a, b)$ is finite, nonnegative, and single valued on S_1; and that $d_1^*(a, b)=0$ if and only if $a=b$. For if $d_1^*(a, b)=0$, then $d_1(a, b)=0$ and thus $a=b$. If $a=b$ then of course $d_1^*(a, b)=0$. Clearly $d_1^*(a, b)=d_1^*(b, a)$. Now

$$
\begin{aligned}
&d_1^*(a, b)+d_1^*(b, c)\\
&\quad=d_1(a, b)+\sum_{i=1}^{\infty}2^{-i}d_2^*(f_i(a), f_i(b))+d_1(b, c)+\sum_{i=1}^{\infty}2^{-i}d_2^*(f_i(b), f_i(c))\\
&\quad=d_1(a, b)+d_1(b, c)+\sum_{i=1}^{\infty}2^{-i}\{d_2^*(f_i(a), f_i(b))+d_2^*(f_i(b), f_i(c))\}\\
&\quad\geqq d_1(a, c)+\sum_{i=1}^{\infty}2^{-i}d_2^*(f_i(a), f_i(c))=d_1^*(a, c).
\end{aligned}
$$

To show that d_1^* is equivalent to d_1, let (y_n) be a sequence in S_1 and y a point in S_1. Suppose $\lim d^*(y_n, y)=0$. Then since $d_1(y_n, y)\leqq d_1^*(y_n, y)$, it follows that $\lim d_1(y_n, y)=0$. Suppose then that $\lim d_1(y_n, y)=0$. Then

$$
\begin{aligned}
d_1^*(y_n, y)&=d_1(y_n, y)+\sum_{i=1}^{\infty}2^{-i}d_2^*(f_i(y_n), f_i(y))\\
&\leqq d_1(y_n, y)+\sum_{i=1}^{N}2^{-i}d_2^*(f_i(y_n), f_i(y))+\epsilon/3 \text{ for some } N.
\end{aligned}
$$

Then

$$
\limsup d_1^*(y_n, y)\leqq\lim d_1(y_n, y)+\lim\sum_{i=1}^{N}2^{-i}d_2^*(f_i(y_n), f_i(y))+\epsilon/3=\epsilon/3.
$$

Thus $\lim d_1^*(y_n, y)=0$. Lastly $2^{-i}d_2^*(f_i(a), f_i(b))\leqq d_1^*(a, b)$ or $d_2^*(f_i(a), f_i(b))\leqq 2^{+i}d_1^*(a, b)$. Hence for $\epsilon>0$, choose $\delta=\epsilon/2^i$. This proves f_i: $S_1\to S_2$ is uniformly continuous relative to d_1^* and d_2^*.

BOUNDS ON THE EXPECTATION FOR SUMS OF POWERS

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In the article *Expectations for sums of powers* (this MONTHLY, vol. 66, 1959, pp. 494–496), Newman and Klamkin consider the following problem:

A sequence of independent random variables with a uniform distribution is chosen from the interval (0, 1). The process is continued until the sum of the nth powers of the chosen numbers exceeds 1. What is the expected number of such choices? They show that this expected number is given by

$$E_n = \sum_{i=0}^{\infty} [\Gamma(1 + 1/n)]^i/\Gamma(1 + i/n). \tag{1}$$

Further, for large n, $E_n \sim C \cdot n$ where

$$C = e^{e^{-\gamma}} - \int_{-\infty}^{\infty} \frac{e^{-e^u}du}{(u + \gamma)^2 + \pi^2} \cdot \tag{2}$$

In this note upper and lower bounds on E_n are derived for all n, and hence a bound on the integral in (2) is obtained. Let

$$C_n = \frac{E_n}{n} = \frac{1}{n} \sum_{i=0}^{\infty} [\Gamma(1 + 1/n)]^i/\Gamma(1 + i/n);$$

then

$$C_n = \frac{1}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} [\Gamma(1 + 1/n)]^{ni+j}/\Gamma(1+i+j/n).$$

Since $\Gamma(1+i) < \Gamma(1+i+j/n)$,

$$C_n < \frac{1}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} [\Gamma(1 + 1/n)]^{ni+j}/\Gamma(1 + i) = \frac{e^{\Gamma^n(1+1/n)}[1 - \Gamma^n(1 + 1/n)]}{n[1 - \Gamma(1 + 1/n)]} \cdot$$

Also $\Gamma(2+i) > \Gamma(1+i+j/n)$ and hence

$$C_n > \frac{1}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{n-1} [\Gamma(1 + 1/n)]^{ni+j}/\Gamma(2 + i) = \frac{[e^{\Gamma^n(1+1/n)} - 1][1 - \Gamma^n(1 + 1/n)]}{n\Gamma^n(1 + 1/n)[1 - \Gamma(1 + 1/n)]} \cdot$$

For large n, $\Gamma(1+1/n) \sim e^{-\gamma/n+O(1/n^2)}$, so that in the limit

$$\frac{(e^{e^{-\gamma}} - 1)(1 - e^{-\gamma})}{\gamma e^{-\gamma}} < C_\infty < \frac{e^{e^{-\gamma}}(1 - e^{-\gamma})}{\gamma},$$

or $1.0192 < C_\infty < 1.3320$, where γ is the Euler's constant.

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OSCILLATORY LIMITING BEHAVIOR OF A RANDOM SEQUENCE

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Let X_n, $n=0, 1, \cdots$, be random variables defined in terms of a given sequence $\{(R_k, W_k): k \geqq 0\}$ of independent, identically distributed random vectors with nonnegative integer components by

$$X_n = R_k \quad \text{if} \quad \sum_{h=0}^{k-1} W_h \leqq n < \sum_{h=0}^{k} W_h.$$

For example, let the R_k be random choices of states of a system and the W_k random waits in these states. The system is in state i at time n if $X_n = i$.

The sequence $\{X_n: n \geqq 0\}$ is a discrete renewal process. For expositions of renewal theory see Smith [1] or Feller [2], Ch. 13. Let

$$P_{in} = \Pr\{X_n = i\}, \qquad p_i = \Pr\{R_k = i\},$$
$$a_{ij} = \Pr\{W_k = j \mid R_k = i\}, \qquad w_j = \Pr\{W_k = j\},$$

for i with $p_i > 0$ and $j=0, 1, \cdots$. The P_{in} can be obtained by induction from the easily derived *renewal equation*,

$$P_{in} = \sum_{j=1}^{n} w_j P_{i,n-j} + p_i \sum_{j>n} a_{ij}.$$

The theory of Feller yields

$$\lim_{n\to\infty} P_{in} = \frac{p_i \mathrm{E}\{W_k \mid R_k = i\}}{\mathrm{E}\{W_k\}},$$

whenever the right member exists.

To treat cases of nonconvergence we define

$$a_i(x) = \sum_j a_{ij}x^j, \qquad w(x) = \sum_j w_j x^j, \qquad P_i(x) = \sum_n P_{in}x^n,$$

for $0 \leqq x \leqq 1$. By well-known properties of generating functions (*i.e.*, of Abel summability methods),

$$\liminf_{n\to\infty} P_{in} \leqq \liminf_{x\to 1} (1-x)P_i(x) \leqq \limsup_{x\to 1} (1-x)P_i(x) \leqq \limsup_{n\to\infty} P_{in}.$$

If $\lim_{n\to\infty} P_{in}$ exists then*

$$\lim_{n\to\infty} P_{in} = p_i \lim_{x\to 1} \frac{1-a_i(x)}{1-w(x)} = p_i \lim_{x\to 1} \frac{1-a_i(x)}{1-\sum_i p_i a_i(x)}.$$

If both $\lim_{n\to\infty} P_{in}$ and $\lim_{x\to\infty} a_i'(x)/w'(x)$ exist then

* Since the Abel method is stronger than any Cesàro method, we may replace $\lim_{n\to\infty} P_{in}$ here by any $C-m$ generalized limit. Cf [3], pp. 19–20. This observation is due to the referee.

$$\lim_{n\to\infty} P_{in} = p_i \lim_{x\to 1} \frac{a_i'(x)}{w'(x)} = \lim_{x\to 1} \frac{p_i a_i'(x)}{\sum_i p_i a_i'(x)}.$$

Consider the case with $p_1 = p_2 = \frac{1}{2}$. We have

$$p_1 \frac{1 - a_1(x)}{1 - w(x)} = \left[1 + \frac{1 - a_2(x)}{1 - a_1(x)}\right]^{-1}.$$

Choose α, β arbitrarily such that $0 \leqq \alpha \leqq \beta \leqq \infty$. There exist $a_i(x)$ such that

$$\liminf_{x\to 1} \frac{1 - a_2(x)}{1 - a_1(x)} = \alpha, \quad \limsup_{x\to 1} \frac{1 - a_2(x)}{1 - a_1(x)} = \beta.$$

Here is the idea of a proof of an even stronger assertion. Neglecting zero terms we write $a_i(x) = \sum_{j=0}^{\infty} c_{ij} x^{n_{ij}}$, where the c_{ij} are arbitrary such that $c_{ij} > 0$ and $a_i(1) = 1$, and the increasing sequences $\{n_{ij} : j \geqq 0\}$ of nonnegative integers are at our disposal. Let $r_{ik} = \sum_{j=m_{ik}}^{\infty} c_{ij}$, where the m_{ik} are chosen such that $m_{ik} < m_{i,k+1}$, $\lim_{k\to\infty} r_{2k}/r_{1k} = \beta$ and $\lim_{k\to\infty} r_{2,k+1}/r_{1k} = \alpha$. There exist n_{ij} and x_{ik} such that $x_{ik} \to 1$ and

$$\lim_{k\to\infty} \left[\frac{1 - a_2(x_{2k})}{1 - a_1(x_{2k})} - \frac{r_{2k}}{r_{1k}}\right] = 0, \qquad \lim_{k\to\infty} \left[\frac{1 - a_2(x_{1k})}{1 - a_1(x_{1k})} - \frac{r_{2,k+1}}{r_{1k}}\right] = 0.$$

Further details are omitted. If $\alpha = 0$ and $\beta = \infty$ we have $\liminf_{n\to\infty} P_{1n} = 0$ and $\limsup_{n\to\infty} P_{1n} = 1$; moreover, $\{P_{1n} : n \geqq 0\}$ does not converge in any Cesàro sense.

By similar methods the n_{ij} can be chosen such that

$$\lim_{x\to 1} [1 - a_2(x)]/[1 - a_1(x)]$$

exists but $\lim_{x\to 1} [a_2'(x)/a_1'(x)]$ does not exist, or, alternatively, such that neither limit exists. The basic principle involved in all of the proofs is that the relative rates of convergence of two series are affected by the insertion of gaps.

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ON THE EVALUATION OF $\int_0^\infty (\sin^m t)/t^m dt$ BY THE TRAPEZOIDAL RULE

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The application of a formula due to Poisson shows that the integral $\int_0^\infty (\sin^m t)/t^m dt$ may be evaluated exactly by the trapezoidal rule using certain ranges of interval, dependent on m. Expansion of the series of terms thus formed

ISOLATED POINTS IN A BANACH SPACE

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It is known* that any subset of the real line has at most a countable number of points that are isolated from either the right or the left. The theorem proved here is a generalization of this result to separable Banach spaces.

In the following B shall be a separable Banach space over the reals and $S'(x, \rho)$ will be the shell of the sphere of radius ρ about the point x. The set $X=\{x_i: i=1, 2, \cdots\}$ shall be a countable dense subset of $S'(0, 1)$.

DEFINITION 1. *A set $P\subset B$ is positive linear if it is contained in a positive half ray; that is, there exists an $x\in B$ such that $P\subset\{\lambda x: \lambda>0\}$.*

DEFINITION 2. *If $P\subset B$, then a point x in P is isolated from the left if there is a number $\epsilon>0$ such that if $0<\lambda<1$ and λx is in P then $\|x-\lambda x\|\geqq\epsilon$.*

LEMMA. *If $P\subset B$ is positive linear then P contains at most a countable number of points isolated from the left.*

Proof. This follows from the fact that a positive half ray is homeomorphic to $(0, +\infty)$.

THEOREM. *If $M\subset B$ then for each $\rho>0$, with at most a countable number of exceptions, we have*

$$S(0, \rho)\cap M - S^0(0, \rho)\cap M = \varnothing. \tag{1}$$

Proof. Let $\mathfrak{N}$ be the collection of all spheres about the origin for which (1) does not hold. The theorem shall be demonstrated by constructing a 1:1 mapping θ of $\mathfrak{N}$ into a countable set I.

If $y\in S'(0, 1)$ and $a>0$ let $N(y, a)=S(y, a)\cap S'(0, 1)$ and if $x\in N(y, a)$ let $M(x, y, a)=\{\lambda y: \lambda x\in M, \lambda>0\}$. Let $M(y, a)=\bigcup_{x\in N(y, a)} M(x, y, a)$. Notice that if $x\in N(y, a)$ and if λx and $\mu x\in M$ then λy and $\mu y\in M(x, y, a)$ and $\|\lambda x-\mu x\|=\|\lambda y-\mu y\|$.

Since $M(y, a)$ is positive linear, let $I(y, a)$ be the countable set of points of $M(y, a)$ which are isolated from the left. If $i, n=1, 2, \cdots$ then $I=\bigcup_{i,n} I(x_i, 1/n)$ is a countable set.

If $S(0, \rho)\in\mathfrak{N}$ then there is a point y such that $\|y\|=1$, $\rho y\in M$, and there is an $\epsilon>0$ such that $\rho>\epsilon>0$ and

$$S^0(0, \rho)\cap M\cap S(\rho y, \epsilon) = \varnothing. \tag{2}$$

Choose n and $x_i\in X$ such that $\|\rho x_i-\rho y\|<\rho/n<\epsilon/4$ and let $\theta(S(0, \rho))=\rho x_i$.

Since $\rho y\in S(\rho x_i, \rho/n)$ and $\rho y\in M$ we see that $y\in S(x_i, 1/n)\cap S'(0, 1)=N(x_i, 1/n)$ and consequently $\rho x_i\in M(x_i, 1/n)$. From the fact that $S(\rho x_i, \rho/n)\subset S(\rho y, \epsilon/2)$ we see that $S(x_i, 1/n)\subset S(y, \epsilon/2\rho)$, $N(x_i, 1/n)\subset N(y, \epsilon/2\rho)$. In order to show that $\rho x_i\in I$ it suffices to show that ρx_i is isolated from the left in

* S. Saks, Theory of the Integral, New York, 1937, p. 206.

$M(x_i, 1/n)$. If $0<\lambda<1$ and $\lambda\rho x_i \in M(x_i, 1/n)$ then there is a $z \in N(x_i, 1/n) \subset N(y, \epsilon/2\rho)$ such that $\lambda\rho z \in M$. Since $z \in S(y, \epsilon/2\rho)$ we have

$$\rho z \in S(\rho y, \epsilon/2). \tag{3}$$

Since $\|z\| = 1$ and $\lambda < 1$ we have $\lambda\rho z \in S^0(0, \rho) \cap M$ and from (2) it follows that $\lambda\rho z \notin S(\rho y, \epsilon)$. Combining this and (3) we obtain $\|\lambda\rho x_i - \rho x_i\| = \|\lambda\rho z - \rho z\| \geqq \epsilon/2$ and it follows that $\rho x_i \in I$.

If $S(0, \rho) \neq S(0, \beta)$ then $\rho \neq \beta$ and $\theta(S(0, \rho)) = \rho x_i \neq \beta x_j = \theta(S(0, \beta))$ from which it follows that θ is 1:1.

COROLLARY. *If $M \subset B$ and x is fixed then for each $\rho > 0$, with at most a countable number of exceptions, we have*

$$S(x, \rho) \cap M - \overline{S^0(x, \rho) \cap M} = \varnothing.$$

Proof. This follows from the fact that translation by $-x$ is a homeomorphism.

The hypothesis that B is separable cannot be dropped. If B is a Hilbert space over the reals with an orthonormal basis $\{\phi_c : c \in [0, 1]\}$ and if $M = \{c\phi_c : c \in [0, 1]\}$ then the theorem is false.

AN IDENTITY OF CAYLEY

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Cayley [1] proved that

$$\begin{vmatrix} a_1^{-1} & a_2^{-1} & a_3^{-1} \\ b_1^{-1} & b_2^{-1} & b_3^{-1} \\ c_1^{-1} & c_2^{-1} & c_3^{-1} \end{vmatrix}^{+} \cdot \begin{vmatrix} a_1^{-1} & a_2^{-1} & a_3^{-1} \\ b_1^{-1} & b_2^{-1} & b_3^{-1} \\ c_1^{-1} & c_2^{-1} & c_3^{-1} \end{vmatrix} = \begin{vmatrix} a_1^{-2} & a_2^{-2} & a_3^{-2} \\ b_1^{-2} & b_2^{-2} & b_3^{-2} \\ c_1^{-2} & c_2^{-2} & c_3^{-2} \end{vmatrix}$$

provided

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

In this note we prove the following generalization.

THEOREM. *If the matrix (a_{rs}) is of order n and of rank $\leqq 2$, and all $a_{rs} \neq 0$, then*

$$|a_{ij}^{-1}|^{+} \cdot |a_{ij}^{-1}| = |a_{ij}^{-2}|. \tag{1}$$

The case $n = 3$ is Cayley's result; the case $n = 4$ has been treated by Levine in a recent paper [2]. The proof in the general case is similar to Levine's proof. By a formula of Muir [3]

$$(2)\qquad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_{+} \cdot \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}_{+} = \sum \pm \begin{vmatrix} a_{11}x_{i1} & \cdots & a_{1n}x_{in} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1}x_{k1} & \cdots & a_{nn}x_{kn} \end{vmatrix},$$

where the summation is extended over all $n!$ permutations $ij \cdots k$ of $12 \cdots n$ and the $+$ or $-$ sign is used according as the permutation is even or odd.

Now in (2) take $x_{rs} = a_{rs}$ and then replace a_{rs} by a_{rs}^{-1}. Then the term on the right corresponding to the identity permutation evidently yields the right member of (1). Consider any other permutation p. Write p as a product of disjoint cycles. If any one of the cycles is a transposition it follows that the corresponding determinant in the right member of (2) has two equal rows and therefore vanishes. For a cycle of length h, where $h \geqq 3$, the corresponding determinant also vanishes.

We may assume that the rows of the matrix (a_{rs}) are linearly dependent on the first two rows and that the cycle is $(12 \cdots h)$, where $n \geqq h \geqq 3$. The first h rows of the corresponding determinant are

$$\begin{bmatrix} a_{11}^{-1}a_{21}^{-1} & \cdots & a_{1n}^{-1}a_{2n}^{-1} \\ a_{21}^{-1}a_{31}^{-1} & \cdots & a_{2n}^{-1}a_{3n}^{-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{h1}^{-1}a_{11}^{-1} & \cdots & a_{hn}^{-1}a_{1n}^{-1} \end{bmatrix} = (a_{ij}^{-1}a_{i'j}^{-1}),$$

where $i' = i+1$ for $1 \leqq i < h$, $i' = 1$ for $i = h$. Then

$$(a_{ij}^{-1}a_{i'j}^{-1}) = \left(k_j \prod_{\substack{s=1 \\ s \neq i,i'}}^{h} a_{sj} \right),$$

where $k_j = \prod_{s=1}^{h} a_{sj}^{-1}$ $(j = 1, \cdots, n)$. Since $k_j \neq 0$ this latter matrix has the same rank as the matrix

$$A = \left(\prod_{\substack{s=1 \\ s \neq i,i'}}^{h} a_{sj} \right) \qquad (i = 1, \cdots, h; j = 1, \cdots, n).$$

By our assumption concerning (a_{rs}), we have

$$a_{sj} = b_s a_{1j} + c_s a_{2j} \qquad (s.\ j = 1, \cdots, n).$$

Hence

$$\prod_{\substack{s=1 \\ s \neq i,i'}}^{h} a_{sj} \prod_{s=1}^{h} (b_s a_{1j} + c_s a_{2j}) = \sum_{s=0}^{h-2} d_{is} a_{1j}^{h-2-s} a_{2j}^{s},$$

so that the row space of A is spanned by $h-1$ vectors and the rank of A is less

than h. Thus the corresponding determinant vanishes. This completes the proof of the theorem.

For a cycle of even length it is easily seen that the corresponding term in (2) is cancelled by the determinant in which this cycle is replaced by its inverse; this does not make use of the hypothesis concerning the rank of (a_{rs}). Consequently we may state the following corollary of Muir's formula (2):

If (a_{rs}) is an arbitrary matrix of order n then

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_{+} \cdot \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}_{+} = \sum \begin{vmatrix} a_{11}a_{i1} & \cdots & a_{1n}a_{in} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n1}a_{k1} & \cdots & a_{nn}a_{kn} \end{vmatrix},$$

where the summation is extended over these permutations $(ij \cdots k)$ that are products of disjoint cycles of odd length.

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ANOTHER PROOF OF THE FOUR VERTEX THEOREM

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The fundamental theorem of Schur [2, 3] has a number of fruitful applications. It seems to be of interest to observe that the four vertex theorem for convex curves is also an easy consequence of Schur's theorem. The latter states: *Let D be a plane arc which together with the chord connecting its end points forms a convex curve. Let D^* be an arc of the same length as D and let s be the arc length parameter for both curves. Suppose that at corresponding points the continuous curvatures $k(s)$ and $k^*(s)$ satisfy $k(s) \geqq k^*(s)$ and that d and d^* denote the lengths of the chords joining the end points of D and D^* respectively. Then $d \leqq d^*$, and equality holds only if for all s, $k(s) = k^*(s)$.*

FOUR VERTEX THEOREM. *Let C be any closed convex curve in the plane with continuous curvature. Then there are at least four points on C at which the curvature assumes a relative maximum or minimum.*

It is a mild convenience in the proof of the theorem to assume that the length of C is 2π and that C is oriented so that the curvature is everywhere nonnegative. Then either C is a circle, in which event there is nothing to prove, or there are points on C where the curvature is strictly greater than the average and points where it is strictly less than average. Thus, the set D_1 of points where the curvature is greater than or equal to 1 is a nonempty proper closed subset of C. We claim that it is not an arc. Suppose that it were; then the end

points of D_1 also determine an arc, D_2^*, the complementary arc to D_1. Now let E be a circle of length 2π, and let D_1^* be an arc of E of the same length as D_1 and let D_2 be the complementary arc to D_1^*. Then, by Schur's lemma, $d_1 < d_1^*$ and $d_2 < d_2^*$. But $d_1 = d_2^*$ and $d_1^* = d_2$. This contradiction shows that D_1 is indeed not an arc. Since D_1 is disconnected, there are two points on C, each not in D_1, which cut C into two arcs, each of which contain points of D_1. Hence the maximum value of the curvature on each of these arcs is assumed at interior points of the arcs. At each of these points the curvature has a relative maximum. The usual observation that between any two relative maxima there exists a relative minimum completes the proof of the theorem.

The idea of the above proof also establishes another characterization of the circle. Namely,

THEOREM 1. *Let C be a closed convex planar curve with continuous curvature. Suppose that there is an arc of C which contains every point where the curvature is strictly greater than the average curvature of C, but contains no point where it is strictly less than the average. Then C is a circle.*

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CLASSROOM NOTES

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A PROOF OF THE VECTOR TRIPLE CROSS PRODUCT IDENTITY

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The purpose of this note is to present a relatively simple proof of the following relation for the triple cross product of three vectors **a**, **b** and **c**

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}). \tag{1}$$

This identity is usually left for the students to verify with the assumption that they will simply write out the two members of the equation in component form and then compare them. An exception to this normal procedure is made by Lass, *Vector and Tensor Analysis*. There he presents a rather nice proof in which he maintains the spirit of the subject by developing the relation in terms of the vector language.

Those students who have been exposed to a course in tensor analysis know

that the notation of that subject makes the proof of (1) almost trivial.* The following proof is an adaptation of that given in terms of the tensor notation but stated in such a way that a minimum number of new concepts need to be introduced.

DEFINITION 1. *Let the components of a vector* **A** *be represented by either* A_i *or* A^i, $i=1, 2, 3$. *Then*

$$A^{[i}B^{j]} = A^iB^j - A^jB^i,$$
$$A_{[i}B_{j]} = A_iB_j - A_jB_i. \tag{2}$$

On the basis of (2) we can make the following correspondences

$$\mathbf{b}\times\mathbf{c} \cdots (b^{[j}c^{k]})_i \qquad i, j, k = 1, 2, 3. \tag{3}$$

That is the components of $\mathbf{b}\times\mathbf{c}$ are determined by those three representations of the right-hand member of (3) for which i, j, k are even permutations of 1, 2, 3. Each odd permutation of i, j, k determines the negative of a component of $\mathbf{b}\times\mathbf{c}$ and in the remaining situations, that is when any pair i, j, k take on the same one of the values 1, 2, 3, we define the expression to be zero. With the same understanding as with respect to (3) we have

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \cdots (a_{[p}(b^{[j}c^{k]})_{i]})_r. \tag{4}$$

The theorem to be proved may then be stated as follows:

THEOREM

$$(a_{[p}(b^{[j}c^{k]})_{i]})_r = b^r\sum_{s=1}^{3}a_sc^s - c^r\sum_{s=1}^{3}a_sb^s. \tag{5}$$

Proof. The three components of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ are completely determined by the choice of r and the fact that r, p, i and i, j, k must each form an even permutation of 1, 2, 3. For example, if $r=1$ then $p=2$, $i=3$. Since $i=3$ it follows that $j=1$, $k=2$. Now suppose that α, β, γ is an even permutation of 1, 2, 3 then if $r=\alpha$ the right number of (4) has the following form

$$(a_{[\beta}(b^{[\alpha}c^{\beta]})_{\gamma]})_\alpha. \tag{6}$$

Careful examination of (6) produces the following two facts:

The components of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ are such that

(a) the indices corresponding to r and j agree,

(b) the indices corresponding to p and k agree.

With (6) in mind we may replace the component representation of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ by

$$\sum_{s=1}^{3}a_{[s}(b^{[r}c^{s]})_{i]}, \tag{7}$$

* Nathaniel Coburn, Vector and Tensor Analysis, New York, 1955, p. 120.

where r has replaced j and been dropped as unnecessary in its original position. Moreover p and k have been replaced by a summation index s. The summation does not introduce any new nonzero terms because of the understanding with respect to the representations (3) and (4). For example if $r=1$ then a component of $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ corresponds to $i=3$ and hence $s=2$. In this instance $s=1$ or 3 produces a zero term.

By definition, (7) can be written

$$\sum_{s=1}^{3} a_s(b^{[r}c^{s]})_i - \sum_{s=1}^{3} a_i(b^{[r}c^{s]})_s. \tag{8}$$

Since the same index appears twice in the expression $(b^{[r}c^{s]})_s$ the right-hand term of (8) is equal to zero. Hence (8) reduces to

$$\sum_{s=1}^{3} a_s(b^{[r}c^{s]})_i. \tag{9}$$

Now in (9) given r and s the value of i such that r, s, i forms an even permutation of 1, 2, 3 is uniquely determined. Hence the subscript i may be dropped and (9) written in the form

$$b^r \sum_{s=1}^{3} a_s c^s - c^r \sum_{s=1}^{3} a_s b^s.$$

This completes the proof.

ON THE IRRATIONALITY OF ROOTS

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THEOREM. *For each integer $m \geqq 2$, there is no positive integer whose mth root is a nonintegral rational number.*

Briefly, roots of positive integers are irrational unless they are integral. This theorem is well known and is an easy consequence of, for example, the unique factorization theorem. The purpose of this note is to call attention to a proof of the nonexistence of nonintegral rational roots of integers which depends neither on the unique factorization theorem nor on any theorem concerning divisibility properties peculiar to integers. The proof is by mathematical induction on m. The first step is the proof that

(1) *there is no positive integer whose square root is a nonintegral rational number.*

This part of the proof can be given, after some preparation, to students who are unacquainted with mathematical induction but have had some experience with inequations. [For example, it can be given near the end of a modern ninth-grade mathematics course.] Aside from the familiar ring-properties of the integers and some elementary theorems on inequations, all that is needed for the proof are the two intuitively acceptable theorems:

(A) *Each nonempty set of positive integers contains a least member;*

(B) *For each nonintegral rational number r, there is an integer k such that $k<r<k+1$.*

Proof of (1). Suppose that n is a positive integer such that $\sqrt{n}$ is a nonintegral rational number. Then, by (A), there is a least positive integer q such that $\sqrt{n}\cdot q$ is an integer, and, by (B), there is an integer k such that $k<\sqrt{n}<k+1$. It follows, from simple theorems on inequations, that, since $q>0$, $0<\sqrt{n}\cdot q-kq<q$. Since, consequently, $\sqrt{n}\cdot q-kq$ is a positive integer smaller than q, $\sqrt{n}(\sqrt{n}\cdot q-kq)$ is not an integer. However, $\sqrt{n}(\sqrt{n}\cdot q-kq)=nq-k(\sqrt{n}\cdot q)$, an integer. This contradiction shows that there is no positive integer whose square root is a nonintegral rational number.

The second step is the proof that

(2) *for each integer $m\geqq 2$, if there is no positive integer whose mth root is a nonintegral rational number then there is no positive integer whose $(m+1)$st is a nonintegral rational number.*

In this part of the proof we use the theorem:

(C) *For each choice of integers m, i, and j, there is an integer N such that $i^m-j^m=(i-j)N$.*

[The case $j=0$ is obvious and the case $j\neq 0$ (the only one needed here) is an easy consequence of the theorem according to which, for each positive integer m and each real number x, $x^m-1=(x-1)\sum_{v=0}^{m-1}x^v$. Students who are taking a modern eleventh-grade mathematics course should be acquainted with this theorem on the sum of a geometric progression, as well as with the method of mathematical induction by which it is proved. The whole content of this note, suitably expanded, makes appropriate material for such students, and furnishes them an interesting application of mathematical induction.]

Proof of (2). Suppose that there is no positive integer whose *mth* root is a nonintegral rational number. It follows that there is no nonintegral rational number whose mth power is a positive integer. Suppose, now, that n is a positive integer such that $\sqrt[m+1]{n}$ is a nonintegral rational number. Then, by (A), there is a least positive integer q such that $\sqrt[m+1]{n}\cdot q$ is an integer, say p, and, by (B), there is an integer k such that $k<p/q<k+1$. It follows [as in the first part of the proof] that $p-kq$ is a positive integer smaller than q, so $\sqrt[m+1]{n}(p-kq)$ is not an integer. Hence, $(p/q)(p-kq)$ is a nonintegral rational number and, as remarked above, it follows [from the "inductive hypothesis"] that

$$[(p/q)(p-kq)]^m,$$

although certainly positive, is not a positive integer. But, by (C), there is an integer N such that $(p-kq)^m-(-kq)^m=pN$. Hence, $[(p/q)(p-kq)]^m=(p^m/q^m)(pN)+(-kq)^m=nqN+(-kp)^m$, an integer. This contradiction shows

that, assuming that there is no positive integer whose mth root is a nonintegral rational number, there is no positive integer whose $(m+1)st$ root is a nonintegral rational number.

The theorem now follows from (1) and (2), by mathematical induction.

It is perhaps interesting to note that the proof of (2) also proves that

(2′) *for each positive integer m, if there is no positive integer whose mth root is a nonintegral rational number then there is no positive integer whose* $(m+1)st$ *root is a nonintegral rational number.*

Now, it is trivial that

(1′) *there is no positive integer whose* 1*st root is a nonintegral rational number,*

and, from (1′) and (2′) one obtains, by mathematical induction, the theorem:

For each positive integer m, there is no positive integer whose mth root is a nonintegral rational number.

Although the proof just outlined is somewhat shorter than that set forth above, the latter has the advantage of showing how little knowledge is required for an understanding of the proof of (1).

THE FUNDAMENTAL THEOREM OF ALGEBRA

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The following proof of the Fundamental Theorem of Algebra has three advantages. It uses methods and concepts which are natural to the algebra courses where the proof is frequently given. Its underlying idea is intuitively clear to the undergraduate. It is easy to follow and not unduly long, once the preliminary spade work is done.

If a polynomial with real coefficients is transformed by dividing all of its roots by the absolute value of one of them, the resulting polynomial will have a root on the unit circle. It will then have a factor in common with the polynomial formed from the reciprocals of its roots and the resultant of the two should vanish. The essence of the proof is to show that for some value of the divisor the resultant does indeed vanish.

The argument is restricted to the real field. For this reason the theorem has been phrased as the factorization of real polynomials. Since the product of a polynomial with complex coefficients and its conjugate is real, the extension to the usual form of the fundamental theorem is immediate.

THEOREM. *Any polynomial with real coefficients may be expressed as a product of linear and quadratic polynomials with real coefficients.*

Proof. The proof is by induction on n, the degree of the polynomial. Write the polynomial as

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

and suppose that neither a_0 nor a_n is zero. Consider the polynomials

$$P_1(x) = a_0r^nx^n + a_1r^{n-1}x^{n-1} + \cdots + a_n,$$

$$P_2(x) = a_nx^n + a_{n-1}rx^{n-1} + \cdots + a_0r^n.$$

Bézout's form of the resultant* of $P_1(x)$ and $P_2(x)$ is a determinant of order n symmetric with respect to its main diagonal and also, in this instance, symmetric with respect to its other diagonal. Denote by D the determinant obtained from the resultant by interchanging the diagonals through successive interchanges of rows. Then D vanishes when $P_1(x)$ and $P_2(x)$ have a common factor. Moreover, D is a symmetric determinant with the following properties. Every element on the main diagonal is a polynomial in r whose leading and constant terms are respectively $a_0^2r^{2n}$ and $-a_n^2$. Every element not on the main diagonal is a polynomial in r of degree less than $2n$ whose constant term is 0. Let r_0 be the largest value of r for which any principal minor of D vanishes. Such a number exists since every principal minor of odd order is negative when $r=0$ and positive for sufficiently large values of r. If D is not zero for $r=r_0$, there is a nest of principal minors of consecutive orders, say D_k, D_{k+1}, D_{k+2}, such that only $D_{k+1}=0$ when $r=r_0$. The existence of this nest of minors is a general property of nonsingular symmetric matrices that contain at least one principal minor that is zero. It is a consequence of the fact that not all the principal minors of two consecutive orders can vanish† and may be proved by a simple induction on the order of the matrix. It is necessary, however, to define a zero-order minor to be $+1$. Since $D_{k+1}=0$, it is well known that D_k and D_{k+2} have opposite signs.‡ As they are both positive for sufficiently large values of r, one of them becomes zero for some $r>r_0$, a contradiction. Hence D vanishes for $r=r_0$. This implies that there are relatively prime polynomials $Q_1(x)$ and $Q_2(x)$ of degree less than n such that for $r=r_0$

$$P_1(x)Q_1(x) = P_2(x)Q_2(x).$$

If $Q_2(x)$ is not a constant, the induction assumption insures that it is a product of linear and quadratic factors which must then divide $P_1(x)$. The quotient is a similar product and thus so also is $P_1(x)$. The factorization of $P(x)$ is obtained by replacing x by x/r_0. If $Q_2(x)$ is a constant, $P_1(x)$ for $r=r_0$ is a reciprocal polynomial. Either $x\pm1$ is a factor and the previous argument applies or $P_1(x)$ may be transformed to a polynomial of half the degree. Therefore $P_1(x)$ is a product of quadratic and quartic reciprocal polynomials. But the factorization of quartic reciprocal polynomials is readily established.

* W. S. Burnside and A. W. Panton, The Theory of Equations, vol. II, Dublin, 1904, pp. 76–78.

† Maxime Bôcher, Introduction to Higher Algebra, New York, 1935, p. 57.

‡ W. S. Burnside and A. W. Panton, op. cit., p. 50. The restriction to leading minors imposed by Burnside and Panton is easily removed.

THE EXTENDED LAW OF THE MEAN BY A TRANSLATION-ROTATION OF AXES

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In the Classroom Note *Proof of the mean value theorem* (this MONTHLY, vol. 65, 1958, pp. 362–364), C. L. Wang uses a translation-rotation of the Cartesian coordinate system to obtain the Mean Value Theorem from Rolle's Theorem. The chief merit of this proof is that it provides a natural geometric motivation for the Mean Value Theorem. However, as the author points out, there are cases where his method of proof breaks down. This occurs whenever the graph of the given curve doubles back upon itself and is not the graph of a *function* in terms of the XY-coordinate system. (See Fig. 1 for example.)

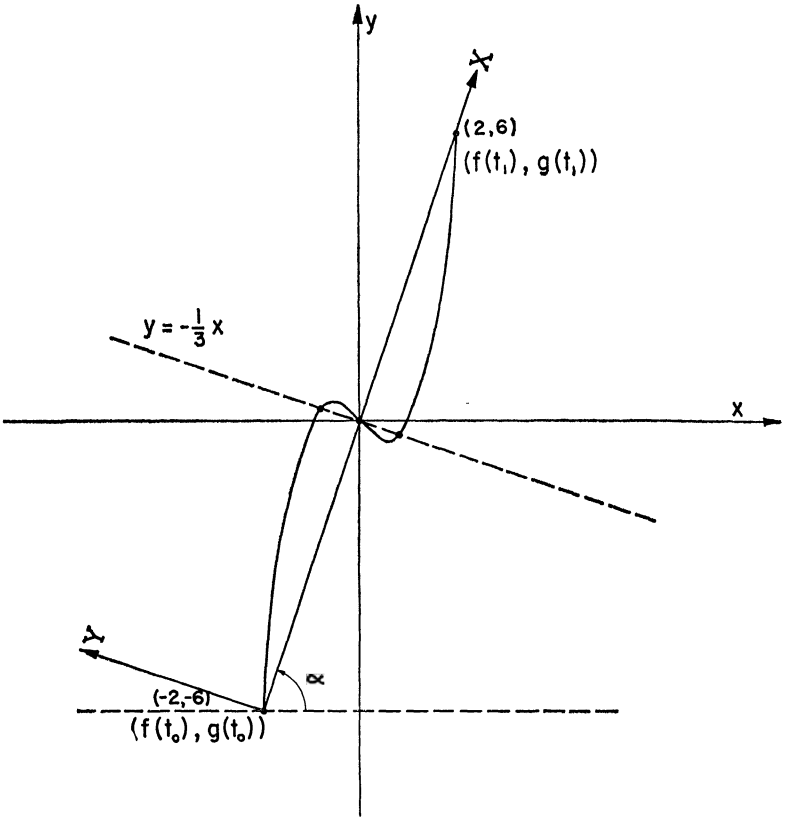


FIG. 1. $y=x^3-x$, $-2\leqq x\leqq 2$; $x=f(t)$, $y=g(t)$, $t_0\leqq t\leqq t_1$.

The method of proof used by Wang can be adapted to cover all cases. When this is done we obtain the Extended Law of the Mean. The proof retains the virtue of yielding a natural motivation and interpretation of this theorem and avoids the use of artificially concocted functions to which Rolle's Theorem is usually applied.

Let $x=f(t)$, $y=g(t)$, $t_0 \leqq t \leqq t_1$, where f and g are continuous for $t_0 \leqq t \leqq t_1$, and $f'(t)$, $g'(t)$ exist for $t_0 < t < t_1$. If $g(t_0)=g(t_1)=0$, there exists, by Rolle's Theorem applied to g, at least one T, $t_0 < T < t_1$, such that $g'(T)=0$. Let the given equations be regarded as the parametric representation of a given curve. If $f'(T) \neq 0$, this curve must have a horizontal tangent. If $f'(T)=0$, we can draw no conclusion without further investigation.

Now assume that $g(t_0) \neq g(t_1)$ but retain the other conditions on f and g. The geometric situation described above can be shifted to a new coordinate system by a translation and rotation of the xy-system into the XY-system. Using the results of Wang, the parametric equations of the given curve, in terms of the new coordinate system, are:

$$F(t) = X = [g(t) - g(t_0)] \sin \alpha + [f(t) - f(t_0)] \cos \alpha,$$
$$G(t) = Y = [g(t) - g(t_0)] \cos \alpha - [f(t) - f(t_0)] \sin \alpha, \qquad t_0 \leqq t \leqq t_1.$$

By Rolle's Theorem applied to G there exists at least one T, $t_0 < T < t_1$, such that $G'(T) = g'(T) \cos \alpha - f'(T) \sin \alpha = 0$, from which it follows that $g'(T)[f(t_1)-f(t_0)] = f'(T)[g(t_1)-g(t_0)]$. This result when interpreted geometrically in terms of the xy-coordinate system simply says that either there is one point on the given curve such that the tangent there is parallel to the secant through the two endpoints of the curve or else there is an indeterminate point. If we make the added assumption that $f'(t) \neq 0$, $t_0 < t < t_1$, the first case occurs and we can write:

$$\frac{g'(T)}{f'(T)} = \frac{g(t_1) - g(t_0)}{f(t_1) - f(t_0)},$$

thus obtaining the Extended Law of the Mean.

A NOTE ON TWO CONVERGENCE TESTS

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In the theory of series of positive terms we use four elementary convergence tests that depend on the following limits;

I. $\lim_{n \to \infty} \sqrt{u_n} = \lambda_1$ (Cauchy's first test),

II. $\lim_{n \to \infty} (u_{n+1}/u_n) = \lambda_2$ (D'Alembert's test),

III. $\lim_{n \to \infty} \dfrac{\ln (1/u_n)}{\ln (n)} = \lambda_3$ (Cauchy's second test [1])

IV. $\lim_{n \to \infty} [n\{1 - (u_{n+1}/u_n)\}] = \lambda_4$ (Raabe-Duhamel's test [2]),

where u_n is the term of order n in the series and ln denotes the natural logarithm.

It is a known result that if λ_2 exists, then λ_1 exists. Moreover, $\lambda_1 = \lambda_2$. The

two tests are then equivalent, the first being more general than the second.

The purpose of this note is to establish a similar relationship between the last two tests. This relationship is embodied in the following:

THEOREM. *If* λ_4 *exists, finite or infinite, then* λ_3 *exists and* $\lambda_3=\lambda_4$.

Proof. We shall use the following theorem:

If $\lim_{n\to\infty}\{(a_{n+1}-a_n)/(b_{n+1}-b_n)\}=\mu$, *where* $b_{n+1}>b_n$ *and* $\lim_{n\to\infty} b_n=\infty$, *then* $\lim_{n\to\infty}(a_n/b_n)=\mu$. ($\mu$ *can be finite or infinite* [3].)

Let us put $a_n=\ln(1/u_n)$, $b_n=\ln(n)$. Then

$$\lim_{n\to\infty}\frac{\ln(1/u_{n+1})-\ln(1/u_n)}{\ln(n+1)-\ln(n)}=\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)].$$

By the theorem quoted above, if $\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)]=\mu$, then

$$\lim_{n\to\infty}\frac{\ln(1/u_n)}{\ln(n)}=\mu.$$

But $\ln(1+x)=x-\frac{1}{2}x^2M(x)$, where $M(x)>0$ and is bounded in $-1<B\leqq x\leqq A$. Now let us assume that λ_4 is finite; then

$$\lim_{n\to\infty}(u_{n+1}/u_n)=\lambda_2 \quad\text{and}\quad \lim_{n\to\infty}[(u_{n+1}/u_n)-1]=0.$$

Hence

$$\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)]=\lim_{n\to\infty}[n\{1-(u_{n+1}/u_n)\}+\tfrac{1}{2}n\{1-(u_{n+1}/u_n)\}^2M(n)].$$

But, since λ_4 is finite, it follows that

$$\lim_{n\to\infty}[-n\ln(u_{n+1}/u_n)]=\lambda_4=\lim_{n\to\infty}\frac{\ln(1/u_n)}{\ln(n)}.$$

Now in the case $\lambda_4=+\infty$, given $N>0$, there exists n_0 such that $n\{1-(u_{n+1}/u_n)\}>N$ for all $n>n_0$. Then $\ln\{1-(N/n)\}>\ln(u_{n+1})-\ln(u_n)$. Hence

$$(N/n)[1+\tfrac{1}{2}(N/n)M(n)]<\ln(1/u_{n+1})-\ln(1/u_n),$$

$$N<\frac{N}{n\ln\{1+(1/n)\}}<\frac{\ln(1/u_{n+1})-\ln(1/u_n)}{\ln(n+1)-\ln(n)}$$

for all $n>n_0$. Then

$$\lim_{n\to\infty}\frac{\ln(1/u_{n+1})-\ln(1/u_n)}{\ln(n+1)-\ln(n)}=+\infty=\lim_{n\to\infty}\frac{\ln(1/u_n)}{\ln(n)}.$$

In the case $\lambda_4=-\infty$, the proof is similar.

Remark. The converse of the theorem is not true. To see this, let us consider the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \cdots$. Then $\lim_{n\to\infty}[n\{1-(u_{n+1}/u_n)\}]$ does not exist, but $\lim_{n\to\infty}[\ln(1/u_n)/\ln(n)]=0$. Hence the relationship between λ_4 and λ_3 is analogous to that between λ_2 and λ_1.

References

1. E. Goursat, Cours d'Analyse Mathématique, Paris, 1943.
2. L. Brand, Advanced Calculus, New York, 1958.
3. T. J. I'A. Bromwich, An Introduction to the Theory of Infinite Series, London, 1949.

A METHOD FOR THE CONSTRUCTION OF SINGLY EVEN MAGIC SQUARES

JEROME ROTHSTEIN, Edgerton, Germeshausen and Grier Inc., Boston, Mass.

Books on mathematical recreations are replete with rules for constructing magic squares* of odd order, *i.e.*, $n=2m+1$. Doubly even magic squares, *i.e.*, those for which $n=4(2m+1)$ are easily constructed in several ways, *e.g.*, by "factoring" into $(2m+1)^2$ four by four blocks, or into sixteen $(2m+1)\times(2m+1)$ blocks. In the first case, the 4×4 magic square of Figure 1, with each number increased by $16(k-1)$ goes in place of k in any $(2m+1)\times(2m+1)$ magic square.

1	15	14	4
11	5	6	8
7	9	10	12
13	3	2	16

FIG. 1

The second construction puts a $(2m+1)\times(2m+1)$ magic square, with each number increased by $(2m+1)^2(k-1)$ in place of k in the 4×4 square. These methods fail in the case of the singly even square, $n=2(2m+1)$, because no magic square exists for $n=2$. This paper shows how to vary the pattern in 2×2 blocks systematically to yield a magic square of order $2(2m+1)$. (We have not discovered an analogous way to alter a block of four odd squares to yield a singly even one.)

The procedure is as follows: (1) Take

1	2
4	3

* By a magic square of order n we mean a square array of positive integers from 1 to n^2 inclusive such that the n rows, n columns, and two diagonals have the same sum. The sum can be shown to be $n(n^2+1)/2$. For an introduction to the field, see, for example, M. Kraitchik, Mathematical Recreations, Chap. VII, New York, 1942.

as a "standard" 2×2 block, and in place of the number k in a $(2m+1)\times(2m+1)$ magic square, put a standard 2×2 block with each number increased by $4(k-1)$. The resulting array will clearly have all its columns with the same sum, and this property is not destroyed if, within a 2×2 block, the upper and lower numbers of a column are interchanged (*e.g.*, 1 & 4 or 2 & 3).

(2) Alternate rows will now have the same sum, and the even rows sum to a value $4(2m+1)$ greater than the odd row sum. Therefore, if $2(2m+1)$ is added to the odd rows while the same amount is subtracted from the even rows, by the expedient of interchanging some upper and lower members of the same column within the same 2×2 block, all the rows will then have the same sum (which will be the same as the column sum). As each "left" column interchange of a 2×2 block shifts 3, while a "right" column shifts 1, $(m+1)$ left column interchanges and $(m-1)$ right column interchanges shift $3(m+1)+m-1=2(2m+1)$, thus leaving only the diagonals to be taken care of.

(3) The diagonal sums before the interchanges of (2) above differ by $2(2m+1)$. A left column interchange in a diagonal 2×2 block raises or lowers a diagonal sum by 3, a right column interchange in a diagonal 2×2 block raises or lowers a diagonal sum by 1. Both interchanges in the same diagonal 2×2 block raises or lowers a diagonal sum by 2. So if, for each diagonal, both interchanges are made in $(m-1)$ 2×2 diagonal blocks, and a left column interchange only is made in one 2×2 diagonal block, then one diagonal can be reduced by $2m+1$ while the other is augmented by the same amount. This would equalize the diagonals and give them the same sum as the columns.

(4) The requirements of (2) and (3) can be carried out simultaneously in many ways, a simple one being as follows: In the 2×2 blocks in the same column as the central 2×2 block and in the 2×2 block immediately to the left of the central block perform left column interchanges only. In the $(m-1)$ columns of 2×2 blocks to the right of the central block perform both right and left interchanges. In the border column of 2×2 blocks on the extreme right perform left interchanges only. The resulting square is magic as (1), (2) and (3) are satisfied. In Figure 2(a), a 5×5 square is exhibited, with a 10×10 square derived therefrom by the above rule shown in Figure 2(b).

Similar rules can be derived for other choices of the standard 2×2 block than that of (1).

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

FIG. 2(a)

65	66	93	94	4	2	32	31	60	58
68	67	96	95	1	3	29	30	57	59
89	90	17	18	28	26	56	55	64	62
92	91	20	19	25	27	53	54	61	63
13	14	24	22	49	50	80	79	88	86
16	15	21	23	52	51	77	78	85	87
37	38	45	46	76	74	84	83	12	10
40	39	48	47	73	75	81	82	9	11
41	42	69	70	99	98	8	7	36	34
44	43	72	71	97	100	5	6	33	35

FIG. 2(b)

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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RECENT RESEARCH IN MATHEMATICS: IMPLICATIONS FOR TEACHER EDUCATION*

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I shall interpret this title quite broadly, as I believe was intended. Research in pure and applied mathematics has long-range implications for teacher education. However, rather immediate and pressing implications follow from recent developments in the uses of mathematics and in experimentation and research in mathematics education. Since it is these implications which are our chief concern, and since that which I say here is intended to be the basis for the even more important discussion which is to follow, I shall turn first to a concise presentation of the current situation and from this deduce a list of implications which can serve as a basis for the following discussion out of which I hope constructive action will develop.

The major features of the current mathematics and mathematics-education landscape are as follows:

1. Mathematics, itself, both pure and applied, is growing at an unprecedented rate.
2. Both the range and amount of the applications of mathematics are ex-

* Presented at the Twelfth Annual Meeting of the American Association of Colleges for Teacher Education, February 11, 1960, Chicago, Illinois.

panding at perhaps an even faster rate and at a wide range of levels of difficulty and sophistication.

3. These two facts have led to an urgent need that more mathematics be understood earlier by those being trained in even the social sciences as well as in science, engineering, and technical vocational programs. In my mind, there is even a corresponding urgency for more mathematical background for all persons as a part of their general education. This latter belief is, however, more founded on philosophical ideas than on documented developments in research and application.
4. An unprecedented amount of study, discussion, experimentation and recommendations for reform in the content, and sequence of the curriculum and in teaching methods is being carried on, using large sums of money, and steadily releasing to teachers and administrators brochures, bibliographies, reports, and experimental classroom materials.

A brief explanation and documentation of these salient features of the mathematics-education landscape may help us to see both them and their implications more clearly. As I go along, I will point out that none of these four features is a post-sputnik eruption, although they have been played up, perhaps unfortunately, by the sputnik hysteria.

There is, of course, no easy way to measure the growth of mathematics. One measure, however, is the growth of its technical and research literature. Today's major mathematical abstracting journal, *Mathematical Reviews*, was founded in 1940 when World War II cut us off from European abstracting services. The 1945 issue contained 334 pages; 1950, 870; 1955, 1338. The number of new journals founded in the last twenty years includes *The Pacific Journal of Mathematics*, *Proceedings* and *Notices of the American Mathematical Society*, *The Arithmetic Teacher*, *Communications* and *Journal of the Association for Computing Machinery*, *Computers and Automation*, *Industrial Quality Control*, *Journal of Mathematics and Mechanics*, *Mathematics Student Journal*, *Journal of the Operations Research Society of America*, *SIAM Newsletter*, *SIAM Review*. In addition to these, several high quality local journals with international circulation have been founded by individual universities such as Illinois and Michigan, but even with an enforced condensation of articles, the flood of productive new ideas still produces a substantial publications backlog.

The titles of some of these new journals provide partial documentation of my second point; namely, that the range and amount of applications of mathematics have also been expanding at a terrific rate. Twenty years ago, and less, what are now substantial areas of application or even university departments were essentially unheard of or largely undeveloped: operations analysis, decision theory, information theory, linear programming, quality control, digital and analogue computation are illustrations. For example, in my school there is now a Professor of Mathematical Biology, who also bears the title "Senior Research Mathematician," located in our Mental Health Institute, a Professor

of Operations Analysis in the College of Engineering. There is an undergraduate course in Quantitative Methods in Biology, there are two courses in psychology which have calculus as a prerequisite, and there are several courses in mathematical economics. Notice that whereas in the past what little mathematics there was in the social sciences was largely statistics, this is no longer true. However, the growth of statistics itself has spurted and the subject has become much more mathematical. All of our engineers now get a little dose of probability and statistics in their freshman-sophomore years, and we now find it difficult to staff the several sections of a year's course in statistics for undergraduate engineers which course did not even exist a few years ago. The Aeronautical Engineering Department now requires all of its students to study theory of functions of a complex variable, and the earlier the better—a course which a few years ago was taken only by mathematicians and was usually a part of the graduate program.

I will leave for the discussion or for your later reading the further elaboration of this point and the fact that the roots of all of this run back before sputnik, and are, I believe, being nourished by the demands of a continuing and irreversible development of several aspects of our culture. I believe that in this discussion I have also documented my third point that there is an urgent need for *more* and *better selected* mathematics to be *understood* by *a much larger segment* of our population. This does not mean merely that we need more mathematicians, but that for both vocational and cultural reasons more persons must be mathematically *literate* and *appreciative* and at higher levels of literacy.

The fourth salient feature which I listed was the amount of study, discussion, reports, and materials relating to the mathematics curriculum and teaching methods which are bombarding teachers and administrators today. I will cite only one basis for this assertion. This little pamphlet, *Studies in Mathematics Education, A Brief Survey of Improvement Programs for School Mathematics*, was published in 1959 by Scott Foresman and Company. It listed only major and extensive programs. It is already out of date, but it listed four programs at the elementary level, thirteen at the secondary level, three at the college level, five under the heading "School Mathematics (Grades K-College)," and one under the heading of "Teacher Training."

All of this is fascinating and important, but we must march on to the major question "What are the implications of this for teacher education?" They are many, but they can be summarized in three statements:

1. Future teachers and administrators must be led to develop the motivation which will cause them throughout their professional careers to be alert to and informed about new programs, and regularly to go farther than this, even to the extent of carrying on continuing curricular and methodological changes and experimentation.
2. Future teachers and administrators must be helped to acquire the fundamental knowledge and understanding and background in both mathe-

matics and pedagogy which is essential to being able to read and critically evaluate the materials and proposals which will rain upon them.

3. These same motivations, knowledge and understanding must somehow be communicated to our in-service teachers and administrators.
4. Experimentation, research, evaluation must be continued in varied places and with varied approaches whether or not the major subsidized programs themselves continue or are replaced by others, or all die out.

These four statements may all seem to be truisms with nothing new in them. However, let us list quickly that which they seem to imply if we consider them in conjunction with the salient features of the present mathematics-education landscape which I listed earlier.

1. *The collegiate mathematics taught to undergraduates preparing to teach must be surveyed and compared with that which is needed to read, understand, and evaluate current proposals.*

This does not necessarily mean that a totally new mathematics program must be designed for those prepararing to teach. It may mean that the mathematics program as a whole for all college students needs revision or that a few special mathematics courses should be designed for and required of prospective teachers. There are three books that illustrate a rather extreme example of what I mean. The first is the experimental text dealing with matrices prepared for the second semester of the twelfth grade by a School Mathematics Study Group writing team last summer. The second is the report of the Commission on Mathematics of the College Entrance Examination Board printed in 1959 which recommended for this semester a choice of abstract algebra, probability and statistics, or introductory ideas of functions and calculus. The third is Volume I of a text, *Modern Mathematical Methods and Models*, which was prepared by the Committee on the Undergraduate Program of the Mathematical Association of America for a college sophomore course for persons with a major interest in the social sciences. It also treats matrices in an elementary way. A college using the CUP program would also have been teaching materials which would have prepared teachers to accept and teach, or to reject, the Commission's and the S.M.S.G.'s proposal. One can not intelligently either accept or reject that of which he knows nothing.

This example of matrices is rather extreme, and I am not sure that I myself favor their introduction into either the high school or as a required element in a teacher training program. However, teachers must somewhere learn more of such things as the foundations of the number system and of mathematics as a whole, of axiomatics, of set theory, of inequalities, of modern definitions of functions and relations, of probability and statistics.

2. *Methods courses in the teaching of mathematics must be given adequate time and be taught by persons who are both informed as to current trends, projects, and literature, and who also have an adequate knowledge of mathematics*

themselves. Mathematics teachers can not be well prepared by individual readings associated with a general methods course.

3. *Undergraduate and graduate programs must be planned with each other in mind.* That is, the undergraduate student should be carried far enough so that he may continue with some graduate study in mathematics. At the same time, graduate programs should begin at a level such that a well-prepared teacher may continue with some mathematics at the graduate level, and graduate offerings should always be planned to offer a varied sequence of courses appropriate for the secondary school teacher.

It is impossible to present all the desirable mathematics in a normal undergraduate teacher's certificate program. Substantial amounts of mathematics should be included in the graduate study program of all mathematics-educators. The development of advanced placement programs, seminars, enriched and accelerated programs from the seventh grade (and even lower) on up says that adequate background has never before been so important and that the junior high teacher must be included when we discuss these matters.

4. *Since many of the current programs lay great stress on the use of pupil discovery and heuristic teaching processes as the means to several ends such as understanding and appreciation of the nature and aliveness of the subject, devices must be found for communicating to students an understanding of and some practice in such techniques.* Care in the selection and training of supervising teachers, increased observation of well-taught classes, and practice in these processes under good close supervision will help. There are some movies in preparation which may help, and a clever and inspiring methods instructor may help teachers actually to write and "rehearse" materials and detailed plans for developmental approaches to new units.
5. *Extensive in-service training projects need to be formulated and staffed.* All of the recommendations made previously also apply to post-graduate and in-service training, but for this type of training to function most effectively special planning and added personnel are needed to meet the requests for extension courses, Saturday classes, continuing consulting aid, and in-service institutes. The college can often supply the initial push, the guidance and the secretarial help needed to initiate and maintain mathematics teachers' study groups and clubs.
6. Since much data is still needed and since nothing inspires both confidence and emulation in teachers more than observing or hearing of first-hand classroom experimentation, *education schools and staffs have a double responsibility to associate themselves with and to share in thoughtfully designed experimentation in mathematics education and in the psychological bases of mathematical learning, problem solving, and creativity.*
7. The roots of most mathematical ideas as well as the development of attitudes toward mathematics are to be found in the elementary schools. *Elementary school teachers must be trained both to understand and to enjoy*

elementary mathematics. In view of the total elementary teacher training program and the tendency of such teachers to be non-mathematical in their interests and background, it is probable that we need special mathematics courses for elementary teachers taught by sympathetic but mathematically sound and well informed persons.

In recent days we have heard much of "modern mathematics" and the lag of secondary school mathematics programs in comparison with the rapid advance of the subject itself. I do not believe that any topic should be deleted from the curriculum because of its age or added because of its modernity. However, I do believe that the curriculum must be the most intelligent selection possible from the totality of all that is important in mathematics today, whether old or new. To make such a selection one must first understand this mathematics and secondly select "intelligently." This latter means the selection must also take into account what types of training are important to the individuals and to the nation and also what of these materials can be taught effectively—meaningfully, for *understanding*—at various stages of pupil maturity. To do this teachers must understand how ideas develop within children and how the teacher functions to facilitate this development. There is much still to be learned of each of these, including how one passes along these insights and procedures to the coming generation of teachers.

West Virginia Study of the Education of Junior High School Mathematics Teachers

West Virginia University, in cooperation with the State Department of Education in West Virginia, is sponsoring an extended study of the preparation of junior high school mathematics teachers. This study has resulted in part from the recognition by the State Department of Education that the most critical shortage of teachers in West Virginia is in the area of junior high school mathematics, and also from the unique program in West Virginia for the certification of teachers for elementary and secondary schools by examination. One of the results of the project is hoped to be the preparation of an examination in mathematics to be used as a part of the West Virginia examination for certification of junior high school mathematics teachers.

The first step in the study was a two-day conference held at West Virginia University on December 7 and 8, 1959. Participants in the conference were leaders in some of the national mathematics curriculum studies, school administrators, and junior high school teachers in the state, as well as representatives of West Virginia University and the State Department of Education. The School Mathematics Study Group (SMSG) has granted permission to several junior high schools in West Virginia to try out the SMSG mathematics courses for the junior high school during the second semester this year and next year as a part of the study.

West Virginia University is sponsoring an in-service course for mathematics teachers during this year and is planning to continue this activity in 1960–61. In the summer of 1960, a study group will meet at West Virginia University to study the new mathematics curriculum materials and the ideas of modern mathematics needed by teachers in order to teach these materials, as a basis for the development of programs for the preparation of junior high school mathematics teachers, both preservice and in-service. The work started during the summer of 1960 will be continued during the academic year 1960–61 and, according to the plan, first steps will be taken in preparation of the examination to which reference was made earlier in this report.

Mathematics in a Science Fair

The Science Fair of Prince Georges County, Maryland for 1960 has chosen the theme "Mathematics." Each participant in the Science Fair is required to produce an exhibit in any field of science which he chooses but which will show the use of mathematics in this field of science. Quite a number of those interested in Science Fairs have been concerned about the lack of attention given to mathematics in these fairs. Prince Georges County appears to have found a unique solution to this problem for one year. For further information on plans in Prince Georges County, an inquiry might be addressed to Mr. Howard Owens, Science Supervisor, Prince Georges County Public Schools, Upper Marlboro, Maryland.

Conference on Elementary Science and Mathematics

The Oklahoma Frontiers of Science, Incorporated, sponsored a Conference on Elementary Science and Mathematics on December 5, 1959. The Conference was intended for school administrators not only in Oklahoma but in all parts of the United States. Sponsoring agencies for the Conference, in addition to the Oklahoma Frontiers of Science, were the American Association for the Advancement of Science, the American Association of School Administrators, and the Council of Chief State School Officers.

A paper on mathematics in the elementary school was presented by Dr. Paul Rosenbloom of the University of Minnesota. Copies of the report of the Conference may be obtained by writing to Dean James Harlow, College of Education, University of Oklahoma, Norman, Oklahoma.

Interest in Elementary School Mathematics Increases

The widespread interest in the improvement of the mathematics education in secondary schools is already bringing about similar interest at the elementary school level. During the past year a number of experimental studies at the elementary school level have been reported in this department. These include reports from the University of Illinois (May, 1959), Stanford University (June–July, 1959), Syracuse University (February, 1960).

Of major importance is the extension of the work of the School Mathematics Study Group to grades 4, 5, and 6. A panel on elementary school mathematics has been appointed. The members of the panel are: E. G. Begle, E. Glenadine Gibb, William T. Guy, Stanley Jackson, Irene Sauble, Marshall Stone, and J. Fred Weaver. A larger group met in Chicago in March for ten days to outline materials which will be written by a still larger group working at Stanford University in the summer. The materials to be prepared this summer probably will consist of a sample textbook for grade 4 and experimental units for grades 5 and 6. There will be teachers' commentaries for all materials prepared.

A conference sponsored by the National Council of Teachers of Mathematics in October has so far largely resulted in a decision by the Board of Directors of the Council to establish advisory committees on elementary and on secondary school mathematics which will formulate long-term plans for the improvement of mathematics education at these levels. The October conference was called to consider the overall future role of the National Council.

Among other recently organized curriculum studies in mathematics, not reported earlier in this Department, are the Washington Academy of Sciences sponsored program and another centered at the University of California, Berkeley, under the direction of Robert Karplus of the Department of Physics. Both of these studies are concerned with the integration of the teaching of mathematics and science, in the first instance at the junior high school level, and in the second, at the elementary school level. With foundation support, Catherine Stern of the New York City Public Schools and Anita Riess of

the University of Bridgeport are continuing some of their earlier work on the teaching of elementary mathematics, which involves in both instances the use of concrete developmental materials.

In the Washington area, the experimental seventh grade mathematics course of the University of Maryland is being tried out with gifted sixth grade pupils; the Washington, D. C., public schools have prepared a new course of study in arithmetic which has been considerably influenced by the courses at the junior high school level of SMSG and the University of Maryland. A parents' committee in Montgomery County, Maryland has requested the school administration to move quickly to the use of special teachers, with adequate mathematics preparation, in all classes in mathematics in the intermediate grades.

The AAAS-sponsored study on the use of special teachers of science and mathematics in grades 5 and 6 has been reported in this Department (October, 1959). A somewhat similar study with support from the Ford Foundation is being carried out in Ossining, New York, and Long Beach, L.I. Representatives of these investigations and other school systems using special mathematics and science teachers in the elementary school met in Washington in May.

In the summer of 1959 the National Science Foundation sponsored two summer institutes for supervisors and teachers of mathematics at the elementary school level. These were held at the University of Michigan and Rutgers University. A report on the Michigan Institute appeared in this Department (vol. 67, pp. 465–466). In 1960 there will be five institutes for teachers at this level. The 1960 institutes are located at Longwood College, the University of Michigan, the College of Saint Catherine, San Jose State College, and the University of Texas. In both summers a number of other institutes for elementary school administrators and teachers were concerned with teacher preparation in both mathematics and science.

AIBS Secondary School Biological Sciences Film Series

The American Institute of Biological Sciences is currently sponsoring a complete secondary school biology course which will make integrated use of more than 120 half-hour film lecture-demonstrations, classroom teaching and printed materials for students and teachers. A unique collaboration of nearly 100 of the nation's leading biologists is responsible for the preparation of the course content. In addition, nearly 100 secondary school teachers are serving as reviewers of the scripts and text materials. Funds to support preparation of the project have been made available by the Ford Foundation. The producer-distributor of the course materials is McGraw-Hill Book Company.

The course materials designed for the 10th grade are expected to be ready for distribution in the fall of 1960. The films will be available in color and black and white, and will be available in any combination desired—single films, topic units, or the entire series. Other materials which will be available are student study guides, teachers' manuals, discussion questions and suggestions for field and laboratory work, examinations, etc.

The Director of the project and continuing teacher in the films is Dr. H. Burr Roney, professor of biology at the University of Houston. Dr. Roney has had perhaps more experience in instructional television and films than any other teacher. The primary aim of the project is to provide maximum aid to the greatest number of secondary school biology teachers. This project is not considered an educational "cure-all" nor is it considered a replacement for the "live" classroom teacher.

More detailed information about the project may be obtained by writing to Mr. Jack Steuerwald, Assistant to the Director, Biological Sciences Film Series, American Institute of Biological Sciences, 2000 P Street, N. W., Washington 6, D. C.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1421. *Proposed by* (1) *S. P. Franklin and G. A. Hutchison, University of California at Los Angeles, and* (2) *W. R. Becker, New York City*

(1) Prove, for any integer $n>1$, $n/(n-1)>\{n^2/(n^2-1)\}^n$.

(2) Prove, for all $a>0$ and $b>0$, $\{(a+1)/(b+1)\}^{b+1}\geqq(a/b)^b$.

E 1422. *Proposed by R. W. Kilmoyer, Jr., Lebanon Valley College*

Find polynomials $f(x)$ such that $f(x^2)+f(x)f(x+1)=0$.

E 1423. *Proposed by Aboulghassem Zirakzadeh, University of Colorado*

Consider a plane closed curve possessing only ordinary points and a finite number of double points. Assign a positive direction to the curve and, starting from an ordinary point A, trace the curve in the given positive direction. Assign number 1 to the first double point met, 2 to the second double point met, and so on until you come back to A. Prove that of the two integers assigned to any double point, one is always even and the other always odd.

E 1424. *Proposed by V. E. Hoggatt and Charles King, San Jose State College*

A set of positive integers W_i, $i=1, 2, \cdots$, is *complete* if for each positive integer N there exists a subset $W_{i_j}, j=1, \cdots, k$, of set W_i such that $N=\sum_{j=1}^{k} W_{i_j}$.

(1) Show that the set of Fibonacci numbers $F_i (F_{i+2}=F_{i+1}+F_i,\ F_1=F_2=1)$ is complete.

(2) Show that if any one Fibonacci number is deleted from the set F_i, the set is still complete, but the deletion of two Fibonacci numbers renders the set incomplete.

E 1425. *Proposed by D. J. Newman, Brown University*

If a square lies within a triangle, prove that the area of the square does not exceed half the area of the triangle.

SOLUTIONS

An Application of the Chinese Remainder Theorem

E 1391 [1959, 915]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $m_0, m_1, \cdots, m_r$ be positive integers which are pairwise relatively prime. Show that there exist $r+1$ consecutive integers $s, s+1, \cdots, s+r$ such that m_i divides $s+i$ for $i=0, 1, \cdots, r$.

Solution by J. E. Vinson, University of British Columbia. The problem is equivalent to showing the existence of a solution of the system of congruences

$$s \equiv -i \pmod{m_i}, \qquad i = 0, 1, \cdots, r.$$

Since the m_i are pairwise relatively prime it follows from the Chinese remainder theorem that solutions exist.

Also solved by R. G. Albert, R. G. Archibald, Ray Authement, D. W. Bailey, J. W. Baldwin, Leon Bankoff, H. F. Bechtell, W. J. Blundon, Brother Joseph Heisler, R. G. Buschman, H. R. Coomes, Underwood Dudley, E. S. Eby, George Glauberman, L. D. Goldstone, J. R. Guard, Virginia S. Hanly, J. H. Hodges, J. Hooley, Anne L. Johannsen, J. P. Jordan, Irving Katz, Donald Knuth, P. S. Landweber, H. R. Leifer, T. G. McLaughlin, Wallace Manheimer, D. C. B. Marsh, Sam Matlin, D. S. Passman, Walter Penney, Nathaniel Queen, R. T. Sandberg, C. M. Sandwick, Sr., Donna J. Seaman, Lawrence Shepley, G. J. Simmons, Charles Wexler, and the proposer. Late solutions by Alan Beal, Bill Emerson, Stanley Franklin and Gerald Hutchinson (jointly), D. P. Giesy, M. S. Klamkin, R. Q. Petersen, and C. F. Pinzka.

Queen established the stronger result that for any integers $d_0, d_1, \cdots, d_r$ there exists a number s such that m_i divides $s+d_i$ for $i=0, 1, \cdots, r$. As the proposer pointed out, the given problem solves, as a particular case, a recent Putnam Competition Problem in which it was required to show the existence of 1,000,000 consecutive positive integers each of which has a square factor.

Postage Stamp Dispensers

E 1392 [1959, 915]. *Proposed by R. K. Guy, University of Malaya, Singapore, Malaya*

I am confronted by stamp machines dispensing supplies of 3 cent and 4 cent stamps. I notice that if I have a sufficient supply of coins I can obtain stamps to the value of any whole number of cents except for 1 cent, 2 cents, and 5 cents. If the machines dispense only stamps of values m cents and n cents, how many exact values in cents can I not make up?

Solution by R. Z. Norman, Dartmouth College. Clearly there are infinitely many such unless we assume $(m, n)=1$. Under this assumption we use the following

LEMMA. *$mx+ny=c$ is solvable in nonnegative integers if and only if its complementary equation $mx+ny=mn-m-n-c$ is not.*

Proof. (1) We note that $mx+ny=c$ has exactly one integral solution (x_0, y_0) (its *principal solution*) for which $0 \leqq x_0 \leqq n-1$, and the solution is nonnegative if and only if $y_0 \geqq 0$.

(2) $mx+ny=mn-m-n$ has $(n-1, -1)$ as its principal solution.

(3) If (x_0, y_0) is the principal solution of $mx+ny=c$, then from (2) we see that the principal solution of its complementary equation is $(n-1-x_0, -1-y_0)$. Exactly one of these solutions is nonnegative.

Now for $c<0$ the equation never has a nonnegative solution, hence for $c>mn-m-n$ it always does. And it fails to have a nonnegative solution for exactly half the integers in $[0, mn-m-n]$, of which there are $(mn-m-n+1)/2 = (m-1)(n-1)/2$.

Also solved by R. G. Buschman, E. S. Eby, John Garnett and Brother T. Wesselkamper

(jointly), Michael Goldberg, L. D. Goldstone, J. P. Jordan, Donald Knuth, Sidney Kravitz, H. R. Leifer, Wallace Manheimer, D. C. B. Marsh, Sam Matlin, R. W. Means, J. K. Moore, D. S. Passman, Walter Penney, C. M. Sandwick, Sr., G. L. Simmons, and the proposer. Late solutions by Victor Baras, Alan Beal, and Bill Emerson.

Two Allied Determinants

E 1393 [1959, 915]. *Proposed by D. S. Mitrinovitch, University of Belgrade, Belgrade, Yugoslavia*

Evaluate the nth order determinant $D=|a_{ik}|$ for the two cases: (1) $a_{ik}=0$ when $i+k$ is even, (2) $a_{ik}=0$ when $i+k$ is odd.

Solution by R. A. Leonard, University of Cincinnati. Let r run through the even integers and s through the odd integers in the sequence $1, 2, \cdots, n$. (1) Using the Laplace expansion on the even rows of D we observe that each term in the expansion has as a factor a determinant with a column of zeros if n is odd and there is only one possibly nonzero term if n is even. Hence

$$D = (-1)^{n(n+1)/2} |a_{rs}| \, |a_{sr}| , \ n \text{ even}; \qquad D = 0, \ n \text{ odd}.$$

(2) In the Laplace expansion on the odd rows of D all terms but one vanish identically and, for $n>1$, $D=|a_{rr}| \, |a_{ss}|$. If $n=1$, then $D=a_{11}$.

Also solved by R. G. Albert, H. F. Bechtell, J. M. Chambers, J. P. Jordan, Donald Knuth, P. S. Landweber, Kenneth Loewen, R. C. Luebbe, D. C. B. Marsh, Walter Penney, Thomas Porsching, Donna J. Seaman, and T. L. Vanden Eynden. Late solutions by D. W. Bailey and Anna and R. G. Thompson (jointly).

A Minimum Tetrahedral Volume

E 1394 [1959, 916]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a trihedral angle and a point within it, construct the plane through the point which intercepts on the trihedral angle the tetrahedron of minimum volume.

Solution by L. D. Goldstone, New York State Public Works Lab., Albany, N. Y. Take oblique cartesian axes in 3-space along the edges of the given trihedral angle and let the given point P have coordinates (a, b, c). Let u, v, w be the intercepts on the x, y, z axes of a plane through P; then $a/u+b/v+c/w=1$. Designate the angle between the y and z axes by θ, and the angle between the x axis and the yz plane by ϕ. It is then an elementary exercise to show that the volume of the tetrahedron cut from the given trihedral angle by the plane through P is $V=\frac{1}{6}kuvw$, where $k=\sin\theta\sin\phi$. Now, by the arithmetic-geometric mean theorem we have

$$(a/u + b/v + c/w)/3 = 1/3 \geqq (abc/uvw)^{1/3},$$

or $uvw \geqq 27abc$. We now see that V will be a minimum when $uvw=27abc$, and this can occur only when $a/u=b/v=c/w=1/3$. We now conclude that the plane through P cutting off the tetrahedron of minimum volume intersects the tri-

hedral angle in a triangle having P for centroid and is easily constructed from its intercepts $u=3a$, $v=3b$, $w=3c$.

Also solved by Brother T. Wesselkamper, Michael Goldberg, C. B. Grosch, Donald Knuth, D. C. B. Marsh, and the proposer. Late solution by M. S. Klamkin.

Editorial Note. A more difficult, and still unsolved, problem is that of determining the plane through P which intersects the given trihedral angle in a triangle of minimum area. The plane analog of this problem is the famous "Philo's line" problem, and is beyond the euclidean tools. The plane analog of E 1394 is easy.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4911. *Proposed by Peter Ungar, New York University*

Let $f(z)=\lambda z+a_2z^2+\cdots$ be an entire function where $|\lambda|>1$. Let N be any neighborhood of the origin. The images of N under successive iterations of f cover the whole plane with the possible exception of one point. The same is true if $\lambda=1$ and $f(z)\not\equiv z$.

4912. *Proposed by Paul Erdös, University of Toronto*

Prove: there exists an absolute constant $c_1>0$ such that, for every n, there always exist integers a_i $(i=1, 2, 3)$, $n<a_i<n+n^{1/2}+c_1n^{1/4}$, for which a_1 is a divisor of a_2a_3. If $c_2>0$ is sufficiently small this is false; in fact, there are infinitely many values of n such that, for every triplet, $n<a_i<n+n^{1/2}+c_2n^{1/4}$ we must have that a_1 does not divide a_2a_3.

4913. *Proposed by Paul Erdös, University of Toronto*

Let w be any positive integer and let a_i, a_j be integers satisfying $w^2\leqq a_i, a_j<(w+1)^2$. Prove that all products a_ia_j are distinct.

4914. *Proposed by Leo Moser, University of Alberta*

For any positive integers m, n, prove that K is an integer, where

$$K=\frac{(mn)!1!2!\cdots(m-1)!1!2!\cdots(n-1)!}{1!2!\cdots(m+n-1)!}.$$

4915. *Proposed by S. W. Golomb, California Institute of Technology*

Show that, if the trinomial $f(x)=x^n+x^a+1$ over GF(2), with $0<a<n$, has any repeated factors, then $f(x)$ is a perfect square.

4916. *Proposed by Olga Taussky, California Institute of Technology*

Fuglede's theorem (*Proc. Nat. Acad. Sci.* 36 (1950), 35–40) specialized to finite matrices over the complex numbers states that, for A normal, the relation $AB=BA$ implies $A^*B=BA^*$. Putnam's generalization (*Amer. J. Math.* 73 (1951), 357–362) states that if A_1, A_2 are normal and if $A_1B=BA_2$ then $A_1^*B=BA_2^*$.

Find, conversely, a necessary and sufficient condition on a nonsingular n by n matrix B such that the relations $A_1B=BA_2$ and $A_1^*B=BA_2^*$ imply that A_1 and A_2 are normal.

SOLUTIONS

Entire Function

4857 [1959, 595]. *Proposed by I. N. Baker, University of Alberta, and D. J. Newman, Brown University (independently)*

What is the most general entire function which takes real values on the real axis only?

Solution by D. J. Newman. Let $f=u+iv$. It follows from the hypothesis that v is of one sign in the upper half plane, say it is positive. When $v=0$ (on the real axis) we conclude then that $\partial v/\partial y\geqq 0$, whence $\partial u/\partial x\geqq 0$. Hence f takes each real value at most once. By Picard's theorem f must be linear.

Also solved by I. N. Baker, J. H. Reill, and M. S. Robertson.

Editorial Note. Robertson generalizes the problem as follows: *Let $f(z)$ be an entire function, real on the real axis, for which $I\,f(z)$ changes sign exactly $2p$ times on every circle $|z|=R\geqq R_0>0$. Then $f(z)$ is a polynomial of degree $\leqq p$ with real coefficients.*

Fredholm Integral Equation

4859 [1959, 595]. *Proposed by Richard Bellman, The RAND Corporation*

Consider the Fredholm integral equation

$$u(x) = v(x) + \int_a^1 k(x, y)u(y)dy, \qquad a < 1,$$

with the solution

$$u(x) = v(x) + \int_a^1 K(x, y, a)v(y)dy.$$

Under appropriate conditions on $k(x, y)$ show that the resolvent kernel satisfies the equation $\partial K(x, y, a)/\partial a = -K(x, a, a)K(a, y, a)$.

Solution by I. I. Kolodner, University of New Mexico. Assume that the resolvent K and its partial derivative K_a exist. As is well known, K is the unique solution of the following Fredholm equations:

$$K(x, y, a) = k(x, y) + \int_a^1 k(x, z)K(z, y, a)dz, \tag{1}$$

$$K(x, y, a) = k(x, y) + \int_a^1 K(x, z, a)k(z, y)dz. \tag{2}$$

From (1) one gets, on differentiating with respect to a,

$$K_a(x, y, a) = -k(x, a)K(a, y, a) + \int_a^1 k(x, z)K_a(z, y, a)dz.$$

This shows that K_a is the solution of the same Fredholm equation with $v(x)$ replaced by $-k(x, a)K(a, y, a)$. Using the resolvent formula one concludes that

$$K_a(x, y, a) = -k(x, a)K(a, y, a) - \int_a^1 K(x, z, a)k(z, a)K(a, y, a)dz.$$

In view of (2), this gives

$$-\int_a^1 K(x, z, a)k(z, a)dz = k(x, a) - K(x, a, a),$$

and the desired result follows on making this substitution.

Also solved by J. L. Brown, Jr., Emil Grosswald, H. J. Zimmerberg, and the proposer.

Editorial Note. Bellman has already shown (*Proc. Amer. Math. Soc.*, vol. 8 (1957), pp. 435–440) that if $k(x, y)$ is symmetric, continuous in both variables on $0 \leqq x, y \leqq 1$, and

$$-\int_0^1 \int_0^1 k(x, y)u(x)u(y)dxdy + \int_0^1 u^2dx > 0,$$

then $K_a(x, y, a) = -K(a, x, a)K(a, y, a)$, where $0 \leqq a \leqq 1$.

Symmetric Functions of Roots

4860 [1959, 596]. *Proposed by Immanuel Marx, Purdue University*

Given the polynomial $F_n(x) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$, and integers p and q, $0 < q \leqq p \leqq n$. From each subset of the roots of F_n taken p at a time suppose formed the elementary symmetric functions of degree q, *i.e.*, the sum of all products of distinct members of the subset taken q at a time. Knowing only the coefficients of F_n, devise a method to construct a polynomial whose roots are the $\binom{n}{p}$ symmetric functions described. Illustrate the method by deriving from the quartic $F_4(x) = x^4 - a_1x^3 + a_2x^2 - a_3x + a_4$, having (unknown) roots r_1, r_2, r_3, r_4, the quartic whose roots are $R_1 = r_2r_3 + r_3r_4 + r_4r_2$, $R_2 = r_1r_3 + r_3r_4 + r_4r_1$, $R_3 = r_1r_2 + r_2r_4 + r_4r_1$, $R_4 = r_1r_2 + r_2r_3 + r_3r_1$; in other words, the case $n = 4$, $p = 3$, $q = 2$.

Solution by the proposer. Divide the polynomial F_n formally by the polynomial

$$D_p = x^p - b_1x^{p-1} + b_2x^{p-2} - \cdots + (-1)^pb_p,$$

not until the remainder is of degree $p-1$, but only until the remainder is of degree p; let this remainder be

$$R_p(x) = c_0x^p - c_1x^{p-1} + c_2x^{p-2} - \cdots + (-1)^pc_p.$$

If $D_p(x)$ is an exact factor of $F_n(x)$—and this means, if the roots of D_p are a subset of the roots of F_n—then $R_p(x) = c_0D_p(x)$, so that $c_1 - c_0b_1 = 0$, $c_2 - c_0b_2 = 0$, $\cdots$, $c_p - c_0b_p = 0$.

The functions c_0, c_1, $\cdots$, c_p are polynomials (of degree at most $n-p$) in the quantities b_1, b_2, $\cdots$, b_p, with coefficients linear in a_1, a_2, $\cdots$, a_n. Elimination of all but b_q among the p relations $c_j - c_0b_j = 0$ gives a polynomial equation in b_q with coefficients that are polynomials in a_1, $\cdots$, a_n, and this equation must be satisfied whenever b_q is the appropriate symmetric function of a subset of p roots of F_n.

For $n=4$, $p=3$, $q=2$, one obtains in this way the polynomial

$$x^4 - 2a_2x^3 + (2a_4 + a_2^2 + a_1a_3)x^2 - (2a_2a_4 - a_1^2a_4 + a_3^2 + a_1a_2a_3)x + (a_4^2 + a_2a_3^2 - a_1a_3a_4).$$

Also solved by Chung-lie Wang, and C. C. Yalavigi.

Editorial Note. The proposer comments that the problem was suggested by Lin's iterative method for factoring polynomials (*Journ. Math. Phys.*, 20 (1941), p. 231). An explicit form for the coefficients c_j is indicated in a discussion of Lin's method by Aitken (*Proc. Roy. Soc. Edinburgh*, 63 (1951), p. 174).

Integral Transform

4861 [1959, 596]. *Proposed by Joseph Lehner, Michigan State University*

Find a function $f(x)$, nonnegative and bounded on the closed interval $[0, \infty]$, such that

$$\limsup_{t\to\infty} \left| \int_0^t f(x)e^{2\pi i\lambda x}dx \right| = +\infty$$

for all real λ with the exception of a countable set.

Solution by the proposer. Define

$$f(x) = \begin{cases} 1, & \nu^2 < x < \nu^2 + 1, \\ 0, & \text{elsewhere.} \end{cases} \qquad \nu = 0, 1, 2, \cdots,$$

Let $\phi(t, \lambda) = \left| \int_0^t f(x)e^{2\pi i\lambda x}dx \right|$. Then for $\lambda \neq 0$,

$$\phi(n^2, \lambda) = \left| \sum_{\nu=0}^{n-1} \int_{\nu^2}^{\nu^2+1} e^{2\pi i\lambda x}dx \right| = \left| \sum_{\nu=0}^{n-1} e^{2\pi i\nu^2\lambda} \right| \cdot \left| \frac{e^{2\pi i\lambda} - 1}{2\pi i\lambda} \right|.$$

Hardy and Littlewood (*Acta Math.*, 1914, p. 225) have shown that the sum in the right member exceeds $C\sqrt{n}$ as $n\to\infty$ on a certain sequence, when λ is irrational. Hence,

$$\limsup_{t\to\infty} \phi(t,\lambda) \geqq \limsup_{t\to\infty} \phi(n^2,\lambda) = +\infty, \ \lambda \text{ irrational.}$$

This is the desired result, but it is also easy to treat the case of rational λ by means of Gaussian sums. When λ is an integer $\neq 0$, $|\phi(t,\lambda)| \leqq 1$.

Distinct Numbers of Prescribed Form

4862 [1959, 596]. *Proposed by D. J. Newman, Brown University*

Given a set S of distinct numbers $a_1, \cdots, a_n$. Define $k=k(S)$ to be the number of numbers of the form $a_i+2^j, j=1, 2, \cdots, n$. What is $\min_S k(S)$?

Solution by D. C. B. Marsh, Colorado School of Mines. Array the numbers, a_i+2^j, as shown:

$$\begin{aligned} &a_1+2,\ a_1+4,\ \cdots,\ a_1+2^n \\ &a_2+2,\ a_2+4,\ \cdots,\ a_2+2^n \\ &\cdots\cdots\cdots\cdots \\ &a_n+2,\ a_n+4,\ \cdots,\ a_n+2^n. \end{aligned}$$

Obviously, no row may contain a repeated value. It is also easy to show that when the a_i are distinct, no pair of rows may contain more than one common number. Thus the maximum number of numbers which may be repeated in any such array will be ${}_nC_2$ at best, if it is possible for each distinct pair of rows to contain a different common element. In such a case, $\min_S k(S)=n^2-{}_nC_2 = {}_{n+1}C_2$. That this minimum may actually be attained is evidenced by the particular set: $a_i=a+2^i$, with a arbitrary.

Also solved by J. H. B. Kemperman, and the proposer.

Extending the Triangle Inequality

4863 [1959, 728]. *Proposed by W. F. Stinespring, University of Chicago*

For what constant k (if any) can the inequality

$$|A+B| \leqq k(|A| + |B|)$$

be established for all n by n real symmetric matrices ($n\geqq 2$)? Here the notation $|A|$ means the positive square root of A^2, and the relation $\leqq$ between two matrices means that their difference is nonpositive definite.

Solution by Chandler Davis and W. F. Stinespring. No such inequality is valid (even if k depends on n). Let

$$A = \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & t \\ t & -t^2 \end{pmatrix}.$$

Because $A \geqq 0$, $|A| = A$; because $B \leqq 0$, $|B| = -B$. Thus $|A| + |B| = A - B$, a diagonal matrix with eigenvalues 2, $2t^2$. On the other hand, $A + B$ has eigenvalues $\pm 2t$, so $|A + B|$ is the constant matrix $2t$. For $|A + B| \leqq k(|A| + |B|)$ here requires $2t \leqq k \cdot 2$ and $2t \leqq k \cdot 2t^2$. No matter how large k is chosen, sufficiently small positive t violates the second inequality.

Remark. It is easy to show that, for any given A, B, there exists some k such that the inequality holds. Even this weaker statement fails, however, in the infinite-dimensional case.

Also solved by Ken Alles, and R. D. Gordon.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Matrix Calculus (2nd ed.) By E. Bodewig. Interscience, New York, 1959. xi+452 pp. $9.50.

This book is true to its title, and almost as current as a seminar with its comparisons of new methods and preoccupation with credit to the proper originators. Its four parts are titled Matrix Calculus, Linear Equations, Inversion of Matrices, and Eigenproblems. Direct and iteration methods are both given in detail with full references, and special cases are discussed. The notation is not completely standard, the author explaining in his preface that his new notation is intended to build on the fact that matrix arithmetic uses the rows and columns of matrices rather than their individual elements, for the most part. His point seems to me well taken and his notation adequate, but what might have been a service in a textbook introducing matrices from the vector space approach is likely to prove merely an annoyance in this compendium of methods, which is a reference work. Several slips in reference numbers got by the editors, and the English is not always idiomatic, though it is always clear enough. In any case, though the notation or possibly a wrong reference number may hold up the user for a few minutes, he will find it worth while to check through this book if he has a hard matrix to deal with.

MARGARET W. MAXFIELD
Naval Ordnance Test Station,
China Lake, California

Irreducible Tensorial Sets. By U. Fano and G. Racah. Academic Press, New York, 1959. vii+171 pp. $6.80.

A unified treatment of quantities (such as vectors, tensors, eigenstates) which experience a *linear* transformation when the system of space coordinates rotates is given for the benefit of physicists interested in the theory of coupling and recoupling of angular momenta. Though scattered examples justify the general title, the burden of the material deals with the quantum mechanics of rotation. The authors give two pages of bibliography from Casimir (1931) to Wigner (1958).

ARTHUR BERNHART
The University of Oklahoma

Analytic Function Theory, Vol. I. By Einar Hille. Ginn, Boston, 1959. xi+308 pp. $6.50.

It would be hard to imagine a better textbook in complex function theory for a senior or first year graduate course than the volume under review. The table of contents is essentially canonical: 1. Number systems; 2. The complex plane; 3. Fractions, powers, and roots; 4. Holomorphic functions; 5. Power series; 6. Some elementary functions; 7. Complex integration; 8. Representation theorems; 9. The calculus of residues; Appendix A. Some properties of point sets; Appendix B. Some properties of polygons; Appendix C. On the theory of integration. However, there are many interesting variations in the book itself. The symbolism of Boolean algebra is used where it will do the most good; the book is studded with historical footnotes which do much to emphasize the evolutionary character of mathematics; the author's own interest in abstract analysis is evidenced by his description of important function spaces and algebras as they appear in his exposition; important facts about complex functions of a real variable are given (often in exercises). The proofs are exact, and for the most part maximally simple. Definitions are precise. Some teachers may wish to rearrange the order somewhat, for example so that students will meet a variety of analytic functions early in the course (e^z is defined on page 138). Fortunately the book is so written that reasonable permutations cause little confusion. For students who already know some real variable theory beyond the usual advanced calculus, the book may be somewhat elementary. An enterprising teacher can still use it when teaching such students, however, omitting some items and supplying more theory *ad libitum* (*e.g.*, the general Jordan curve theorem, elliptic functions, Riemann surfaces, the great Picard theorem, etc.).

Professor Hille brings to the writing of this book a seldom-rivalled depth of scholarship in both the classical and abstract parts of his subject, together with a lucid and witty prose style. Accordingly his book must go into the select library of "What every young analyst should know."

EDWIN HEWITT
University of Washington

An Introduction to Differential Geometry. By T. J. Wilmore. Oxford University Press, New York, 1959. 314 pp. $5.60.

This book offers much more than its modest title would lead one to expect. Although it is not intended to be a comprehensive account of classical differential geometry, a sizeable amount of the metric theory of curves and surfaces in three-dimensional Euclidean space is packed into Part 1, all done very neatly by use of vector methods. Following three chapters on curves and intrinsic and extrinsic properties of surfaces, Part 1 is concluded by a fourth chapter on geometry of surfaces in the large which includes material heretofore not seen in an English text. For an understanding of this material on global geometry the reader needs some familiarity with such concepts as compactness, Hausdorff space, homeomorphism, and covering space (which are not defined or explained here). However, this chapter is independent of the remainder of the book and may be omitted. In the author's view the material of Part 1 should be covered before the reader encounters the more general tensor theory in Part 2.

Under the general heading of differential geometry in n-dimensional space in Part 2, there appears a chapter each on tensor algebra, tensor calculus, Riemannian geometry, and applications of tensor methods to surface theory. Vector spaces and the dual space are introduced and then follows the tensor product of vector spaces. Differentiable manifolds, connections, and generalized covariant differentiation constitute part of the chapter on tensor calculus. The following chapter on Riemannian geometry portrays the tensor analysis of a special object—that of a field of symmetric, second-order, covariant tensors. In the concluding chapter Riemannian geometry is specialized to a very brief account of the use of tensors in ordinary surface theory. In the progressive specialization of the latter part of the book, the author appears to reverse his earlier conviction on order of presentation.

The book is admirably written by one who is familiar with current research in the field. Mention is made of various unsolved problems. Useful references are given for further reading. Much material is covered in the rich lists of exercises. Most of the sixty-six exercises in the final list are taken from examination papers of the University of Liverpool. The book is highly recommended as a textbook for a two-semester sequence on the essential ideas and methods of differential geometry.

C. E. SPRINGER
The University of Oklahoma

Introduction to Algebraic Geometry. By Serge Lang. Interscience, New York, 1958. xi+260 pp. $7.25.

This book is written not only as an introduction to algebraic geometry, but as an introduction to Weil's *Foundations of Algebraic Geometry.* The author says that the purpose of the book is to give a self-contained introduction to the algebraic aspects of the "algebraico-geometric" method of studying algebraic

geometry without presupposing any extensive knowledge of algebra. It fulfills its purpose very well.

The book begins with a discussion of places, valuations, and extensions of places. Then algebraic varieties (affine, projective, and abstract) are introduced. Here are included the Hilbert Nullstellensatz and classical elimination theory. There is essentially no intersection theory in the text. The third chapter, on absolute theory of varieties, contains much of the material in the first chapter of Weil's *Foundations*. There are chapters on products, projections, and correspondences; on normal varieties (included here is Zariski's original proof of his Main Theorem on birational correspondences); on divisors and linear systems; on differential forms; and on simple points. There is a short chapter on connected algebraic groups, which includes theorems of a general nature which have been used recently in laying the foundation of the theory of algebraic groups. The final chapter contains Weil's proof of the Riemann-Roch theorem for curves and a sketch of the construction of the Jacobian variety.

The book is written in a concise and readable style. There are no exercises, but the proofs of some theorems are left to the reader. However, there is plenty of encouragement for the student to delve further into the subject; at the end of nearly every chapter there are references to pertinent papers, many very recent.

ALICE T. SCHAFER
Connecticut College

Mathematics of Finance. By Edwin D. Mouzon, Jr. and Paul K. Rees. Allyn and Bacon, Boston, 1959. xii+464 pp. $7.95.

This is a well-written book designed for a three-semester-hour course for beginning students in the mathematics of finance. In addition to the topics usually covered in such a course the authors have provided a substantial amount (about thirty per cent of the text) of background material in percentage and in elementary algebra through geometric progressions, enough for an introductory course in the rudiments of algebra and business arithmetic. The topics included in the main part of the text are statistics (14 pages); simple and compound interest; simple and general annuities; amortization and sinking funds; depreciation and capitalized cost; bonds; life annuities and life insurance.

The authors have exercised great care in the formulation of definitions and in the exposition of the theory, illustrating them copiously with good examples worked out in detail. The stated aim of the authors to make this textbook suitable for self-study as well as for classroom use is, in the reviewer's estimation, fully achieved.

There are numerous well-graded exercises with answers supplied for about three-fourths of them. The book is also provided with ten tables, including a five-place table of logarithms, eight financial tables and a mortality table.

J. M. FELD
Queens College

Analysis of Linear Systems. By David K. Cheng. Addison-Wesley, Reading, Massachusetts, 1959. xiii+431 pp. $8.50.

Professor Cheng has written an excellent text for engineers who need to know the useful mathematical techniques for solving linear ordinary differential equations. He spends considerable effort on the translation from the physical problem to the mathematical problem. This translation is, of course, the principal source of error in the analysis of physical systems. He has many excellent examples and good sets of problems at the end of each chapter.

The student who has mastered this book will have a working knowledge of Fourier series, Laplace transforms and inverse transforms, and the mathematical tools for determining stability of linear systems, such as the Routh-Hurwitz criterion and the Nyquist diagram. The mathematician will have little to complain about with respect to rigor, for the author gives ample references for complete proof of the points he wishes to treat only briefly.

The reviewer recommends this as an excellent textbook for the electrical and mechanical engineer and certainly a desirable reference for the physicist or mathematician who is interested in applying his theory.

JOHN E. MAXFIELD
Naval Ordnance Test Station
China Lake, California

Analysis of Straight-Line Data. By Forman S. Acton. Wiley, New York, 1959. xiii+267 pp. $9.00.

This book has been written with enthusiasm, clarity and care. The author has a pleasant breeziness of description that helps to emphasize important ideas. Well-chosen examples are used to illustrate and explain the text.

Due to his conviction that more detailed expositions of classical and modern statistical techniques are needed the author has selected topics pertinent to the engineer or physical scientist interested in data containing one or more lines entrapped within his experimental variability. The author hopes that the experimentalist himself will be able to use the analytical techniques introduced. The following quotation is from the author's preface. "The structure of the experiment determines the proper method for analyzing its data, and the experimenter himself knows that structure best. Since his statistical background is apt to be shaky, he needs to be guided—at times in considerable detail—and he usually prefers concrete examples from which general principles may be inferred. I have therefore tried to introduce each subject with data from a real experiment—examining them at first broadly, and then again with an increase in detail that occasionally they do not deserve. When I have felt that they will not mislead I have sometimes relied on explanations that are heuristic, but in general I have tried to emphasize the essential questions that are posed by the mathematical models our experimenter may choose in analyzing his data. It is with the influence of the model on the nature of the extractable information that

I have been chiefly concerned." The reader of this work will be convinced that the author has succeeded most admirably in carrying out his intentions.

The text discusses clearly and carefully with examples and illustrations the following topics: the classical model in which x is known without error and the variance of y is constant, regression with several values of y for each known x, samples from bivariate normal populations, regression with both x and y in error, several lines and the analysis of variance, the exposure of curvature and the use of orthogonal polynomials, use of transformations, rejection of unwanted data, and cumulative data. In addition there is a good list of current references plus a bibliography which includes helpful comments. The appendix contains in 22 pages the necessary statistical tables.

JOHN C. BRIXEY
The University of Oklahoma

Probability and Related Topics in Physical Sciences. By Marc Kac with special lectures by G. E. Uhlenbeck, A. R. Hibbs, and B. van der Pol. Interscience, New York, 1959. xiii+266 pp. $5.60.

This monograph is Volume I in the series *Lectures in Applied Mathematics* and grew out of lectures delivered by the author at the Summer Seminar, Boulder, Colorado, 1957. To sum up, the book is a notable contribution in the general area of probability and statistical mechanics written in a fresh, appealing style and represents an auspicious beginning for the series. Kac succeeds altogether in capturing the reader's fancy with his choice of examples, frequently unfamiliar and ingenious, and his methods of proof, often not quite standard, often interlarded with burdensome detail, but always challenging.

In a sense, no review of this book is necessary since the author includes an excellent account of its content and purpose in his preface. It is well to point out additionally that, although the subjects treated seem to change almost kaleidoscopically at times, there is an underlying idea which gives the book a kind of unity, and this idea is, of course, measure theory. Indeed, perhaps the greatest weakness of the book is that the theory of measure as such is bypassed, and it is reasonable to suppose that a reader without this background could be partially overwhelmed in consequence. It is to be hoped that this omission will not make the book inaccessible to physicists who could certainly derive as much profit from it as mathematicians, but the going does get a bit rough in Chapter IV. (Integration in Function Spaces.) As a compensation, there is a brief article on quantum mechanics by A. R. Hibbs among the appendices which is quite elementary and charmingly written.

Kac has annotated his text with a number of footnotes, mostly historical, which are likewise most engaging. One wishes there were even more of them! In fact the only quarrel to be picked with the author is his statement on page 23 that the problem of constructing Hadamard matrices is still unsolved. This ignores the beautiful work done by Paley (1933) on this problem as well as more

recent contributions. Curiously, the partial solution by Paley has a distinctively Kacian flavor!

KENNETH A. BUSH
University of Idaho

BRIEF MENTION

Calculus with Analytic Geometry (2nd ed.). By Richard E. Johnson and Fred L. Kiokemeister. Allyn and Bacon, Boston, 1960. xiii+709 pp. $8.25.

The most notable change in the second edition of this excellent text is the inclusion of work on vectors and the addition of over 400 supplementary exercises. If you plan to give a course in analytic geometry and calculus, this book is certainly one which merits consideration as a possible text.

Measurement: Definitions and Theories. Edited by C. West Churchman and Philburn Ratoosh. Wiley, New York, 1959. viii+274 pp. $7.95.

A collection of authoritative essays derived from the A.A.A.S. symposium held in New York City in 1956.

The Scientific American Book of Mathematical Puzzles and Diversions. By Martin Gardner. Simon and Schuster, New York, 1959. xi+178 pp. $3.50.

The rich and varied contents of this volume under the skilled pen of Martin Gardner will delight the mathematician and nonmathematician, alike. It is too bad that the article on flexagons fails to recognize the excellent paper by Oakley and Wisner which appeared in the March 1957 issue of this MONTHLY.

An Introduction to Statistical Mechanics. By J. S. R. Chisholm and A. H. de Borde. Pergamon Press, New York, 1958. ix+160 pp. $6.00.

Designed for an undergraduate course in statistical mechanics, this book would be more suitable for American use if it contained a reasonable selection of problems and exercises for student consideration.

Magnets: The Education of a Physicist. By Francis Bitter. Doubleday, New York, 1959. 155 pp. 95¢.

Echoes of Bats and Men. By Donald R. Griffin. Doubleday, New York, 1959. 156 pp. 95¢.

Soap Bubbles and The Forces Which Mould Them. By C. V. Boys. Doubleday, New York, 1959. 156 pp. 95¢.

How Old is the Earth? By Patrick M. Hurley. Doubleday, New York, 1959. 160 pp. 95¢.

The Neutron Story. By Donald J. Hughes. Doubleday, New York, 1959. 158 pp. 95¢.

Mathematics Tests Available in the United States. By Sheldon S. Myers. National Council of Teachers of Mathematics, Washington D. C., 1959. 12 pp. 50¢.

The Supervisor of Mathematics, His Role in the Improvement of Mathematics Instruction. Secondary School Curriculum Committee, National Council of Teachers of Mathematics, Washington D. C., 1959. 10 pp. 15¢.

A Guide to the Use and Procurement of Teaching Aids for Mathematics. By E. J. Berger and D. A. Johnson. National Council of Teachers of Mathematics, Washington, D. C., 1959. iii+41 pp. 75¢.

Nomography, Second Edition. By Alexander S. Levens. Wiley, New York, 1959. viii +296 pp. $8.50.

An extensive revision and expansion of Levens' 1948 work on alignment charts and concurrency nomograms.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., *University of Buffalo*

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Colorado: Assistant Professor L. W. Rutland has been promoted to Associate Professor in the Department of Applied Mathematics; Assistant Professor P. F. Hultquist has been promoted to Associate Professor in the Department of Applied Mathematics and appointed Executive Director of the Numerical Analysis Center.

Olivet Nazarene College: Assistant Professor B. F. Hobbs has been promoted to Acting Chairman of the Department of Mathematics; Mr. P. F. Tomlinson has been promoted to Assistant Professor.

Western Michigan University: Professor J. H. Powell has been appointed Head of the Mathematics Department, effective July 1, to succeed Professor Charles Butler. Professor Butler has asked to be relieved of administrative duties in order to return to full-time teaching.

Mr. R. L. Arms, University of Michigan, has accepted a position as Mathematician with the Bendix Aviation Corporation Computer Division, Los Angeles, California.

Mr. R. V. Borchers, U. S. Naval Proving Ground, Dahlgren, Virginia, has accepted a position as Mathematician with the National Aeronautics and Space Administration, Washington, D. C.

Mr. C. M. Bruen, I.B.M. Scientific Computation Laboratory, Endicott, New York, has accepted a position as Research Specialist with the Aerospace Division of the Boeing Airplane Company, Seattle, Washington.

Mr. N. D. Cohen, Los Angeles State College, has accepted a position as Mathematician with the Burroughs Corporation, Pasadena, California.

Mr. C. E. Diesen, Bell Aircraft Corporation, Buffalo, New York, has accepted a position as Dynamics Engineer with the Hughes Aircraft Company, Culver City, California.

Miss Joyce B. Friedman, Technical Operations, Inc., Washington, D. C., has been appointed a Member of the Technical Staff with the MITRE Corporation, Bedford, Massachusetts.

Mr. R. A. Gammill, Jr., Philco Corporation, Philadelphia, Pennsylvania, has been appointed Engineer with the Philco Corporation, Willow Grove, Pennsylvania.

Mrs. Carolen M. Jackson, U. S. Air Force Aeronautical Chart and Information Center, St. Louis, Missouri, has accepted a position as Mathematician with the Dugway Proving Ground, Dugway, Utah.

Mr. G. R. Kuhn, Battelle Memorial Institute, has accepted a position as Mathematician with the Bendix Aviation Corporation Research Laboratory, Detroit, Michigan.

Mr. C. P. Lecht, MITRE Corporation, Lexington, Massachusetts, has been appointed Second Lieutenant in the United States Army Ordnance Corps., Aberdeen Proving Ground, Maryland.

Dr. W. W. Leutert, Tidewater Oil Company, Los Angeles, California, has accepted a position as Manager of Advanced Programming and Applied Mathematics with Remington Rand UNIVAC, New York.

Miss Ethelyne L. McBee, Dade County Schools, Miami, Florida, has been appointed Mathematics and Science Teacher at the Tucson Public Schools, Tucson, Arizona.

Mr. William McKay, Franklin Institute, Philadelphia, Pennsylvania, has accepted a position as Research Programmer with the International Business Machines Corporation, White Plains, New York.

Mr. R. H. Miller, University of Minnesota, has been appointed Instructor at Wisconsin State College, Whitewater.

Mr. N. C. Mitrowsis, Fordham University, has accepted a position as Assistant Project Engineer with the Curtiss-Wright Corporation, Wright Aeronautical Division.

Mr. Otto Mond, General Electric Company, Cincinnati, Ohio, has accepted a position as Associate Mathematician with the International Business Machines Corporation, Research Center, Yorktown Heights, New York.

Professor E. P. Northrop will be on leave from the University of Chicago for eighteen months beginning April 1, 1960 to serve as the Ford Foundation's resident representative in Turkey and as Consultant to the Turkish Ministry of Education, with special attention to mathematics and the natural sciences.

Mr. R. R. Parker, HRB—Singer, Inc., State College, Pennsylvania, has accepted a position as Associate Electronic Engineer with Cornell Aeronautical Laboratory Inc., Buffalo, New York.

Mr. Perry Scheinok, Wayne State University, has been promoted to Assistant Professor.

Mr. W. R. Slinkman, Bemidji State College, has been promoted to Assistant Professor.

Mr. C. J. Smith, Washington University, has accepted a position as Methods Technician with Remington Rand UNIVAC, St. Louis, Missouri.

Mr. Irwin Stoner, Service Bureau Corporation, New York, has been promoted to Senior Analyst in the Special Projects Division, Poughkeepsie, New York.

Mr. B. A. Tague, Massachusetts Institute of Technology, has been appointed a Staff Member at Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. R. S. Varga, Bettis Atomic Power Laboratory, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has been appointed Professor at Case Institute of Technology, Cleveland, Ohio.

Miss Janice E. Wilshire, Baylor University, has accepted the position of Mathematician with the Holloman Air Force Base, Alamagordo, New Mexico.

Mr. J. L. Wulff, Sacramento State College, has been promoted to Assistant Professor.

Mr. Rene-Victor Goormaghtigh, La Brugeoise Steelworks, Belgium, died February 10, 1960. He was a member of the Association for fourteen years.

FREE BOOK LIST

The National High School and Junior College Mathematics Club, Mu Alpha Theta, has prepared an annotated bibliography of mathematical books and periodicals suitable for the high school library. The books have been carefully selected to provide a well-rounded library on a limited budget and grouped into six sets of about $30 valuation per

set. The entire library, including all six sets of books and the ten recommended periodicals may be purchased for less than $250.

Any person interested in having a copy may obtain one free by sending a stamped, self-addressed, *number 10 envelope* to: HIGH SCHOOL BOOK LIST, c/o Richard V. Andree, The University of Oklahoma, Norman, Oklahoma.

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Summer Meeting in East Lansing, Michigan, on August 29–September 3, 1960. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on August 29–September 2, and from 9:00 A.M. to 12:00 noon on September 3.

The Employment Register Desk will be located in the Snyder Dormitory, the same building in which the Registration Desk will be maintained. Interview tables in communal rooms will be assigned for interviews. There will be no charge for registering to either job applicants or to employers, except when the Late Registration Fee for employers is applicable. Provision will be made for anonymity of applicants upon request.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position description forms, *which must be completed and returned to Providence not later than August 5, 1960, in order to be included free of charge in the listings at the meeting in East Lansing.* Forms which arrive after this closing date, but before August 22, will be included in the listings at the meeting for a Late Registration Fee of $3.00, and will also be included in the sold listings, but not until ten days after the meeting. Printed listings will be available for distribution both during and after the meeting. The prices are as follows: Position descriptions, $2.00; listing of applicants, academic only, $5.00; comprehensive listing of applicants, academic, industrial, and government, $20.00.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 267 persons have been elected to membership by the Board of Governors on applications duly certified.

ALFRED T. ANDERSON, M.A. (Michigan) Teacher, Sewanhaka High School, Floral Park, New York

MARC ARONSON, Student, University of Florida

PARVIZ AYUBI, Student, San Jose State College

RICHARD A. ALO, Student, Gannon College

ROBERT H. ANGLIN, Cost Accountant, Dan River Mills, Inc.

DANIEL ASHLER, Ed.M. (Temple) Teacher, School District of Philadelphia

RICHARD H. BALOMENOS, M.A. (N.Y.U.) Teaching Fellow, Graduate School of Education, Harvard

PAUL W. BARCUS, B.S. (M.I.T.) Major, United States Marine Corps
JONATHAN C. BARRON, Student, College of South Jersey, Rutgers, The State University
ROBERT L. BATES, B.A. (Gallaudet Coll.) Senior Mathematician, Dept. of Navy, Washington, D. C.
RICHARD G. BAUER, M.S. (Rutgers) Higgins Fellow, Columbia University
STUART D. BAXTER, Ph.D. (Toronto) Mathematician, National Research Council of Canada
THOMAS L. BEATTY, Student, North Dakota Agricultural College
ALBERT B. BENNETT, JR., M.A. (Maine) Educational Specialist, Dept. of Education, Augusta, Maine
SCOTT A. BENSON, Student, Knox College
JAMES K. BIDWELL, M.S. (Illinois) Teacher, Central High School, Bay City, Michigan
FRANK A. BLOOD, Student, Case Institute of Technology
JAMES R. BOCK, B.A. (San Fernando Valley S.C.) Grad. Student, University of California, Los Angeles
CHARLES H. BOLL, Ph.D. (Stanford) Research Scientist, Lockheed Missile Systems Division
RICHARD G. BRADSHAW, M.A. (Columbia) Asst. Prof., Clarkson College
LINCOLN E. BRAGG, B. S. (Carnegie Inst. Tech.) Grad. Student, Carnegie Institute of Technology
VINCENT BRANT, M.Ed. (Duke) Supervisor, Board of Education of Baltimore County, Maryland
JAMES B. BRENNAN, Student, Fordham University
ROBERT T. BRESNIN, Student, Los Angeles State College
MARJORIE BRICKMAN, B.S. (Creighton) Grad. Asst., University of Nebraska
KENNETH L. BRINKMAN, M.A. (U.C.L.A.) Member, Technical Staff, Hughes Aircraft: Grad. Student, University of California, Los Angeles
MARY S. BROCK, B.S. (Eastern S.C.) Teacher, Harlan High School, Kentucky
BROTHER BENEDICT VIRGIL, B.A. (De LaSalle Normal) Head of Dept., De LaSalle High School, New Orleans, Louisiana
BROTHER L. JOHN, M.A. (DePaul) Head of Chemistry Dept., St. Mel High School, Chicago, Illinois
BROTHER FREDERICK JAMES MCGLYNN, B.S. (Notre Dame) Head of Dept., Catholic Central High School, Monroe, Michigan
G. B. BROWN, B.S. (Texas) Instr., Mills College
ANDREW M. BRUCKNER, Ph.D. (U.C.L.A.) Asst. Prof., University of California at Santa Barbara
MARSHALL J. BURGH, Student, Roosevelt University
WILLIAM A. E. BUTLER, Student, Beamsville District High School, Ontario, Canada
F. ANN CALLAHAN, Student, George Pepperdine College
NORMA L. CAMP, Student, Lindenwood College for Women
RICHARD L. CAMPBELL, Student, Case Institute of Technology
JOSEPH F. CARLS, Student, St. Bonaventure University
J. W. CARMICHAEL, Jr., Student, Western New Mexico University
HENRY E. CASTRO, B.S.E.E. (Texas) Research Scientist, Applied Physics Laboratory, University of Texas
WILLIAM L. CAUDLE, Student, New Mexico State University
ANSON E. CHAPMAN, Student, Roosevelt University
HARRY S. P. CHIN, M.S. (Rensselaer Polytech. Inst.) Senior Mathematician, Bendix Pacific
YIU HUNG CHAN, Student, Queen's University
VICTOR CH'IU, Student, Kent State University
CHANG CHOI, Student, The University of the South
FRANCIS J. COLLINS, Ed.M. (Boston S.T.C.) Head of Dept., Natick High School, Massachusetts
WILLIAM C. CONNETT V, Student, Georgetown University
F. JOE CROSSWHITE, M.Ed. (Missouri) Instr., Keokuk Community College
MRS. ANNE W. CUNNINGHAM, A.M. (Michigan) Teacher, Ann Arbor Senior High School, Michigan
DOYLE O. CUTLER, B.A. (Texas Christian) Grad. Student, Texas Christian University
ROBERT F. DAIGNEAULT, Student, Lebanon Valley College
RICHARD H. DALE, B. S. (Maryland) Mathematician, White Sands Missile Range
JAMES W. DEEGAN, Student, St. Bonaventure University
RUSSELL E. DENMAN, B.S.E.E. (Purdue) Engr., Lockheed Missile Systems Division
CALVIN DESPORTES, B.S.E.E. (Georgia Inst. Tech.) Retired, Apalachicola, Florida
HERBERT C. DIMLICH, B.S. (Boston) Head of Dept., West High School, Pawtucket, Rhode Island
RAYMOND J. DISTLER, M.S. (Kentucky) Instr., College of Engineering, University of Kentucky
JAMES S. DOMBEK, B.S.E. (Arkansas S.T.C.) Asst. Instr., University of Kansas
JOHN P. DOWNES, M.A. (New York S.T.C., Albany) Asst. Prof., State University of New York, College of Education at Oneonta
KLAUS E. ELDRIDGE, Student, Hardin-Simmons University
WILLIAM R. EMERSON, Student, California Institute of Technology
ROBERT M. EWART, Jr., Student, Case Institute of Technology
MRS. BESSIE V. FAIRLEY, M.A. (Southwestern Louisiana Inst.) Head of Dept., Opelousas High School, Louisiana
LEONARD M. FALTZ, Student, College of the City of New York
RICHARD A. FELLOWS, B.S. (San Jose S.C.) Statistician, Fairchild Semiconductor Company
ARTHUR C. FLECK, B.S. (Western Michigan) Grad. Asst., Michigan State University
CHUNG FONG, B.A. (California) Grad. Student, Sacramento State College
RICHARD D. FREEMAN, JR., Student, Michigan State University
FRANCIS R. FROLA, B.A. (San Jose S.C.) Grad. Asst., Washington University
GREGORIO FUENTES, Licenciado (Madrid) Instr., Gonzaga University
CHARLES D. GALLANT, Student, St. Francis Xavier University
PAUL J. GAMBAIANA, B.A. (Iowa S.T.C.) Chairman of Dept., West Sioux High School, Hawarden, Iowa
JOHN B. GARNETT, Student, University of Notre Dame
N. HUGH GEARY, Student, Purdue University
ABOLGHASSEM GHAFFARI, D.Sc. (Sorbonne), Ph.D. (London) Aeronautical Research Scientist, National Bureau of Standards, Washington, D. C.
DENNIS M. GIRARD, Student, University of Detroit
EUGENE C. GLUESING, M.A. (Minnesota) Advanced Programmer, Remington Rand UNIVAC
JEROME X. GOLDSCHMIDT, B.S. (Dayton) NDA Fellow, Michigan State University
MARGARET J. GORDON, Student, Cornell University
JOAN A. GOTTLIEB, Student, Barnard College, Columbia University
BARRY GRANOFF, Student, Fairleigh Dickinson University
GEORGE L. GROSS, Ph.D. (Iowa S.U.) Lecturer, Adelphi College: Staff Consultant, Grumman Aircraft Engineering Corporation
FRANK A. GUGINO, M.E. (Rensselaer Polytech. Inst.) Consulting Engr., Tonawanda, New York
PHIL W. HAGEN, Student, St. John's University
JAMES E. HALL, A.M. (Harvard) Instr., Northern Illinois University
JOHN H. HANCOCK, M.A. (Southern California) Senior Mathematician, Marquardt Corporation
LAWRENCE E. HANSON, M.S. (California, Davis) Instr., Chico State College
WESLEY E. HARTMAN, Student, University of California, Los Angeles
PAUL M. HASSE, B.S. (S.U. South Dakota) Junior Statistician, North American Aviation
WILLIAM S. HATCHER, M.A. (Vanderbilt) Teaching Asst., Vanderbilt University
REV. ROBERT A. HAUS, M.S. (Notre Dame) Instr., Canisius College
JOHN C. HAWKINS JR., M.B.A. (Harvard) Teacher, Hastings High School, Hastings-on-Hudson, New York
CHARLES F. HICKS, M.A. (Polytech. Inst. Brooklyn) Senior Structural Engr., Republic Aviation Corporation
ROGER L. HIGDEM, M.S. (North Dakota) Asst. Prof., College of Idaho
LEONA L. HIRZEL, M.A. (Columbia) Asso. Prof., State University of New York, College of Education at Oneonta
DENNIS L. HOBDE, Student, Sacramento State College
MRS. BELVA H. HOPKINS, B.A. (Western Maryland Coll.) Teacher, High Point Senior High School, Beltsville, Maryland

WILBUR F. HOPPE, M.S. (Wisconsin) Instr., Western Illinois University
ROBERT G. HOSEA, M.A. (Stanford) Teacher, Rosemead High School, California
WILLIAM H. HOUPT, B.A. (Dickinson Coll.) Engr., Western Electric Company
JOSEPH T. HOWSON, JR., B.S. (Rensselaer Polytech. Inst.) Grad. Asst., Rensselaer Polytechnic Institute
FRANCIS M. HUDSON, M.A. (Texas) Instr., Western State College
BRUCE K. HURLBERT, M.A. (Long Beach S.C.) Instr., Mt. San Antonio Junior College
CLINT L. HYATT, M.A. (Louisiana S.U.) Auditor, Ponca City, Oklahoma
LARON J. JACOBSEN, Student, University of Utah
ROBERT C. JAMES, JR., Student, Texas Christian University
THOMAS M. JENKINS, A.B. (Kenyon Coll.) Asst. in Instruction & Grad. Student, Yale University
MRS. DORRIS H. JOHNSON, M.A. (Florida) Chairman of Dept., Clearwater High School, Florida
PAUL H. JOHNSON, Aircraft worker, Convair
ROBERT A. JOHNSON, B.S. (Iowa S.C.) Engr., Lockheed Aircraft Corporation
ROBERT W. JOHNSON, B.A. (Amherst Coll.) Grad. Student, Massachusetts Institute of Technology
WARD D. JONES, M.B.A. (Harvard) Grad. Student, American University
JAMES P. KALLENBORN, Student, Kent State University
DWIGHT E. KECK, B.S.Ed. (Ball S.T.C.) Grad. Asst., Michigan State University
JOHN F. KEPLER, M.A. (Vanderbilt) Grad. Student, Vanderbilt University
IRVING J. KESSLER, Student, Brooklyn College
GENE A. KEMPER, M.S. (North Dakota) Instr., University of North Dakota
LAEL F. KINCH, Student, University of Kentucky
JOHN R. KING, M.A. (Boston Coll.) Asst. Prof., Providence College
DAVID E. KOLTENBAH, Student, Kent State University
LAWRENCE J. KRATZ, Student, Xavier University
HUDSON VANE. KRONK, B.S. (Rensselaer Polytech. Inst.) Grad. Asst., Michigan State University
ROBERT L. KRUSE, Student, Pomona College
JOHN P. LAMB, Student, University of Notre Dame
CAPT. WILLIAM R. LAMBERT, M.S. (Colorado) Instr., United States Air Force Academy
RICHARD B. LANE, Student, Carleton College
JOHN D. LAWRENCE, Student, Washington State University
JOSEPH R. LEROY, M.S. (Alabama) Senior Technologist, Shell Chemical Company
RAYMOND P. LEROY Sr., B.S. (Troy S.C.) Head Coach & Instr., Hayneville High School, Alabama
MARTIN LEVY, A.B. (Sacramento S.C.) Lecturer, Sacramento State College
GENE L. LEWIS, Student, Harvard University
KARL G. LIEBLONG, JR., B.S.Ed. (Westchester S.T.C.) Head of Dept., Collingdale High School, Pennsylvania
GEORGE E. LINDAMOOD, Student, Wittenberg University
EDWARD B. LIPTON, M.A. (Columbia) Mathematical Analyst, Lockheed Aircraft Corporation
BETTY LIDE, M.A. (North Carolina) Instr., Temple University
JAMES B. LISK, A.B. (Northwest Nazarene) Instr., Columbia Basin College
GEORGE W. LOFQUIST, M.Ed. (North Carolina) Instr., Louisiana State University in New Orleans
WILLIAM A. LOPEZ, Student, LeMoyne College
MARY LOU LYONS, Student, University of Kentucky
CHARLES R. MACCLUER, Student, Ohio State University
NEIL A. MACKIE, Student, St. Joseph's College
ROBERT A. MACLEOD, Student, University of Alberta
JAMES T. MADEJ, Student, Knox College
JOHN MAHONEY, M.A., (Ohio S.U.), Instr., Yuba College
FREDERICK R. MAIOCCO, Student, West Virginia University
FRED MARER, M.A. (Southern California) Chairman of Dept., Los Angeles City College
DUDLEY R. MARSH, B.S. (Massachusetts) Instr., Worcester Junior College
MORRIS L. MARX, B.S. (Tulane) Grad. Student, Tulane University
CHARLES E. MAYO, Student, Croft High School, Waterbury, Connecticut
DAVID A. MCBLAIN, Student, Grinnell College
LOUIS R. MCCREERY, B.A. (U.C.L.A.) Head of Dept., San Marino High School, California
ELLIS R. MCDANIELS, Student, University of Wichita
BOBBY K. MCGUIRE, Student, North Central College
ERNEST E. MCINTOSH, JR., Student, San Francisco State College
ANTHONY J. MCKEE, Student, St. Bonaventure University
ROBERT W. MEANS, Student, University of Santa Clara
ROBERT K. MEANY, Ph.D. (Wisconsin) Asst. Prof., Texas Christian University
JOSEPH P. MEISSNER, Student, Xavier University
RUTH C. MERKLE, M.A. (Western Reserve) Chairman of Dept., Avon Lake High School, Ohio
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THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-seventh annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi on February 19–20, 1960 with Centenary College acting as host institution. The Friday afternoon session was held in two concurrent sessions. Professor W. M. Sanders, Mississippi Vice-Chairman and Professor T. K. Maddox, Louisiana Vice-Chairman, presided. Professor S. B. Murray of Mississippi State University, Chairman of the Section, presided at the Friday evening and Saturday morning sessions. There were 162 persons registered, including 75 members of the Association.

The following officers were elected for the coming year: Chairman, Professor Z. L. Loflin, Southwestern Louisiana Institute; Vice-Chairman for Louisiana, Professor J. H. Wahab, Louisiana State University at New Orleans; Mississippi Vice-Chairman, Pro-

fessor Eleanor Walters, Delta State College; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

Reports on the High School Mathematics Contest were given by Professor H. T. Karnes of Louisiana State University and Professor N. A. Childress of the University of Mississippi showing increased participation by the high schools. The section instructed these chairmen to extend invitations to all schools in both states next year.

The invited speaker for the meeting was Professor F. E. Ulrich of The Rice Institute. He presented the two following addresses:

1. *Non-Euclidean metrics from the standpoint of Cayley-Klein-Poincairé and their use in studying properties of analytic functions.*

After general remarks concerning geometries defined by subgroups of the projective group, the Cayley-Klein definition of distance and angle was given and the three classical geometries defined. The Klein model for the hyperbolic case was discussed and from it the Poincairé model obtained. The rigid motions were then expressed as nonsingular linear transformations of a complex variable. The interpretation of Schwarz's lemma in this metric was given and Pick's formulation was discussed. It was shown that this metric may also be defined by means of the differential form $|dz|/(1-|z|^2)$. More general metrics were considered and their role in the Riemann surface type problem indicated.

2. *The behavior of functions harmonic and positive in an angular domain.*

Functions $u(x, y)$ were considered which are harmonic and positive in a simply-connected domain D and zero on the boundary B of D except at one point O. The function must become infinite as $(x, y)\to O$ if $u(x, y)\not\equiv 0$. Theorems were considered which relate the growth of $u(x, y)$ as $(x, y)\to O$ in D to the character of the boundary B in the neighborhood of O. The general situation is: The more pointed D is in the neighborhood of O, the more rapidly $u(x, y)$ becomes infinite. If the boundary has a cusp at O, $u(x, y)$ becomes infinite more rapidly than any power of $1/r$ where r is the distance from (x, y) to O. The precision which may be obtained depends on the regularity of B in the neighborhood of O.

Other papers presented are as follows:

3. *The Undergraduate Honors Program at Louisiana State University*, by Professor R. D. Anderson, Louisiana State University.

The methods of choosing the superior student in mathematics and the courses taught for these students were discussed, together with the impact of the undergraduate honors program on the graduate program.

4. *The National Science Foundation summer program for high school juniors*, by Professor H. S. Butts, Louisiana State University, introduced by the Secretary.

The type of student that took part in the National Science Foundation summer program for high school juniors at Louisiana State University in 1959 and the type of mathematics courses designed for these students were discussed, together with the objectives of such a course.

5. *Two geometrical interpretations of the scalar product of two vectors*, by Professor B. E. Mitchell, Belhaven College.

Let $\mathbf{a}$ and $\mathbf{b}$ be two positional vectors, common origin O, termini A and B respectively. *Theorem* I: The scalar product $\mathbf{a}\cdot\mathbf{b}$ is the power of O with respect to the circle constructed on AB as a diameter. *Theorem* II: The product $\mathbf{a}\cdot\mathbf{b}$ is the radius of a circle, center O, with respect to which A or B and the projection of B on OA, or the projection of A on OB are inverse points.

6. *Semigroups in introductory modern algebra*, by Professor T. F. Mulcrone, S.J., Loyola University.

In view of the growing prominence of the theory of semigroups and the more frequent ref-

erences to them in group, ring and field theory, the author suggests that the treatment of groups in introductory modern algebra courses might profitably be preceded by the definition of the semigroup and the use of examples of semigroups to illustrate the closure, associativity, identity and inverse laws. Several elementary semigroups were given to illustrate these laws and to provide examples of such elements as zeros, idempotents, nilpotents and ideals. The importance of the semigroup of the totality of transformations (mappings) of a set into itself was pointed out.

7. *Non-Euclidean numbers*, by Professor E. B. Moyers, University of Mississippi, introduced by the Secretary.

That the postulates of a non-Euclidean geometry form the basis of a consistent logical system is shown by comparison with the set of real numbers that have the property of being less in absolute value than a fixed positive number. To facilitate the discussion these numbers are shown to form a field under operations defined in terms of elementary functions.

8. *On properties weaker than countable compactness*, by Mr. Leon Weill, Mississippi Southern College, introduced by the Secretary.

In a completely regular space X, several statements were shown to be equivalent.

9. *On convergence properties in $C(X)$ equivalent to $C(X)=C^*(X)$*, by Mr.C. T. Hathorn, Mississippi Southern College, introduced by the Secretary.

The equivalence of several statements in any topological space and a completely regular space were considered.

10. *Pseudo-compact spaces*, by Professor R. W. Bagley, Mississippi Southern College.

Some recent results concerning pseudo-compact spaces are discussed. In particular results concerning product spaces and those concerning convergences in $C(X)$ are considered.

11. *Quantum statistics*, by Professor Margaret M. LaSalle, Southwestern Louisiana Institute.

This paper presents the fundamental reason for quantum statistical theory. It is shown by rules for independent probabilities, linear combinations of the wave functions $\sum C_i\psi_i$ and by the probability of the set of eigenvalues $|C_i|^2$, that in general two or more wave functions are not equivalent to a single wave function. Since quantum statistical mechanics has to do with many quantum mechanical states, measures of previous states and expected future states are essential.

12. *Continuous solutions of a functional equation*, by Professor R. D. Boswell, Jr., Mississippi State University.

The author discusses the continuous real-valued functions which satisfy the equation $f(x+y)=f(x)[1+af^2(y)]^{1/2}+f(y)[1+af^2(x)]^{1/2}$, where a is a real number.

13. *On solving systems of linear equations*, by Professor B. E. Mitchell, Louisiana State University.

A method similar to Chio's pivotal condensations method for evaluating determinants for matrices yields a method for solving systems of linear equations similar to Crout's method.

T. L. REYNOLDS, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fortieth regular meeting of the Southern California Section of the Mathematical Association of America was held at Los Angeles State College, Los Angeles, California, on March 12, 1960. Professor D. H. Hyers, Chairman of the Section, presided. The registered attendance was 150, including 118 members of the Association.

At the business meeting Professor J. D. Swift, Chairman of the Nominating Committee, reported that the following officers were elected by mail ballot for the next academic

year: Chairman, Professor R. C. James, Harvey Mudd College; Vice-Chairman, Professor Clifford Bell, University of California, Los Angeles; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. Professor Swift reported also that the members of the Program Committee for the coming year were: Professors P. A. White, University of Southern California, chairman; C. A. Seekins, Occidental College; R. M. Redheffer, University of California, Los Angeles; P. J. Kelly, University of California, Santa Barbara; L. T. Black, Long Beach City College.

The following program was presented:

1. *Devil's curve and witch of Agnesi*, by Mr. T. E. Sydnor, Pasadena City College.

Research indicates that the Devil's curve got its name from a resemblance to a spindle shaped top called a diabolo, popular as a toy in the early twentieth century. The speaker exhibited a diabolo and indicated how the graph of the Devil's curve may easily be sketched. To honor the scholarly humanitarian Agnesi, the speaker suggested that the so called witch be known henceforth as the curve of Agnesi.

2. *A property of tetrahedrons*, by Professor Morgan Ward, California Institute of Technology.

The speaker showed that the triangle inequality for the chordal metric in the complex plane used by Carathéodory in his *Theory of Functions* (New York, 1954) is implied by a theorem of plane geometry stating that the product of the diagonals of a plane quadrilateral is less than or equal to the sum of the products of the pairs of opposite sides. A simple proof was given of the latter theorem; the case of equality is Ptolemy's theorem on quadrilaterals inscribed in a circle.

3. *A problem in communication theory*, by Dr. A. V. Balakrishnan, Space Technology Laboratories, introduced by the Secretary.

The following problem was discussed: Given N points on the unit sphere in Euclidean space, dimension D, find the distribution of points which maximizes the minimum distance. For $D=2$, one divides the circumference into N equal parts. For $D=3$, solutions have been given for $N<9$ and $N=12$, by Schutte and Van Der Waerden (*Math. Ann.*, 1951), most of the solutions being semiregular polyhedra. For general N and D, one has the theorem: *If* $D \geqq N-1$, *then the optimal polyhedron is the* $N-1$ *dimensional regular simplex with* N *vertices, and* $d_{\min}=\sqrt{(2N/(N-1))}$. In Hilbert space ($D$ and N both $+\infty$) the solution is any set of orthonormal unit vectors, and $d_{\min}=\sqrt{2}$.

4. *Finite projective planes*, by Professor Marshall Hall, Jr., California Institute of Technology.

A finite projective plane is said to be of order n if a line contains exactly $n+1$ points. Every known finite plane is of prime power order. The speaker discussed the known types of finite planes. The fact that the conjectures of Euler and Mann on orthogonal squares are now known to be false might mean that the Bruck-Ryser necessary conditions for the existence of a finite plane are also sufficient.

5. *Regular graphs*, by Professor D. M. Merriell, University of California, Santa Barbara.

The graphs considered were nonoriented finite graphs, with each two vertices joined by at most one arc, and no vertex joining itself. A graph is regular if every vertex has the same degree, the degree being the number of arcs which meet at the vertex. Properties of regular graphs were discussed, including possible degrees for a given order and a criterion for isomorphism. Some unsolved problems and conjectures were stated.

6. *Evaluation of double integrals*, by Professor G. B. Price, California Institute of Technology and the University of Kansas.

The Fundamental Theorem of the Integral Calculus states that $\int_a^b f'(x)dx=f(b)-f(a)$. The usual corresponding theorem in two dimensions states that a double integral can be evaluated by

iterated integrals, but the results in the two cases do not appear to be analogous. This paper contains a treatment of the evaluation of double integrals by methods which are entirely similar to those used in the one dimensional case. Several standard formulas appear as special cases of a general theory.

7. *The Pasadena experiment with School Mathematics Study Group materials*, by Mr. W. H. Glenn, Jr., Mathematics Coordinator, Pasadena City Schools.

Pasadena teachers have been experimenting for two years with School Mathematics Study Group materials for seventh- and eighth-grade mathematics, as one of several groups financed by the National Science Foundation. The general philosophy has been "to think of grades 7 and 8 not as the end of elementary school mathematics, but rather as a foundation for the work of the senior high school." Professor Richard Dean, California Institute of Technology, has served as the local consultant. The test centers have been evaluating the materials while teaching them in the classroom, with an ultimate goal of developing improved textbook material.

R. B. HERRERA, *Secretary*

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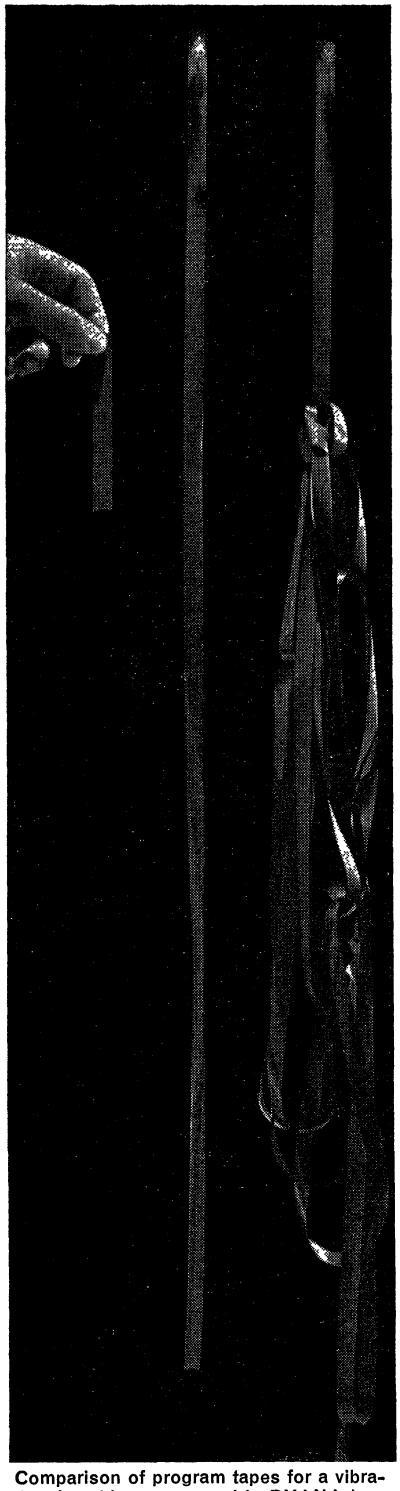
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LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY
NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.
NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.
OHIO
OKLAHOMA, Oklahoma City University, October 21–22, 1960.
PACIFIC NORTHWEST
PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.
ROCKY MOUNTAIN
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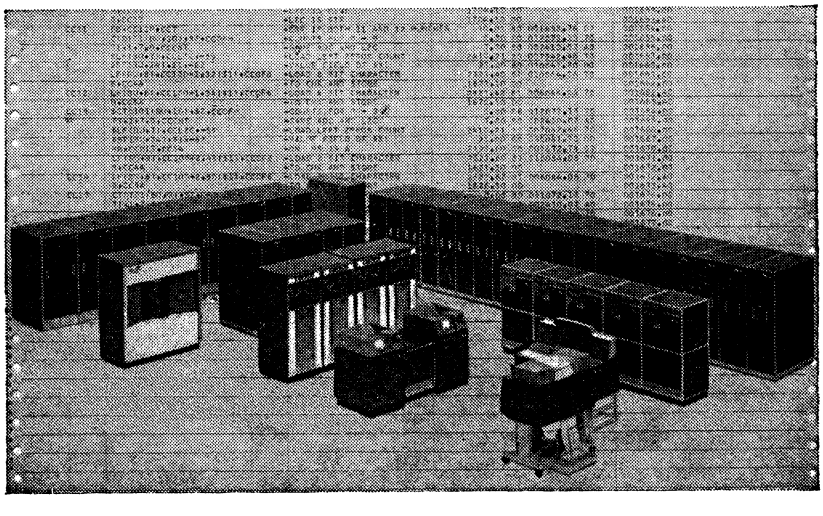
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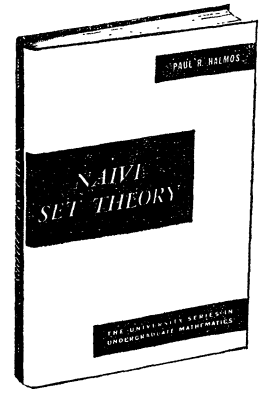
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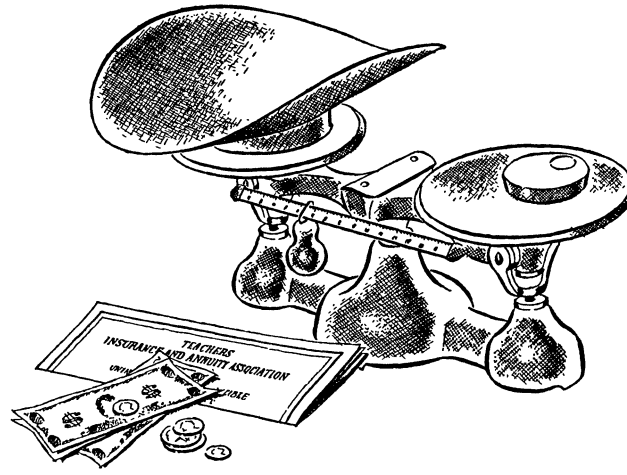
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AUGUST-SEPTEMBER 1960

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AN ALTERNATIVE TO SET THEORY

P. C. GILMORE, IBM Research Center, Yorktown Heights, N. Y.

1. Introduction. Much of modern research into the foundations of mathematics is a continuation of nearly 250 years of research following upon the invention of the calculus. It has been directed to finding some coherent conceptual basis for the calculus in particular and analysis in general, a conceptual basis which would provide a foundation for analysis in the sense of providing a motivation to assist in suggesting new theorems and a criterion of rigor to assist in judging the proofs of theorems.

The set theory of Cantor, despite the difficulties inherent in its concepts, has been partly successful in providing such a conceptual basis. Further, because the concept of set is so elegant, convenient, powerful and intuitively compelling, set theory has become a universal mathematical language used in almost all of modern mathematics. Nevertheless, the difficulties inherent in set theory remain.

In this paper an alternative to set theory is discussed which provides not only a conceptual basis for analysis free from the difficulties that trouble set theory, but also a mathematical language with some technical advantages over set theory. The alternative to set theory is presented in terms of symbol theory: a language for expressing certain fundamental relationships between symbols just as set theory is a language for expressing relationships between sets. Like naive set theory, the conceptual scheme of symbol theory can motivate axiomatic theories which are adequate for the development of analysis. The purpose of this paper is not, however, to present such an axiomatic theory but rather to describe symbol theory and the conceptual picture of mathematics suggested by it. Incidental to the presentation will appear the advantages of symbol theory over set theory as a mathematical language.

Logical notation is used throughout the paper only as a convenient shorthand means for stating sentences. The variables 'x', 'y', etc., are to be thought of as ranging over the particular universe under discussion. The universal quantifiers '(x)', '(y)', etc., are to be understood as asserting that no matter what objects in the universe the variable denotes, the sentence following the quantifier is true. The existential quantifiers '(Ex)', '(Ey)', etc., assert that there is an object in the universe such that when the variable denotes that object, the sentence following the quantifier is true. The logical connective '$\supset$' expresses the conditional "if . . . then . . . ", the connective '$\equiv$' the equivalence " . . . if and only if . . . ", the connective '&' the conjunction " . . . and . . . ", the connective '$\vee$' the disjunction " . . . or . . . "' and '$-$' expresses the negation.

2. Symbol theory. It is the intention in this section to introduce naive symbol theory in much the same manner that naive set theory is introduced to novices, namely, by discussion of simple examples.

Consider a universe of symbols (finite or infinite, it does not matter at the moment) which includes the following symbols:

(2.1) $$1, 2, 3, 4, \text{even}, \text{odd}, +, <.$$

The members of the universe are not to be considered as symbols without meaning but it is presumed that the correct use of the symbols is known (in some vague sense of "use" which will be clarified later). It is possible to recognize two relationships between members of the universe; or put in another way, two relations, which will be called the epsilon and the nu relations, have the universe of symbols as their common domain. Thus it is possible to recognize from the manner in which *any* two symbols of the universe are used whether or not one is epsilon related to the other and whether or not one is nu related to the other.

In order to express easily the epsilon and nu relationships between symbols of the universe the following simple language will be used. For any two symbols s and t of the universe "$s \epsilon t$" will be said to be true if s is epsilon related to t, and "$s \nu t$" will be said to be true if s is nu related to t. Further the convention will be maintained that an instance of a symbol from the universe is a name for that symbol. Thus, for example, when it is said that "$2 \epsilon \text{even}$" is true it can be concluded that the symbol '2' is epsilon related to the symbol 'even' and when it is said that "$-2 \nu \text{even}$" is true it can be concluded that '2' is not nu related to 'even'.

The two relations are incompatible. The epsilon relationship is expressed in English by sentences such as "two is even", while the nu relationship is expressed by sentences such as "two is not odd". Thus the epsilon relation between symbols is such that the following sentences are true:

$$2 \epsilon \text{even}, \quad 4 \epsilon \text{even}, \quad 1 \epsilon \text{odd}, \quad 3 \epsilon \text{odd};$$

and such that the epsilon relationships expressed by these sentences are the only ones holding between the symbols (2. 1); that is, that all the following sentences, and more, are true:

$$-1 \epsilon \text{even}, \quad -3 \epsilon \text{even}, \quad -2 \epsilon \text{odd}, \quad -4 \epsilon \text{odd}, \quad -1 \epsilon 1, \quad -1 \epsilon 2,$$
$$-\text{even} \epsilon 2, \quad -\text{odd} \epsilon \text{even}, \quad -+ \epsilon <, \quad - < \epsilon +, \quad -3 \epsilon <.$$

Of these latter sentences the first four are true for apparently different reasons than the others; one concludes that '1' is not epsilon related to 'even' for a different reason than one concludes that 'odd' is not epsilon related to 'even', or that '3' is not epsilon related to '<'. The difference can be more clearly understood in terms of the nu relation.

The nu relation between symbols is such that the following sentences are true:

$$2 \nu \text{odd}, \qquad 4 \nu \text{odd}, \qquad 1 \nu \text{even}, \qquad 3 \nu \text{even}.$$

And again also that the following sentences are true:

$$-2\ \nu\ \text{even}, \qquad -4\ \nu\ \text{even}, \qquad -1\ \nu\ \text{odd}, \qquad -3\ \nu\ \text{odd}.$$

But, further, just as it can be concluded that 'odd' is not epsilon related to 'even' so can it also be concluded that 'odd' is not nu related to 'even', or that '3' is not nu related to '<' and so on. That is, it can be concluded that the only nu relationships holding between the symbols (2.1) are those indicated above and that therefore all of the following sentences, and more, are true:

$$-1\ \nu\ 1, \qquad -1\ \nu\ 2, \qquad -\text{even}\ \nu\ 2, \qquad -\text{odd}\ \nu\ \text{even},$$
$$-\ <\ \nu\ +, \qquad -3\ \nu\ <.$$

The symbols that are epsilon or nu related to a given predicate symbol form, so to speak, the domain of application of the given symbol.* Thus, for example, the symbols '1', '2', '3', '4', and other numerals form the domain of application of the predicate symbols 'even' and 'odd'. The symbols that are in the domain of application of a given symbol may have the symbol positively or negatively applied to them (are epsilon or respectively nu related to the given symbol). Some symbols such as '+' and '<' are not predicate symbols since there are no symbols in the universe of symbols epsilon or nu related to them. Although there are no symbols among those of (2.1) either epsilon or nu related to the symbols '1', '2', '3', and '4', in some enlarged universe of symbols there might be such symbols; for example, a symbol to which exactly two objects are epsilon related would be epsilon related to '2'.

It is important to distinguish between "$s\ \nu\ t$" and "$-s\ \epsilon\ t$", and between "$s\ \epsilon\ t$" and "$-s\ \nu\ t$". For any universe of symbols

$$(2.2) \qquad (x)(y)\ -\ [x\ \epsilon\ y\ \&\ x\ \nu\ y]$$

should certainly be true; symbols are never used in such a way that one is epsilon and nu related to another. Sentences logically equivalent with (2.2) are:

$$(x)(y)[x\ \epsilon\ y\ \supset\ -\ x\ \nu\ y] \quad \text{and} \quad (x)(y)[x\ \nu\ y\ \supset\ -\ x\ \epsilon\ y].$$

Hence from the truth of "$s\ \epsilon\ t$" can be concluded the truth of "$-s\ \nu\ t$", and from the truth of "$s\ \nu\ t$" can be concluded the truth of "$-s\ \epsilon\ t$". But the converse is not the case; the sentences

$$(x)(y)[-x\ \nu\ y\ \supset\ x\ \epsilon\ y] \quad \text{and} \quad (x)(y)[-x\ \epsilon\ y\ \supset\ x\ \nu\ y]$$

are not true since they are contradicted in the given universe of symbols, for example, when both x and y are '1', for '1' is neither epsilon or nu related to itself. Hence "$s\ \nu\ t$" is not equivalent to "$-s\ \epsilon\ t$", and "$s\ \epsilon\ t$" is not equivalent to "$-s\ \nu\ t$".

* In [3] in place of 'ν' appeared 'm' and in place of (2.2) appeared: $(x)\ (y)\ [x\ \epsilon\ y \supset x\ m\ y]$. Thus the symbols that are m-related to a symbol form its domain of application. However "$s\ m\ t$" can be understood to be "$s\ \epsilon\ t \vee s\ \nu\ t$"; or conversely, "$s\ \nu\ t$" can be understood to be "$s\ m\ t\ \&\ -s\ \epsilon\ t$". In (4) '$\nu$' and '$\epsilon$' replaced '$m$' and '$\epsilon$'.

It can be said that "$-1\ \epsilon$ even" is true for a different reason than "$-3\ \epsilon <$" because the former sentence follows by (2.2) from "$1\ \nu$ even", while not only is "$-3\ \epsilon <$" true but so also is the sentence "$-3\ \nu <$".

Assume now that the universe of symbols being considered is enlarged so as to contain not only the symbols (2.1) but also all concatenations of members of the universe with one another; that is, consider now a universe of symbols containing the symbols (2.1) that is closed under concatenation. The enlarged universe will therefore contain, among others, the following symbols:

$$2 \text{ even}, \quad \text{odd } 3 \text{ even}, \quad 1<, \quad 1+1, \quad 2+1, \quad <1, \quad <2, \quad <3, \quad <4.$$

The domain of the epsilon and nu relations can be extended to the enlarged universe; between any two members of the universe the epsilon relationship either does or does not hold and the nu relationship either does or does not hold. Clearly some of the concatenations of symbols from (2.1) have nothing either epsilon or nu related to them; for example, no symbol is either epsilon or nu related to the symbol '2 even' or to the symbol 'odd 3 even'. But the following sentences are true:

$$1\ \epsilon < 2, \quad 1\ \epsilon < 3, \quad 2\ \epsilon < 3, \quad 1\ \epsilon < 4, \quad 2\ \epsilon < 4, \quad 3\ \epsilon < 4,$$

as well as the following sentences:

$$1\ \nu < 1, \quad 2\ \nu < 1, \quad 3\ \nu < 1, \quad 4\ \nu < 1, \quad 2\ \nu < 2, \quad 3\ \nu < 2, \quad 4\ \nu < 2, \text{ etc.}$$

Thus, although nothing is epsilon or nu related to the symbol '$<$', there are symbols epsilon and nu related to the concatenation of '$<$' with numerals; such a symbol is called a (two place) relation symbol.

Relation symbols of more than two places are also possible. The most elegant manner of considering them is to presume that instead of the universe being closed under concatenation it is closed rather under the simultaneous concatenation of two symbols and the enclosing of the concatenate in brackets; consequently, that for any two symbols s and t in the universe, the symbol (st), the concatenation of left bracket, s, t and right bracket in that order, is a member of the universe. Then '$<$' is a relation symbol of two places because there are symbols epsilon or nu related to, for example, the symbol '(<2)'. A relation symbol of three places is a symbol 'R', for example, for which there are symbols epsilon or nu related to, for example, the symbol '$((R1)2)$'. Similarly relation symbols of more than three places can be introduced. However, for the convenience of avoiding the writing of extra brackets, instances of symbols like '<1' can be assumed to denote '(<1)', and instances of symbols like '$1R2$' or '$(1R2)$' can be assumed to denote the symbol '$((R1)2)$'.

3. Symbol theory and conventionalism. Assuming that the epsilon and nu relations are relations between symbols which are available for the specification of the usages of symbols, then a plausible conventionalist view of mathematics can be described. It can be assumed of each primitive predicate symbol appearing in a universe of symbols that it has been added to the universe and its use

for that universe specified by a statement describing what objects are to be epsilon related to it and what objects are to be nu related to it. Thus, for example, a predicate symbol 'L' could be added to a universe of symbols by specifying that it is to be used in the universe in such a way as to satisfy:

(3.1) $(x)\{[x \epsilon L \equiv (u) - u \epsilon x \,\&\, (Eu) u \nu x] \,\&\, [x \nu L \equiv (Eu) u \epsilon x]\}$.*

'L' is a symbol to which those symbols are to be epsilon or nu related which themselves have something epsilon or nu related to them; if objects are only nu related to a symbol then that symbol is epsilon related to 'L', and if objects are epsilon related to a symbol then that symbol is nu related to 'L'; 'L' is thus a symbol corresponding to the empty set of set theory. Since 'L' has been added to the universe of symbols, the universal quantifiers in (3.1) must be assumed to include 'L' within its range; but (3.1) is the only specification needed for the use of 'L'. Thus, for example, assuming that there is another symbol appearing in the universe of symbols that is epsilon related to 'L' then from (3.1) can be concluded that 'L' is nu related to itself.

When a symbol is added to the universe of symbols it must be assumed that other symbols are also added so as to close the new universe with respect to the simultaneous concatenation of two symbols and enclosing in brackets. It can then be assumed of each primitive relation symbol appearing in a universe of symbols that it has been added to the universe and its use for that universe specified by a statement describing what objects are epsilon or nu related to the concatenation of it with other symbols. For example, the usage of a three place relation symbol 'U' can be specified by:

(3.2) $(x)(y)(u)\{[u \epsilon (xUy) \equiv u \epsilon x \vee u \epsilon y] \,\&\, [u \nu (xUy) \equiv u \nu x \,\&\, u \nu y]\}$.

Here it is not specified what symbols are epsilon and nu related to 'U' (although a sentence similar in form to (3.1) could specify that no symbols are epsilon or nu related to 'U') but rather it is specified what symbols are epsilon and nu related to complex symbols involving 'U'. For example, (3.2) specifies that those symbols are to be epsilon related to '(LUL)' which are epsilon related to 'L', and specifies that those symbols are to be nu related to '$(LU(LUL))$' which are nu related to 'L' and nu related to '(LUL)'.

The statements which specify the usages of symbols may, actually, have to assert more about the usage of the symbols than (3.1) does for 'L' or than (3.2) does for 'U', for no mention has been made yet of the important distinction between extensional and intensional symbols or of the definition of identity.

The sentence

(3.3) $2 = (1 + 1)$

cannot be understood in the usual sense of identity as asserting that '2' denotes

* Here and below the logical connectives '&' and '$\vee$' are presumed to bind more strongly than the connectives '$\supset$' and '$\equiv$'. This means here, for example, that the sentence is to be read as though '$(u) - u \epsilon x \,\&\, (Eu)\, u \nu x$' were enclosed in brackets.

the same object as '(1+1)'. For, from the convention that has been adopted that an instance of a symbol of the universe is a name for that symbol, one would conclude that '2' and '(1+1)' were the same symbols while they are clearly not. (3.3) can be satisfactorily interpreted, however, if identity is understood as extensional identity; where x and y are extensionally identical (predicate symbols), in symbols $x=y$, if and only if

$$(3.4) \qquad (u)\{[u \in x \equiv u \in y] \ \& \ [u \nu x \equiv u \nu y]\}$$

is true. Hence (3.3) is understood as asserting that to '2' and to '(1+1)' exactly the same things are epsilon and nu related.

It is evident, however, that extensional identity does not have all the properties of identity. In particular it is not the case that for any z, the following is true:

$$(3.5) \qquad (x)(y)(x = y \supset (x \in z \equiv y \in z) \ \& \ (x \nu z \equiv y \nu z));$$

for example, should z be the symbol 'A-shaped' then it is possible for two symbols to be extensionally identical although only one is epsilon related to 'A-shaped'. In set theory a sentence similar to (3.5) is accepted as an axiom, the axiom of extensionality. But since the acceptance of (3.5) as an axiom in symbol theory would exclude the possibility of introducing intensional predicate symbols, (3.5) can better be seen as a definition of "z is extensional", and the assumption that a symbol is extensional be seen as a possible additional assumption to be made of its use. A symbol 'L' whose usage is specified by (3.1) can be shown to be extensional, but a symbol 'B' whose usage is specified by

$$(3.6) \qquad (x)\{[x \in B \equiv (Eu)x \in u] \ \& \ [x \nu B \equiv (u)x \nu u)\}$$

cannot be shown to be extensional; for "$(Eu)\ x \in u$" could be true because x is epsilon related to 'A-shaped' and therefore "$(Eu)\ y \in u$" could be false even though x and y are extensionally identical. If it is necessary that there be an extensional symbol 'B' satisfying (3.6), then 'B' must be assumed to be extensional; that is, 'B' is not only assumed to satisfy (3.6) but also to satisfy (3.5) with 'z' replaced by 'B'. With the additional assumption of extensionality it is possible to conclude in particular

$$[2 \in B \equiv (1 + 1) \in B] \ \& \ [2 \nu B \equiv (1 + 1) \nu B]$$

from (3.3); extensional identity has, therefore, all the properties required of identity with respect to extensional predicate symbols.

Extensional identity for relation symbols of any number of places can be defined similarly to extensional identity for predicate symbols. For example, extensional identity of two place relation symbols x and y is defined by:

$$(3.7) \qquad (u)(v)\{[u \in (xv) \equiv u \in (yv)] \ \& \ [u \nu (xv) \equiv u \nu (yv)]\},$$

and of three place relation symbols x and y is defined by:

(3.8) $(u)(v)(w)\{[u \in ((xv)w) \equiv u \in ((yv)w)] \,\&\, [u\ v\ ((xv)w) \equiv u\ v\ ((yv)w)]\}$.

For any given number, the extensionality of predicate symbols with respect to the extensional identity of relation symbols of that number of places can be defined by (3.5) when the identity in (3.5) is understood as the identity of relation symbols of the given number of places. But it is not possible to give a single definition of extensional identity for relations of any number of places, nor is it possible to give a single definition for the extensionality of predicate symbols with respect to the identity of relation symbols of any number of places. Therefore it is a great convenience, although not a necessity, to be able to assume that only relation symbols of up to three places occur as "primitive" and that relation symbols of more than three places can be constructed from them. For then a general definition of extensional identity can be given: x and y are extensionally identical if and only if they satisfy the conjunction of (3.4), (3.7) and (3.8). Further a definition of "z is an extensional predicate or relation symbol" can be given which is the conjunction of (3.5) and other sentences asserting that anything in either of the two places of a two place relation symbol, or in one of the three places of a three place relation symbol, can be replaced by anything else which is extensionally identical with it. With a general definition of extensional identity it is possible to have relation symbols of more than three places by compounding relation symbols of up to three places. It is only necessary to introduce a relation symbol 'P' of three places, the ordered pair relation symbol, in much the same manner that the ordered pair is defined in set theory. From the statement specifying the use of 'P' one should be able to conclude:

$$(x)(y)(u)(v)[(xPy) = (uPv) \supset x = u \,\&\, y = v].$$

Then a symbol y satisfying, for example,

$$(Eu)(Ev)(Ew)(Ex)[u \in (vy)(wPx)) \vee u\ v\ (vy)(wPx))]$$

is a relation symbol of four places.

Symbols are ideal oriented shapes which do not of themselves bear epsilon or nu relationships with symbols or any other objects. Only when a symbol has been selected for membership in a universe of symbols (and there are no *a priori* dictates as to which symbols should be selected) and a statement specifying its usage has been given can anything be said to be epsilon or nu related to the symbol or can it be said to be epsilon or nu related to any other symbol or itself. The uses that are to be specified for the symbols selected will depend upon the purpose to which the universe of symbols is to be put. Certainly it is not to be expected that some particular universe of symbols together with the sentences specifying the usage of the symbols would, for some intrinsic reason, be *the* universe. Thus, for example, although it is possible to show that the simple theory of types can be understood to be a universe of symbols with their usages specified by sentences similar to (3.1) and (3.2) with an additional assumption of

extensionality,* it cannot be concluded that the universe of the simple theory of types is in any sense the correct one, or that the universe cannot be extended. In brief, mathematical truth is seen as truth relative to the particular universe of symbols with its statements of usage under consideration and not as truth independent of all universes of symbols.

4. Set theory. Unless one is only interested in formal systems or axiomatic theories and looks upon these systems as the mathematician's only legitimate interest, one must be interested also in the conceptual scheme which motivates the axiomatic set theories. Indeed, the concept of set—a collection of definite, distinct objects of our intuition or of our intellect, to give the best known "definition" from [2],†—and the relationship of membership between members of a set and the set, are what are central to set theory, not any particular axiomatic set theory. Axiomatic set theories are but attempts to describe a universe of sets; they are motivated by the conceptual scheme of naive set theory and derive any importance that they may possess from the conceptual scheme that motivates them.

As indicated in (6), for example, the manner in which sets are regarded varies from an extreme platonism to a conceptualism. From the platonist view sets are abstract objects existing independently of man, unchanging in some ideal world. From the conceptualist view sets are objects which are somehow constructed by man and maintain an existence after being constructed. But as Weyl shows in (8) with regard to the particular conceptualism that he espoused, the acceptance of the view leads to a weakening of analysis.

The conceptual basis provided for mathematics by the platonic view of set theory is unrivaled in elegance and simplicity. The truths of mathematics are discovered through the examination of the ideal world of sets; a theorem of analysis, for example, is a factual statement describing a state of affairs in the world. But despite the elegance, simplicity and power of this naive set theory it must be discarded as a foundation for mathematics because of the inconsistency of its concepts. The paradoxical sets of set theory can of course be excluded by accepting any one of a variety of ad hoc restrictions on the concept of set which have been investigated. Yet, no matter how the restrictions are formulated, the generality of the concept of set permits one to legitimately ask "but why does

* More precisely it can be shown that the simple theory of types without an axiom of infinity is equivalent with a theory S formalized within the first order predicate calculus. The theory S has ϵ and ν as its only primitive relations, (2.2) as its only axiom and an axiom scheme each instance of which is a sentence specifying the usage of a single symbol. Details will be given in my forthcoming paper *An interpretation of the simple theory of types.* A similar result was stated without proof in [3]. The result is stated in more detail and in a slightly different form in [4]; an interpretation is provided in [3], however, for only the monadic simple theory of types rather than the full simple theory of types.

† It is a translation of "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unserer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) an einem Ganzen," taken from [1].

not one of the forbidden sets exist?" The question is not answered by pointing out that otherwise an inconsistency can be obtained, for that can only be an answer to the question "Why cannot this sentence be added as an axiom to this theory?" The question can be better worded "If a set is a collection of definite distinct objects of our intuition or of our intellect conceived of as a whole, then why cannot this particular collection of definite distinct objects be conceived of as a whole?" The paradoxes of set theory *are* paradoxical and are the main objection to using set theory as a conceptual basis for mathematics.

5. Symbol theory as an alternative to set theory. To the view of mathematics suggested by set theory there exists many alternatives: intuitionism and formalism in all their various forms are two examples and the conventionalist view described in Section 3 is another. But symbol theory is also an alternative to set theory in the immediate sense of being able to replace set theory in the many uses made of the latter in mathematics. However, as the paradoxes of set theory are considered to be the main obstacle to the acceptance of set theory as a conceptual basis for mathematics, it is necessary to examine the paradoxes from the point of view of symbol theory.

The epsilon and nu relationships that hold between symbols of any universe of symbols that has been selected, hold only by virtue of the specified conventions of usage and not by virtue of any intrinsic properties of the symbols. Thus, for example, that the symbol 'L' whose use is specified by (3.1) will be epsilon related to itself in some universes of symbols is a fact which is a consequence of the convention (3.1) and conventions for other symbols of the universes and does not follow from the nature of the symbol 'L' itself.

It is of course possible to specify conventions that cannot possibly be satisfied; for example, it is possible to specify that 'A' be used as indicated by:

$$(5.1) \qquad (x)\{[x \in A \equiv x = x] \,\&\, [x \nu A \equiv x = x]\}.$$

But from (2.2) it is impossible that any symbol satisfy (5.1)—it can only be through a misunderstanding that one would insist that 'A' do so. So also it is possible to insist that the use of the symbol 'R' is specified by:

$$(5.2) \qquad (x)\{[x \in R \equiv - x \in x] \,\&\, [x \nu R \equiv x \in x]\}$$

although one would do so in ignorance, for by substituting 'R' for x,

$$'[R \in R \equiv - R \in R]'$$

can be concluded from (5.2) and therefore "$-R \in R \,\&\, R \in R$" is a consequence of (5.2). It is therefore not paradoxical that the symbol 'R', nor any other symbol, cannot satisfy (5.2) since there is a simple explanation of why it cannot do so: namely, the sentence (5.2) is contradictory.

Although it is not possible for any symbol to satisfy (5.2) it is possible for there to exist a predicate symbol with properties analogous to the properties of Russell's paradoxical set of all sets which are not members of themselves. For

the usage of such a predicate symbol 'R' is specified by:

$$(5.3) \qquad (x)\{[x \in R \equiv x \nu x] \,\&\, [x \nu R \equiv x \in x]\}.$$

This specification of usage is not contradictory nor does it contradict (2.2). By substituting 'R' for x, "$[R \in R \equiv R \nu R]$" can be concluded from (5.3), and therefore "$-R \in R \,\&\, -R \nu R$" can be concluded from (5.3) and (2.2). Thus it is possible for there to be a Russell predicate symbol, although one must conclude that such a symbol is neither epsilon nor nu related to itself.

The fact that the usage of a Russell symbol 'R' can be consistently specified by (5.3) although not by (5.2), and that the most important difference between these two statements is in the replacement of '$-x \in x$' by '$x \nu x$' might suggest that symbol theory provides an explanation of the Russell (and other) paradoxes in the distinction between "$-s \in t$" and "$s \nu t$". The validity of the conclusion depends upon the sense of the word "explanation" intended. It cannot be denied that the replacement of '$-x \in x$' by '$x \nu x$' in (5.2) converts it into a satisfiable sentence; but that is not to say that the paradoxes are avoided by introducing the nu relation in addition to the epsilon relation. The paradoxes would not be avoided in set theory by recognizing in some way the nu relationship as well as the epsilon relationship between sets; for the paradoxical set of sets that are not epsilon related to themselves would persist in plaguing such a theory and would be joined by the paradoxical set of sets that are not nu related to themselves. Symbol theory avoids the difficulties of the paradoxes not by virtue of the nu relation, although the possibility of consistently introducing a Russell predicate symbol argues for symbol theory; rather the difficulties are avoided by recognizing that the epsilon and nu relationships between the basic objects of mathematics are not intrinsic to the objects but are ones that arise only from assumptions of conventional usage, and that therefore contradictory assumptions of usage are the cause of contradictory relationships between the objects. If by "explanation of the paradoxes" is meant, however, insight into how to avoid specifying usages for symbols that cannot be fulfilled, then the distinction between "$-s \in t$" and "$s \nu t$" is a partial explanation of the paradoxes; but it is, of course, only partial.

An objection to symbol theory, however, might be raised. It might be argued that if symbol theory is able to replace set theory as a mathematical language then one is led to a contradiction which provides an argument against symbol theory as a foundation for mathematics. For by Cantor's diagonal method of proof it can be shown that the universe of sets is nondenumerable. If symbol theory is able to replace set theory then it must be possible to duplicate this proof in some specified symbol theory and show that the universe of symbols for that theory is nondenumerable. But clearly at most denumerably many primitive symbols can be added to the universe and the universe of all concatenations of these symbols will also be denumerable. Here, apparently, is a contradiction of a different kind than the paradoxes of set theory, but one which nevertheless

makes symbol theory unsuitable as a foundation for mathematics, just as the paradoxes make set theory unsuitable.

A careful examination of the above argument can turn it equally well against set theory. From the point of view of set theory denumerability is a property of sets which is intrinsic to sets since there either does or does not exist in the universe of sets a one-one correspondence between a given set and the set of positive integers. From the point of view of symbol theory, on the other hand, 'denumerable' is a symbol which can be added to the universe of symbols by specifying a convention of usage. The convention will specify that a given symbol be epsilon related to 'denumerable' if and only if there exists in the universe of symbols a symbol which is a one-one correspondence between the symbols epsilon related to the given symbol and the symbols epsilon related to 'positive integer' (assuming some commonly accepted usage of this symbol has been given). But there will exist such a correspondence in the universe of symbols if and only if one has been added; thus although a given symbol may not be epsilon related to 'denumerable' in some universe of symbols, it is possible that the universe can be enlarged by the addition of new symbols so that in the enlarged universe the given symbol is epsilon related to 'denumerable.' From the point of view of set theory this relativism that is inherent in symbol theory is a fundamental flaw which argues against the acceptance of symbol theory as a conceptual basis for mathematics. But equally well from the point of view of symbol theory the absolutism inherent in set theory is a fundamental flaw. Neither theory is, on this point, of itself a sufficient argument for discarding the other.

The contrast between the two points of view is reminiscent of what is generally referred to as Skolem's paradox, discussed in detail in [5]. In [7], Skolem extended a theorem of Lowenheim to prove that no matter what intuitive conceptual scheme motivates a formal theory, an interpretation or model for the formal theory can be obtained with a denumerable universe of discourse. Thus in particular for any formal set theory whatsoever there can be found a denumerable model. It is possible to conclude from Skolem's result that no formal set theory is adequate for the description of the reality of the world of sets and that in particular a formal description of nondenumerable universes is impossible. However in view of the extraordinary richness of the formal theories that are available to the mathematician and in view of the contradictions inherent in set theory, it might be more reasonable to conclude that denumerability and nondenumerability are concepts that are more subtle than is suggested by naive set theory. Indeed Skolem suggested this conclusion and proposed a relativization of the concepts which, taking into account that he spoke of set theory, is very similar to that just outlined.

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QUADRICS IN R_n

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Techniques of vector and matrix algebra have been used in geometry of R_n, a real unitary n-dimensional space. We shall combine these techniques with use of certain homogeneous systems to treat properties of a quadric, *i.e.*, a complete equation of second degree in n variables. Then the results developed can be applied to the cases of two and three dimensions which actually cut down considerably the arithmetic of many problems.

We assume that the reader is familiar with elementary ideas of linear spaces, matrices, proper values and directions of a matrix, and reduction of quadratic forms. We denote any square matrix by a capital letter. The determinant of a matrix A is denoted by $\det A$; the inner product of two vectors ξ and η is denoted by (ξ, η).

1. Straight line. We begin with a short review of the vector and matrix equation of a straight line in R_n. Let $\eta \in R_n$ be fixed, and end on a straight line d. Let δ be a direction for d, *i.e.*, difference of two vectors ending on d. Then $\xi = \eta + t\delta$ is the equation of the line d. Clearly we can write this equation in matrix form such as

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} + t \begin{pmatrix} l_1 \\ \cdot \\ \cdot \\ \cdot \\ l_n \end{pmatrix},$$

where $\xi = (x_1, \cdots, x_n)$, $\eta = (a_1, \cdots, a_n)$, and $\delta = (l_1, \cdots, l_n)$, and t is a parameter. But for reasons to be seen later we write this equation as

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n \\ 1 \end{pmatrix} + t \begin{pmatrix} l_1 \\ \cdot \\ \cdot \\ \cdot \\ l_n \\ 0 \end{pmatrix},$$

or in row matrices. We note that to each point, or vector ξ in R_n corresponds a matrix $(x_1 \cdots x_n)$ or $(x_1 \cdots x_n\ 1)$ and to each direction δ corresponds a matrix $(l_1 \cdots l_n)$ or $(l_1 \cdots l_n 0)$. For convenience we denote these matrices by ξ and δ, and we use whichever form is suitable.

An equation of the first degree in $x_1, \cdots, x_n$ is called a plane in R_n, $n \geqq 3$.

2. Quadric. Let $\xi=(x_1 \cdots x_n\ 1)$ and A be a symmetric $(n+1)\times(n+1)$ matrix. Then a quadric can be written as $(A\xi, \xi)=0$. For example in R_3,

$$ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz + 2px + 2qy + 2rz + d = 0$$

can be written as

$$(x\ \ y\ \ z\ \ 1)\begin{pmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0.$$

Of course the use of these particular homogeneous coordinates $(x_1, \cdots, x_n, 1)$ made it possible to express a quadric as $(A\xi, \xi)=0$. For later uses we call the matrix of the quadratic part of the quadric Q.

3. Intersection of a straight line and a quadric. Let $(A\xi, \xi)=0$ be a quadric and $\xi=\eta+t\delta$ be a straight line in R_n. Then the points of intersection are solutions of the system

$$(A\xi, \xi) = 0, \qquad \xi = \eta + t\delta.$$

Eliminating ξ we get

$$(A\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0. \tag{3.1}$$

Here since R_n is a real space and A is symmetric $(A\delta, \eta)$ has been replaced by $(A\eta, \delta)$. We also observe that $(A\delta, \delta)=(Q\delta, \delta)$. Thus (3.1) is written as

$$(Q\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0. \tag{3.2}$$

Clearly the following cases can be considered:

(a) If $(Q\delta, \delta)\neq 0$, the line intersects the quadric in two points, real or complex conjugate or coincident.

(b) If $(Q\delta, \delta)=0$, $(A\eta, \delta)\neq 0$, the line and quadric intersect in one point.

(c) If $(Q\delta, \delta)=0$, $(A\eta, \delta)=0$, and $(A\eta, \eta)\neq 0$, the line does not intersect the quadric.

(d) If (3.2) is an identity, then the line lies on the quadric.

4. THEOREM (Diametral plane) *The midpoints of all chords of a quadric parallel to a given direction δ lie in a plane called the diametral plane corresponding to that direction.*

Proof. (Indeed this theorem has been proved in many ways; most of them quite tediously). Here we shall demonstrate the use of Section 3. Assume that $(Q\delta, \delta) \neq 0$, so that there will be two points of intersection [see Sec. 3 (a)], and let η end at the midpoint of the corresponding chord. Then the chord lies along the line $\xi = \eta + t\delta$. Now let t_1 and $t_2 = -t_1$ (since η ends at the midpoint of the chord) be the values of t corresponding to the ends of the chord. Since $t_1 + t_2 = 0$, we conclude from (3.2) that $(A\eta, \delta) = 0$. Thus η ends on the plane $(A\xi, \delta) = 0$.

5. COROLLARY. *If the diametral plane is orthogonal to corresponding direction δ in Section 4, then δ is a characteristic direction of Q.*

Proof. We easily see that $(A\delta, \eta) = 0$ can be written in the form $(Q\delta, \eta) + M = 0$, where M is a constant and $\eta = (x_1 \cdots x_n)$. The normal direction to $(A\delta, \eta) = 0$ is $Q\delta$ and δ lies in this direction. Hence $Q\delta = k\delta$, which proves the corollary.

In this case the plane $(A\delta, \eta) = 0$ is called a plane of symmetry for the quadric.

6. A center of a quadric. A vector $\eta = (x_{10}, \cdots, x_{n0})$ is said to end at a center of a quadric $(A\xi, \xi) = 0$ if for any ξ_1 satisfying $(A\xi_1, \xi_1) = 0$ there is a ξ_2 satisfying $(A\xi_2, \xi_2) = 0$ such that

$$\eta = \tfrac{1}{2}(\xi_1 + \xi_2). \tag{6.1}$$

Now to obtain a center let

$$\xi = \eta + t\delta \tag{6.2}$$

be a line through the end of η, where

$$\delta = (l_1, \cdots, l_n) = (x_1 - x_{10}, \cdots, x_n - x_{n0}).$$

The points of intersection of (6.2) and the quadric are obtained by

$$(Q\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0, \qquad (Q\delta, \delta) \neq 0$$

[see (3.2)]. If (6.1) is true, then

$$(A\eta, \delta) = 0. \tag{6.3}$$

Now let

$$A = \begin{pmatrix} h_{11} \cdots h_{1n} & p_1 \\ \cdot \quad\quad \cdot & \cdot \\ \cdot \quad\quad \cdot & \cdot \\ h_{1n} \quad\quad h_{nn} & p_n \\ p_1 \cdots p_n & d \end{pmatrix}.$$

Then (6.3) can be written as

$$(x_{10} \cdots x_{n0}) \begin{pmatrix} h_{11} & \cdots & h_{1n} & p_1 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ h_{1n} & & h_{nn} & p_n \\ p_1 & \cdots & p_n & d \end{pmatrix} \begin{pmatrix} l_1 \\ \cdot \\ \cdot \\ l_n \\ 0 \end{pmatrix} = 0.$$

This implies that

$$\alpha = \left(p_1 + \sum_{i=1}^{n} h_{1i}x_{i0}, \cdots, p_n + \sum_{i=1}^{n} h_{in}x_{i0} \right)$$

is orthogonal to $\delta = (x_1 - x_{10}, \cdots, x_n - x_{n0})$ for all $\xi = (x_1, \cdots, x_n)$ satisfying $(A\xi, \xi) = 0$. Either α is a fixed direction, and $(A\xi, \xi) = 0$ is a double plane perpendicular to α at the end point of η, or $\alpha = 0$, that is,

$$p_j + \sum_{i=1}^{n} h_{ji}x_{i0} = 0, \qquad j = 1, \cdots, n,$$

which means $(x_{10}, \cdots, x_{n0})$ is the solution of

$$(x_1 \cdots x_n 1) \begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ h_{1n} & & h_{nn} \\ p_1 & \cdots & p_n \end{pmatrix} = (0 \cdots 0).$$

Of course the problem of the existence of center is the problem of n equations and n unknowns. We only mention that if $\det Q \neq 0$, there is a unique center and in the case $\det Q = 0$ we have to consider other determinants for obtaining centers.

7. Vertex of a quadric without center. For a quadric to lack a center the rank k of the matrix Q must be less than n. Let $\alpha_1, \cdots, \alpha_k$ be k orthonormal characteristic directions of Q corresponding to k nonzero characteristic roots of Q, and the corresponding planes of symmetry be

$$(A\alpha_i, \xi) = 0, \qquad i = 1, \cdots, k. \tag{7.1}$$

By well-known results on reduction of quadratic forms, the quadratic terms of the original quadric are a linear combination of the squares of the first degree terms of the left members of the equations (7.1).

We consider the system of $k+1$ equations

$$(A\xi, \xi) = 0; \qquad (A\alpha_i, \xi) = 0, \qquad i = 1, \cdots, k. \tag{7.2}$$

Since $(A\alpha_i, \xi)^2 = 0$, we may subtract some expression of the form $\sum_k c_i (A\alpha_i, \xi)^2$ from the equation of the quadric, leaving a linear equation, and the system

(7.2) reduces to

$$(A\alpha_i, \xi) = 0, \qquad i = 1, \cdots, k$$

$$(A\xi, \xi) - \sum_k c_i(A\alpha_i, \xi)^2 = \sum_n b_i x_i + c = 0. \tag{7.3}$$

Since the k planes of symmetry in (7.1) are orthogonal, and $k<n$, the system (7.3) has rank either k or $k+1$. We consider three possible cases:

(a) If the rank is k and the system has solutions, the last equation is a consequence of the first k equations. Then the system $(A\alpha_i, \xi)=0$, $i=1, \cdots, k$, has solutions which are centers of the quadric, and lie in the quadric.

(b) If the rank is k, and there are no solutions, again the solutions of the system $(A\alpha_i, \xi)=0$, $i=1, \cdots, k$, are centers, but do not lie on the quadric.

(c) If the rank is $k+1$, the set of solutions is a hyperplane of dimension $n-k-1$. Any point of this hyperplane is called a vertex of the quadric. In particular, if $k=n-1$, there is a unique point which is the vertex of the quadric.

8. Polar of a point with respect to a quadric. Let the point be the end point of $\eta=(x_{10}, \cdots, x_{n0})$, and let δ be any direction. Then the equation of any line through the end point of η is

$$\xi = \eta + t\delta. \tag{8.1}$$

The points of intersection of (8.1) and a quadric $(A\xi, \xi)=0$ are obtained by

$$(Q\delta, \delta)t^2 + 2(A\eta, \delta)t + (A\eta, \eta) = 0 \tag{8.2}$$

[see (3.2)]. Let $(Q\delta, \delta)\neq 0$. Then (8.2) has two solutions and (8.1) intersects $(A\xi, \xi)=0$ in two points. The locus of the harmonic conjugate of the end point of η with respect to these points is a plane, called the polar of the end point of η with respect to $(A\xi, \xi)=0$.

Proof. Let t_1 and t_2 be the roots of (8.2) and let t correspond to ξ which ends at the harmonic conjugate of the end point of η with respect to two points of intersection. Then

$$\frac{2}{t} = \frac{1}{t_1} + \frac{1}{t_2} \quad \text{or} \quad t = \frac{2t_1t_2}{t_1 + t_2} = -\frac{(A\eta, \eta)}{(A\eta, \delta)}.$$

If $(A\eta, \delta)=0$, then η ends at a center of $(A\xi, \xi)=0$ and the polar is a plane at infinity. Let $(A\eta, \delta)\neq 0$. Then we have

$$(A\eta, \delta)\xi = (A\eta, \delta)\eta - (A\eta, \eta)\delta.$$

Taking inner product of both sides by $A\eta$ we get $(A\eta, \delta)(A\eta, \xi)=0$. Since $(A\eta, \delta)\neq 0$ we get $(A\eta, \xi)=0$, which is a plane.

If η ends on $(A\xi, \xi)=0$, then this plane is the tangent plane at the end point of η.

COMPLEX GROUPS

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1. Introduction. The notion of quotient groups is usually arrived at through consideration of normal subgroups. In this paper, quotient groups are characterized by means of three postulates. The subsequent removal of two of these postulates gives rise to a class of mathematical systems, called complex groups, possessing greater set-theoretic generality than quotient groups. Since each complex group is itself a group, a technique for constructing new groups from a given group is implicitly involved.

2. Complexes. A *complex* of a group G is any subset of G. In particular, the null set $\varnothing$ is a complex of G. If A and B are complexes of G, then the *complex product* AB of A and B is the set of all elements ab, where $a\in A$ and $b\in B$. The formation of a complex product AB is termed *complex multiplication*, and A and B are called *factors* of AB. If a factor of a complex product is the null set, then the complex product itself is the null set. Braces are suppressed when a factor is a single-element set. Thus, for example, the complex product of $\{a\}$ and B is written aB instead of $\{a\}B$. The cardinal number of a complex B is denoted $c(B)$. A complex B is said to be *torsion* if each of its elements has finite period. B is *idempotent* if $BB=B$.

For the sake of completeness, a few of the elementary rules governing complex multiplication are presented. Notice that, in general, complex multiplication is not distributive over set-theoretic intersection. If A, B, and C are complexes of a group G, then

(i) $A(BC)=(AB)C$,
(ii) $A(B\cup C)=AB\cup AC$ and $(B\cup C)A=BA\cup CA$,
(iii) $A(B\cap C)\subseteq AB\cap AC$ and $(B\cap C)A\subseteq BA\cap CA$,
(iv) if $A\subseteq B$, then $CA\subseteq CB$ and $AC\subseteq BC$, and
(v) if $A\neq\varnothing$ and $B\neq\varnothing$, then $c(AB)\geqq c(A)$ and $c(AB)\geqq c(B)$.

3. Postulates for a quotient group. A set $\mathfrak{R}$ of complexes of a group G is a quotient group of G if and only if $\mathfrak{R}$ satisfies the following postulates:

(P1) $\mathfrak{R}$ is a group under complex multiplication,
(P2) $\mathfrak{R}$ covers G, and
(P3) members of $\mathfrak{R}$ are pairwise disjoint.

These postulates are consistent and independent. Of course, it must be verified that they actually characterize quotient groups.†

Let e and E denote respectively the identity elements of G and $\mathfrak{R}$. It will first be shown that E is a subgroup of G. Since $\mathfrak{R}$ covers G, there exists an $X\in\mathfrak{R}$

* Presently with Librascope General Precision, Glendale, California.

† J. E. Eaton, making use of different definitions and concepts, has proved a theorem from which this would follow almost at once. Amer. J. Math., vol. 62, 1940, pp. 222–226.

such that $e \in X$. Clearly, $X \subseteq XX$, and in view of (P3), $X = XX$. But $X = XX$ implies that $X = E$, therefore $e \in E$. Now, suppose that $x \in E$ and $x^{-1} \in X \in \mathfrak{R}$. Since $xx^{-1} \in EX = X$, it follows from (P3) and the fact that $e \in E$ that $X = E$. Thus E contains the identity and the inverse of each of its elements, and is a subgroup of G.

To see that E is normal in G, notice that for each $x \in G$ there exists an $X \in \mathfrak{K}$ such that $x \in X$ and $xEx^{-1} \subseteq XEX^{-1} = E$. From the relations $xEx^{-1} \subseteq E$ and $x^{-1}Ex \subseteq E$ it easily follows that E is a normal subgroup of G.

It remains to show that each $X \in \mathfrak{R}$ is a left, and hence a right, coset of E. Let $x \in X \in \mathfrak{R}$, and observe that $x^{-1} \in X^{-1}$, $xE \subseteq XE = X$, and $x^{-1}E \subseteq X^{-1}E = X^{-1}$. The relation $xE = X$ ensues, and the demonstration is complete.

4. Complex groups. A *complex group from a group* G, or more simply, a *complex group*, is a set of complexes of G which satisfy (P1) but not necessarily (P2) or (P3). Thus, a complex group is merely a set of complexes which themselves form a group under complex multiplication. Obviously, any quotient group of any subgroup of G is a complex group from G.

The notation $\mathfrak{G}$ is used to denote any complex group from group G. In order to avoid confusion, elements of $\mathfrak{G}$ are called *members*. The symbols X, Y, and Z are used to denote members of $\mathfrak{G}$, while the symbol E is reserved for the identity member of $\mathfrak{G}$. Similarly, x, y, and z denote elements of G, and e denotes the identity element of G. The set-theoretic union of the members of $\mathfrak{G}$ is denoted by $\overline{\mathfrak{G}}$. A complex group is *disjunctive* if its members are pairwise disjoint. A complex group is *linearly ordered* if its members are pairwise comparable. A complex group is *finite* if it has finitely many members. Since complex groups are not necessarily abelian, the multiplicative notation for ordinary group theory is used. Thus, X^{-1} is the unique member of $\mathfrak{G}$ such that $XX^{-1} = X^{-1}X = E$, and as usual, $X^0 = E$ and $X^{-n} = (X^n)^{-1}$, where n is a positive integer.

Example 1. Let A be the additive group of integers, and let $\mathfrak{A}$ be the set of all complexes of A of the form $\{n\} \cup \{x \mid x \in A \text{ and } x \geqq n+5\}$, where n is an integer. $\mathfrak{A}$ is neither disjunctive nor linearly ordered, and the identity member $E = \{0\} \cup \{x \mid x \in A \text{ and } x \geqq 5\}$ is not a subgroup of A.

Example 2.* Let M be the multiplicative group of positive rational numbers, and let $\mathfrak{M}$ be the set of all complexes of M of the form $\{x \mid x \in M \text{ and } x > r\}$, where r is a positive real number. $\mathfrak{M}$ is linearly ordered, and the identity element 1 of M is not contained in the identity member $E = \{x \mid x \in M \text{ and } x > 1\}$ of $\mathfrak{M}$. Furthermore, since $\mathfrak{M}$ is nondenumerable, we see that it is possible for a group G to have a complex group $\mathfrak{G}$ which is isomorphic with no quotient group or subgroup of G.

Example 3. Let G be the group obtained by forming the direct product $A \otimes M$, where A and M are the groups defined in the previous two examples.

* I am indebted to the referee for this interesting example.

Construct in G the three complexes $K_1=\{(x, y)\,|\,x\in A$ and $y\in M$ and $y>1\}$, $K_2=\{(x, 1)\,|\,x\in A$ and x is odd$\}$, and $K_3=\{(x, 1)\,|\,x\in A$ and x is even$\}$. Set $E=K_1\cup K_3$ and $X=K_1\cup K_2$. The set $\mathfrak{G}$ whose members are E and X is a complex group from G. $\mathfrak{G}$ is finite and cyclic, but not disjunctive. Furthermore, $\mathfrak{G}$ is not isomorphic with any subgroup of G.

5. Preliminary results. It is clear from the examples that there exist complex groups which are not quotient groups. For convenience, a *komplex group* is defined to be a complex group which is not a quotient group. The first theorem and its corollaries indicate where one must look to find komplex groups.

THEOREM 1. *If $\mathfrak{G}$ is a complex group from group G, then $\mathfrak{G}$ is a quotient group of $\overline{\mathfrak{G}}$ if and only if E is a subgroup of G.*

Proof. It is sufficient to show that $\overline{\mathfrak{G}}$ and $\mathfrak{G}$ satisfy the conditions given for a quotient group in Section 3. First, it will be shown that $\mathfrak{G}$ is disjunctive. Suppose that X, $Y\in\mathfrak{G}$ and $X\neq Y$. Since $(XY^{-1})(YX^{-1})=E$, it follows that either $XY^{-1}\cup YX^{-1}\subseteq E$ or $(XY^{-1}\cup YX^{-1})\cap E=\varnothing$. The first alternative does not hold because $XY^{-1}\subseteq E$ if and only if $E\subseteq YX^{-1}$, and $XY^{-1}\neq E$. From the second alternative it follows that $X\cap Y=\varnothing$, since otherwise, $X\cap Y\neq\varnothing$ implies that $XY^{-1}\cap E\neq\varnothing$. $\mathfrak{G}$ is therefore disjunctive, and it remains only to show that $\overline{\mathfrak{G}}$ is a subgroup of G. Clearly, $\overline{\mathfrak{G}}$ is a semigroup with identity e. Let $x\in X\in\mathfrak{G}$, and let $y\in X^{-1}\in\mathfrak{G}$. Then $xy\in XX^{-1}=E$, and $x^{-1}=y(xy)^{-1}\in X^{-1}$. Thus, $\overline{\mathfrak{G}}$ contains the inverse of each of its elements, and the proof is complete.

A complex group $\mathfrak{G}$ from group G is *elementary* if each of its members is a finite complex of G. It is easy to show that all members of an arbitrary complex group have the same cardinal number, therefore a complex group is elementary if and only if each of its members contains n elements, where n is a fixed natural number.

COROLLARY 1. *Every elementary complex group is a quotient group.*

COROLLARY 2. *$\mathfrak{G}$ is a quotient group of $\overline{\mathfrak{G}}$ if E is torsion.*

COROLLARY 3. *$\mathfrak{G}$ is a torsion quotient group of $\overline{\mathfrak{G}}$ if $\overline{\mathfrak{G}}$ is torsion.*

Komplex groups can never be constructed from torsion groups. On the other hand, any group which is either mixed or torsion free possesses komplex groups. This is evident from Example 1 and the fact that any group which is not torsion contains an infinite cyclic subgroup.

Not every disjunctive complex group is a quotient group. This may be seen by returning to Example 2, and considering the complex group from M whose sole member is $E=\{x\,|\,x\in M$ and $x>1\}$. The next theorem gives a necessary and sufficient condition for a complex group to be disjunctive. A member of a complex group is said to be *isolated* if its intersection with each of the remaining members is null.

THEOREM 2. *A complex group* $\mathfrak{G}$ *from group* G *is disjunctive if and only if* $\mathfrak{G}$ *possesses an isolated member.*

Proof. Suppose that X, Y, $Z \in \mathfrak{G}$ such that $X \neq Y$ and Z is an isolated member. Since $Z \cap ZX^{-1}Y = \varnothing$, and $X \cap Y = \varnothing$ if and only if $(ZX^{-1})X \cap (ZX^{-1})Y = \varnothing$, it follows that $X \cap Y = \varnothing$. Therefore, $\mathfrak{G}$ is disjunctive, and the theorem is proved.

An interesting feature of complex groups which are not disjunctive is revealed by the next theorem.

THEOREM 3. *The intersection of two distinct members of a complex group is either null or infinite.*

Proof. Let X, $Y \in \mathfrak{G}$ such that $X \neq Y$ and $X \cap Y \neq \varnothing$. It follows that $E \cap X^{-1}Y \neq \varnothing$ and $Y^{-1}X \cap E \neq \varnothing$. Since $e \notin X^{-1}Y \cap Y^{-1}X$, assume without loss of generality that $e \notin X^{-1}Y$, and hence that $e \notin E \cap X^{-1}Y$. Now observe that for each positive integer n, $(E \cap X^{-1}Y)^n \subseteq E \cap X^{-1}Y$, because $(E \cap X^{-1}Y)^n \subseteq E \cap X^{-1}Y \cap (X^{-1}Y)^2 \cap \cdots \cap (X^{-1}Y)^n$. Thus each element in $E \cap X^{-1}Y$ has infinite period, and it follows that $X \cap Y$ is infinite, as was to be shown.

6. Finite complex groups. As might be expected from consideration of Example 3, the requirement that $\mathfrak{G}$ possess finitely many members does not guarantee simple set-theoretic structure for $\mathfrak{G}$. The best set-theoretic result for general finite complex groups is obtained as a corollary to the next theorem.

THEOREM 4. *Two distinct members of a torsion complex group* $\mathfrak{G}$ *are never comparable.*

Proof. Suppose that X, $Y \in \mathfrak{G}$ such that $X \subseteq Y$. Then $XY^{-1} \subseteq E$, and since $\mathfrak{G}$ is torsion, XY^{-1} has finite period n. But $(XY^{-1})^n \subseteq XY^{-1} \subseteq E$ implies that $XY^{-1} = E$ and $X = Y$. The proof is complete.

COROLLARY 4. *Two distinct members of a finite complex group are never comparable.*

Let $\mathfrak{G}$ be a complex group with identity member E. The *core* of $\mathfrak{G}$ is defined to be the set of all members of $\mathfrak{G}$ which have a nonnull intersection with E. It is easy to show that the core of $\mathfrak{G}$ is a subgroup of $\mathfrak{G}$. Occasionally, this fact can be used to help determine the structure of a complex group. For example, if $\mathfrak{G}$ is a finite complex group of prime order, then either $\mathfrak{G}$ is disjunctive or $\mathfrak{G}$ is equal to its core.

7. Maximal complex groups. Let $\mathfrak{G}$ be a complex group from group G with identity member E. If each complex group from G having E as its identity member is a subgroup of $\mathfrak{G}$, then $\mathfrak{G}$ is said to be *maximal*. The next theorem, which follows at once from the lemma, states that there is a one-to-one correspondence between the set of all idempotent complexes of G and the set of all maximal complex groups from G.

LEMMA 1. *If $\mathfrak{H}$ and $\mathfrak{K}$, are complex groups from group G having the same identity member E, then there exists a complex group $\mathfrak{G}$ from G such that $\mathfrak{H}$ and $\mathfrak{K}$ are subgroups of $\mathfrak{G}$, and E is the identity member of $\mathfrak{G}$.*

Proof. Define $\mathfrak{G}$ to be the set of all complex products of finitely many members of $\mathfrak{H} \cup \mathfrak{K}$. It is now an easy matter to verify that $\mathfrak{G}$ has the properties mentioned in the lemma.

THEOREM 5. *Associated with each idempotent complex K of a group G is a unique maximal complex group from G which has K for its identity member.*

8. Divisible properties. A property P is said to be *divisible* if whenever a group G possesses property P then all complex groups from G possess property P. The property of being abelian is an obvious divisible property. From Corollary 3 it may be seen that the property of being torsion is a divisible property. The final theorem mentions a third divisible property. The proof is omitted because, although straightforward, it is lengthy.

THEOREM 6. *The property of being cyclic is a divisible property.*

GULLIVER'S RETURN TO THE LAND WITHOUT ONE, TWO, THREE

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Gulliver's interest in advanced mathematics can be traced to adventures not mentioned by Swift and only recently recorded.* They took place on an island which, because of the current sophisticated symbols for numbers, the traveller called the Land without One, Two, Three. Upon his return to Europe, Gulliver studied, both in England and on the Continent, what then (that is, in the early 1700's) was the most modern topic of investigation: the functions

$$x, x^2, x^3, \sqrt{x}, \cos x, \log x, \cdots .$$

The more the traveller penetrated into analysis, the more he admired the ingenious theory. Yet he was puzzled by several features of those sophisticated symbols for functions, which somehow reminded him of the numerals in the Land without One, Two, Three—features of analysis that, incidentally, are still apparent today.

(a) Each symbol for a function† includes the letter x and is equivocal in that it also designates the value assumed by that function for x. For instance, the function $\log x$ assumes the value $\log x$ for x.

* Karl Menger, Gulliver in the land without one, two, three, Math. Gaz., vol. 43, 1959, pp. 241–250.

† The only exceptions are the constant functions. The same numeral designates the number 3 and the function assuming the value 3 for each number x.

(b) Power functions are denoted by symbols that differ in structure from those in which x is *preceded* by signs such as $\sqrt{}$, cos or log. In each symbol for a higher power, x is *followed* by a raised small numeral. The first power, also called *identity function*, is denoted simply by x.

(c) Symbols for the results of substitutions into a function, say into log x, include merely the x-free part of its symbol. For instance, mathematicians write log cos x and not log x(cos x) when substituting cos x into the function log x. Similarly, the x-free part of a function symbol is all that is used in applying the function to what Newton called *fluents*, *e.g.*, the time t and the gas pressure p; that is to say, mathematicians write, for instance, t^2 and log p, and not $x^2(t)$ or log $x(p)$.

(d) The symbol x, because of its equivocal use, is unfit to express in a simple way some of the most important properties of the identity function. For instance, the law that substitution of the function x into the function x yields the function x cannot be expressed by writing $xx=x$ or $x(x)=x$. In order to formulate it, one must ad hoc describe the identity function by a symbol that is better than x, say by $h(x)$ and write:

$$\text{If } h(x) = x, \quad \text{then} \quad h(h(x)) = h(x) \quad \text{or} \quad h(h(x)) = x.$$

This important law about the so-called function x cannot be rendered in a simple formula. What is needed is an implication. The same is true for certain left-side distributive laws involving substitution.§ They, too, can only be expressed in the form of implications, such as

$$\text{If } s(x) = f(x) + g(x) \quad \text{and} \quad p(x) = f(x)\cdot g(x),$$
$$\text{then } s(h(x)) = f(h(x)) + g(h(x)) \quad \text{and} \quad p(h(x)) = f(h(x))\cdot g(h(x)).$$

But rarely are those important laws mentioned at all. In comparison with addition and multiplication, analysis underemphasizes substitution, the functional operation par excellence, and ignores important laws about the functions possessing wayward symbols—the powers and the identity. These were the facts that struck Gulliver most.

Yet it appeared to him that the current theory of *functions of x* could easily be transformed into a (largely substitutive) theory of *functions*, namely, by

(1) introducing a permanent symbol, say j, for the identity function and $j^2, j^3, \cdots$ for the higher power functions—for the functions *themselves* and not for function *values*;

(2) shedding the x in symbols for other functions; in other words, by using function symbols such as $j, j^2, j^3, \sqrt{}$ or $j^{1/2}$, cos, and log.

(A) No function symbol would then include the letter x or be equivocal. Followed by x, it would unequivocally designate the value assumed by that function for x.

§ The corresponding right-side distributive laws are not valid for any three functions. For instance, there are functions such that $h(f(x)+g(x))\neq h(f(x))+h(g(x))$.

(B) The discrepancy between the symbols for power functions and other function symbols would disappear. The value that j^2 assumes for x might also be denoted by $j^2(x)$ or even j^2x, instead of by x^2, just as $\sqrt{x}$ and $\log x$ denote the values of other functions.

(C) The results of substitutions into a function f might be expressed by appending to the letter f the symbol for the function that is being substituted, for instance, by writing fg, $\log\cos$, $\cos j^2$, $j^2\cos$, j^2j^3. Similarly, instead of t^2, one might write j^2t for the square of the time in analogy to $\log p$ for the logarithm of the pressure.

Clearly, mere juxtaposition of function symbols must not be used to express both substitution and multiplication. If it denotes substitution, then in designating the product of functions one needs some sign, say a dot between the symbols for the factors, as in*

$$j^2\cdot j^3 = j^5 \quad \text{and} \quad \log\cdot\cos = \cos\cdot\log \text{ in contrast to } j^2j^3 = j^6 \quad \text{and} \quad \log\cos \neq \cos\log.$$

(D) A permanent symbol such as j would make it possible to express the basic properties of the identity function in simple formulas such as†

$$jj = j \quad \text{and} \quad jf = f = fj \qquad \text{for any function } f.$$

Transparent formulations of the left-side distributive laws involving substitution would read:

$$(f + g)h = fh + gh \quad \text{and} \quad (f\cdot g)h = fh\cdot gh \qquad \text{for any functions } f, g, h.$$

Then, at long last, those laws would appear on a well-deserved equal footing with the traditional additive-multiplicative distributive law§

$$(f + g)\cdot h = f\cdot h + g\cdot h \qquad \text{for any functions } f, g, h.$$

Gulliver was sensitive to the afore-mentioned difficulties concerning symbols for functions and ready to solve them because he had found the inhabitants of the Land without One, Two, Three experiencing difficulties with their symbols for numbers and he had proposed solutions by simplifying their system of numerals. After his study of analysis in Europe, the traveller, therefore, got curious about whatever had become of arithmetic on that island. So, the next time he was in the vicinity, several years after his first stay, Gulliver revisited the Land without One, Two, Three.

Most islanders still counted the way they had done before; that is to say,‡

* If juxtaposition denotes multiplication, then substitution must be indicated by some symbol, say, by $\circ$ or $\leftarrow$. For instance, one would write $j^2j^3 = j^5$ in contrast to $j^2 \circ j^3 = j^6$ or $j^2 \leftarrow j^3 = j^6$.

† If juxtaposition denotes multiplication, then the formulas about substitution would read $j \circ j = j$ or $j \leftarrow f = f = f \leftarrow j$ for any function f.

§ In conjunction with the commutative law of multiplication, this law implies its right-side counterpart $h\cdot(f+g) = h\cdot f + h\cdot g$ for any functions f, g, h.

‡ Cf. Math. Gaz., loc. cit. (footnote*, p. 641), p. 241.

$$|,\ |^{p},\ |^{t},\ 4|,\ 5|,\ 6|,\ \cdots$$

(pronounced: stix, stix-pair, stix-triple, four stix, five stix, six stix, · · ·). And, as before, the word *stix* (written |) also denoted the island's monetary unit.

Gulliver thus still was puzzled by the following features of the insular system:

(a′) Each symbol for a number† included the sign | and was equivocal in that it also designated a number of stix in the sense of monetary units. For instance, the symbol 5| indicated the number of such units in the amount of 5| as well as the amount itself.

(b′) The symbols for the three lowest numbers differed in structure from those in which | was *preceded* by signs such as 4, 5, 6. In the symbols for the second and third numbers, | was *followed* by small raised letters, while the first number was denoted simply by |.

(c′) The symbol for a product of two numbers included merely the stix-free part of the symbol of the first factor. For instance, arithmeticians wrote 4.5| (pronounced: four times five stix) and not 4| ·5| (or four stix times five stix), which appeared inappropriate. Similarly, the stix-free part of a numeral was all that they used in counting objects other than stix; for instance, they wrote 5 fingers and eye^{p} (pronounced: five fingers and eye-pair) rather than 5| fingers and |^{p} eyes (or five stix fingers and stix-pair eyes), since this, too, would have looked (and sounded) awkward.

(d′) The symbol |, because of its equivocal use, was unfit to express in a simple way some of the most important properties of the number that it denoted. For instance, the law that the product of the number | and the number | is the number | could not be expressed by writing $| \cdot | = |$. *Stix times stix equals stix* would have sounded to the islanders as awkward as *dollar times dollar equals dollar* would to us. In order to formulate that law they had to introduce ad hoc a symbol that was better than |, say $h|$, and to write:

$$\text{If } h| = |, \quad \text{then} \quad h \cdot h| = h| \quad \text{or} \quad h \cdot h| = |.$$

This important law about the so-called number | could not be rendered in a simple formula. What was needed was an implication. The ordinary distributive law, too, could only be expressed in the form of an implication:

$$\text{If } s| = m| + n|, \quad \text{then} \quad s \cdot q| = m \cdot q| + n \cdot q|.$$

But rarely were those important laws mentioned at all—in line with the general neglect of multiplication, which, in contrast to addition, was taught only in advanced university courses.

During his first visit, Gulliver had come to the conclusion that what the island was lacking were symbols for the lowest numbers. (This was why he called it the Land without One, Two, Three.) In a way, even in the higher cases only

† The only exception was the number zero. The same numeral designated that number and the amount of no stix.

numbers of stix and not numbers themselves had symbols; and, indeed, many an insular arithmetical text was entitled *Theory of Numbers of Stix*. Yet he realized that the only requirement for the development of a *Theory of Numbers* would have been a minor notational change, namely,

(1′) introducing the symbols 1, 2, 3 for the three lowest numbers—for the numbers *themselves*, in contrast to numbers *of stix*;

(2′) shedding the stix in the higher numerals; in other words, using numerals

$$1, 2, 3, 4, 5, 6, \cdots.$$

(A′) No numeral would then include the sign | or be equivocal. Followed by |, it would designate the corresponding number of stix.

(B′) The discrepancy between the three lowest numerals and the others would disappear. The amounts of one stix, two stix, three stix (*i.e.*, monetary units), instead of by |, $|^p$, $|^t$, might also be denoted by 1|, 2|, 3|, just as 4| and 5| denoted larger amounts of monetary units.

(C′) The products of two numbers might be expressed by sandwiching a dot between the two numerals, as in $4\cdot5$, $5\cdot2$, $2\cdot5$, $2\cdot3$. Similarly, instead of eye^p, one might also write 2 eyes, in analogy to 5 fingers.

(D′) The symbol 1 would make it possible to express basic properties of the number one in simple formulas such as

$$1\cdot1 = 1 \quad \text{and} \quad 1\cdot n = n = n\cdot1 \quad \text{for} \quad \text{any number } n.$$

Similarly, a transparent formulation of the additive-multiplicative distributive law would read:

$$(m + n)\cdot q = m\cdot q + n\cdot q \quad \text{for} \quad \text{any numbers } m, n, q.$$

In such simple terms, multiplication would assume its proper place.

These ideas* Gulliver had propounded in a pamphlet *Arithmetic without Stix*, which circulated on the island widely, if, for various reasons, only in hand-written copies. The Immortals† were unimpressed and declared Gulliver's ideas to be trivial. The publishers' referees, on the contrary, complained about their abstruseness—some called stix-free numerals outright incomprehensible—and they all considered it as completely out of the question that, in terms of 1, 2, 3, numbers could ever be *taught*. Nor, obviously, could publishers regard a work whose title included the word *arithmetic* (in contrast, say, to the *Abstract Stix Theory and its Generalizations*) as a prestige book.

Toward the end of Gulliver's first stay, however, two little-known professors —mortal but alive—reported about experimental courses on stix-free multiplication. Their experiences had been satisfactory—in fact, so much so that they considered using the method in college courses. Approaching the pedagogical problem in a strictly scientific spirit, they designed large-scale experiments

* And other ideas; cf. Math., Gaz., loc. cit. (footnote*, p. 641).

† Immortals were the members of the all-powerful organization of the Incontestably Major Mathematicians of Real Talent and Learning.

which, they felt, should be arranged by the Insular Arithmetic Foundation—an organization sponsored by the island's ruler for various purposes including the improvement of arithmetical education. So they wrote to the director of the I. A. F. pointing out that Gulliver's ideas would make multiplication available to wider strata of the learning publication; but when Gulliver left they had not yet received an answer.

Even though on Gulliver's return to the Land without One, Two, Three most islanders still counted stix, stix-pair and so on, he found arithmetic in a state of fermentation. The island was swamped with beautifully printed new textbooks most of which discussed, in one form or another, the insular numerals and tried to improve them. (Even *Arithmetic without Stix* had been planographed.) Besides the symbol $n|$, there appeared various other notations for the amount of n monetary units, *e. g.*,

$$(n, |), \qquad (|, n), \qquad |n, \qquad \cdot|n.$$

"Whatever other difficulties may beset analysis," Gulliver thought, "the value that the function f assumes for x is denoted by $f(x)$; and, thank goodness, analysts have not to cope with other symbols such as

$$(f, x), \qquad (x, f), \qquad xf, \qquad \cdot xf."$$

(Of course, Gulliver was thinking of his contemporaries only, without anticipating all future developments.)

More important than those notational efforts was the fact that many new textbooks stressed the conceptual distinction between amounts of stix and numbers, *e.g.*, the *amounts* $|^p$ and $5|$, and the *numbers* $|^p$ and $5|$. Only some of the island's Major Mathematicians kept emphasizing that the traditional equivocation, being perfectly obvious, had never in the least confused any Immortal, whence they concluded in an impressive, if unusual, display of humility that it could not confuse anyone. Then there were a few who declared that using a symbol in discrepant meanings required courage which, they proclaimed, mathematicians ought to display.

Most mortal arithmeticians who had recognized the equivocal usage as such, however, tried to avoid it; and they attempted to express the distinction between numbers and numbers of stix in visible form. In fact, they tried very hard. Some reformers proposed to write $(|)^p$, $5(|)$, and $n(|)$ for the amounts and $(\cdot)^p$, $5(\cdot)$, and $n(\cdot)$ for the numbers. Others rendered the distinction by writing $|^p$, $5|$, $n|$ for the amounts, and $[|]^p$, $5[|]$, $n[|]$ for the numbers.

"Who back home would want to write $(x)^2$, $\log(x)$, and $f(x)$ for the values, and $(\cdot)^2$, $\log(\cdot)$, and $f(\cdot)$ for the functions?" Gulliver wrote in his diary. "Or x^2, $\log x$, $f(x)$ for the values, and $[x]^2$, $\log[x]$, $f[x]$ for the functions? If they wanted to improve things, they would rescue the identity function from its anonymity by introducing a symbol for it, say j. Thereafter they would write $j^2(x)$ or x^2, $\log x$, and $f(x)$ for the function values,

and j^2, log, and f for the functions. In fact, if they decided to discriminate between functions and numbers *typographically*, say, by using roman type for references to numbers, and italics or boldface letters for functions (as in j, ^{2} *log*, f or **j**, ^{2} **log**, **f**), then they might even save the parentheses in many symbols for function values and write j^2x, *log* x, and fx or **j**^{2}x, **log** x, and **f**x."

Still other insular arithmeticians claimed that the words *five stix* (or 5|) were abbreviations of the words

the number of the amount of five stix (or 5|),

but that this phrase was so cumbersome that people got tired of repeating it and rather referred to the number, elliptically and with mental reservations, as *five stix*. Gulliver, on the other hand, was an outspoken man with an idiosyncrasy against mental reservations, particularly in mathematics. Besides, from his point of view, the number in question could be satisfactorily denoted by the single word *five*, wherefore the words *five stix* were redundant or hyperbolic rather than abbreviated and elliptic.

"Back home," the traveller wrote, "the symbol log x might be considered as an abbreviation or elliptical rendering of the phrase

the function whose value for x *is* log x.

But at the same time the symbol is also redundant and hyperbolic as a designation of the logarithmic function or the function log. Certain lacunae in a symbolism create *hyperbolic ellipses*."

A last group of reformers treated pure numbers as *ratios*. They replaced the traditional numerals 5| and $|^p$ by the symbols

$5| \to |$ and $|^p \to |$ (pronounced: five stix to stix and stix-pair to stix)—

a notation that they regarded as very suggestive, if in need of a supplementary explanation, namely, that

$$(|^p \to |)| = |^p \quad \text{and} \quad (|^p \to |) \text{ canoes} = \text{canoe}^p.$$

The followers of this line of thought trusted that the elegant symbols $|\to|$, $|^p\to|$, and $|^t\to|$ (for the numbers that Gulliver proposed to denote by the uncouth numerals 1, 2, 3) would undoubtedly be adopted as standard.

"Some of the proposals in the Land without One, Two, Three aiming at an improvement of the traditional numerals are sophisticated indeed," Gulliver wrote in his diary. "Their analogue in the realm of functions would be the proposal that, instead of designating the logarithmic and square and identity functions by simple symbols such as log and j^2 and j (and their values for x, by log x and $j^2(x)$ and $j(x)$), one should rather denote the functions by

$$x \to \log x \quad \text{and} \quad x \to x^2 \quad \text{and} \quad x \to x$$

and explain that

$$(x \to \log x)(x) = \log x \quad \text{and} \quad (x \to x^2)(t) = t^2 \quad \text{and} \quad (x \to x)(c) = c$$—

an inconceivable proposal."

(In this case, too, Gulliver spoke only about his contemporaries and failed to foresee all that would be written 250 years later.)

The two lively mortals still had not heard from the I. A. F., but had, of course, nonetheless gone ahead since they considered their educational experiments to be in the public interest. They wrote a second report which, in contrast to the first, was deemed worthy by the *Arithmetical Reviews* of a mention of its title. In this report the authors described how they made some bright youngsters in a high school (yes, high school!) learn something until then unheard of on the island—a multiplication table:

$$\begin{aligned} &1\cdot 1 = 1, \quad 1\cdot 2 = 2, \quad 1\cdot 3 = 3, \cdots \\ &2\cdot 1 = 2, \quad 2\cdot 2 = 4, \quad 2\cdot 3 = 6, \cdots \\ &3\cdot 1 = 3, \quad 3\cdot 2 = 6, \quad 3\cdot 3 = 9, \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Students trained in the traditional way, on the other hand, only in their senior year in college, caught a few glimpses of the advanced arithmetic of stix, such as:

$$\begin{aligned} &\text{If } h| = |, \quad \text{then } h\cdot h| = |, \quad h\cdot |^p = |^p, \quad h\cdot |^t = |^t, \cdots \\ &\text{If } h| = | \text{ and } k| = |^p, \quad \text{then } k\cdot h| = |^p, \quad k\cdot |^p = 4|, \quad k\cdot |^t = 6|, \cdots \\ &\text{If, moreover, } m| = |^t, \quad \text{then } m\cdot h| = |^t, \quad m\cdot |^p = 6|, \quad m\cdot |^t = 9|, \cdots \end{aligned}$$

Such profound results naturally could not be degraded to the level of memorizing material; nor were students expected to remember those sophisticated formulations.

Besides the new books that completely ignored the numerals 1, 2, 3, the two independent mortals showed Gulliver also some texts freely using them as though 1, 2, 3 had always been current. Besides, the discoverers of the celebrated formula $|+|^p = |^p + |$, who called *Arithmetic without Stix* trivial, had begun a reform of the insular symbolism—of course a nontrivial reform. The *Arithmetical Reviews'* report about their work was condensed in the *Insular Daily Press* under the headlines: OUR NUMERALS REVOLUTIONIZED. IMMORTALS INTRODUCE II AND III INSTEAD OF $|^p$ AND $|^t$.

The situation thus was not without promising signs. So, after a week's stay, the traveller wrote in his diary:

"Leaving the Land with embryonic One, Two, Three. Returning to the World without Identity and Power Functions."

ON NONUNIFORM DIFFERENTIABILITY

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In a recent paper [3], Robert Weinstock proved an interesting result regarding the continuity of the derived function $f'(x)$, where it is assumed that $f(x)$ possesses a finite differential coefficient at every point in its domain of definition. He proved that if the derivative $f'(x)$ exists on an open interval I of which $a \leqq x \leqq b$ is a closed subinterval, then a necessary and sufficient condition that $f'(x)$ be continuous on $a \leqq x \leqq b$ is that $f(x)$ be uniformly differentiable on $a \leqq x \leqq b$.

The main result which we propose to establish in this paper is embodied in the following

THEOREM. *If $f(x)$ is differentiable at every point of the open interval I, of which $[a, b]$ is a closed subset, then the set of points of nonuniform differentiability of $f(x)$ on $[a, b]$ is of first category and is an F_σ.*

To prove this result we introduce the notion of the points of "uniform" and "nonuniform" differentiability of $f(x)$ defined as follows:

Let

$$\phi(x, h) = \frac{f(x + h) - f(x)}{h} - f'(x).$$

By definition $f(x)$ is uniformly differentiable on $a \leqq x \leqq b$ if for each $\epsilon > 0$ there exists an x independent $\delta > 0$ such that $|\phi(x, h)| < \epsilon$ whenever $0 < |h| < \delta$ $(a \leqq x \leqq b)$ and $x + h \in I$. Since $f(x)$ is differentiable in $a \leqq x \leqq b$ we have that if $\epsilon > 0$ is preassigned, then to every point x of $[a, b]$ there corresponds a $\delta(x) > 0$ such that $|\phi(x, h)| < \epsilon$ whenever $|h| \leqq \delta(x)$. For a fixed $\epsilon > 0$ it may happen that $\delta(x)$ has no positive lower bound in $[a, b]$. Then we can find at least one point ξ of $[a, b]$ and a neighborhood of ξ in which $\delta(x)$ has no positive lower bound.

A point at which for each $\epsilon > 0$ there exists a neighborhood in which $\delta(x)$ has a positive lower bound is said to be a point of uniform differentiability of $f(x)$.

A point in a neighborhood of which $\delta(x)$ has no positive lower bound for some sufficiently small $\epsilon > 0$ is said to be a point of nonuniform differentiability of $f(x)$.

The above definitions are equivalent to the following:

A point ξ is a point of uniform or nonuniform differentiability of $f(x)$ according as $\phi(x, h)$ has or has not the unique double limit zero, as $x \to \xi$, $h \to 0$.

Let ξ be an arbitrary point in $[a, b]$ and $(\xi - d, \xi + d)$, $d > 0$, be a neighborhood of ξ; let p be any positive number. Let U be the least upper bound of $|\phi(x, h)|$ for $\xi - d \leqq x \leqq \xi + d$ and $|h| \leqq p$. We denote the greatest lower bound of

* The author is indebted to the referee for a number of very valuable suggestions.

U as d and p tend to zero independently by $\alpha(\xi)$. At a point of uniform differentiability $\alpha(x)=0$; if $\alpha(x)\neq 0$, then x is a point of nonuniform differentiability of $f(x)$. In the following we establish some properties of $\alpha(x)$ analogous to the corresponding properties of the function defining the measure of nonuniform convergence of a series [2].

THEOREM 1. *A necessary and sufficient condition that $f'(x)$ be continuous at ξ is that $\alpha(\xi)=0$.*

Assume first that $f'(x)$ is continuous at $x=\xi$ and suppose that $\alpha(\xi)=p>0$. Let $0<\epsilon<p$. Since $f'(x)$ is continuous at $x=\xi$, corresponding to this ϵ there is an $h_1>0$ such that

$$|f'(x_1)-f'(x_2)|<\epsilon \tag{1}$$

whenever x_1 and x_2 are points of $(\xi-h_1, \xi+h_1)$. Again, since $\alpha(\xi)=p$, in every neighborhood of ξ, for an arbitrary $h(\neq 0)$, there exists a point x and a number h', $0<|h'|\leqq|h|$, such that $\phi(x, h')>\epsilon$, *i.e.*,

$$\left|\frac{f(x+h')-f(x)}{h'}-f'(x)\right|>\epsilon,$$

$$|f'(x+\theta h')-f'(x)|>\epsilon, \qquad 0<\theta<1. \tag{2}$$

With the neighborhood $(\xi-h_1, \xi+h_1)$ already chosen with regard to (1), we may choose $|h|$ so small that x and $x+\theta h'$ are points of this neighborhood. But (2) is a contradiction of (1) and hence $\alpha(\xi)=0$.

Conversely, if $\alpha(\xi)=0$, then corresponding to any $\epsilon>0$, we can find a neighborhood of ξ such that $|\phi(x, h)|<\frac{1}{3}\epsilon$ for every point x of that neighborhood and for all h such that $|h|\neq 0$ is sufficiently small. Choose a particular value of h, say h_1, for which this is true. Then $|\phi(x, h_1)|<\frac{1}{3}\epsilon$ for every x of this neighborhood. Denote this neighborhood by D_1 and let $\psi(x)=\{f(x+h_1)-f(x)\}/h_1$. Then $\psi(x)$ is continuous at ξ. So, corresponding to the above $\epsilon>0$ it is possible to determine a neighborhood of ξ, say D_2, such that $|\psi(x_1)-\psi(x_2)|<\frac{1}{3}\epsilon$ for every x_1, x_2 of D_2.

Let $D=D_1\cap D_2$. Then D is a neighborhood of ξ. Let α_1, α_2 be any two points of D. Then $|f'(\alpha_1)-f'(\alpha_2)|<\epsilon$. Hence $f'(x)$ is continuous at ξ.

Theorem 1 may be restated as

THEOREM 2. *If $f'(x)$ exists in an open interval I of which $a\leqq x\leqq b$ is a closed subinterval then the points of discontinuity of $f'(x)$ in $a\leqq x\leqq b$ are precisely the points of nonuniform differentiability of $f(x)$ in $a\leqq x\leqq b$.*

It has been shown [1] that $\alpha(x)$ is upper semicontinuous in $[a, b]$. Now, since $\alpha(x)$ is upper semicontinuous, we have that for any positive number β the set of points of $[a, b]$ where $\alpha(x)\geqq\beta$ is closed.

THEOREM 3. *For arbitrary $\beta>0$, the closed subset of $[a, b]$ on which $\alpha(x)\geqq\beta$ is nondense in $[a, b]$.*

Proof. Let G_β be the set of points of $[a, b]$ at which $\alpha(x) \geqq \beta > 0$. Suppose, if possible, that G_β is not nondense in $[a, b]$. Then there exists at least one subinterval of $[a, b]$ at which G_β is everywhere dense and, since G_β is closed, every point of it is a point of G_β. Let us denote this interval by $[a', b']$ and let ξ be any point of this interval. Since $\alpha(\xi) \geqq \beta$, if $\beta_1 (< \beta)$ is any positive number and $\{\delta_n\}$ any positive null sequence, then there exists a point ξ' in some neighborhood of ξ contained in $[a', b']$ and a number h_1, where $|h_1| < \delta_1$, such that

$$\left| \frac{f(\xi' + h_1) - f(\xi')}{h_1} - f'(\xi') \right| > \beta_1.$$

Since $f'(\xi')$ exists, we can find an h_2, where $|h_2| < |h_1|$, such that

$$\left| \frac{f(\xi' + h_2) - f(\xi')}{h_2} - \frac{f(\xi' + h_1) - f(\xi')}{h_1} \right| > \beta_1.$$

Again, since $\{f(x+h_2) - f(x)\}/h_2$ and $\{f(x+h_1) - f(x)\}/h_1$ are continuous at ξ', there exists a neighborhood D_1 of ξ' contained in $[a', b']$ such that for every point x of D_1 we have

$$\left| \frac{f(x + h_2) - f(x)}{h_2} - \frac{f(x + h_1) - f(x)}{h_1} \right| > \beta_1.$$

Now, D_1 is a neighborhood of ξ' and $\alpha(\xi') \geqq \beta$. So, corresponding to the positive number δ_2', where $\delta_2' = \min [\delta_2, |h_2|]$, there exists a point ξ'' in D_1 and a number h_3, where $|h_3| < \delta_2'$, such that

$$\left| \frac{f(\xi'' + h_3) - f(\xi'')}{h_3} - f'(\xi'') \right| > \beta_1.$$

In a similar manner, we can find a neighborhood D_2 of ξ'' contained in D_1 such that for every point x of D_2

$$\left| \frac{f(x + h_3) - f(x)}{h_3} - \frac{f(x + h_4) - f(x)}{h_4} \right| > \beta_1,$$

where $|h_4| < |h_3|$.

If the process is continued we get a decreasing sequence of neighborhoods D_m such that for every point x of D_m

$$\left| \frac{f(x + h_{2m-1}) - f(x)}{h_{2m-1}} - \frac{f(x + h_{2m}) - f(x)}{h_{2m}} \right| > \beta_1,$$

where $|h_{2m}| < |h_{2m-1}| < \delta_m'$ and $\delta_m' = \min [\delta_m, |h_{2m-2}|]$.

The neighborhoods D_m can be so chosen that there exists a point η in all of them and at this point

$$\left|\frac{f(\eta + h_{2m-1}) - f(\eta)}{h_{2m-1}} - \frac{f(\eta + h_{2m}) - f(\eta)}{h_{2m}}\right| > \beta_1$$

holds for every m.

Now, $\{h_{2m}\}$ is a null sequence and so from the above we arrive at a contradiction to the fact that $f(x)$ is differentiable at η. Hence G_β must be nondense in $[a, b]$.

Let $\{\beta_n\}$ be any positive null sequence. Then any point of nonuniform differentiability of $f(x)$ will belong to the set $\alpha(x) \geqq \beta_n$ for some n and this set is nondense in $[a, b]$. Hence the points of nonuniform differentiability of $f(x)$ are points of a set which is the outer limiting set of a sequence of nondense closed sets. Hence we obtain

THEOREM 4. *If $f(x)$ is differentiable at every point of the open interval I, of which $[a, b]$ is a closed subset, then the set of points of nonuniform differentiability of $f(x)$ on $[a, b]$ is of first category and is an F_σ.*

In conclusion, I express my earnest gratitude to Dr. H. M. Sengupta, for his kind help in the preparation of the paper.

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ON PERIODICITIES OF CERTAIN SEQUENCES OF RESIDUES

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Let H be the class of polynomials $F(x)$ such that $F(m)$ is an integer whenever m is an integer. It is not difficult to show that H is identical with the class of polynomials of the form

$$F(x) = \sum_{j=0}^{K} a_j \binom{x}{j},$$

where a_j is an integer for $0 \leqq j \leqq K$. Let M be an arbitrary positive integer:

$$M = \prod_{i=1}^{n} P_i^{k_i},$$

where $P_1, \cdots, P_n$ are distinct primes and $k_i \geqq 1$ $(1 \leqq i \leqq n)$. Let K be a positive

integer, and $r_1, \cdots, r_n$ be integers chosen so that

$$P_i^{r_i-1} \leqq K < P_i^{r_i} \qquad (1 \leqq i \leqq n). \tag{1}$$

Finally, define

$$N = \prod_{i=1}^{n} P_i^{k_i+r_i-1}.$$

The symbols M, N, and K will retain these meanings throughout the paper.

Since $\Delta^{K+1}F(m)=0$ for every integer m,* it is clear that the sequence $\{F(m) \pmod M\}$ must have a period which does not exceed M^K. It is our purpose to determine the exact periods of such sequences.

LEMMA 1. *The sequence* $\{\binom{m}{K} \pmod M\}$ $(m = \cdots, -1, 0, 1, \cdots)$ *is periodic with fundamental period* N.

Proof. Expand $\binom{m+N}{K}$ about $N=0$ to obtain

$$\binom{m+N}{K} = \binom{m}{K} + \sum_{j=1}^{K} \binom{N}{j}\binom{m}{K-j}. \tag{2}$$

We first show that $\binom{N}{j}$ is divisible by M for $j=1, \cdots, K$. Write

$$\binom{N}{j} = \frac{N}{i} \cdot \frac{N-1}{1} \cdot \frac{N-2}{2} \cdots \frac{N-(j-1)}{j-1}. \tag{3}$$

If $1 \leqq R \leqq j-1$, the highest power of P_i which divides R is $P_i^{r_i-1}$, since $j \leqq K < P_i^{r_i}$. Hence, since $P_i^{r_i-1}$ is a factor of N, $N-R$ is divisible by $P_i^{q_i}$ $(q_i \leqq r_i-1)$ if and only if R is. On the other hand, since j is not divisible by $P_i^{r_i}$, the factor $P_i^{k_i}$ in the numerator of N/j is not cancelled. As this argument holds for $i=1, \cdots, n$, we can conclude that

$$\binom{N}{j} \equiv 0 \pmod M \qquad (j = 1, \cdots, K),$$

and the periodicity follows from (2).

If N' is a second period, then the greatest common divisor of N and N' is also a period. Let $(N, N') = \prod_{i=1}^{n} P_i^{q_i}$. If N is not the fundamental period, then there is a subscript i such that $q_i < r_i + k_i - 1$. Without loss of generality, assume that $i=1$. Then

$$N_1 = P_1^{r_1+k_1-2} \prod_{i=2}^{n} P_i^{r_i+k_i-1}$$

is also a period. We will show that this is impossible for any $K \geqq P_1^{r_1-1}$. First we assert that

* Δ is the forward difference operator. That is, $\Delta F(m) = F(m+1) - F(m)$.

$$\binom{N_1}{P_1^{r_1-1}} \not\equiv 0 \pmod{M}. \tag{4}$$

This is obvious if $r_1=1$. If $r_1>1$, expand as in (3). For $R=1, 2, \cdots, P_1^{r_1-1}$, it can again be seen that the powers of P_1 in factors of the form $(N_1-R)/R$ are cancelled. Now $N_1/P_1^{r_1-1}$ is not divisible by $P_1^{k_1}$, and (4) follows.

Next let $K=P_1^{r_1-1}+j$ with $j>0$. If $\{(\binom{m}{K}) \pmod{M}\}$ has period N_1, it follows that

$$\binom{N_1+\nu}{P_1^{r_1-1}+j} \equiv 0 \pmod{M} \qquad (\nu = 0, 1, \cdots, j).$$

However, this leads to a contraction of (4), because we could then write

$$\binom{N_1}{P_1^{r_1-1}} = \sum_{\nu=0}^{j} (-1)^{j-\nu}\binom{N_1+\nu}{P_1^{r_1-1}+j}\binom{j}{\nu} \equiv 0 \pmod{M}.$$

This completes the proof of Lemma 1.

We can immediately generalize to

THEOREM 1. *Let*

$$F(x) = \sum_{j=0}^{K} a_j\binom{x}{j} \tag{5}$$

be in H. If $(a_K, M)=1$, *the sequence* $\{F(m) \pmod{M}\}$ *is periodic, with fundamental period N.*

Proof. If K is the least integer which satisfies (1), we can infer from Lemma 1 that $\{a_K(\binom{m}{K}) \pmod{M}\}$ has fundamental period N, while all lower degree terms have periods which are proper divisors of N. Thus the conclusion follows for this case. Assume that $K-r-1$, $(r\geqq 0)$, is the least integer which satisfies (1), and that the theorem is true when $K-r$ is the smallest such integer. Consider

$$\Delta F(x) = F(x+1) - F(x) = \sum_{j=0}^{K-1} a_{j+1}\binom{x}{j}.$$

In this equation, $K-1$ plays the role of K in (5). From the induction assumption, the sequence $\{\Delta F(m) \pmod{M}\}$ has the fundamental period N. From this it follows that the fundamental period of $\{F(m) \pmod{M}\}$ is not less than N, while from Lemma 1 it follows that it is not greater than N.

COROLLARY. *If* $F(x)$ *is any polynomial in H, of degree K, then the sequence* $\{F(m) \pmod{M}\}$ *has a fundamental period of the form*

$$N_1 = \prod_{i=1}^{n} P_i^{j_i}, \quad \textit{where} \quad 0 \leqq j_i \leqq r_i + k_i - 1 \ (i = 1, \cdots, n).$$

As a partial converse to Theorem 1, we have

THEOREM 2. *Let* $\{f_m\}$ $(-\infty < m < \infty)$ *be a sequence of integers, and let*

$$\Delta^{K+1} f_m \equiv 0 \pmod{M} \qquad (m = 0, \pm 1, \pm 2, \cdots).$$

Then there is in H a polynomial $F(x)$ *of degree not exceeding* K, *such that*

$$F(m) \equiv f_m \pmod{M} \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{6}$$

Consequently, the sequence $\{f_m \pmod{M}\}$ *has a fundamental period which divides* N.

Proof. Define

$$F(x) = \sum_{r=0}^{K} \Delta^n f_0 \binom{x}{r}.$$

Then

$$F(m) = f_m \qquad (m = 0, 1, \cdots, K). \tag{7}$$

Since $F(x)$ is of degree not greater than K, we have

$$0 = \Delta^{K+1} F(m) \equiv \Delta^{K+1} f_m \pmod{M} \qquad (-\infty < m < \infty),$$

and (6) follows from (7) by a trivial induction.

It can also be stated that, if in addition to the hypothesis of Theorem 2, there is an integer m such that $(\Delta^K f_m, M) = 1$, then the fundamental period of $\{f_m \pmod{M}\}$ is precisely N.

In the case where M is a prime, we can obtain a stronger result.

THEOREM 3. *Let* P *be a prime, and* $F(x)$ *a polynomial in H, of degree* K, *such that the coefficient of* $\binom{x}{K}$ *is not divisible by* P. *Then, if* $P^{r-1} \leqq K < P^r$, *the sequence* $\{F(m) \pmod{P}\}$ *is periodic with fundamental period* P^r. *Conversely, if a sequence of integers* $\{f_m\}$, $(-\infty < m < \infty)$, *is such that* $\{f_m \pmod{P}\}$ *has fundamental period* P^r, *then there is a polynomial* $F(x)$ *in H, with* $P^{r-1} \leqq \deg F(x) < P^r$, *and*

$$F(m) \equiv f_m \pmod{P} \qquad (-\infty < m < \infty). \tag{8}$$

Proof. The first statement is a special case of Theorem 1. For the converse, let $\{f_m \pmod{P}\}$ have the assumed periodicity, and consider the linear system in the P^r unknowns $\{a_i\}$:

$$\sum_{n=0}^{m} a_n \binom{m}{n} \equiv f_m \pmod{P} \qquad (m = 0, 1, \cdots, P^r - 1).$$

Since this is a diagonal system, with coefficients on the diagonal equal to unity, there is a unique solution $\{a_n\}$ in the field of integers modulo P. Define

$$F(x) = \sum_{n=0}^{P^r - 1} a_n \binom{x}{n}.$$

Then $F(x)$ satisfies (8) for $m=0, 1, \cdots, P^r-1$. By the corollary to Theorem 1, $\{F(m) \pmod P\}$ also has period P^r, and therefore $F(x)$ satisfies (8) for all m. Deg $F(x) \geqq P^{r-1}$, since if not $\{F(m) \pmod P\}$ would have period P^{r-1}, and so would $\{f_m \pmod P\}$, contrary to assumption.

For an alternate proof of the converse, one can observe that $\Delta^{P^r} f_m \equiv 0 \pmod P$ for all m, and the conclusion essentially follows from Theorem 2.

Acknowledgement. The work reported here was stimulated by an attempt to prove [1], which follows from Theorem 3 for $P=2$.

The referee has pointed out that Lemma 1 has appeared previously in [2]. The result is obtained there for $m=0, 1, \cdots$, by a lengthy argument involving a chain of six lemmas and two theorems which are weaker than Lemma 1.

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THE DEFINITE INTEGRAL SYMBOL

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Smith [1] states that Leibniz refers to "the integral calculus as the *calculus summatorius*, a name connected with the summa (s) sign." Child [2] has directed attention to the invention of the $\int$ by Leibniz. Histories of mathematics [1, 3, 4], encyclopedias [5, 6] and some textbooks refer to the symbol $\int$ as an extended or elongated S, as a medieval S, as an old-fashioned S, the first letter of the word summa or sum, meaning the total amount, summation, the whole, etc. [7, 8, 9, 10]. Others ignore explanation of its morphology, etymology, or derivation [11, 12, 13, 14, 15].

Since Leibniz is generally credited with inventing the symbolism of modern calculus [1, 2, 6], I cannot refrain from remarking about an alternative derivation of the integral sign; one equally plausible as that generally stated.

Almost every botanist knows that if an apple is peeled by hand with a knife, beginning at the stem end and circling about the central axis without breaking the peel until the opposite end is reached, a regular spiral is obtained which forms an elongated S when placed on a flat surface with the inside of the peel up. Integration of the surfaces of certain solids having the shape of any of the three classes of spheroids, spherical, oblate, or prolate, yield such elongated S's, and continuous peelings of many fruits and melons that can be classified as spheroids take on the shape of the integral sign. Of these classes, all three may be found naturally occurring among citrus fruit, and the representative of any class which when peeled as shown in Figure 1 presents a satisfactory medieval S.

Now Bell [3] says of Leibniz (1646–1716), that "much of his mathematics,

to say nothing of his other wonderings . . . was written out in the jolting, draughty rattletraps that bumped him over the cow trails of the 17th century Europe . . . at his employers' erratic bidding." And, our own experiences in travel suggest that such conditions favor a firm acquaintanceship with apples. For what could have afforded a more convenient lunch to obtain, to carry or to prepare under such circumstances? Or, a more interesting one to contemplate when the road was smooth enough to permit peeling the apple? And the apple may have been bought beforehand in a convenient market for "by the end of the 17th century, fruit for market was recognized as a definite industry" [16] or picked fresh from cultivated apple trees, or wild apple trees indigenous to central Europe [17].

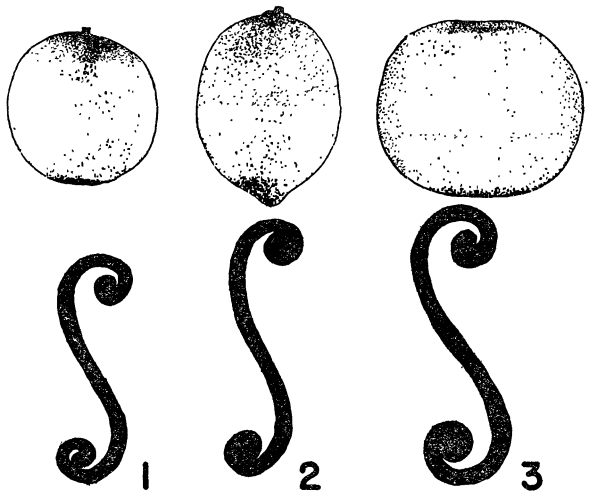


FIG. 1. Tracing of photographs of three kinds of citrus fruit and their respective peelings. 1, spherical Valencia orange; 2, prolate Eureka lemon; 3, oblate Marsh grapefruit.

The late Bing C. Wong, Professor of Mathematics at the University of California, Berkeley, I am informed by one of his former students, may have envisioned a similar derivation of the integral sign. But, I could not verify this through those of his associates I could reach.

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CORRECTION

W. R. Andress, *Basic properties of pandiagonal magic squares*, this MONTHLY, vol. 67, 1960, pp. 143–152. In the second line of Theorem 2, page 146, $a_{r+2,s+2}=a_{r,s}$ should be replaced by $a_{r,s}+a_{r+2,s+2}=0$.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

A DETERMINANT REPRESENTATION FOR THE CLASSICAL ORTHOGONAL POLYNOMIALS

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It is well known [1] that for each of the classical families of orthogonal polynomials (Hermite, Laguerre and Jacobi) there is a generalized Rodrigues formula through which the Nth member of the family is given (except for a normalization factor) by the relation

$$P_N = \frac{1}{w}\frac{d^N}{dx^N}(wF^N).$$

A particular family of polynomials is characterized by the choice of the functions w and F. For the Hermite polynomials, $w=e^{-x^2}$ and $F=1$; for the (generalized) Laguerre polynomials $w=e^{-x}x^\alpha$; and, for the Jacobi polynomials

$$w = (1-X)^\alpha(1+X)^\beta \quad \text{and} \quad F = 1-X^2.$$

It was shown [2] that the Nth derivative of a function $f(x)$ is given by

$$\frac{d^N f(x)}{dx^N} = \Delta_N f(x),$$

where the symbol Δ_N denotes the Nth ordered determinant

$$\begin{bmatrix} D_1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ D_2 & D_1 & -2 & 0 & 0 & \cdots & 0 \\ D_3 & D_2 & D_1 & -3 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & 1-N \\ D_N & \cdots & & D_3 & D_2 & & D_1 \end{bmatrix}$$

in which the elements $D_1, \cdots, D_N$ are defined by

$$D_K = \frac{1}{(K-1)!} \frac{d^K \log f(x)}{dx^K}.$$

Upon using this formula for evaluation of the Nth derivative which occurs in the Rodrigues formula, one obtains $P_N = F^N \Delta_N$ where the elements of Δ_N are given by

$$D_K = \frac{1}{(K-1)!} \frac{d^K}{dx^K} [\log w + N \log F].$$

Now, it is easily shown by induction that when a determinant of the form Δ_N is multiplied by F^N, the result is the same as that of multiplying each element D_K by F^K. Thus it is seen that the Nth member of a family of orthogonal polynomials is given by $P_N = \Delta_N$ where the elements of Δ_N are defined by

$$D_K = \frac{F^K}{(K-1)!} \frac{d^K}{dX^K} [\log w + N \log F].$$

For the Hermite polynomials, one obtains

$$D_1 = -2x, \qquad D_2 = -2, \quad \text{and} \quad D_K = 0 \quad \text{for} \quad K > 2;$$

for the (generalized) Laguerre polynomials,

$$D_1 = N + \alpha - X \quad \text{and} \quad D_K = (-1)^{K-1}(N + \alpha) \quad \text{for} \quad K > 1;$$

for the Jacobi polynomials, $D_K = -[(N+\alpha)(x+1)^K + (N+\beta)(x-1)^K]$. Thus it is seen that the classical orthogonal polynomials may be expressed in an especially simple way in terms of the determinant Δ_N.

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A NOTE ON EVALUATING DETERMINANTS*

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In a recent note, Lotkin [1] compares two methods for the evaluation of determinants, namely "the pivotal method" and "the method of contractants"; see [1] for details. He asserts that the method of contractants is somewhat of an improvement over the pivotal method. We compare in this note the method of contractants with the well-known very natural triangular method; see Scarborough [2], pp. 498–501. In the following table we list the formulas for the number of arithmetic operations, as stated by Lotkin, for the method of contractants and the corresponding formulas, easily verified, for a straightforward application of the triangular method.

Operation	*Contractants*	*Triangular*
Addition	$n(n-1)(2n-1)/4$	$n(n-1)(2n-1)/6$
Multiplication	$n(n-1)(2n-1)/3$	$n(n+1)(2n+1)/6-1$
Division	$(n-2)(n-1)(2n-3)/6$	$(n-1)(n+2)/2$

Let $C+$, $C\times$, $C\div$ be the number of respective operations for contractants acting on a determinant of order n, and let $T+$, $T\times$, and $T\div$ be the numbers for the triangular method. We note the following observations:

1. *$C+$ is asserted in [1] to be approximate.*
2. *$T+$ is exactly $\frac{2}{3}$ of $C+$.*
3. *$T\times$ is asymptotically $\frac{1}{2}$ of $C\times$.*
4. *$T\div$ is quadratic in n while $C\div$ is cubic.*
5. *The numbers of C operations are considerably increased by the presence of zeros in the given determinant, while the numbers of T operations are considerably decreased.*
6. *On new fast computers such as STRETCH and LARC, division takes about three times as long as multiplication.*

We conclude by listing for $n=10$, 50, and 100 the numbers of operations for these two methods. We do not assert that there do not exist methods faster than the triangular method.

n	$C+$	$T+$	$C\times$	$T\times$	$C\div$	$T\div$
10	$427\frac{1}{2}$	285	570	384	204	54
50	$60,637\frac{1}{2}$	40,425	80,850	42,924	38,024	1274
100	492,525	328,350	656,700	338,349	318,549	5049

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§ On leave from the University of Michigan, Summer 1959.

SUCCESSIVE APPROXIMATION APPLIED TO QUADRATURE FORMULAS*

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The conventional treatment of quadrature formulas emphasizes a discrete nature of the independent variable. In most cases, however, it is desirable to interpret the increment as a continuous variable also. Indeed, error estimates generally are established using the latter interpretation. Quite naturally, one is then led to differential equation concepts rather than to mere quadrature concepts and the former may suggest alternative techniques. For example, the familiar Simpson's rule may be used to define an operator S, where

$$Sf(x) = \tfrac{1}{6}x[f(0) + 4f(\tfrac{1}{2}x) + f(x)], \tag{1}$$

which in turn yields a continuous approximation to the solution of the differential equation

$$\frac{dy}{dx} = Dy = f(x) \tag{2}$$

subject to the boundary condition, $y=0$ for $x=0$.

If we denote by y_1, the function defined in (1) and by y_2, the difference between the exact solution of (2) and the approximate y_1, then y_2 satisfies the differential equation

$$\frac{dy_2}{dx} = (1 - DS)f(x). \tag{3}$$

Since $y=y_1$ for $x=0$, (3) is subject to the boundary condition, $y_2=0$ for $x=0$. Thus (3) depicts the *growth* of the error in the approximate y_1 from its initial value 0 for $x=0$.

However, (3) is also a differential equation not unlike (2) itself and one might apply the operator S to obtain an estimate of the error y_2. Such a procedure is reminiscent of the familiar Picard iterative process of successive approximation used in the study of differential equations. Symbolically, the estimate of the error is given by $y_3=S(1-DS)f(x)$ and the sum y_1+y_3 might be expected to be an improvement over y_1. If we put $y_4=y_2-y_3$, then $y=y_1+y_3+y_4$, where now y_4 is the error and is seen to satisfy the differential equation $dy_4/dx=(1-DS)^2f(x)$. More generally, if for each positive integer j we denote by y_{2j} the solution of the differential equation

$$\frac{dy_{2j}}{dx} = (1 - DS)^j f(x), \tag{4}$$

subject to the boundary condition $y_{2j}=0$ for $x=0$, and if we denote by y_{2j+1}, the Simpson's rule approximation to the solution of (4), *i.e.*,

* Work sponsored by the Office of Ordnance Research, U. S. Army.

$$y_{2j+1} = S(1 - DS)^j f(x), \tag{5}$$

then the exact solution of (2) may be expressed as

$$y = \sum_{j=0}^{k-1} y_{2j+1} + y_{2k}, \qquad k = 1, 2, \cdots, \tag{6}$$

where y_{2k} is interpreted as the error after $k-1$ iterations of the Simpson's rule operator. Note that (6) may be expressed as

$$y = S \sum_{j=0}^{k-1} (1 - DS)^j f(x) + y_{2k} \tag{7}$$

and if, as might be anticipated, $\lim_{k\to\infty} y_{2k} = 0$, (7) suggests an explicit solution of the form

$$y = S \sum_{j=0}^{\infty} (1 - DS)^j f(x). \tag{8}$$

If we formally "sum" the geometric series

$$\sum_{j=0}^{\infty} (1 - DS)^j = \frac{1}{1 - (1 - DS)} = \frac{1}{DS},$$

then (8) becomes $y = S(DS)^{-1} f(x)$ which, by a proper interpretation of the operators and the order in which they apply, yields the exact solution $y = D^{-1} f(x) = If(x)$. Clearly, what is required to justify (8), however, is the condition $\lim_{k\to\infty} y_{2k} = 0$.

An interesting illustration of the above is afforded by the power function $f(x) = x^n$, $n \geqq 0$. It is easily verified that for $n > 0$,*

$$(1 - DS)^j x^n = [1 - \tfrac{1}{6}(n + 1)(1 + 2^{2-n})]^j x^n, \qquad j = 0, 1, \cdots$$

and thence by (4) and (5) (the latter together with (1)),

$$y_{2j} = [1/(n + 1)][1 - \tfrac{1}{6}(n + 1)(1 + 2^{2-n})]^j x^{n+1},$$
$$y_{2j+1} = [(1 + 2^{2-n})/6][1 - \tfrac{1}{6}(n + 1)(1 + 2^{2-n})]^j x^{n-1}.$$

Thus it follows that

$$y = \tfrac{1}{6}(1 + 2^{2-n})(1 + e_n + \cdots + e_n^j)x^{n+1} + [1/(n + 1)]e_n^{j+1} x^{n+1},$$

where $e_n = 1 - \frac{1}{6}(n+1)(1+2^{2-n})$ and clearly $\lim_{j\to\infty} y_{2j} = 0$† if and only if

$$|e_n| < 1. \tag{9}$$

When (9) is satisfied, the exact solution of (2) is indeed given by the geometric series

* For $n = 0$ we define $x^n \equiv 1$ and thus $(1 - DS)^j x^0 \equiv 0$.

† Uniformly in some finite interval, say, $0 \leqq x \leqq X$.

$y = \frac{1}{6}(1 + 2^{2-n})(1 + e_n + \cdots)x^{n+1} = \frac{1}{6}(1 + 2^{2-n})(1 - e_n)^{-1}x^{n+1} = [1/(n + 1)]x^{n+1}$

and the above formalism is justified. On the other hand, (9) requires that n be less than 10.96 (approx.).

It is important to note that the fact of convergence does not depend on the range (or increment) of integration. In fact, e_n is the *relative* error inherent in this application of Simpson's rule (*i.e.*, the ratio of error to exact integral) and is a constant independent of the increment. (This illustrates the fallacy in a notion that quadrature formulas yield arbitrarily good results if the increment of the independent variable is chosen sufficiently small.) For $n > 10.96$, the absolute value of the relative error exceeds unity and the iterates, using Simpson's rule, merely introduce larger and larger errors.

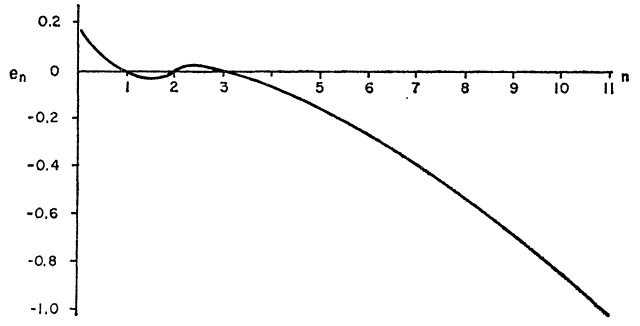


FIG. 1

Figure 1 is a plot of the relative error versus the exponent for the select range. At $n = 0$, the relative error is not continuous but jumps from zero to the limiting value $\frac{1}{6}$. This plot illustrates the well-known fact that Simpson's rule integrates polynomials of degree 3 or less exactly, *i.e.*, $e_n = 0$ for $n = 0$, 1, 2, and 3. Probably not so well known are the evidently good results for fractional exponents in the interval, $1 \leqq n \leqq 3$.

Finally, let us examine the case where $f(x)$ is a simple step function (*i.e.*, a step function with at most a finite number of steps). Since the Simpson's rule operator is additive, it will be sufficient to consider a function with but one step, say, $f(x) = 0$ for $0 \leqq x < a$ and $f(x) = c$ for $x > a$. The exact, continuous integral of $f(x)$ is given by $y = 0$ for $0 \leqq x < a$ and $y = c(x - a)$ for $x > a$, while the Simpson's rule approximation becomes $y_1 = 0$ for $0 \leqq x < a$, $y_1 = \frac{1}{6}cx$ for $a < x < 2a$, and $y_1 = \frac{5}{6}cx$ for $x > 2a$, and is not continuous. Hence the error y_2 is given by $y_2 = 0$ for $0 \leqq x < a$, $y_2 = -\frac{1}{6}ca + \frac{5}{6}c(x - a)$ for $a < x < 2a$, and $y_2 = -\frac{1}{3}ca + \frac{1}{6}c(x - 2a)$ for $x > 2a$, and, except for $x = a$ and $x = 2a$ satisfies the differential equation

$$Dy_2 = (1 - DS)f(x) = \begin{cases} 0, & 0 \leqq x < a, \\ \frac{5}{6}c, & a < x < 2a, \\ \frac{1}{6}c, & x > 2a, \end{cases} \tag{10}$$

where the absolute value of the right member does not exceed $\frac{5}{6}c$.

Now the right member of (10) may be expressed as the sum of two step functions, namely, $f_1(x)=0$ for $0\leqq x<a$ and $f_1(x)=\frac{5}{6}c$ for $x>a$; $f_2(x)=0$ for $0\leqq x<2a$ and $f_2(x)=(\frac{1}{6}-\frac{5}{6})c$ for $x>2a$. For these functions we have, using (10),

$$(1-DS)f_1(x)=\begin{cases}0, & 0\leqq x<a,\\ \frac{5}{6}(\frac{5}{6})c, & a<x<2a,\\ \frac{1}{6}(\frac{5}{6})c, & x>2a;\end{cases}$$

$$(1-DS)f_2(x)=\begin{cases}0, & 0\leqq x<2a,\\ \frac{5}{6}(\frac{1}{6}-\frac{5}{6})c, & 2a<x<4a,\\ \frac{1}{6}(\frac{1}{6}-\frac{5}{6})c, & x>4a.\end{cases}$$

Thus, the second generation error y_4, except for $x=a$, $x=2a$ and $x=4a$, satisfies the differential equation,

$$Dy_4=(1-DS)^2f(x)=(1-DS)f_1(x)+(1-DS)f_1(x)$$
$$=\begin{cases}0, & 0\leqq x<a,\\ (\frac{5}{6})^2c, & a<x<2a,\\ \frac{5}{6}[\frac{1}{6}c+\frac{5}{6}(\frac{1}{6}-\frac{5}{6})c], & 2a<x<4a,\\ \frac{5}{6}[\frac{1}{6}c+\frac{1}{6}(\frac{1}{6}-\frac{5}{6})c], & x>4a,\end{cases}$$

where now the absolute value of the right member does not exceed $(\frac{5}{6})^2c$.

Similarly, it may be shown by mathematical induction that $y_{2j}(j=2, 3, \cdots)$, except for discontinuities at $x=a$, $x=2a$, $x=4a$, $\cdots$, $x=2ja$, satisfies a differential equation of the form (2) wherein the absolute value of the right member does not exceed $(\frac{5}{6})^jc$. Clearly, then, the continuous parts of y_{2j} satisfy the condition $\lim_{j\to\infty} y_{2j}=0$ (uniformly in every finite interval $0\leqq x\leqq X$) and the integral of $f(x)$ is given by the continuous part of (8).

GREATEST COMMON DIVISOR OF SEVERAL INTEGERS AND AN ASSOCIATED LINEAR DIOPHANTINE EQUATION

ROBERT WEINSTOCK,* University of Notre Dame

1. Introduction. This note calls attention to a procedure—apparently overlooked in the past—for computing the greatest common divisor of a set of n integers and for solving, at the same time, a linear diophantine equation in which these n integers appear as coefficients. The procedure stems from the constructive aspect of a standard proof of the well-known [1]

THEOREM. *For given positive integers* $a_1, \cdots, a_n$, *the least positive number of the form* $a_1x_1+\cdots+a_nx_n$ ($x_1\cdots, x_n$ *integers*) *is* $d=(a_1\cdots, a_n)$, *the greatest common divisor of* $a_1, \cdots, a_n$.

* Now in the Department of Physics, Oberlin College.

The proof involves showing that if integers $x_1, \cdots, x_n$ are chosen so that $0<a_1x_1+\cdots+a_nx_n=s\neq d$, then there exist integers $x_1', \cdots, x_n'$ such that $0<a_1x_1'+\cdots+a_nx_n'=r<s$. This may be accomplished as follows: Since $s\neq d$, and since d necessarily divides s, there is at least one of the a_j—which, for notational simplicity, we take to be a_1—such that $a_1=qs+r$ with $0<r<s$; it then follows that

$$a_1 = q(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + r, \tag{1}$$

whence

$$0 < a_1(1 - qx_1) + a_2(-qx_2) + \cdots + a_n(-qx_n) = r < s. \tag{2}$$

Thus, with $x_1' = 1-qx_1$, $x_j' = -qx_j$ $(j=2, \cdots, n)$, the assertion is proved.

2. Greatest common divisor. The proposed procedure for computing $d=(a_1, \cdots, a_n)$ is the following: With $x_1^{(0)}, \cdots, x_n^{(0)}$ chosen arbitrarily, we use the technique of the above proof to determine $x_1^{(1)}, \cdots, x_n^{(1)}$ so that

$$r_1 = a_1x_1^{(1)} + \cdots + a_nx_n^{(1)} < a_1x_1^{(0)} + \cdots + a_nx_n^{(0)} = r_0 \neq d;$$

repetition of the process—in which we proceed from the $x_j^{(k)}$ and r_k to the $x_j^{(k+1)}$ and r_{k+1}—is carried out until finally we achieve an r_t that divides all of $a_1, \cdots, a_n$; then, clearly, $r_t=d$.

From the proof we note that each r_k $(k\geqq 1)$ is obtained directly as the least positive remainder on division of one of the a_j by r_{k-1}, so that in our quest for d there is no need actually to compute the $x_j^{(k)}$ for $k\geqq 1$. (See Sec. 3 below, however.) If, in carrying out the procedure, we find that a given r_{k-1} divides a_1, say, then r_k must be obtained through division of one of the other a_j by r_{k-1}; if, however, r_{k-1} divides *all* the a_j, then $r_{k-1}=d$.

The process may be shortened (*i.e.*, is not lengthened) through the use of least absolute remainders for the successive r_k (as in the example worked out in Sec. 4 below) and through a reasonably sensible choice of the arbitrary $x_1^{(0)}, \cdots, x_n^{(0)}$. Only slight modification of the analysis is required to justify the use of least absolute remainders: Indeed, if in (1) we have $0<-r<s$, we obtain in place of (2)

$$0 < a_1(qx_1 - 1) + a_2(qx_2) + \cdots + a_n(qx_n) = -r < s. \tag{3}$$

3. Linear diophantine equation. If in the computation of d delineated above we also compute the quantities $x_j^{(k)}$, we eventually arrive at a solution of the equation

$$a_1x_1 + \cdots + a_nx_n = d. \tag{4}$$

Indeed, if $r_t=d$, then (4) is solved by $x_1=x_1^{(t)}, \cdots, x_n=x_n^{(t)}$.

Confronted with the trivially more general problem of solving

$$a_1y_1 + \cdots + a_ny_n = N, \tag{5}$$

we proceed first to determine $d=(a_1, \cdots, a_n)$. If it turns out that d does not divide N, we know that (5) has no solution. If, however, d divides N, we next proceed to compute a set of $x_1, \cdots, x_n$ satisfying (4); then, clearly, $y_j=(N/d)x_j$ $(j=1, \cdots, n)$ provides a solution of (5).

The rule for computing the successive $x_j^{(k)}$ is embodied in (2) if, for the moment, we employ least positive remainders only:

$$x_j^{(k+1)} = -q_k x_j^{(k)} \ (j \neq p), \qquad x_p^{(k+1)} = 1 - q_k x_p^{(k)}, \tag{6}$$

where $a_p=q_k r_k+r_{k+1}$ $(0<r_{k+1}<r_k)$ and a_p is any one of the coefficients not divisible by r_k.

Introducing the Kronecker delta ($\delta_{jp}=0$ if $j\neq p$, $\delta_{pp}=1$) and using (3) to take advantage of least absolute remainders, we revise (6) to obtain for the successive computations of the $x_j^{(k)}$:

$$x_j^{(k+1)} = -(r_{k+1}/|r_{k+1}|)(q_k x_j^{(k)} - \delta_{jp_k}) \qquad (j = 1, \cdots, n), \tag{7}$$

where $a_{p_k}=q_k|r_k|+r_{k+1}$ $(0<|r_{k+1}|\leqq\frac{1}{2}|r_k|)$. To give the procedure a unique determination (once the $x_j^{(0)}$ are arbitrarily assigned for $j=1, \cdots, n$), we may (i) provide that if $|r_{k+1}|=\frac{1}{2}|r_k|$, we choose $r_{k+1}>0$, and (ii) let p_k be the least integer p such that a_p is not divisible by r_k.

If our problem is merely to solve the diophantine equation (5), a possible short cut may present itself in the fortuitous result $|r_u|=N$ for some $u\geqq 0$. In this event we have $y_j=x_j^{(u)}$ $(j=1, \cdots, n)$ as a solution of (5).

4. Numerical example. We illustrate the method by obtaining a solution of the diophantine equation

$$1302y_1 + 1911y_2 + 5460y_3 + 15015y_4 = N, \tag{8}$$

where N is a given positive integer; at the same time we compute $d=(1302, 1911, 5460, 15015)$, which—as pointed out in Section 2 above—can be accomplished without consideration of (8).

In the solution table that follows, the figures in the final three columns are set down directly after the assignment of $x_j^{(0)}$ $(j=1, 2, 3, 4)$ and prior to computation of the $x_j^{(k)}$ for $k\geqq 1$; the reasons for proceeding so are implicit in the second and final paragraphs of Section 3 above.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	r_k	$-(r_{k+1}/\lvert r_{k+1}\rvert)q_k$	p_k
0	5	2	1	−1	777	2	1
1	9	4	2	−2	−252	−5	1
2	−44	−20	−10	10	42	−45	2
3	1980	901	450	−450	21		

Since $r_3=21$ divides each of 1302, 1911, 5460, and 15015, we have the result $(1302, 1911, 5460, 15015)=21$. Also, a solution of (8)—provided N is a multiple of 21—is $y_1=1980(N/21)$, $y_2=901(N/21)$, $y_3=450(N/21)$, $y_4=-450(N/21)$.

5. Remark. For $n\geq 4$, the above method for obtaining a solution of (5) in integers is considerably more direct than the method one finds in text books ([2], for example). The latter, however, has the advantage of leading to a general solution, while the method outlined in Section 3 provides only particular solutions. (Distinct choices of the sets of $x_j^{(0)}$ generally lead to distinct solutions.)

References

1. I. M. Vinogradov, Elements of Number Theory, New York, 1954, p. 17.
2. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, New York, 1939, pp. 62–65.

A NOTE ON CONVERGENCE IN TOPOLOGICAL SPACES

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Let $\{x_n\}$ be an infinite sequence of points in a topological space S.

DEFINITION. *The sequence $\{x_n\}$ is said to be an $\mathfrak{D}$-sequence if and only if for every O open in S such that $x_n\in O$ for an infinite number of n, then $x_n\in O$ for $n\geq N$ for some integer N.*

Example 1. A convergent sequence is not necessarily an $\mathfrak{D}$-sequence. To show this let $x_n\equiv(-1)^n(1/n)$ in the space of reals. This is a convergent sequence, but if $O\equiv\{x\,|\,x<0\}$, then $x_n\in O$ for an infinite number of n, but $x_n\notin O$ for an infinite number of n. Thus $\{x_n\}$ is not an $\mathfrak{D}$-sequence.

Example 2. An $\mathfrak{D}$-sequence is not necessarily convergent. Let S be the set of all positive integers; we topologize S by letting the open sets be $\varnothing$, S, and O_i where $O_i\equiv\{1, \cdots, i\}$. Let $x_n\equiv n$ for all n. Then $\{x_n\}$ is clearly an $\mathfrak{D}$-sequence which is not convergent. Note that this space is not a T_1-space.

THEOREM 1. *Let S be a T_1-space and $\{x_n\}$ an $\mathfrak{D}$-sequence in S. Then $\{x_n\}$ is a convergent sequence.*

Proof. Let $X\equiv\bigcup_n x_n$. Case 1: Suppose X is finite. Let $y=x_{n_1}=x_{n_2}=\cdots=x_{n_i}=\cdots$ where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$. Then $\lim x_n=y$; for if $y\in O$ where O is open in S, then $x_n\in O$ for $n\geq N$ since $\{x_n\}$ is an $\mathfrak{D}$-sequence. Case 2: Suppose X is infinite and $X'\neq\varnothing$, where X' is the derived set of X. Take $p\in X'$. Then $\lim x_n=p$; for if $p\in O$ is an open set, then $O\cap X$ is infinite since S is T_1 and thus $x_n\in O$ for $n\geq N$. Case 3: X is infinite and $X'=\varnothing$. We will show that this case cannot occur. Let $y_1, y_2, \cdots, y_i, \cdots$ be an infinite sequence of distinct points in X. Then $y_i\notin X'$ for all i. Then for each i, there exists an open set O_i such that $y_i\in O_i$ and $X\cap O_i=y_i$. Let $O^*=\bigcup_{i=1}^{\infty} O_{2i}$. Then $O^*\cap X$ is infinite and $x_n\in O^*$ for an infinite number of n, but $x_n\notin O^*$ for an infinite number of n since $y_{2i+1}\notin O^*$ for all i.

Verification of these facts follows from Glaisher's tables [2].

I do not know whether equations (3) and the second equation of Moser are simultaneously solvable in composite integers.

III. On the generalization amicable numbers.

L. E. Dickson [1] and Th. E. Mason [3] defined amicable k-tuples as sets of k positive integers $n_1, \cdots, n_k$ such that $\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1+n_2+\cdots+n_k$ and they gave some numerical examples of amicable triples. I examined the Glaisher tables [2] and found the following amicable triples:

(a) $2^2\cdot3^2\cdot5\cdot11$, $2^5\cdot3^2\cdot7$, $2^2\cdot3^2\cdot71$

(b) $2^3\cdot3\cdot5\cdot31$, $2^2\cdot3\cdot5\cdot29$, $2^2\cdot3\cdot5\cdot29$.

There exist pairs of "almost amicable numbers" satisfying the equations $\sigma(m) = \sigma(n) = m+n+1$, *e.g.* $\langle m, n\rangle = \langle 48, 75\rangle$, $\langle 140, 195\rangle$, $\langle 1050, 1925\rangle$.

References

1. L. E. Dickson, Amicable number triples, this MONTHLY, vol. 20, 1913, pp. 84–92.
2. J. W. L. Glaisher, Number-divisor Tables. Cambridge 1940.
3. Th. E. Mason, On amicable numbers and their generalizations, this MONTHLY, vol. 28, 1921, pp. 195–200.
4. L. Moser, Some equations involving Euler's totient function, this MONTHLY, vol. 56, 1949, pp. 22–23.

A THEOREM ON FINITE ABELIAN GROUPS

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Let G_m be the multiplicative group of integers less than and relatively prime to m, with multiplication defined modulo m.

THEOREM. *Given a finite Abelian group* $\mathfrak{G}$, *there exists an integer* m *such that* $\mathfrak{G}$ *is isomorphic to a subgroup of* G_m.

Proof. By the fundamental theorem of Abelian groups, $\mathfrak{G}$ is isomorphic to the direct product of cyclic groups of prime power order; *i.e.*, $\mathfrak{G} \cong \mathfrak{G}_1 \otimes \cdots \otimes \mathfrak{G}_n$ where the order of $\mathfrak{G}_i$ is $p_i^{\alpha_i}$.

From Dirichlet's theorem on the infinitude of primes in an arithmetic series, it follows that n distinct primes q_i can be found such that $q_i = a_i p_i^{\alpha_i} + 1$. As there exist primitive roots for every prime, there is an element y_i in G_{q_i} such that y_i generates G_{q_i}. Let $x_i = y_i^{a_i}$. Then x_i generates a cyclic subgroup H_i of G_{q_i} whose order is $p_i^{\alpha_i}$.

Let $\mathfrak{F} = G_{q_1} \otimes \cdots \otimes G_{q_n}$ and $\mathfrak{H} = H_1 \otimes \cdots \otimes H_n$. $\mathfrak{H}$ is isomorphic to $\mathfrak{G}$. By a known theorem,* $\mathfrak{F}$ is isomorphic to G_m with $m = \prod_{i=1}^{n} q_i$. Thus the theorem is proved.

* H. S. Vandiver and Milo W. Weaver, Introduction to Arithmetic Factorization and Congruences from the Standpoint of Abstract Algebra, this MONTHLY, vol. 65, October, Part II, 1958, pp. 39–41.

CORRECTIONS TO "VOLUMES AND AREAS OF CROSS-SECTIONS"*

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In the second line from the bottom on page 249, the factor $\cos u$ is missing. Therefore $e^{2|u|}$ and $e^{-2|u|}$ in the definition of $F(u, h)$ must be replaced by $e^{t|u|}$ and $e^{-t|u|}$ with a suitable $t>2$ to attain

$$(4\pi)^{-1}V'(B_0) = \int_0^{1/2\pi} e^{tu}g(u)du < 0, \qquad g(u) = (\cos 2u + \tfrac{1}{2})\cos u.$$

The inequalities $g(u) \leqq b = 3/2$, $g(u) \leqq 0$ in $[\frac{1}{3}\pi, \frac{1}{2}\pi]$, $g(u) \leqq -c < 0$ in $[\alpha, \beta]$ if $\frac{1}{3}\pi < \alpha < \beta < \frac{1}{2}\pi$ give

$$(4\pi)^{-1}V'(B_0) < b\int_0^{\alpha} e^{tu}du - c\int_{\alpha}^{\beta} e^{tu}du < t^{-1}e^{\beta t}[(b + c)e^{(\alpha-\beta)t} - c],$$

which tends to $-\infty$ for $t \to \infty$.

* This MONTHLY, vol. 67, 1960, pp. 248–250.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

All material for this department should be sent to C. O. Oakley, Department of Mathematics, Haverford College, Haverford, Pa.

A CLASSROOM PROOF OF $\lim_{t\to 0}(\sin t)/t = 1$

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The usual proof of $\lim_{t\to 0}(\sin t)/t = 1$ requires that the inequalities: $\sin t < t < \tan t$ be established for sufficiently small (positive) values of t. The latter part of this string of inequalities is usually determined by comparing the area of a unit-circular sector of angle t with the area of a triangle. If the finding of the area of a circular sector exhausts the students' reservoir of thinking, then an understanding of the remainder of the proof is lost. The following method of proof avoids the use of area.

Let it already have been established (either by definition or by some kind of geometric-trigonometric methods) that the point with coordinates $(\cos t, \sin t)$ is at a distance, measured appropriately along the circumference of the unit circle, of $|t|$ units from the point $(1, 0)$.

THEOREM 1.

$$\lim_{t\to 0}(1 - \cos t)/t = 0.$$

Proof. Since the straight-line distance between $(\cos t, \sin t)$ and $(1, 0)$ does

not exceed the arc length along the circle between the same two points, we have

$$0 \leqq \sqrt{\{(\cos t - 1)^2 + (\sin t - 0)^2\}} \leqq |t| .$$

Squaring, simplifying, and dividing by $2t$ yields

$$0 \leqq (1 - \cos t)/t \leqq \tfrac{1}{2}t, \qquad t > 0; \qquad \tfrac{1}{2}t \leqq (1 - \cos t)/t \leqq 0, \qquad t < 0.$$

By the "pinching theorem" for limits, the theorem is proved.

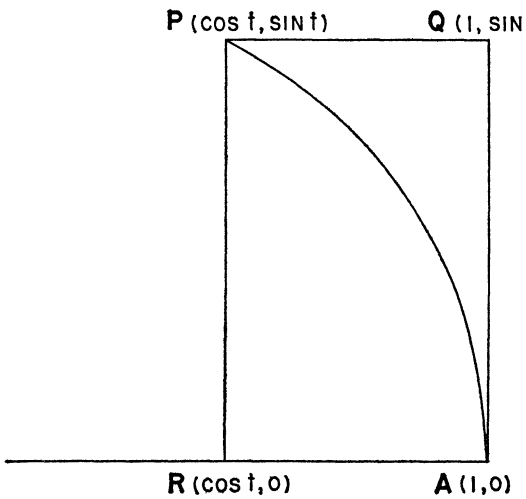


FIG. 1

THEOREM 2.

$$\lim_{t \to 0} (\sin t)/t = 1.$$

*Proof.** If $0 < t < \frac{1}{2}\pi$, let P be the point $(\cos t, \sin t)$, Q, the point $(1, \sin t)$, A, the point $(1, 0)$, and R, the point $(\cos t, 0)$. (Fig. 1) Then $\overline{PR} \leqq \text{arc } PA \leqq \overline{PQ} + \overline{QA}$ and therefore

$$\sin t \leqq t \leqq 1 - \cos t + \sin t.$$

But $\sin t < t$ implies $1 - \cos t + \sin t < 1 - \cos t + t$ so that

$$t \leqq 1 - \cos t + \sin t \leqq 1 - \cos t + t.$$

Hence, dividing by t,

$$1 \leqq \frac{1 - \cos t}{t} + \frac{\sin t}{t} \leqq \frac{1 - \cos t}{t} + 1.$$

By Theorem 1 and the "pinching theorem" for limits, the theorem is proved.

* For $t \to 0+$, the proof for $t \to 0-$ being similar.

DERIVATIVES OF sin θ AND cos θ

C. S. OGILVY, Hamilton College

The derivatives of the sine and cosine functions may be presented handily with the aid of vectors, following a suggestion of R. T. Coffman of Richland, Washington.

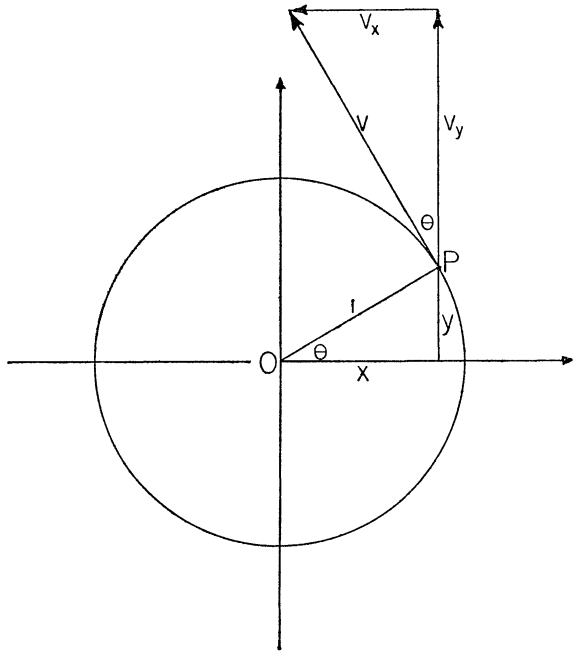


FIG. 1

Consider the point $P(x, y)$ moving around the unit circle (Fig. 1) in the direction of increasing θ; then $x=\cos\theta$ and $y=\sin\theta$. We note that $ds/d\theta=1$ if θ is measured in radians. The direction of motion of P is tangential to the circle; hence the vector v is perpendicular to OP at P. From the vector triangle,

$$\sin\theta = \frac{|v_x|}{|v|} = \left|\frac{dx}{dt}\,\frac{dt}{ds}\right|\frac{ds}{d\theta} = \left|\frac{dx}{d\theta}\right|,$$

and we must make it minus because it is in the direction of decreasing x. Exactly similarly, $dy/d\theta=\cos\theta$.

EQUIANGULAR COMPLEX NUMBERS

J. P. BALLANTINE, University of Washington

1. Introduction. Consider the complex number,*

$$z = 0.959913 \text{ cis } 1.440881 \text{ radians.} \tag{1}$$

Rough calculation shows that

$$z^4 = 0.849039 \text{ cis } (-0.5196619 + 2\pi),$$
$$z^{13} = 0.587509 \text{ cis } (-0.1181050 + 6\pi).$$

The numbers z^0, z^4, and z^{13} plotted in the complex plane are the vertices of a nearly equilateral triangle (Fig. 1). The same holds for z^k, z^{k+4}, and z^{k+13}.

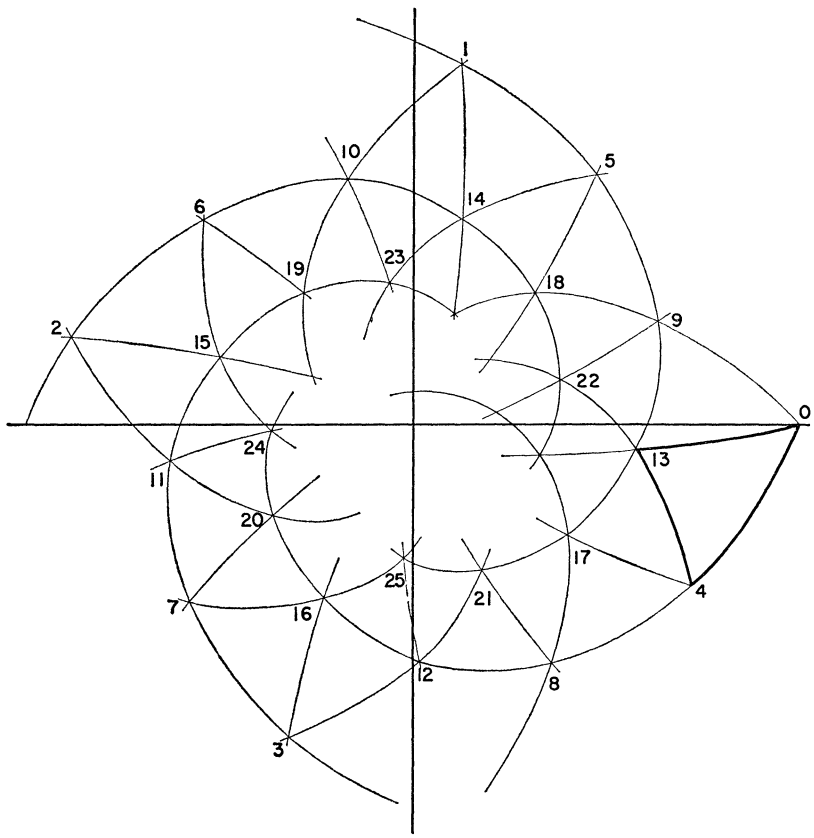


FIG. 1. Graphs of the values of z^n for integral values of n, where z is the equiangular complex number for $p=4$, $q=13$.

More generally, suppose that

* Cis θ is a common abbreviation for $\cos\theta + i \sin\theta$.

$$z = r \operatorname{cis} \theta, \tag{2}$$

where r and θ are fixed and $|r| \neq 0$ or 1. Let n assume the values $n_1, n_2, \cdots$, of an integral-valued arithmetic progression. Then the complex numbers z^n fall along a spiral whose formula in the polar coordinates ρ, ϕ is of the form

$$\rho = Re^{h\phi}. \tag{3}$$

The values of R and h can be determined by substituting z^n for $n = n_1$ and n_2 into (3). The formula for h is

$$h = \frac{(n_2 - n_1) \log r}{\theta(n_2 - n_1) + 2\pi(m_2 - m_1)}.$$

R is readily found from h. Because of the ambiguity in $(m_2 - m_1)$, many different value of h are possible, giving rise to many different possible spirals. For simplicity, we choose $m_2 - m_1$ so that $|h|$ is as large as possible. Then there is only one spiral, except in uninteresting cases where there may be two. So z and any integral-valued arithmetic progression of two or more terms determines one spiral.

A spiral triangle is determined by its three vertices. For sides it has the spirals determined by taking its vertices two at a time. If it should happen that the three numbers z^0, z^4, and z^{13} determine a spiral triangle that is equiangular, then the spiral triangle z^0, z^9, and z^{13} also is equiangular, and the configuration showing all the integral powers of z comprises an infinitude of equiangular spiral triangles covering the entire complex plane except the points 0 and ∞. This would be an idealization of designs frequently found in nature.

DEFINITION. *If p and q are any assigned relatively prime positive integers, and if z^0, z^p, and z^q are the vertices of an equiangular spiral triangle, then z is called an equiangular complex number for p and q.*

If z is equiangular for p and q, then so are z^{-1}, $\bar{z}$ the conjugate of z, and $\bar{z}^{-1}$. The values $\theta = 0$ or π and $r = 0$ or 1 are impossible. (The three cube roots of unity determine an equiangular spiral triangle whose angles are π, which we exclude, since we demand that the angles of an equiangular spiral triangle be $\pi/3$.) One of the four numbers, $z, \bar{z}, z^{-1}, \bar{z}^{-1}$ will fall in the upper half of the unit circle, and that one we take as z.

2. The logarithm transformation. Let z of (2) be transformed into Z by

$$Z = \log z = \log r + i(\theta - 2\pi m). \tag{4}$$

Let

$$Z_p = \log z^p = p \log r + i(p\theta - 2\pi p'),$$
$$Z_q = \log z^q = q \log r + i(q\theta - 2\pi q').$$

The values of p' and q' are chosen so as to make the imaginary terms in Z_p and

Z_q small. Let

$$\Delta = \begin{vmatrix} p & q \\ p' & q' \end{vmatrix}.$$

The Z-plane is bounded by the lines $|Y| = \pi$. By our choice of p' and q', the transform of the spiral triangle O, Z_p, Z_q will be so small that it will not be broken by said bounding lines, and will be an ordinary equilateral triangle, since the transformation is conformal.

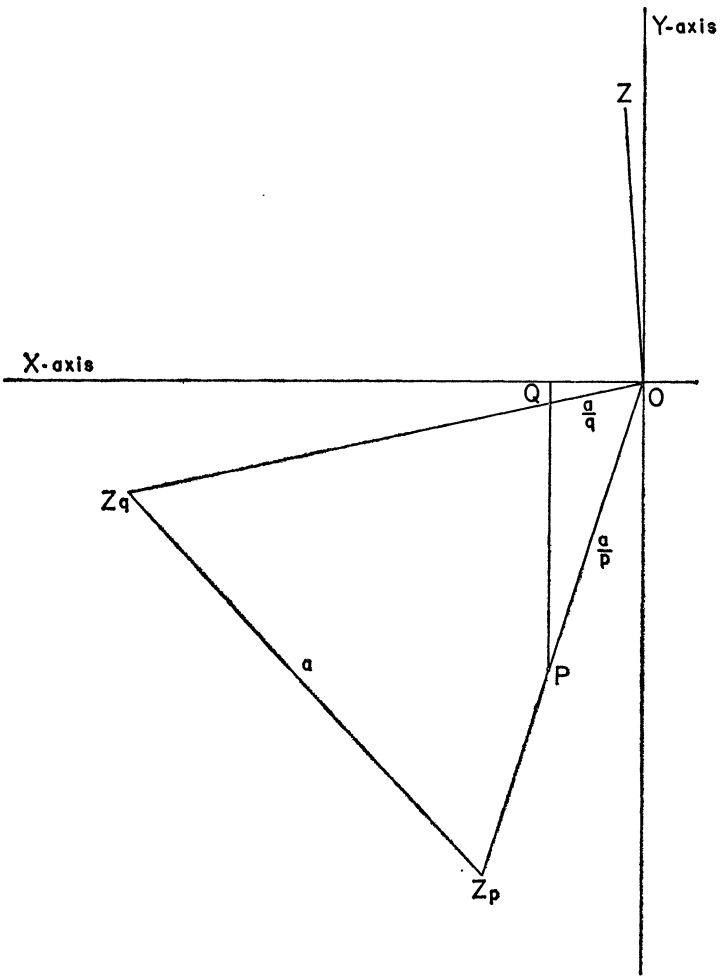


FIG. 2*

* To save space, the figure is purposely drawn out of scale. The line OZ should be many times the length shown and P, Q should be much nearer to O.

3. Formulas for finding equiangular complex numbers. Figure 2 shows the Z-plane, with origin at O. It shows the points Z, and O, Z_p, and Z_q, forming an equilateral triangle of side a. Also shown are the construction points, P and Q. P is on line OZ_p at a distance a/p from O. Q is on OZ_q at a distance a/q from O.

TABLE OF XY-COORDINATES OF FIVE POINTS

POINT	X	Y
Z	$\log r$	θ
Z_p	$p \log r$	$p\theta - 2\pi p'$
Z_q	$q \log r$	$q\theta - 2\pi q'$
P	$\log r$	$\theta - 2\pi p' \div p$
Q	$\log r$	$\theta - 2\pi q' \div q$

The value of a is found by equating two different expressions for the distance PQ. By the two point distance formula,

$$PQ = \left| 2\pi \left(\frac{q'}{q} - \frac{p'}{p} \right) \right| = \frac{2\pi}{pq} \, |\, \Delta \,| \, .$$

PQ occurs in the triangle OPQ, $\angle POQ = 60°$, $OP = a/p$ and $OP = a/q$. By the law of cosines, $PQ = (aS)/(pq)$, where $S^2 = p^2 - pq + q^2$. $a = (2\pi/S)\,|\Delta|$. Now, applying the law of sines to the triangle OPQ,

$$\sin P = (p \sin 60°)/S, \qquad \cos P = (2q - p)/(2S).$$

The line PQ is parallel to the Y-axis. So the X- and Y- coordinates of Z_p can each be found by two different methods, and the results equated to give $\log r$ and θ.

$$X_p = p \log r = a \sin P,$$

$$Y_p = p\theta - 2\pi p' = a \cos P,$$

$$\log r = \frac{a \sin P}{p} = \frac{\sqrt{3}\pi\Delta^*}{S^2} \, .$$

$$\theta = \frac{1}{p} (2\pi p' + a \cos P) = \frac{1}{p} \left(2\pi p' + \frac{\pi\Delta(2q - p)}{S^2} \right).$$

4. Suggested procedure. Given p and q, the value of $z = r \operatorname{cis} \theta$ may be found by the following steps. Select p' and q', and compute Δ. The sign of Δ can be changed by interchanging p and q. The resulting z will be outside or within the unit circle according as Δ is plus or minus. Find S^2, $J = (\pi\Delta)/S^2$, $\log r = J\sqrt{3}$,

$$\theta = \frac{1}{p} (2\pi p' + (2q - p)J).$$

* Without the ambiguous sign, since $\sqrt{3}$ comes from sin 60°.

Example. Let $p=4$, $q=13$, take $p'=1$ and $q'=3$. Then $\Delta=-1$ and $S^2=133$;

$$J = -\pi \div 133 = -0.0236210,$$
$$\log r = J\sqrt{3} = -0.0409128,$$
$$\theta = \tfrac{1}{4}(2\pi + 22J) = \tfrac{1}{2}(\pi + 11J) = 1.4408808.$$

The resulting value of z appeared in (1).

Note that the choice $\theta=(2\pi-22J)\div 4$ does not lead to an equiangular complex number.

SOME SAMPLE EQUIANGULAR COMPLEX NUMBERS

p	q	p'	q'	Δ	r	θ
4	13	1	3	−1	0.95991286	1.4408808
8	21	3	8	1	0.98398308	2.3958140
21	73	2	7	1	0.99871657	0.6028121
19	103	7	38	1	0.99939645	2.3182883
19	103	2	11	3	0.99819046	0.6716797

5. Conclusion. A routine is to be designed for a digital computer to find complex roots by Newton's method. The first approximation is found by trying several scattered complex numbers within the unit circle. One easy way to tell the machine which numbers to try is to feed in one complex number z and direct the machine to use successive integral powers of z. For best results, z should be an equiangular complex number.

Nature comes to the same conclusion in arranging the points on a pine cone, or the inner blossoms of a daisy.

ON EULER'S EQUATION

HUGH A. THURSTON, University of British Columbia

I. Destructive

1. If u is a function of x and y, the equation $x(\partial u/\partial x)+y(\partial u/\partial y)=0$ is known as Euler's equation. In many books (*e.g.*, on p. 109 of Courant's *Differential and Integral Calculus*, vol. II) it is said to be a sufficient and (tacitly assuming u to be well enough behaved) necessary condition for the function to be homogeneous of degree 0. This is not exactly true.

2. A common "proof" is as follows. Let u be $\phi(x, y)$; and for a given x and y define ψ by

(i) $$\psi(t) = \phi(x, y) - \phi(x\cdot t, y\cdot t) \text{ for every } t.$$

Then (assuming that ϕ obeys the chain rule)

$$\psi'(t) = -x\cdot\phi_1(x\cdot t, y\cdot t) - y\cdot\phi_2(x\cdot t, y\cdot t) \text{ for every } t.$$

Thus $t\cdot\psi'(t)=0$, by Euler's equation for ϕ; hence, if $t\neq 0$, $\psi'(t)=0$.

Courant omitted the proviso "if $t\neq 0$," and deduced that ψ is a constant, whence $\psi(t)=\psi(1)$ for every t, *i.e.*, $\phi(x, y)=\phi(x\cdot t, y\cdot t)$ for every t.

3. In fact, the best that we can deduce without knowing more about $\psi'(0)$ is that $\psi(t)=\psi(1)$ when $t>0$, and $\psi(t)=\psi(-1)$ when $t<0$. Each of these shows that $\phi(x, y)=\phi(x\cdot t, y\cdot t)$ whenever $t>0$: *i.e.*, ϕ is positively-homogeneous of degree zero.

4. A positively-homogeneous solution of Euler's equation is

(ii) $\phi(x, y)=\xi(x/y)$ whenever $y>0$, $\eta(x/y)$ whenever $y<0$, a whenever $y=0$, for some differentiable functions ξ, η, and some number a. ϕ is not homogeneous unless $\xi=\eta$.

5. As a particular solution, I shall choose

$$\text{(iii)}\qquad \xi(z) = z/(e^z + e^{-z}), \qquad \eta(z) = 2z/(e^z + e^{-z}), \qquad a = 0.$$

It is easy to verify that $\phi_1(x, y)$ and $\phi_2(x, y)$ exist and Euler's equation is satisfied for every x and every nonzero y.

Now let us suppose that $y=0$. Firstly, $\phi_1(x, 0)=0$. Secondly,

$$\lim_{h\to 0^+}\frac{\phi(x, h) - \phi(x, 0)}{h} = \lim_{h\to 0^+}\frac{\xi(x/h) - a}{h} = \lim_{h\to 0^+}\frac{x}{h^2(e^{x/h} + e^{-x/h})} = 0.$$

Similarly, $\lim_{h\to 0^-}[\phi(x, h)-\phi(x, 0)]/h=0$. Then $\phi_2(x, 0)$ exists (and is zero) for every x. Then $x\cdot\phi_1(x, 0)+0\cdot\phi_2(x, 0)=0$, and so the equation is satisfied always.

6. Why bother to verify that $\phi_2(x, 0)$ exists? Because although $0\cdot\phi_2(x, 0)$ cannot equal anything other than 0, it cannot truly equal zero unless it exists. From the practical aspect, too, if we were trying to solve the equation with a ϕ such that $\phi_2(x, y)$ fluctuated more wildly than y^{-1} for y near to 0, we could scarcely trust the mere "$x\cdot\phi_1(x, 0)=0$".

Equations (iii) were chosen with this in mind; (ii) and (iii) make $\phi_2(x, 0)$ exist for every x. If we relaxed this requirement, any differentiable ξ and η would make (ii) a solution.

7. It is perhaps unusual to think about what is really meant by "solve the equation $x\cdot\phi_1(x, y)+y\cdot\phi_2(x, y)=0$." It is (very naturally) more usual to concentrate on the techniques of solution. Nevertheless, let us continue. All, or nearly all, methods of solving partial differential equations use the chain rule. This rule is not universally valid; the usual condition under which it is proved is differentiability, which is much stronger than the mere existence of $\phi_1(x, y)$ and $\phi_2(x, y)$.

8. Now suppose that we solve the equation using the chain rule. Then we are unlikely to have the most general solution. But we might, if we are lucky, have the most general differentiable solution. In practice, this is likely to be general enough.

However, we use differentiability only to justify the chain rule. Why not, then, simply look for all solutions which obey this rule? This is more general than looking for differentiable solutions. We can go still further. We use only a special case of the chain rule. Let us then define ϕ to be *amenable on* R if

$$(d/dt)\phi(x\cdot t, y\cdot t) = x\cdot\phi_1(x\cdot t, y\cdot t) + y\cdot\phi_2(x\cdot t, y\cdot t)$$

whenever $(x\cdot t, y\cdot t)\in R$. If R is the set of all pairs of numbers, ϕ is simply *amenable*.

9. We can now find the most general amenable solution of Euler's equation. Returning to the proof in Section 3, we have $\psi'(t)=0$ whenever $t\neq 0$. $\psi'(0)$ exists (it equals $-x\cdot\phi_1(0, 0)-y\cdot\phi_2(0, 0)$). Then $\psi'(0)=0$, and so ψ really is constant. Then $\psi(1)=\psi(0)$, *i.e.*, $\phi(x, y)=\phi(0, 0)$. Thus ϕ is constant. Because a constant is differentiable, this is also the most general differentiable solution.

10. Thus neither strictly nor loosely is it true to say that the general solution of Euler's equation is a homogeneous function of degree zero. Strictly, the general differentiable (or amenable) solution is a constant; loosely the general solution includes nonhomogeneous functions.

II. Constructive

11. If E_+ denotes $\{(x, y)\,|\,y>0\}$, any of the usual proofs (including that in Sec. 3) yield:

THEOREM. *If* $x\cdot\phi_1(x, y)+y\cdot\phi_2(x, y)=0$ *for every* (x, y) *in* E_+, *and if* ϕ *is amenable on* E_+, *then there is a differentiable* ξ *such that* $\phi(x, y)=\xi(x/y)$ *for every* (x, y) *in* E_+.

The theorem remains true if we replace E_+ by $\{(x, y)\,|\,y<0\}$. We fill the gap with the trivial

THEOREM. *If* $x\cdot\phi_1(x, y)+y\cdot\phi_2(x, y)=0$ *whenever* $y=0$, *then there is a number* a *such that* $\phi(x, 0)=a$ *for every* x.

Proof. $x\cdot\phi_1(x, 0)=0$ for every x. Therefore $\phi_1(x, 0)=0$ for every nonzero x. $\phi_1(0, 0)$ exists (by hypothesis). Therefore $\phi_1(x, 0)=0$ for every x.

12. Together, the theorems of Section 11 yield:

THEOREM. *If* $x\cdot\phi_1(x, y)+y\cdot\phi_2(x, y)=0$ *for every* x *and* y, *and if* ϕ *is amenable except perhaps on* $\{(x, 0)\}$, *then there are differentiable functions* ξ *and* η *such that*

(iv)
$$\phi(x, y) = \begin{cases} \xi(x/y) & \text{whenever } y > 0, \\ \eta(x/y) & \text{whenever } y < 0, \\ \phi(0, 0) & \text{whenever } y = 0. \end{cases}$$

13. Now what about the converse: *i.e.*, what conditions on ξ and η are needed

for (iv) actually to give a solution? It is easily seen that (iv) satisfies the equation when $y \neq 0$; and it satisfies the equation when $y = 0$, provided that $\phi_2(x, 0)$ exists. So what we need are necessary and sufficient conditions that $\phi_2(u, 0)$ exist for every u. These are found with little trouble to be

$$\text{(v)} \qquad \begin{aligned} \xi(0) &= \eta(0) = \phi(0, 0), \\ \lim_{x \to \infty} x \cdot (\xi(x) - \xi(0)) &= \lim_{x \to -\infty} x \cdot (\eta(x) - \eta(0)), \\ \lim_{x \to -\infty} x \cdot (\xi(x) - \xi(0)) &= \lim_{x \to \infty} x \cdot (\eta(x) - \eta(0)). \end{aligned}$$

14. Perhaps the most important case is where ϕ is amenable everywhere except at the origin. The solution here will be given by (iv) and (v) together with whatever other conditions are needed to make ϕ amenable on $\{(x, 0) \mid x \neq 0\}$. $(d/dt)\phi(x \cdot t, 0) = (d/dt)\phi(0, 0) = 0$. $x \cdot \phi_1(x, 0) + 0 \cdot \phi_2(x, 0) = 0$. Therefore no extra condition is required.

15. A slightly different problem is to find the most general ϕ, amenable except at the origin, for which

$$x \cdot \phi_1(x, y) + y \cdot \phi_2(x, y) = 0 \quad \text{whenever} \quad (x, y) \neq (0, 0).$$

We argue as in Section 11, except that now we no longer know that $\phi_1(0, 0)$ exists. We find, in place of (iv):

$$\phi(x, y) = \xi(x/y) \quad \text{if} \quad y > 0; \qquad \eta(x/y) \quad \text{if} \quad y < 0;$$

$$a \quad \text{if} \quad y = 0 \quad \text{and} \quad x > 0; \qquad b \quad \text{if} \quad y = 0 \quad \text{and} \quad x < 0.$$

(*i.e.*, ϕ is the general positively-homogeneous function of degree zero).

In place of (v) we get:

$$\begin{aligned} \lim_{x \to \infty} x \cdot (\xi(x) - a) &= \lim_{x \to -\infty} x \cdot (\eta(x) - a), \\ \lim_{x \to -\infty} x \cdot (\xi(x) - b) &= \lim_{x \to \infty} x \cdot (\eta(x) - b). \end{aligned}$$

16. This is about as much as one can expect without a radically different type of attack, and without producing solutions which would be little appreciated—for few of us would be happy with functions which did not obey the chain rule.

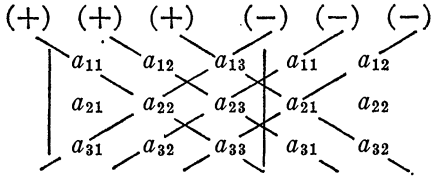
Of course, if ϕ is wanted only over some special region, we may get a simple answer. *E. g.*, if R is an open convex set, then ϕ is differentiable in R and $x \cdot \phi_1(x, y) + y \cdot \phi_2(x, y) = 0$ for every (x, y) in R if and only if ϕ is differentiable and homogeneous of degree zero on R (*i.e.*, $\phi(x, y) = \phi(x \cdot t, y \cdot t)$ whenever $t \neq 0$ and both $(x, y) \in R$ and $(x \cdot t, y \cdot t) \in R$).

CONCERNING FOURTH-ORDER DETERMINANTS

ROGER OSBORN, University of Texas

A problem which arises periodically in intermediate (and college) algebra courses is that of expanding fourth-order determinants. Many students have been indoctrinated in the diagrammatic method of expanding third-order determinants, and they wonder (silently mostly, but sometimes aloud) why the same type diagram cannot be used to expand a fourth-order determinant. Quite usually the student's question is answered merely by, "It can't be done." This note is devoted to a proof that it can't be done by a single diagram of the same type used for third-order determinants.

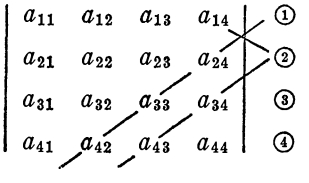
A diagrammatic scheme for expanding a third order determinant requires rewriting the first two columns and taking products down diagonals, some positive and some negative as shown in the diagram:



An obvious extension of this diagram for a fourth-order determinant would be to rewrite the first three columns and take diagonal products. This obviously does not yield the correct result since there are only eight such diagonal products instead of the twenty-four product terms given by the standard definition of the fourth-order determinant.

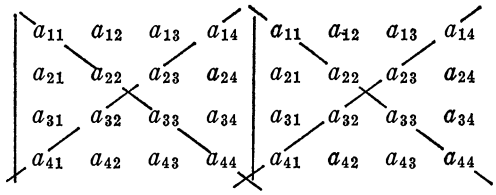
A question now arises. Can additional columns be added to the diagram, in sufficient number to give twenty-four diagonal products, and of such nature that the products will be the ones desired—*i.e.*, no two column or row indices will be alike in any single product and no two products will be alike?

The answer to this question is no. Suppose we append additional elements, in columns, to the diagram. At the outset we keep an open mind about which elements are to be appended. Add elements to the diagram as illustrated.



Added element ① must be a_{11} since its row and column subscripts may neither repeat a row or column subscript found on the diagonal shown terminating at ①. Element ② will lie on two diagonals, and since along these diagonals the only row subscript not already used is 2, and since the only column subscript not already used is 1, the element must be a_{21}. Similarly, element ③ must be a_{31}, and element ④ must be a_{41}.

Having determined the nature of the elements of the first added column (so that no diagonal product will contain any two elements from the same row or column), use the same device to determine the elements of the second added column. The elements of this column, from top to bottom, will be found to be the same as the elements of the second column of the given determinant, namely, a_{12}, a_{22}, a_{32}, a_{42}. Proceeding similarly, the next two added columns will be identical with the third and fourth columns, respectively, of the given determinant. As soon as the fourth column is repeated, though, it is observed that two sets of identical diagonal products have been created, violating the principle that no two products may be identical.



Hence, the diagonal scheme is found not to be valid for the expansion of fourth-order determinants since the additional elements which must be appended to avoid violating the principle that no product may contain two elements from the same row or column cause the other principle to be violated.

It might be of interest to note, before closing, that of thirty textbooks chosen at random, each of which included a discussion of determinants, sixteen (or more than half) gave the diagonal scheme for evaluating a third-order determinant. Of these sixteen, ten (or a third of the total number) did not even mention the impossibility of the use of the diagonal scheme for evaluating higher order determinants. Of the remaining six books, five stated that the diagonal scheme was invalid for $n>3$, but gave no justification for the statement. Only one book stated that the scheme was invalid for $n>3$ and gave a reason for the statement, and the only reason given was that the repetition of only three columns would not give a sufficient number of product terms. It is unfortunate that so many books give a scheme which cannot be generalized and then do not even mention that generalization is impossible.

ON A NOTE OF N. S. MENDELSOHN

K. VENKANNAYAH, Government College, Mercara, India

N. S. Mendelsohn in his note (this MONTHLY, vol. 58, 1951, p. 563) has proved the following results:

(1) *The sequence* $S_n=(1+1/n)^n$ *is a monotone increasing function of* n *for positive integral* n.

(2) *The sequence* $T_n=(1+1/n)^{n+1}$ *is a monotone decreasing function of* n *for positive integral* n.

The proofs of these results are based on the well-known inequality

$$(3) \qquad (a_1 + \cdots + a_n)/n > (a_1 \cdots a_n)^{1/n}.$$

We should note that Mendelsohn obtains (1) and (2) by applying (3) to the two sets of numbers

$$(4) \qquad \alpha_1 = 1, \quad \alpha_i = (n+1)/n, \quad i = 2, \cdots, n+1;$$

$$(5) \qquad \beta_1 = 1, \quad \beta_i = n/(n+1), \quad i = 2, \cdots, n+2.$$

Here we are interested in obtaining the results (1) and (2) by applying the inequality

$$(6) \qquad (p_1a_1 + \cdots + p_ka_k)/(p_1 + \cdots + p_k) > (a_1^{p_1} \cdots a_k^{p_k})^{1/(p_1+\cdots+p_k)}$$

where $p_1, \cdots, p_k$ are positive rationals, to either of the sets (4) and (5). In effect this amounts to applying (3) and

$$(7) \qquad (a_1 + \cdots + a_n)/n < (a_1^{a_1} \cdots a_n^{a_n})^{1/(a_1+\cdots+a_n)},$$

which emerge as particular cases of (6), to either of the sets (4) and (5).

Consider the set (4) and let $k=n+1$, $p_1= \cdots =p_{n+1}=1$, $a_1=1$, $a_2= \cdots =a_{n+1}=(n+1)/n$ in (6). We get $S_{n+1}>S_n$, which proves (1). If, on the other hand, we set $k=n+1$, $p_1=1$, $p_2= \cdots =p_{n+1}=(n+1)/n$, $a_1=1$, $a_2= \cdots =a_{n+1}=n/(n+1)$ in (6), we obtain $T_{n+1}<T_n$, which proves (2).

If, however, we wish to consider (5) instead of (4), we can set in (6), $k=n+2$, $p_1= \cdots =p_{n+2}=1$, $a_1=1$, $a_2= \cdots =a_{n+2}=n/(n+1)$ to get (2) and set $k=n+2$, $p_1=1$, $p_2= \cdots =p_{n+2}=n/(n+1)$, $a_1=1$, $a_2= \cdots =a_{n+2}=(n+1)/n$ to recover (1).

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

QUALIFICATIONS AND TEACHING LOADS OF MATHEMATICS AND SCIENCE TEACHERS

KENNETH E. BROWN, U. S. Office of Education

Not long ago President Eisenhower asked citizens to scrutinize their schools to see whether they were meeting "the stern demands of the era we are entering." Many of the citizens following the President's suggestion began their scrutiny with teachers and pupils—for they are the heart of the school.

Because they realize that the school's success depends in large measure on the competence of the teachers, many citizens are looking closely at teachers—their preparation, experience, and work loads. And because they hear on every

side from leaders in government, education, business, and industry that to meet the stern demands of the era ahead men will need thorough training in mathematics and science, they are particularly interested in teachers of these subjects.

Visitors to some schools are encouraged by the information they get; visitors to others are discouraged.

To get a better picture of the mathematics and science teachers' preparation, the U. S. Office of Education in cooperation with New Jersey, Virginia, and Maryland, studied rather intensively the qualifications of teachers in these three states.* The purpose of the study was threefold: (1) To determine the background preparation of science and mathematics teachers in the selected states, (2) to determine the teaching load and the teaching combinations of these teachers, (3) to develop a technique for securing data which might serve as a pattern for other states in making their own studies.

In this study a mathematics or science teacher is defined as any teacher who taught one or more classes in one of these subjects in grades 9–12. There were 1393 such teachers in the study, of whom 662 taught mathematics, but no science, 594 taught science, but no mathematics, and 137 taught both mathematics and science.

The range in average semester hours of college courses per teacher was from 170 in one state to 143 in another state. The highest average number of semester hours (166) was for the teachers in the 3-year senior high schools and the lowest number (142) was in the 5-year junior-senior high schools. When one considers that 120 semester hours is the number required for the bachelor's degree in many colleges, in terms of number of hours the teachers are well prepared. In fact fewer than 2% reported no degree.

Mathematics Teachers. In this study only 7% of the mathematics teachers were without preparation in college mathematics. A study of individual teachers' schedules indicated that the mathematics course most frequently taught by teachers who had taken no college mathematics was one class in general mathematics; and in no case was he devoting full time to the subject.

The mathematics courses studied were divided in two categories: (1) the mathematics below, or prerequisite to, the calculus and (2) the calculus and more advanced courses. Although the mathematics teachers had an average of 23 semester hours in mathematics and 33% had masters' degrees, 39% had not had mathematics as advanced as the calculus. There was a large group of teachers well prepared in number of hours of mathematics and another large group with meager preparation. In many cases the teachers had not had recent courses in mathematics; one teacher in five had taken all their courses prior to 1950. The average number of hours in education was 31 compared with 23 in mathematics.

* Kenneth E. Brown and Ellsworth S. Obourn, Qualifications and Teaching Loads of Mathematics and Science Teachers, Superintendent of Documents, Government Printing Office, Washington 25, D.C.

Science Teachers. Of the science teachers, 64% had the baccalaureate degree with an average of 47 semester hours in science. The average number of hours in one state was 56 and in another state 39. The semester hours of science credit were distributed as follows: chemistry, 17; biology, 14; physics, 9.5. The science teachers had taken 36 semester hours in education.

Teachers' Marks. Approximately 70% of both the science teachers and mathematics teachers had marks of "B" or above. About one-third had marks of "A−" or "B+." This study indicates that teachers' marks in general are quite respectable.

THE UICSM* TEACHER TRAINING FILMS†

GERTRUDE HENDRIX, University of Illinois, AND BYRL SIMS, NETRC§

The UICSM mathematics teacher-training films are designed to replace twenty sessions of a live demonstration class as a part of an intensive four-week teacher-training conference. The series consists of twenty half-hour films, prefaced by a shorter film which will introduce the class and show how the films were made. The films are *not* designed to teach the subject matter of the course to the teachers. They will study the course content in other conference sessions each day. The films, like live demonstrations, are intended for two major purposes:

1) An existence proof that the materials are teachable to junior high school children.
2) Vivid and repeated illustrations of a teaching method and teaching devices known to be successful with the UICSM FIRST COURSE materials.

Observers' familiarity with the content being taught is presupposed.

Eighteen of the films each trace the development of one topic through the entire course, showing important incidents of teacher and class caught on documentary film. Intervening time is covered by brief inserts of explanatory narrative. Such films were made possible by a specially-designed studio built into a regular classroom, with booths for three hidden cameras. One camera covers the teacher at all times; another provides a front view of the class; and the third takes a profile view. Each camera can "move in" for close-ups at appropriate moments. Each sequence used in the completed films is an edited combination from three rolls shot simultaneously.

* University of Illinois Committee on School Mathematics.

† These films are made possible by a grant from the U. S. Office of Education to the National Educational Television and Radio Center, in cooperation with the University of Illinois. They were tested by use in a National-Science-Foundation-sponsored training conference during the past summer, 1960. The original grant finances the film making and the research study only. Provisions for distribution have not yet been made.

§ National Educational Television and Radio Center.

Such films are possible when (1) there is an outstanding teacher thoroughly familiar with good course content and with the kinds of situations likely to arise when one is teaching it; (2) there is a programmer also familiar with the course and aware of the topics with which a teacher new to the material is most likely to need help; (3) there is a class strong enough in prerequisite background not to require costly spur-of-the-moment digressions for remedial teaching of arithmetic; (4) there is another class in the same course taught by the same teacher and kept about two weeks ahead of the film class; and (5) there is a technical staff sensitive enough to what is needed to get camera coverage of the most significant spots, and to edit the film so that the things most valuable for teachers to see are retained.

Accounts of the programming process evolved for these films, and designing of the technical equipment to make them possible, are stories too long for a five-hundred word article.

THE MATHEMATICS PROGRAM AT THE UNIVERSITY OF PARANÁ

JAYME MACHADO CARDOSO, University of Paraná, Brazil

In the University of Paraná, as in all the Brazilian universities, the teaching of mathematics is done by each faculty (Sciences, Engineering, Economics) without following the American pattern of centralizing the teaching in one Department of Mathematics. Each faculty has its separate Department of Mathematics directly subordinated to the faculty's administration, practically without any connection among the different faculties.

The title of Bachelor in Mathematics is obtained in the Faculty of Sciences, where the student must be approved in the following fourteen subjects:

First year: 1) Mathematical analysis I, 2) Analytical and projective geometry; 3) Experimental physics I, 4) Vector calculus, 5) Philosophy.

Second year: 6) Mathematical analysis II, 7) Descriptive geometry, 8) Experimental physics II, 9) Tensor calculus, 10) Rational mechanics.

Third year: 11) Higher mathematical analysis, 12) Higher geometry (Topology), 13) Mathematical physics, 14) Celestial mechanics.

To remedy this inconvenience of the lack of connection and to give better possibilities of research, the Brazilian Government has given great incentive to the creation of Research Institutes in the universities. In 1959 there was founded in the University of Paraná, among others, the Institute of Mathematics. This institute is formed of four divisions: a) Division of Pure Mathematics, b) Division of Applied Mathematics, c) Division of Statistics, d) Division of Teaching and Divulgation. The institute is conducted by a director and by a council constituted of full professors in mathematics and statistics of all the faculties of the University.

Thanks to the existence of the Institute of Mathematics, it will be possible to give different courses which are not normally offered by the faculties of the University, especially in the modern part of mathematics.

THE METHODS COURSE IN MATHEMATICS FOR PROSPECTIVE SECONDARY SCHOOL TEACHERS

The editors of this department have invited a number of persons to describe the methods course for prospective secondary school teachers of mathematics that is offered by their institution. Recommendations of the Commission on Mathematics and other groups lay stress on the importance of the mathematics methods course. On the other hand, there is a great deal of discussion and considerable disagreement about what should be in a course. It is felt that these contributions will be helpful to those planning to establish such courses, or to improve those that are now in operation. Other contributions of this kind would be welcome and could be considered for later publication in this department.

Iowa State Teachers College. Among the several possible and defensible emphases the course might be given, we have chosen curriculum. This gives us the opportunities we seek to (a) review the objectives of instruction in mathematics; (b) look at typical organization of content, kindergarten through college; (c) question relative emphases on topics in light of changing emphases in mathematics and its applications; (d) identify contradictions, at least from the learner's point of view, in commonly taught language and concepts; (e) re-examine the roles of the teacher and the learner in light of new psychological insights into the learning process; (f) build hypothetical teaching sequences to illustrate and try out the principles evolved; (g) study critically the currently available experimental or recommended materials of instruction.

Of course one cannot set up these items, like hurdles in a track event, to be cleared one-by-one in the order *a* through *g*. A more gestalt approach is essential. Dewey said the man educated in a field is the one prepared to make sound judgment in that field. This is the sort of teacher we wish to incubate. The heat we use for this purpose is the active group, confronted with ideas and provided with situations that encourage critical thought.

H. C. Trimble

State University of Iowa. The undergraduate course in *Methods of Teaching High School Mathematics* as taught at the State University of Iowa consists, roughly, of two parts. The first part deals with the secondary curriculum—past, present, and future. The second part is devoted to special methods and techniques involved in teaching a variety of topics, *e.g.* development of operations with directed numbers. Emphasis is given to study and evaluation of current literature, particularly the journals and yearbooks of the National Council of Teachers of Mathematics and materials developed by the Commission on Mathematics, the School Mathematics Study Group, and committees at the Universities of Illinois and Maryland.

The trend in assigned reading is toward more content and less method. For majors this presents little or no problem; minors, on the other hand, often find it difficult to understand the new vocabulary with which they are confronted. As a result, it has become necessary to include in the course a varying amount of mathematical content. We believe, however, that a methods course should emphasize *method* rather than *subject matter.*

H. Vernon Price

University of California, Los Angeles. Fundamental Concepts of Mathematics—Algebra. This course was designed to give secondary teacher candidates the necessary maturity in algebra to make good high school teachers of algebra. Although it is not primarily a methods course, much attention is given to developing the topics discussed in ways which are suitable for high school presentation.

A thorough development of the number system is given, including a brief historical

account of some of the early number systems. A considerable amount of time is given to the consideration of number bases other than ten, including the changing of a numeral written in one base to the corresponding numeral written in another base. This is followed by the consideration of the eleven field postulates of algebra followed in turn by the postulational development of integers, positive and negative, rational numbers, and complex numbers. A brief treatment of irrational numbers completes the study of the number development of the number system.

Some consideration is given to mathematical proof including direct and indirect proof, and mathematical induction. A brief discussion is included on function and its representations.

A rather complete treatment of determinants and their applications is included, followed by an introduction to matrices, including the definition of rank and the multiplication theorem for matrices. The general theory of the solution of simultaneous linear equations, both homogeneous and non-homogeneous, is developed.

The last unit in the course treats the theory of algebraic equations in which are developed those theorems needed to determine the rational roots of a polynomial equation, together with rational approximations of the irrational roots.

Clifford Bell

University of Minnesota. Mathematics with its abstractness, its logical structure, its wide application, has unique learning problems. At one extreme it involves simple skills where memory and practice are essential. At the other extreme, the solution of a problem, the proof of a theorem, the application of a generalization requires a high level of creative thinking. Thus, the teacher of mathematics will need to know when and what concepts to teach, when and why students are having difficulty, when and how to make concepts meaningful, when and how to practice skills and to stimulate productive thinking. The better the teacher's background in mathematics, the greater his problem in selecting what he can hope to present to adolescents—hence, the need of a methods course.

The methods course in mathematics at the University of Minnesota includes topics such as the following: Motivating the learning of mathematics; guiding the student to discover new concepts; teaching the student how to study mathematics; building skill in solving problems; providing a program of acceleration and enrichment for gifted students; planning an effective review, remedial, and practice program; evaluating the learning of facts, skills, and problem solving; developing desirable attitudes and appreciations of mathematics; evaluating new curriculum recommendations; procuring and using new instructional materials; planning and demonstrating a variety of ways to present a mathematical topic; learning the goals of mathematics instruction; reading research in mathematics education.

The explosion of knowledge in the field of education, the expansion in the resources and materials for mathematics teaching, and the many new mathematics curriculum proposals make a methods course more important than ever. If as the Commission on Mathematics says, "A poor curriculum well taught is better than a good curriculum badly taught," then a methods course which improves teaching techniques is an essential course for every prospective mathematics teacher.

Donovan A. Johnson

University of Virginia. The methods course at the University of Virginia for prospective teachers of secondary school mathematics is built around four units: (1) Number and Computation, (2) Measurement, (3) Relationships and (4) Proof. There is no assigned textbook. The required readings at present include sections from the following: 13th, 20th, 22nd, 23rd and 24th *Yearbooks* of the N.C.T.M., the C.E.E.B.

Report, certain methods textbooks, and certain selected articles. For individual projects, a list of supplementary references is included with each unit.

The purposes of the course are as follows: (1) to trace briefly the development of these major concepts in grades one through twelve, with particular emphasis on their development in the upper six grades, (2) to illustrate methods which lead to a developmental (discovery) approach to learning mathematics, (3) to reorient these college students to secondary mathematics, and (4) to introduce them to various teaching materials.

Activities include written assignments on the units, reports, discussion, and sample lessons given by the students themselves. The students review some high school textbooks; special emphasis is placed now on their becoming acquainted with new text materials, such as those of the S.M.S.G., the U.I.C.S.M., and the Maryland Program. Near the end of the course, attention is given to evaluation of pupil progress. All student teachers at the University of Virginia, in addition to their work in methods courses, receive six hours instruction and laboratory practice in the use of visual aids.

William C. Lowry

CONTINENTAL CLASSROOM

Continental Classroom, the network TV program for college credit, will continue to focus on improving science instruction when it goes on the air for the third consecutive year September 26. The 1960–61 course, to be televised over NBC-TV, will be *Contemporary Mathematics.* The first semester will be devoted to *Modern Algebra;* the second semester, to *Probability and Statistics.*

To encourage credit enrollment, a new format for Continental Classroom has been created. *Modern Algebra* will be divided into two sections. College and university students seeking undergraduate credit will be required to view the lessons telecast on Monday, Wednesday, and Friday from 6:30 to 7 A.M. Teachers and others enrolled for graduate credit in the School of Education will be required to view the telecasts five days a week. The additional TV sessions on Tuesday and Thursday will be devoted to the teaching of modern algebra in secondary schools. The same pattern will be followed during the second semester when *Probability and Statistics* is offered.

Co-sponsors of *Contemporary Mathematics* are the Conference Board of the Mathematical Sciences, Learning Resources Institute, and the National Broadcasting Company. An eight-man advisory committee, headed by Dr. E. G. Begle of Yale University has been appointed by the Conference Board.

National teacher of *Modern Algebra* (selected by the advisory committee) will be Dr. John L. Kelley, professor of mathematics and head of the mathematics department at the University of California, Berkeley. Dr. Frederick Mosteller of the statistics department at Harvard University will teach *Probability and Statistics.* For the teacher-education programs of each course, Dr. Julius H. Hlavaty, chairman of the mathematics department at the DeWitt Clinton High School, New York, and Dr. Paul C. Clifford, associate professor at Montclair State College, Upper Montclair, N. J., will assist the national teachers.

As outlined by Dr. Kelley, *Modern Algebra* will include the fundamental concepts underlying the recent changes and developments in both the approach to and teaching of this subject. It will supply all prerequisites for the second semester offering in *Probability and Statistics.* A textbook, *Modern Algebra,* specially written by Dr. Kelley for this TV course, is available in bookstores.

ETS Cooperative Plan for Admission and Guidance*

A new plan for communication between high schools and colleges, enabling schools

* From a news release by the Educational Testing Service, Princeton, N. J.

to convey to the college a student's total high school record and other essential information about him, has been developed by Educational Testing Service of Princeton, N. J., and adapted for their own needs by educators in the state of Georgia.

The new "Cooperative Plan for Guidance and Admission" was announced jointly by Henry Chauncey, President of Educational Testing Service; Claude Purcell, State Superintendent of Schools in Georgia; and Harmon W. Caldwell, Chancellor of The University System of Georgia.

Recent developments in electronic data processing at Educational Testing Service now make it possible to report a student's total school performance—grades, activities, interests, health record, personal characteristics—in a form that is meaningful to employers and college admissions officers. The same process also makes feasible a reporting back to high schools of how well their graduates are doing in college and on the job.

As the first state to adapt the plan for its own needs, Georgia is a pilot state in implementing the system described by Dr. Chauncey as "holding promise for nationwide improvement of guidance in the high school and of the transition from school to college." Educators in several other states are currently studying the plan and considering ways to put it to work in their schools and colleges.

Trends in Engineering Enrollments

A recent report from the U. S. Office of Education indicates that undergraduate enrollment in the nation's 234 engineering colleges and universities declined 5.4% in the fall of 1959 in the face of an all-time high in general college enrollment. General college enrollments for this academic year show an increase of over 4%. On the other hand, the number of Master's degrees awarded in engineering in the United States increased 16% in 1959 over the preceding academic year. This is substantially higher than most advance estimates. The total number of Master's degrees in engineering awarded in 1959 is 6,761, compared with 5,788 in the year previous.

Bachelor's degrees in engineering totaled 38,162 in the academic year 1958–59, an increase of 10% over the preceding year. Doctor's degrees totaled 713, which is also an increase of 10%.

CBA Chemistry Project

A new chemistry course is being tried out in nine of the nation's high schools this year. The course has been under development by a group of high school and college chemistry teachers working on it for the past year. The present trial has its headquarters at Earlham College, Richmond, Indiana, and is supported by a grant from The National Science Foundation.

A major emphasis of the new course is concern with the atomic and molecular structures of substances. In these structures, the atoms are held together by chemical bonds which arise from the electronic nature of atoms. For this reason, the project is called The Chemical Bond Approach. Considerable attention is given to the relation between the geometrical properties of the underlying structure of a substance and its observable properties. This is proving to be not only a powerful conceptual tool for theoretical chemistry, but also an effective teaching device for introducing students to chemistry.

This year's trial is providing a basis for a revision of classroom and laboratory materials. One hundred teachers will be trained in the new course this coming summer. An expanded trial will be carried on next year.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1426. *Proposed by C. F. Pinzka, University of Cincinnati*

Professor E. P. B. Umbugio is trying to supplement his meager academic salary by entering soap contests. One such contest requires the contestants to find the number of paths in the following array which spell out the word MATHEMATICIAN:

```
                    M
                  M A M
                M A T A M
              M A T H T A M
            M A T H E H T A M
          M A T H E M E H T A M
        M A T H E M A M E H T A M
      M A T H E M A T A M E H T A M
    M A T H E M A T I T A M E H T A M
  M A T H E M A T I C I T A M E H T A M
M A T H E M A T I C I C I T A M E H T A M
M A T H E M A T I C I A I C I T A M E H T A M
M A T H E M A T I C I A N A I C I T A M E H T A M
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Umbugio has counted 1587 paths which originate from one of the first five rows. With the deadline for submitting entries approaching, he is distraught to say the least. Help the Professor out by finding the number of paths with a minimum of computation.

E 1427. *Proposed by F. Leuenberger, Zuoz, Switzerland*

If a_1, a_2, a_3 are the sides and h_1, h_2, h_3 the altitudes of a triangle T, show that

$$\sqrt{3}\sum_{1}^{3} a_i \geqq 2\sum_{1}^{3} h_i,$$

with equality if and only if T is equilateral. (Dedicated to the memory of Victor Thébault.)

E 1428. *Proposed by Aaron Lieberman, The G. C. Dewey Corporation, N. Y.*

Evaluate

$$\prod_{0}^{\infty} [1 + (1/2)^{2^n}].$$

E 1429. *Proposed by J. S. Brock, David Taylor Model Basin, Washington, D. C.*

If

$$f(\theta) = \frac{1 + a^2b - 2ab\cos\theta + 2b\cos 2\theta - a(1+b)\cos 3\theta}{1 + a^2 - 2a\cos 3\theta},$$

where $-1 < a < 1$, show that $\int_0^{\pi} f(\theta)d\theta = \pi$.

E 1430. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development*

What is the highest order of multiplicity a root can have for the equation

$$x(x-1)(x-2)\cdots(x-n+1) = \lambda?$$

SOLUTIONS

An Inequality for the Triangle

E 1272 [1957, 432; 1958, 123]. *Proposed by the late Victor Thébault, Tennie, Sarthe, France*

If A, B, C are the angles of a triangle, show that

$$(\sin A/2 + \sin B/2 + \sin C/2)^2 \leqq \cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2.$$

II. *Solution by Leon Bankoff, Los Angeles, Calif.* If the vertices are suitably named, all plane triangles belong to one or the other of the classes $A \geqq 60° \geqq B \geqq C$ or $C \geqq B \geqq 60° \geqq A$. In either event,

$$(\sin B/2 - 1/2)(\sin C/2 - 1/2) \geqq 0,$$

with the result that

$$4\sin B/2 \sin C/2 \geqq 2(\sin B/2 + \sin C/2) - 1.$$

Consequently,

$$1 + 4\Pi \sin A/2 \geqq 2(\sin A/2)(\sin B/2 + \sin C/2) + (1 - \sin A/2).$$

Now, if angle A remains constant, $\sin B/2 \sin C/2$ reaches a maximum when $B/2 = C/2 = (\pi - A)/4$. Hence

$$\sin B/2 \sin C/2 \leqq \sin^2[(\pi - A)/4] \equiv (1 - \sin A/2)/2.$$

It follows that

$$\Sigma \cos A \equiv 1 + 4\Pi \sin A/2 \geqq 2\Sigma \sin A/2 \sin B/2$$

or $\Sigma \cos^2 A/2 \geqq (\Sigma \sin A/2)^2$.

Editorial Note. Bankoff pointed out that the earlier solution [1958, 123] to this problem is faulty. This earlier solution states that "the minimum value for $\cos^2 A/2+\cos^2 B/2+\cos^2 C/2$ for $A/2+B/2+C/2 \leqq \pi/2$ is attained when $A/2=B/2=C/2$." The counter-example $A=10°$, $B=10°$, $C=160°$ shows this statement to be false; here $\Sigma \cos^2 A/2=2.0149$, whereas with $A=B=C$ we have $\Sigma \cos^2 A/2=2.25$.

Some Properties of the Fibonacci Numbers

E 1396 [1960, 81]. *Proposed by* (1) *Dustan Everman, Arizona State College,* (2) *A. E. Danese, Union College,* (2) *K. Venkannayah, Government College, Mercara, India*

For the Fibonacci sequence $\{f_n\}$, where $f_1=f_2=1$, $f_n=f_{n-1}+f_{n-2}$, $n \geqq 3$, show that

(1) Every fifth number in the sequence is divisible by 5.

(2) $f_{n+h}f_{n+k}-f_nf_{n+h+k}=(-1)^nf_hf_k$. (All the identities in Ganis, *Notes on the Fibonacci sequence,* this MONTHLY, vol. 66, p. 129, are special cases of this.)

(3) $$H(f_1, \cdots, f_{2n-1}) = \begin{vmatrix} f_1 & f_2 & \cdots f_n \\ f_2 & f_3 & \cdots f_{n+1} \\ \vdots & \vdots & \cdots \vdots \\ f_n & f_{n+1} & \cdots f_{2n-1} \end{vmatrix} = 0, \qquad n \geqq 3.$$

Solution by E. M. Scheuer, Space Technology Laboratories, Los Angeles, Calif.

(1) This follows since $f_5=5$ and $f_n=5f_{n-4}+3f_{n-5}$.

(2) Using the well-known formula $f_N=(a^N-b^N)/2^N\sqrt{5}$, where $a=1+\sqrt{5}$ and $b=1-\sqrt{5}$, and noting that $a^Nb^N/2^{2N}=(-1)^N$, one easily establishes the desired identity.

(3) Since the third row is the sum of the first two rows, this is obvious.

Also solved by A. N. Aheart, Alan Beal, J. H. Biggs, A. P. Boblétt, D. A. Breault, Brother Joseph Heisler, J. L. Brown, Jr., P. L. Chessin, F. B. Correia, D. F. Criley, A. E. Danese, T. N. Delmer, C. W. Dodge, Underwood Dudley, S. H. Eisman, Jane Evans, Dustan Everman, Evelyn Frank, Stanley Franklin and Gerald Hutchison (jointly), B. K. Gold, Michael Goldberg, L. D. Goldstone, H. W. Gould, S. H. Greene, Robert Heller, A. F. Horadam, Geoffrey Horrocks, A. R. Hyde, Shigeru Ishii and Joel Levy (jointly), Anne L. Johannsen, John Jordan, C. H. King, M. S. Klamkin, Donald Knuth, P. S. Landweber, Harry Langman, Barry MacKichan, D. C. B. Marsh, Marvin Mielke, E. P. Miles, Jr., John Moon and Ted Petrie (jointly), D. A. Moran, D. L. Muench, Katherine E. O'Brien, C. S. Ogilvy, Walter Penney, J. L. Pietenpol, C. F. Pinzka, Thomas Porsching, Joel Rascoff, B. E. Rhoades, R. M. Rippey, Barbara Sakitt, Benjamin Sapolsky, R. E. Shafer, R. T. Shannon, Vencil Skarda, C. V. L. Smith, Mrs. K. L. Smith, M. F. Stilwell, E. A. Sturley, Anna and R. G. Thompson (jointly), K. Venkannayah, Julius Vogel, Chih-yi Wang, T. West, C. T. Whyburn, Alan Wilson, Dale Woods, A. W. Yonda, and David Zeitlin.

Other proofs are possible. Thus (1) follows from (2) by taking $n=h=5$, $k=5a$, and then using induction. Identity (2) may be established by induction on n with h fixed followed by induction on h. Results (1) and (2) are not new (see Dickson, *History of the Theory of Numbers,* vol. 1, Chap. XVII).

Another Inequality for the Triangle

E 1397 [1960, 82]. *Proposed by Leon Bankoff, Los Angeles, California*

Show that $\Sigma AI \leqq \Sigma AH$, where I is the incenter and H the orthocenter of a triangle ABC.

Solutions by the proposer. I. It has been proved that $(\Sigma \sin A/2)^2 \leqq \Sigma \cos^2 A/2$ (see above solution to Problem E 1272). Hence

$$2\Sigma \sin A/2 \sin B/2 \leqq \Sigma \cos^2 A/2 - \Sigma \sin^2 A/2 = \Sigma \cos A = 4(\Pi \sin A/2) + 1.$$

Then

$$2\Sigma \csc A/2 \leqq 4 + \Pi \csc A/2 = 4 + 4R/r,$$

or

$$\Sigma AI \leqq 2(R + r) \leqq \Sigma AH.$$

II. Let P denote the orthogonal projection of I upon BC, and I_a the excenter relative to the vertex A. D, the midpoint of II_a lies on the circumference. The vertices may be named so that $C \geqq B \geqq 60° \geqq A$ or $A \geqq 60° \geqq B \geqq C$, depending on the species of triangle under consideration. In either case, we have $(BI-2r)(CI-2r) \geqq 0$, or $2r+(BI)(CI)/2r \geqq BI+CI$. Now, in the similar right triangles BIP and II_aC, we have $(BI)(CI)/2r = ID$. Hence $2r+ID \geqq BI+CI$. But $2R-ID \geqq AI$. It follows that $\Sigma AI \leqq 2R+2r \equiv \Sigma AH$, where, if the triangle is obtuse, the segment joining H to the obtuse angle is taken in the negative sense.

Also solved by L. D. Goldstone and D. C. B. Marsh.

Editorial Note. The proposer's Solution I led him to examine critically the earlier proof of E 1272 [1958, 123]. The detection of an oversight in this proof then led him to construct Solution II (which is independent of the result of E 1272) and, finally, in order to validate his Solution I, to the construction of the correct proof of E 1272 given above in this issue. Just as E 1397 can be obtained from E 1272, so can E 1272 be obtained form E 1397.

The Spice of Life

E 1398 [1960, 82]. *Proposed by A. N. Aheart, West Virginia State College*

If A, B, C are the angles of a triangle, show that $\cos A + \cos B + \cos C < 2$.

I. *Solution by W. O. J. Moser, University of Manitoba.* We have $A+B+C=\pi$ and two of the angles, say A and B, acute. Hence

$$\begin{aligned}\cos A + \cos B + \cos C &= 2\cos (A + B)/2 \cos (A - B)/2 + \cos C \\ &\leqq 2\cos (A + B)/2 + \cos C \\ &= 2\cos (\pi - C)/2 + \cos C \\ &= 2\sin C/2 + 1 - 2\sin^2 C/2 \\ &= 3/2 - 2(\sin C/2 - 1/2)^2 \leqq 3/2,\end{aligned}$$

with equality only if $\cos(A-B)/2=1$ and $\sin C/2=1/2$, that is, only when the triangle is equilateral.

II. *Solution by A. R. Hyde, Bronxville, N. Y.* Let $y=\cos A+\cos B+\cos C=\cos A+\cos B-\cos(A+B)$. For any specified B, y is a maximum when $A=(\pi-B)/2$. The locus of these maxima is $y=2\cos(\pi-B)/2+\cos B$, for which, in turn, the maximum occurs when $B=\pi/3$. Hence maximum y is $3\cos(\pi/3)=3/2$, which occurs only when the triangle is equilateral.

III. *Solution by G. E. Forsythe and G. Szegö, Stanford University.* Consider A, B, C as variables in the closed interval $[0, \pi]$ with side condition $A+B+C=\pi$. Let A, B, C be a system for which $\max(\Sigma \cos A)$ is attained. If, say, $A=0$, then $B+C=\pi$ and $\Sigma \cos A=1$; if $A=\pi$, $B=C=0$ and again $\Sigma \cos A=1$. So we may assume that A, B, C are in the open interval $(0, \pi)$. Hence, using Lagrange multipliers, $-\sin A=-\sin B=-\sin C=\lambda$, so that $A=B=C$. (We cannot have $A=\pi-B$, since that would make $C=0$.) It follows that $\Sigma \cos A \leqq 3/2$, with equality only when the triangle is equilateral.

IV. *Solution by Wayne Shepherdson, Modesto Junior College.* One need consider only acute triangles. Set $\alpha=\pi-2A$, $\beta=\pi-2B$, $\gamma=\pi-2C$. In a semicircle of unit radius draw two radii such that three central angles α, β, γ are formed. Label the chords opposite these angles u, v, w, respectively. In the three isosceles triangles thus formed, each angle adjacent to u is A; adjacent to v is B; adjacent to w is C. Then $u=2\cos A$, $v=2\cos B$, $w=2\cos C$. Now $u+v+w$ is a maximum when u, v, w are three consecutive sides of a regular hexagon. It follows that $\Sigma \cos A \leqq 3/2$, with equality only when the triangle is equilateral.

V. *Solution by P. D. Thomas, Coast and Geodetic Survey, Washington, D. C.* On pages 186, 191 of R. A. Johnson, *Modern Geometry*, are found the relations

$$d^2 = R(R-2r) \quad \text{and} \quad \cos A+\cos B+\cos C = 1+r/R,$$

where r, R are the inradius and circumradius and d is the distance between the incenter and the circumcenter. From the first relation it follows that $2r \leqq R$, or $1+r/R \leqq 3/2$. Hence, from the second relation, $\Sigma \cos A \leqq 3/2$, with equality only when $d=0$, in which case the triangle is equilateral.

VI. *Solution by David Zeitlin, Remington Rand Univac.* One need consider only acute triangles. Applying the Erdös-Mordell inequality (this MONTHLY, [1958, 521]) to the center O of the circumscribed circle of radius R of the triangle ABC, we obtain $3R \geqq 2(r_1+r_2+r_3)$, where $r_1=R\cos A$, $r_2=R\cos B$, $r_3=R\cos C$. Thus $\Sigma \cos A \leqq 3/2$, with equality only for $A=B=C$.

VII. *Solution by D. C. B. Marsh, Colorado School of Mines.* Consider the sides of triangle ABC as vectors α, β, γ with vector sum zero and lengths a, b, c. Using the definition of "dot product" (*e.g.*, $\alpha \cdot \beta = -ab\cos C$) one finds

$$\cos A+\cos B+\cos C = (3-|\alpha/a+\beta/b+\gamma/c|^2)/2 \leqq 3/2.$$

Also solved by R. G. Albert, H. D. Arnett, D. W. Bailey, J. W. Baldwin, Leon Bankoff, Edward Barbeau, J. G. Bâron, Alan Beal, Ralph Beals, H. F. Bechtell, E. E. Bosman, Marian Brashears, D. A. Breault, A. M. Broshi, C. K. Brown, J. L. Brown, Jr., W. J. Carpenter, F. B. Correia, R. W. Cottle, Thomas Curry, B. R. Davis, G. DiAntonio, Jane Evans, J. A. Faucher, M. P. Fobes, C. E. Franti, George Glauberman, Michael Goldberg, J. H. Goldman, L. D. Goldstone, R. H. Gramann, S. H. Greene, C. A. Grimm, Emil Grosswald, David Haley, Edward Harris, Geoffrey Horrocks, Shigeru Ishii and Joel Levy (jointly), R. A. Jacobson, John Jordan, Geoffrey Kandall, Irving Katz, J. M. Kingston, Eudon Kirkland, M. S. Klamkin, Donald Knuth, Joseph Konhauser, L. C. Kurtz, M. A. Laframboise, Harry Langman, J. F. Leetch, F. Leuenberger, R. J. Lewyckyj, H. C. Machin, Marvin Mielke, Q. G. Mohammad and M. R. Puri (jointly), C. S. Ogilvy, Walter Penney, W. M. Perel, J. L. Pietenpol, C. F. Pinzka, Bob Ramsey, Joel Rascoff, B. E. Rhoades, R. M. Rippey, Barbara Sakitt, Benjamin Sapolsky, R. E. Shafer, C. M. Sidlo, Vencil Skarda, L. L. Sleizer, Philip Smedley, E. L. Spitznagel, Jr., M. F. Stilwell, D. D. Strebe, Dmitri Thoro, J. A. Todd, J. H. Wahab, Chih-yi Wang, Chung-lie Wang, T. West, Charles Wexler, Bill Wise, Roscoe Woods, C. C. Yalavigi, and the proposer. Late solution by William Becker.

There were many ingenious and short solutions establishing the weaker inequality stated by the proposer. Several solvers located the problem (with the stronger inequality) as Prob. 25, 7th ed. of Hobson's *A Treatise on Plane and Advanced Trigonometry*. Many solvers offered more than one solution. Forsythe and Szegö, in alternative solutions, showed that $\Sigma \cos A = 3/2 - \Sigma(b-c)^2(b+c-a)/2abc$ and $\Sigma \cos A = 3/2 - d^2/2R^2$ (see Solution V for notation).

Counterfeit Coins

E 1399 [1960, 82]. *Proposed by H. S. Shapiro, New York University*

Counterfeit coins all weigh 9 grams and genuine coins all weigh 10 grams. One is given five coins of unknown composition, and an accurate scale (not a balance). How many weighings are needed to isolate the counterfeit coins?

Solution by N. J. Fine, Institute for Advanced Study. We shall show how to determine four coins in three weighings. In the following matrix, a 1 in the ith row and jth column denotes that the jth coin appears in the ith weighing:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Let x_j be the weight of the jth coin, less 9, so that $x_j = 0$ or 1. Thus we have the equations

$$\begin{aligned} x_2 + x_3 + x_4 &= s_1, \\ x_1 + x_3 &= s_2, \\ x_1 + x_2 &= s_3. \end{aligned}$$

Adding, we determine x_4 from the congruence

$$x_4 \equiv s_1 + s_2 + s_3 \pmod 2.$$

We can then solve for x_1, x_2, x_3. To determine five coins in four weighings we need only weigh the fifth coin separately.

It can be proved that the determination of five coins in three weighings is impossible if the weighings are predetermined, that is, if we require a non-sequential solution. It is highly likely that it is impossible even without this restriction.

The ratio 3/4 of weighings to coins can be lowered to 5/7, using the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is an attractive conjecture that this ratio tends to 0 as the number of coins increases.

Also solved by C. W. Dodge, Michael Goldberg, M. S. Klamkin, Donald Knuth, D. C. B. Marsh, C. S. Ogilvy, Barbara Sakitt, and L. L. Sleizer.

Red and Green Balls

E 1400 [1960, 82]. *Proposed by S. D. Pratico, Iona College, New Rochelle, N. Y.*

There are 6 red balls and 8 green balls in a bag. Five balls are drawn at random from the bag and placed in a red box; the remaining 9 balls are put in a green box. What is the probability that the number of red balls in the green box plus the number of green balls in the red box is not a prime number?

Solution by C. W. Trigg, Los Angeles City College. If there are g green balls, $0 \leqq g \leqq 5$, in the red box, then there are $6-(5-g)$ or $g+1$ red balls in the green box, and $1 \leqq 2g+1 \leqq 11$. The only composite number in this span is 9, so the desired probability is $C(8, 4)C(6, 1)/C(14, 5)$ or 30/143, if unity is considered as prime. If unity is considered as not prime, the probability is $30/143+C(8, 0)C(6, 5)/C(14, 5)$ or 213/1001.

Also solved by A. N. Aheart, D. W. Bailey, J. W. Baldwin, Alan Beal, Ralph Beals, H. F. Bechtell, Roy Bertoldo, A. P. Boblétt, Julian Braun, A. W. Brunson, M. L. Cantor, R. W. Cottle, D. F. Criley, Monte Dernham, Underwood Dudley, Jane Evans, G. T. Fox, Charles Franti, B. K. Gold, Michael Goldberg, L. D. Goldstone, S. H. Greene, Emil Grosswald, W. C. Guenther, Geoffrey Horrocks, A. R. Hyde, John Jordan, Donald Knuth, Deena Ann Koniver, Morton Kupperman, W. E. Lawrence, R. J. Lewyekyj, L. I. Lowell, D. C. B. Marsh, Joseph Mayer, R. W. Means, Q. G. Mohammad and M. R. Puri (jointly), C. S. Ogilvy, J. T. Parent, Bart Park, Walter Penney, J. L. Pietenpol, C. F. Pinzka, Joel Raskoff, R. E. Shafer, Vencil Skarda, L. L. Sleizer, Philip Smedley, D. D. Strebe, Arthur Torell, Julius Vogel, J. K. Ward, Chung-lie Wang, W. R. Westphal, Charles Wexler, W. Zayachkowski, David Zeitlin, and the proposer.

Not all these solutions agreed.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

It is with real sorrow that we report the passing of a good friend and prolific contributor to the Problem Departments. His ingenious and challenging results in geometry and number theory have so identified him for many years with these columns that his name is the first to occur to many readers when the Department is spoken of. M. Victor Thébault died on March 19, 1960, at the age of 79, at his home in Tennie, Sarthe, France.

PROBLEMS FOR SOLUTION

4917. *Proposed by L. Lewin, Enfield, Middlesex, England*

Prove

$$\int_0^{\pi/6} \log^2(2\sin x)\,dx = 7\pi^3/216. \tag{1}$$

$$\int_0^{\pi/6} x\log^2(2\sin x)\,dx = 17\pi^4/25920. \tag{2}$$

4918. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let α_k, β_k be complex numbers, $k=1, \cdots, n$, with $|\alpha_k| = |\beta_k| = 1$. Let $0 \leqq \arg \alpha_1 < \cdots < \arg \alpha_n < 2\pi$, $0 \leqq \arg \beta_1 < \cdots < \arg \beta_n < 2\pi$, where $0 \leqq \arg \gamma < 2\pi$ for any nonzero complex number γ. Also let $\sum_1^n \alpha_k = \sum_1^n \beta_k = 0$. Prove that $\sum_1^n \alpha_k\bar{\beta}_k \neq 0$.

4919. *Proposed by J. L. Ullman and C. J. Titus, University of Michigan*

Let $\phi(\theta)$ be real valued for $0 \leqq \theta \leqq 2\pi$, let $\phi'(\theta)$ be continuous and positive, let $\phi(0) = 0$, $\phi(2\pi) = 2\pi$ and let $\int_0^{2\pi} e^{i\phi(\theta)}d\theta = 0$. Prove that

$$\left| \int_0^{2\pi} e^{i(\phi(\theta)-\theta)}d\theta \right| > 4.$$

4920. *Proposed by Smbat Abian, University of Pennsylvania*

An Ab-integration in a ring A is a mapping $a \to a^*$ of A into itself satisfying the following conditions:

$$(a + b)^* = a^* + b^*, \qquad (a^*b + ab^*)^* = a^*b^* \qquad \text{for any pair } a, b \in A.$$

Is there a nontrivial Ab-integration in the field of real numbers?

4921. *Proposed by David Gale, Brown University*

Prove that the numbers $w_1, \cdots, w_8$ can be the numbers of games won by the eight teams at the end of a major league baseball season if and only if

$$\sum_{i=1}^{8} w_i = 616, \qquad \sum_{i=1}^{k} w_i \leqq 11k(15 - k), \qquad k = 1, \cdots, 7.$$

(During a major league season every two teams play each other 22 times.)

4922. *Proposed by Peter Crawley, California Institute of Technology*

In the solution of a recent problem (4761 [1959, 67]) it was shown that the additive group of rationals has no maximal subgroups. Prove the generalization: An Abelian group G has no maximal subgroups if and only if G is divisible (*i.e.*, $nG = G$ for all integers n).

SOLUTIONS

A Coin Tossing Probability

4864 [1959, 728]. *Proposed by D. J. Newman, Brown University*

Let a coin be tossed repeatedly, and let p be the probability that at some point the number of heads exceeds twice the number of tails. Prove

$$p = \tfrac{1}{2}(\sqrt{5} - 1). \tag{1}$$

I. *Solution by J. S. Frame, Michigan State University.* More generally, given the integers $a > b > 0$, let $f(t)$ be the probability that at some time in an infinite succession of coin tosses the number of heads exceeds a/b times the number of tails by more than t/b. If $z_1, \cdots, z_b$* denote the roots z such that

$$z^{-b} + z^a = 2 \qquad |z| < 1, \tag{2}$$

and if h_t denotes the sum of the $\binom{b+t}{t}$ distinct products of t factors with repetitions allowed, formed from the $b+1$ quantities $z_0 = 1, z_1 \cdots, z_b$, we shall prove that

$$p = f(0) = 1 - \prod_{i=1}^{b} (1 - z_i), \qquad f(t) = 1 - h_t(1 - p), \qquad t = 0, 1, 2, \cdots. \tag{3}$$

* The fact that the roots of (2) which are inside the unit circle are precisely b in number can be shown without difficulty by use of Rouchè's theorem.

Clearly for $a=2$, $b=1$, $t=0$, we shall then have

$$p^{-1}+p^2=2, \quad \text{or} \quad p^3-2p+1=(p-1)(p^2+p-1)=0, \tag{4}$$

and the given special case will have the solution $p=(\sqrt{5}-1)/2$.

Proof. The probability $f(t)$ is changed to $f(t-b)$ after a throw of heads, and to $f(t+a)$ after a throw of tails. Hence

$$f(t)=\tfrac{1}{2}[f(t-b)+f(t+a)]. \tag{5}$$

This difference equation of degree $a+b$ has $a+b$ linearly independent solutions of the form z_i^t corresponding to the $a+b$ roots of equation (2). However, since the probability $f(t)$ tends to 0 as t becomes positively infinite, only those solutions for which $|z_i|<1$ can be used. We write

$$f(t)=\sum_{i=1}^{b} c_i z_i^{b+t}, \qquad t \geqq -b; \qquad f(0)=p. \tag{6}$$

and determine the constants c_i by the boundary conditions

$$f(-1)=f(-2)=\cdots=f(-b)=1, \tag{7}$$

since for t negative the inequality is satisfied at the start. Elimination of the c_i from equations (6) and (7) gives the determinantal equation

$$\begin{vmatrix} f(t) & z_1^{b+t} & \cdots & z_b^{b+t} \\ 1 & z_1^{b-1} & \cdots & z_b^{b-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & z_1 & \cdots & z_b \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 0. \tag{8}$$

For $t=0$, $p=f(0)$, equation (8) expresses $1-p$ as the ratio of two Vandermonde determinants

$$1-p=V(1, z_1, z_2 \cdots z_b)/V(z_1, z_2, \cdots z_b)=\prod_{i=1}^{b}(1-z_i). \tag{9}$$

For $t=1, 2, \cdots$ equation (8) can be changed to

$$\frac{1-f(t)}{1-f(0)}=\begin{vmatrix} 1 & z_1^{b+t} & \cdots & z_b^{b+t} \\ 1 & z_1^{b-1} & \cdots & z_b^{b-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & z_1 & & z_b \\ 1 & 1 & & 1 \end{vmatrix} \div \begin{vmatrix} 1 & z_1^{b} & \cdots & z_b^{b} \\ 1 & z_1^{b-1} & \cdots & z_b^{b-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & z_1 & \cdots & z_b \\ 1 & 1 & \cdots & 1 \end{vmatrix} = h_t. \tag{10}$$

This function h_t is known from determinant theory to be the sum of the distinct products of t factors formed (with repetitions allowed) from $1, z_1, \cdots z_b$. This proves equation (3) and solves the problem.

It is interesting to note [see this MONTHLY, vol. 64, 1957, p. 238] that

powers of the b roots z_i which we have used are given by the convergent series

$$(11) \qquad z^q = w^q\left(1 + \sum_{k=1}^{\infty} \frac{q}{q+ks}\binom{(q+ks)/b}{k} w^{ks}\right), \qquad s = a + b, \qquad w^b = 1/2,$$

where the different values of w give the b different roots z.

II. *Solution by W. A. O'N. Waugh, McGill University.* Consider a particle which performs a random walk in accordance with the results of the coin-tossing game, by moving two places to the right for a tail and one place to the left for a head. If it starts from position $+1$, it reaches or goes to the left of the origin when the number of heads exceeds twice the number of tails. In this form the problem is a special case of an exercise in W. Feller, *An Introduction to Probability Theory and Its Applications* (2nd ed.) 1957, Ch. XIV, q. 3.

For a solution, let q_x be the probability of ultimate absorption at the origin if the particle starts from a position $x>0$. Then, with the convention that $q_0=1$, we have

$$q_x = \tfrac{1}{2}q_{x+2} + \tfrac{1}{2}q_{x-1} \qquad x = 1, 2, \cdots.$$

A trial solution $q_x=\lambda^x$ leads to $\lambda=1$ or $\frac{1}{2}(\pm\sqrt{5}-1)$. Obvious conditions on the solution are $q_0=1$, and $q_x\to 0$ as $x\to\infty$, and these give $q_x=2^{-x}(\sqrt{5}-1)^x$, whence the solution to the given problem is $q_1=\frac{1}{2}(\sqrt{5}-1)$.

Also solved by George Glauberman, Roger Pinkham, D. C. Stevens, and one whose name is missing.

The Inverse Tangent Integral

4865 [1959, 728]. *Proposed by L. Lewin, Enfield, England*

Defining the inverse tangent integral of the second order by

$$Ti_2(x) = \int_0^x \frac{\tan^{-1}(x)}{x}\,dx = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2} \cdots,$$

prove that $6Ti_2(1) - 4Ti_2(1/2) - 2Ti_2(1/3) - Ti_2(3/4) = \pi \log 2$.

Solution by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass. The proposer has shown that

$$Ti_2\left(\frac{y^2}{2}\right) + \frac{1}{2}Ti_2\left\{\frac{y(2+y)}{2(1+y)}\right\} + \frac{1}{2}Ti_2\left\{\frac{2(y-1)}{y(2-y)}\right\} + Ti_2\left\{\frac{y}{2+y}\right\}$$
$$- Ti_2\left(\frac{y}{2-y}\right) + Ti_2\left(\frac{1}{1+y}\right) - Ti_2(y-1) = 2Ti_2(1) + \frac{\pi}{4}\log\left\{\frac{y(2-y)}{2(1+y)}\right\}.$$

(See L. Lewin, *Dilogarithms and Associated Functions*, London, 1958, p. 37). The desired result follows upon letting $y=1$.

Editorial Note. The problem and the proposer's solution were received before the publication of the book. A variety of similar formulas may be obtained, *e.g.*

$$Ti_2(7/24) + 2Ti_2(1/7) + 6Ti_2(1/3) - 8Ti_2(\tfrac{1}{2}) + Ti_2(3/4) + (\pi/2)\log(3/2) = 0.$$

Semi-magic Matrices

4866 [1959, 729]. *Proposed by F. D. Parker, University of Alaska*

Show that a semi-magic matrix A (the sums of the rows and columns are all equal) can be decomposed into a sum $B+C$ such that for integral K,

$$(B + C)^K = B^K + C^K.$$

Solution by Leonard Carlitz, Duke University. Setting aside the trivial solution $B=A$, $C=0$, let $A=(a_{rs})$ with $\sum_r a_{rs} = \sum_s a_{rs} = \lambda$, $r, s = 1, 2, \cdots, n$; and let L be the n-rowed square matrix with every element unity. We get $AL = LA = \lambda L$, $L^2 = nL$. Now let

$$B = \frac{\lambda}{n} L, \qquad C = A - \frac{\lambda}{n} L.$$

Then $BC = (\lambda/n)LA - (\lambda^2/n^2)L^2 = (\lambda/n)\lambda L - (\lambda^2/n^2)nL = 0$. Similarly $CB = 0$. Hence by an easy induction

$$(*) \qquad A^k = B^k + C^k \qquad (k \geqq 1).$$

Note 1. The condition $k \geqq 1$ is necessary. For if (*) holds with $k = -1$, we get $(B+C)(B^{-1}+C^{-1}) = I = (B^{-1}+C^{-1})(B+C)$, so that $I + BC^{-1} + CB^{-1} = 0$, and $I + C^{-1}B + B^{-1}C = 0$, and so $2I + (BC^{-1} + C^{-1}B) + (CB^{-1} + B^{-1}C) = 0$.

On the other hand, (*) with $k = 2$ yields $BC + CB = 0$, so that $B^{-1}C + CB^{-1} = 0$, $C^{-1}B + BC^{-1} = 0$ and therefore $2I = 0$.

Note 2. The above construction yields the trivial solution if and only if $\lambda = 0$ or A is a constant multiple of L. It is not certain that in these cases a nontrivial solution exists.

Also solved by Gerald Goertzel, R. D. Gordon, and the proposer.

An Integral Inequality

4867 [1959, 729]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let f be a measurable function such that $0 < f(x) \leqq 1/2$ on $[0, 1]$. Prove

$$\frac{\exp \int_0^1 \log f(x)\,dx}{\int_0^1 f(x)\,dx} \leqq \frac{\exp \int_0^1 \log\{1 - f(x)\}\,dx}{\int_0^1 \{1 - f(x)\}\,dx}.$$

Solution by Tadeusz Stanisz, Jagiello University, Krakow, Poland. The inequality to be proved may be written in the form

$$(*) \qquad \log\left[\frac{\int_0^1 \{1 - f(x)\}\,dx}{\int_0^1 f(x)\,dx}\right] \leqq \int_0^1 \log\left[\frac{1 - f(x)}{f(x)}\right] dx.$$

Since the function $\phi(u) = \log\{(1-u)/u\}$ is convex for $0 < u \leqq \frac{1}{2}$ (because $\phi''(u) \geqq 0$ for $0 < u \leqq \frac{1}{2}$) the inequality (*) is an immediate consequence of a known form of Jensen's inequality. See, *e.g.*, Pólya u. Szegö, *Aufgaben und Lehrsätze*, II, 77.

Also solved by George Glauberman, and the proposer.

Editorial Note. The proposer's solution makes use of backward induction (see Hardy-Littlewood-Pólya, *Inequalities*, p. 20) and illustrates one of the rare situations where this method is naturally applicable.

A Special Infinite Series

4868 [1959, 729]. *Proposed by Ronald Pyke, Columbia University*

Show that for all $|\beta| \leqq e^{-1}$ and for all real x the function f defined by $f(0, \beta) = 1$ and

$$f(x, \beta) = x \sum_{j=0}^{\infty} \beta^j \frac{(j + x)^{j-1}}{j!} \qquad (x \neq 0)$$

is finite and satisfies the relationship

$$f(x, \beta) = e^{x\beta f(1,\beta)}.$$

Solution by Robert Weinstock, Oberlin College. We define

$$F(x, \beta) = \sum_{j=0}^{\infty} \beta^j \frac{(j + x)^j}{j!} \qquad (|\beta| < e^{-1}),$$

from which we obtain directly

$$f(x, \beta) = F(x, \beta) - \beta F(x + 1, \beta). \tag{1}$$

But (*e.g.*, Pólya-Szegö: *Aufgaben und Lehrsätze*, III, 214)

$$F(u, \beta) = \frac{e^{uz}}{1 - z} \qquad (|\beta| < e^{-1}),$$

where z is a certain solution of the equation $z = \beta e^z$. Thus, from (1),

$$f(x, \beta) = \frac{e^{xz}}{1 - z}(1 - \beta e^z) = e^{xz}.$$

Since, therefore $\beta f(1, \beta) = \beta e^z = z$, the desired result follows.

Also solved by S. H. Greene, M. S. Klamkin, Y. Matsuoka, and the proposer.

Editorial Note. The result has been noted before. Solvers supplied the following references: Bromwich, *Infinite Series*, p. 160, ex. 4; problems 4776 [1958, 783], 4489 [1953, 485].

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Theory of Relativity. By W. Pauli, Pergamon Press, New York, 1959. xiv+241 pp. $6.00.

This is an English translation (G. Field) of the article "Relativitäts Theorie" in *Encyklopädie der Mathematischen Wissenschaften*, vol. V19, which first appeared in 1921. The original monograph has been supplemented by 25 pages of notes indicating later developments. Physical applications are kept to the fore, and the mathematical treatment is mature. Not suitable for a text (except in graduate seminars), this is an indispensable document for the physics bookshelf.

ARTHUR BERNHART
The University of Oklahoma

Advanced Complex Calculus. By Kenneth S. Miller. Harper, New York, 1960. 235 pp. $5.75.

This text is for a first course in the theory of functions of a complex variable for mathematics majors or engineers, and is written as a sequel to the author's *Advanced Real Calculus* (reviewed in this MONTHLY, vol. 64, 1957, p. 519). Titles of the chapters are: (1) Numbers and Convergence, (2) Topological Preliminaries, (3) Functions of a Complex Variable, (4) Contour Integrals, (5) Sequences and Series, (6) Calculus of Residues, (7) Some Properties of Analytic Functions, (8) Conformal Mapping, (9) Method of Laplace Integrals.

The first eight chapters furnish a compact coverage of the usual subject matter; the main theorems are carefully stated and proved, and are well arranged and indexed. Well-chosen, but not very numerous, exercises occur at each chapter end; they are more frequently extensions than illustrations of the text. One helpful feature of the book is the discussion of real multi-valued functions preliminary to the complex case. A less promising innovation is the last chapter. It appears that space of 14 pages is inadequate to discuss the solution of linear differential equations by contour integrals.

In general the exposition seems aimed at brevity rather than the large view. Many interrelations are left to the instructor to bring out, and he will need additional illustrations and exercises. Either as a reference book or text this volume should be a useful addition to the literature in this field.

E. H. CUTLER
Lehigh University

Real Analysis. By Edward James McShane and Truman Botts. Van Nostrand, Princeton, N. J., 1959. ix+272 pp. $6.60.

This excellent book presents, from the modern point of view, a large body of the fundamental theorems of real function theory. It contains eight chapters and three appendices. Chapter 0 is introductory and Chapter I introduces the real number system as a complete ordered field and ends with a rather nice discussion of various forms of the maximality principle. Chapter II starts immediately with the notion of directed convergence which plays a fundamental and unifying role throughout the book. Topological spaces are introduced here and are carried over to the next chapter where continuity is nicely treated. The next three chapters are centered mainly in the general Euclidean space and treat, among other things, functions of bounded variation, absolute continuity, differentiation, and integration. Of the two chapters on integration, the first introduces the Lebesgue-Stieltjes integral (via the Daniell-Darboux method) and concludes with the Riemann-Stieltjes integral. The second of these chapters presents the basic decomposition and substitution theorems. Concerning these two chapters, the reviewer feels that some improvement could have been made in pedagogy and selection; however, this is a matter of taste. The last chapter is devoted to linear spaces—chiefly the L_p spaces, and is, in fact, a good geometrical introduction to linear space theory. The treatment of Hilbert space is quite nice, especially the last section which presents the simultaneous spectral resolution of a finite number of commuting hermetian operators. Three appendices treat inductive definitions, six equivalent formulations of the maximality principle, and Tychonoff's Theorem (the latter is *not* presented as an equivalent formulation of the maximality principle). On the whole, the book is comprehensive, well written, and would make a good text for a solid course in analysis.

PASQUALE PORCELLI
Louisiana State University

Logic In Elementary Mathematics. By Robert M. Exner and Myron F. Rosskopf. McGraw-Hill, New York, 1959. ix+274 pages. $6.75.

Fundamentally, this is a textbook in symbolic logic covering the usual topics; the statement calculus, the restricted predicate calculus and proof and demonstration. In particular, the book uses examples from mathematics and illustrates the applications of logic in mathematics. The result is a book of considerable value for advanced undergraduates and graduate students. In a number of respects, the development is, in my opinion, too difficult for anyone not at least partially familiar with the concepts. There is some confusion in the discussion of the difference between names and names of names (pp. 12–13).

LOUIS O. KATTSOFF
Boston College

An Introduction to Mathematics. By Donald W. Western and Vincent H. Haag. Holt, Rinehart and Winston, New York, 1959. xi+580 pp. $7.50.

Writing "new-approach" books for freshmen has apparently become as compelling as writing calculus texts. The motivation for this activity seems clear enough: one adopts a text and another gets written as an outgrowth of notes prepared to explain the one adopted. This seems a reasonable way for new texts to emerge, and the result in this instance is reasonable, though we can well expect that someone—by the same inductive process—will produce another after adopting this.

The chapter titles pretty well tell the story of what the authors think freshmen should sample from, and the order in which the sampling should occur: Logic, The Natural Numbers, The Complex Number System, Extensions of the Logic of Algebra, Numerical Calculation, Algebra of Sets, Analytic Geometry and Functions, The Probability Function, Circular Functions, Lines and Vectors, Conic Sections, Limits of Sequences and the Definite Integral, Limits of Functions and the Derivative, The Calculus, Curve Tracing, Matrices, Analytic Geometry of Three Dimensions.

The presentation is generally good and at times novel. This remark, along with the table of contents, is meant to indicate that this is a fairly well-written, all-things-to-all-people textbook. It clearly merits consideration by anyone teaching modern freshman courses.

There are the usual displays of typographical errors and obvious omissions found in first printings. The detection and correction of these can be assigned as a valid class exercise.

Additionally, however, one finds things which will cause the student to wonder whether the authors were serious in their preface remark to the effect that an understanding of logic is a necessary tool for mathematics. As examples (by no means a complete list) of what occurs, one concludes from an example on page 26 that "Pope John is President of the United States" is a *meaningless* statement. An unfortunate example on page 53 indicates that each multiple of 2 is a multiple of 6, and from an example following a discussion of what it means to factor a polynomial, one can deduce that $x+2$ is a polynomial of degree less than one (pages 96–97). From the definition of a function (page 182) and an explanatory example (page 183), it follows that some week of December, 1959, is blessed with 31 days. Page 233 contains a discussion of combinations and tells us that "Two *combinations* are *different* if all their elements are not the same," and page 271 presents a delightful example of how logical reasoning aids mathematical study: "Many geometric and physical problems involve triangles, and every triangle can be considered as a pair of right triangles obtained by erecting a suitable perpendicular from a vertex to the opposite side of the triangle. Hence, we shall focus attention on right triangles."

Surely there is a great deal to say for the increasing practice of presenting mathematics with some unity and some attention to logical detail. In doing

this, however, is it really necessary to exclude or hide some important notions? (The idea of a prime and the unique factorization theorem are left as a library exercise.) And is it necessary that easy things become complicated? (The text uses induction to prove that $n(n+1)$ is even.)

It will, then, be necessary for the teacher to prepare complementary notes for use with this as a text. We can expect that this will result in a really good, corrected second edition as well as other texts by other authors.

ROBERT J. WISNER
Haverford College and
Institute for Advanced Study

Introduction to the Laplace Transform. By Dio L. Holl, Clair G. Maple and Bernard Vinograde. Appleton-Century-Crofts, New York, 1959. viii+174 pp. $4.25.

This material has been prepared as a text for a three-hour course on the quarter system. It is designed to follow a first course in differential equations and is more complete than the one-chapter or appendix coverage usually appearing in an engineering mathematics book. The theory is kept fairly elementary by restricting the transform variable to the reals. The theorems and proofs are carefully worded. Some harder theorems borrow results from the advanced calculus. References are given where proofs are omitted. The usual material is well covered. There is a good formal treatment of impulse functions and superposition theorems for the responses to a unit function input or to a unit impulsive function input. The applications are interesting and clearly discussed. References are given for the derivations of the differential equations involved. The second chapter of applications uses partial differential equations. The final chapter extends some theorems so that they hold for some functions having infinite discontinuities.

The problems appear to be well selected. Seven pages of transforms appear in the appendixes. It looks like a good text.

EARL LAFON
University of Oklahoma

Introduction to Probability and Statistics. By B. W. Lindgren and G. W. McElrath. Macmillan, New York, 1959. xiii+277 pp. $6.25.

Introduction to Probability and Statistics emphasizes ideas in statistics and statistical methods and their understanding. It is neither a cook-book of statistical recipes nor a text on mathematical statistics. The book provides an interesting, stimulating, and refreshing approach to the first course in statistics. Probabilities are considered as mathematical models, concepts of samples and populations are carefully discussed, elementary presentations of theories of testing hypotheses and estimation are accurate and readable.

Topics in this book are well organized and tied together without frequent

annoying back references. It is evident that the authors have given much thought to the presentation of most of the fundamental ideas and methods of statistics to the beginning student. Arguments are often intuitive and heuristic but convincing without being verbose. Emphasis in discussing statistical methods is placed on assumptions necessary for their valid applications. The usual topics are covered and a large number of procedures are included that are not usual in the introductory book. Subject matter includes basic probability, distributions, sampling, estimation and testing, tests on location and dispersion, introductory analysis of variance, regression, nonparametric methods, quality control and acceptance sampling, sequential analysis, and decision theory. Examples and problems are well chosen and graduated and relate to realistic applications. Beginning chapters have review problems at the end.

Introduction to Probability and Statistics should be considered seriously for the first course in statistics for the undergraduate. It is particularly suited for students of the physical sciences. Beginning calculus would assist the student but should not be a prerequisite. Most of the material could be covered in a one-semester course. A course based on this book should interest more students in the advanced study of statistics and this is needed.

RALPH A. BRADLEY,
The Florida State University

Introduction to Mathematical Statistics. By Robert V. Hogg and Allen T. Craig. Macmillan, New York, 1959. ix+245 pp. $6.75.

This new text is intended for a two-semester first course in mathematical statistics.

The authors state that their experience with a preliminary version indicates that it is desirable for the student to have some work beyond a first course in calculus, but that the content of this additional mathematical training is unimportant. Since elementary calculus will not suffice to derive much statistical theory, more advanced results are stated and used as appropriate. These results are given without proof save for about six pages devoted to set theory and set functions. The latter notions are exploited throughout the book.

Most of the "standard" topics of a first course are covered and there is no need to reproduce the table of contents but the following selection of items may indicate the "flavor" of the book. There is a good chapter on finding the distribution of a function of one or more random variables, including situations where the function is not one-to-one. The material on estimation is very good for a book at this level. Among the topics in this area one finds a discussion of sufficient statistics, complete classes of probability density functions, uniqueness, and the Koopman-Pitman class of probability density functions. Limiting distributions and limiting moment-generating functions are broached. Distributions of order statistics and functions thereof are discussed, the probability integral transform theorem is proved, confidence intervals for quantiles using order statistics are evaluated, and tolerance limits are introduced.

A good set of problems follows each section. Rather laconic tables of the normal, chi-square, "t," and "F" distributions are appended.

This is a carefully written text. The reviewer feels that if will prove successful in the classroom and that it can also be used profitably for independent study.

ERNEST M. SCHEUER
Space Technology Laboratories, Inc.

BRIEF MENTION

Homology Theory on Algebraic Varieties. By Dr. A. H. Wallace. Pergamon Press, New York, 1958. viii+115 pp. $5.50.

This book will be reviewed in the Spring 1960 *Bulletin of the American Mathematical Society* by Professor Stewart S. Cairns.

University Mathematics, (2nd ed.). By Joseph Blakey. Philosophical Library, New York, 1959. 581 pp. $10.00.

Not a book which will meet the current accepted American standards, either of rigor or teachability, but still a possible source for the tricky problems which apparently are still common on English degree-examination papers.

A Treatise on Algebraic Plane Curves. By Julian Lowell Coolidge. Dover, New York, 1959. xxiv+510 pp. $2.45.

A reprint of Coolidge's work written for Oxford Press, apparently about 1929.

The Theory of Numbers and Diophantine Analysis. By Robert D. Carmichael. Dover, New York, 1959. 118 pp. $1.35.

Carmichael's two books (1914 and 1915) bound as one volume.

Algebraic Theories. By Leonard E. Dickson. Dover, New York, 1959. ix+276 pp. $1.50.

This reprint of Dickson's *Modern Algebraic Theories*, first published in 1926, still contains much of interest. Our thanks to the publisher for very appropriately dropping the first word of its former title. The main concepts discussed center around matrices, invariance, and groups.

An Elementary Treatise on Fourier's Series. By William Elwood Byerly. Dover, New York, 1959. ix+287 pp. $1.75.

Byerly's 1893 treatise still may have historical interest for students of the history of mathematics. These students should also be reminded of the very excellent account of Fourier's series available as the first Slaught Paper, *Fourier's series: A genesis and evolution of a theory*, by R. E. Langer.

Handbook of Automation, Computation and Control, Vol. II. Edited by Eugene M. Grabbe, Simon Ramo and Dean E. Wooldridge. Wiley, New York, 1959. xxiii+1070 pp. $17.50.

This extensive volume is devoted to computers and data processing. Both digital and analog computers, their use and construction, are discussed by a staff of specialists.

Journal of the Association for Computing Machinery. The Association for Computing Machinery, 2 E. 63rd St., New York 21, N. Y. Quarterly, $6 per year, including ACM membership.

An increasing amount of applied mathematics is finding its way into this journal, and hence we call its attention to our readers.

Numerical Methods for Nuclear Reactor Calculations. By G. I. Marchuk. Consultants Bureau, Inc., New York, 1959. 295 pp. $60.00.

Although the price of this paperback, lithoprinted volume will put it beyond the price range for most, still it is interesting to note how much mathematics including Jacobians and partial differential equations is included in this Russian translation.

Quantum Chemistry Integrals and Tables. By J. Miller, J. M. Gerhauser and F. A. Matsen. University of Texas Press, Austin, 1959. 1221 pp. $15.00.

Designed to be used in Quantum Mechanical calculations involving modern internally programmed computers. These tables, computed and printed directly from IBM 650 records, should prove a valuable adjunct to any library.

A Table of the Incomplete Elliptic Integral of the Third Kind. By R. G. Selfridge and J. E. Maxfield. Dover, New York, 1959. xiv+805 pp. $7.50.

A set of 6-place tables prepared by the Research Department U. S. Naval Ordnance Test Station on their IBM 704 computer, carefully verified both by echo-checking and by means of a check integral at the completion of each value of a. Libraries will be well advised to obtain this set of tables while it is still available.

Sampling Inspection Tables (2nd ed.). By H. F. Dodge and H. G. Romig. Wiley, New York, 1959. xi+224 pp. $8.00.

In addition to the sampling tables, a collection of "probability-of-acceptance" curves is included. The table portion appears identical with a similar work published in 1944 by these authors.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., *University of Buffalo*

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Eckford Cohen, University of Tennessee, represented the Association at the inauguration of Dr. A. D. Holt as President of the University of Tennessee on May 13 and 14, 1960.

Professor H. B. Curry of the Pennsylvania State University mathematics department was honored by being named Evan Pugh Research Professor, effective February 1, 1960. These professorships, named after the first president of the University, were established by the Board of Trustees to recognize outstanding research by members of the faculty, and to provide support for the continuation of this research.

Professor William Feller, Princeton University, and Professor Antoni Zygmund, University of Chicago, have been elected members of the National Academy of Sciences.

Professor Magnus Hestenes, University of California, Los Angeles, was awarded an honorary Doctor of Science Degree by Wartburg College on June 7, 1960.

Mr. Stephen Lichtenbaum, Harvard University, has been awarded the William Lowell Putnum Prize Scholarship for the twentieth competition.

Dr. R. F. Rinehart, Director of Special Research and Operations Research at Duke University, was awarded the honorary degree of Doctor of Science by Wittenberg University on June 6, 1960.

Millsaps College: Professor T. L. Reynolds has been appointed Professor and Acting Head of the Department of Mathematics at William and Mary College; Professor S. R. Knox has been granted a sabbatical leave for the academic year 1960–61 to do advanced graduate work in Mathematical Statistics at Virginia Polytechnic Institute.

Western Michigan University: Assistant Professor Stanislaw Leja has been promoted to Associate Professor; Mr. J. D. Vollmer, Wayne State University, has been appointed Instructor.

University of Arizona: Drs. H. M. Lieberstein, Army Research Center, University of Wisconsin and Paul Slepian, Hughes Research Laboratories, Culver City, California, and Assistant Professor Berthold Schweizer, University of Southern California, have been appointed Associate Professors; Drs. J. D. Brooks, University of Southern California, and D. E. Myers, Millikin University, have been appointed Assistant Professors.

University of North Carolina: Associate Professor W. R. Mann has been promoted to Professor and will be on leave at the University of California, Berkeley, on a National Science Foundation Faculty Fellowship for the year 1960–61; Dr. C. W. Patty, University of Georgia, has been appointed Research Instructor; Assistant Professor R. L. Davis, University of Virginia, has been appointed Assistant Professor; Professor Rafael Artzy, Israel Institute of Technology, Haifa, Israel, has been appointed Associate Professor; Professor R. H. Sorgenfrey, University of California, Los Angeles, has been appointed Visiting Professor for the fall semester, 1960; Professor M. E. Shanks, Purdue University, has been appointed Visiting Professor for the spring semester, 1961; Assistant Professor J. H. Wells will be on leave at the University of California, Berkeley, for the year 1960–61.

University of South Carolina: Drs. E. E. Enochs, University of Chicago, and J. L. Boal, Massachusetts Institute of Technology, have been appointed Assistant Professors; Assistant Professor H. T. LaBorde, University of Cincinnati, has been appointed Associate Professor.

Mr. R. E. Blewster Jr., CONVAIR, Fort Worth, Texas, has accepted a position as Mathematician with the Air Force Flight Test Center, Edwards Air Force Base, California.

Mr. D. W. Bowan, Millikin University, has accepted a position as Assistant Research Mathematician at the University of Dayton Research Institute, Data Processing Division.

Mr. Juliun Braun, Seattle, Washington, has accepted a position as Electronic Programming Supervisor with the Chrysler Corporation, Missile Division, Detroit, Michigan.

Professor C. F. Brumfiel, Ball State Teacher's College, has been appointed Associate Professor at the University of Michigan.

Assistant Professor R. J. Buehler, Iowa State University, has been promoted to Associate Professor.

Mr. J. F. Burke, University of Maine, has been appointed Assistant Professor at Norwich University.

Dr. Harold Chatland, Montana State University, has accepted a position as Engi-

neering Specialist in the Electronic Defense Laboratory of Sylvania Electric Products, Mountain View, California.

Associate Professor J. G. Christiano, Northern Illinois University, has been promoted to Professor.

Mr. C. L. Conner, University of Colorado, has accepted a position as Systems Analyst with the Radio Corporation of America, Moorestown, New Jersey.

Mr. T. E. Depkovich has accepted a position as Technical Specialist—Dynamics with the Aerojet General Corporation, Sacramento, California.

Mr. D. C. Fried, General Electric Company, Phoenix, Arizona, has accepted a position as Scientist with Technical Operations, Incorporated, Washington, D. C.

Dr. Seymour Geisser has returned to his position as Mathematician with the National Institute of Health, Washington, D. C., after having served as Visiting Associate Professor at Iowa State University during the spring semester, 1960.

Assistant Professor John Hilzman, Harpur College, has been appointed Assistant Professor at Idaho State College.

Professor B. W. Jones, University of Colorado, has returned from his leave at the University of Puerto Rico.

Mr. Irving Kay, Burroughs Research Center, Paoli, Pennsylvania, has accepted a position as Project Leader with the International Electric Corporation, Paramus, New Jersey.

Mr. S. M. Keathley, Chance Vought Aircraft Company, Dallas, Texas, has accepted a position as Senior Numerical Analyst with Pratt & Whitney Aircraft, West Palm Beach, Florida.

Assistant Professor L. H. Lange, University of Notre Dame, has been appointed Assistant Professor at San Jose State College.

Associate Professor Anne L. Lewis, University of North Carolina, Woman's College, has been promoted to Professor and appointed Head of the Mathematics Department.

Mr. I. J. Lieberman, Philco Corporation, Philadelphia, Pennsylvania, has accepted a position as Manager of the Digital Systems Laboratory with the Kearfott Company, Clifton, New Jersey.

Dr. D. O. McKay, University of Buffalo, has been appointed Instructor at the University of Western Ontario, London, Ontario, Canada.

Associate Professor R. C. Meacham, University of Florida, has been appointed Professor at Florida Presbyterian College.

Mr. Kenneth Moosman, Dayton Air Force Depot, Dayton, Ohio, has accepted a position as Senior Mathematical Analyst with the Lockhead Missiles & Space Division, Sunnyvale, California.

Mr. R. F. Norris, Arizona State University, has accepted a position as Associate Physicist with the International Business Machines Research Laboratories, San Jose, California.

Associate Professor Ingram Olkin, Michigan State University, has been appointed Associate Professor at the University of Minnesota.

Associate Professor G. W. Patterson, University of Pennsylvania, has been appointed Vice-Chairman of the Electronic Computers Committee of the Institute of Radio Engineers.

Mr. Sylvester Reese, University of Wisconsin, has accepted a position as Research Engineer with the Autonetics Division of North American Aviation, Incorporated, Downey, California.

Associate Professor Charles Riggs, Texas Technological College, has been promoted to Professor.

Dr. A. S. Said, Columbia University, has accepted a position as Senior Chemist with Beckman Instruments, Incorporated, Fullerton, California.

Associate Professor Charles Saltzer, Case Institute of Technology, has been appointed Professor at the University of Cincinnati.

Dr. Harold Shulman, Institute of Mathematical Sciences, New York University, has accepted a position as Consultant with the Service Bureau Corporation, Brooklyn, New York.

Mr. C. D. Sise has accepted a position as Analyst in the Machine Computation Laboratory of the United Aircraft Corporation Research Laboratory, East Hartford, Connecticut.

Sister Mary Bonaventure, Immaculate Conception Junior College, has been appointed Teacher of Mathematics and Sciences at the St. Joseph High School, Camden, New Jersey.

Mr. M. M. Sluyter, Armour Research Foundation, Chicago, Illinois, has accepted a position as Research Scientist with the Advanced Reaserch Division of THERM, Incorporated, Ithaca, New York.

Mr. F. L. Thompson, University of California, Berkeley, has accepted a position as Mathematician at the Naval Ordnance Test Station, China Lake, California.

Mr. L. J. Watson, Jr., Phoenix Union High School, Phoenix, Arizona, has accepted a position as Head of Presentations with the System Development Corporation, Santa Monica, California.

Mr. S. H. Cunha, Quebec Hydro-Electric Commission, died March 21, 1960. He was a member of the Association for ten years.

Professor E. J. Finn, Catholic University of America, died May 9, 1960. He was a member of the Association for twenty-four years.

Professor A. L. Nelson, Wayne State University, died April 5, 1960. He was a charter member of the Association.

M. Victor Thébault, Tennie, Sarthe, France, died March 19, 1960. He was a member of the Association for twenty-eight years.

CONFERENCE ON MECHANICS

The Seventh Midwestern Conference on Fluid Mechanics and Solid Mechanics will be held at Michigan State University, East Lansing, September 6–8, 1961. Authors wishing to present papers should send in their abstracts preferably by January 1, 1961. For further information, address inquiries to: Professor J. E. Lay, Mechanical Engineering Department, Michigan State University, East Lansing, Michigan.

THE MATHEMATICAL ASSOCIATION OF AMERICIA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 292 persons have been elected to membership by the Board of Governors on applications duly certified.

CHARLES H. ACKERKNECHT, Student, Los Angeles State College

GORDON D. ADAMS, B.A. (Swarthmore) Asst., Cornell University

EDA S. ALTER, Student, Barnard College

EDWARD Z. ANDALAFTE, M.A. (Missouri) Instr., University of Missouri

MRS. PATRICIA K. ANDRESEN, M.A. (Missouri) Instr., University of Missouri

LOUIS S. ARONICA, B.S. (Pennsylvania S.U.) Instr., Western College

DONALD G. ARONSON, Ph.D. (M.I.T.) Asst. Prof., University of Minnesota

DOUGLAS R. AUGUSTINE, B.A. (Washington) Engineering Technician, Diamond Ordnance Fuze Lab.

GEORGE L. BALDWIN, M. A. (Eastern New Mexico) Asst. Prof., New Mexico State University

MRS. RUTH BECK, B. A. (Stern Coll.) Grad. Asst., University of Pennsylvania

MICHAEL G. BECKER, Student, Xavier University

WILLIAM R. BECKER, Student, New York University; Programmer, Service Bureau Corporation

JOSEPH BEER, M. A. (Syracuse) Grad. Asst., Syracuse University

BENJAMIN R. BEH, Student, St. Ambrose College

JAMES C. BEIDLEMAN, M.S. (Bucknell) Research Asst., Ordnance Research Laboratory

PETER J. BICKEL, Student, University of California, Berkeley

THOMAS E. BLACK, JR., Teaching Asst., Purdue University

MRS. PATRICIA L. BLACKER, Student, San Jose State College

GEORGE R. BLAKLEY, M.A. (Maryland) NSF Fellow, University of Maryland

J. MITCHELL BLOSE, M.Ed (Pittsburgh) Chairman, Senior High School, Kittanning, Pennsylvania

WALTER BLUMBERG, M.A. (Brooklyn Coll.) Teacher, Flushing High School, Queens, New York

JAN L. BOAL, Ph.D. (M.I.T.) Asst. Prof., University of South Carolina

ELDON C. BOES, Student, St. Ambrose College

ROY B. BOGUE, B.E.E. (Auburn) Instr., Auburn University

ARVAL W. BOHN, M.A. (Texas) Instr., Louisiana State University in New Orleans

ROBERT E. BOHRER, Student, South Dakota School of Mines and Technology

VIROOL BOONYASOMBUT, B.S. (Chulalongkorn U., Thailand) Grad. Student, Ohio State University

R. MURRAY BOOTHROYD, M.Ed. (Minnesota) Instr., Sherburn High School, Minnesota

WARREN L. BOSCH, Student, Wittenberg University

MRS. DELORES H. BRIGHT, M.A. (Long Beach S.C.) Head of Dept., Narbonne High School, Long Beach, California

MARY A. BRIGHT, Student, Ohio State University

J. ROBERTS BRITTON, Supt., Hawaiian Electric Company; Lecturer, Dept. of Engineering, University of Hawaii

AVIAD M. BROSHI, Student, Yeshivah of Flatbush High School, Brooklyn, New York

ROLAND BROSSARD, M.Sc. (Montreal) Asso. Prof., University of Montreal

CALVIN K. BURGE, M.S. (Illinois) Instr., Lansing Community College

ENOCH BURTON, M.A. (Indiana) Head of Dept., Shortridge High School, Indianapolis, Indiana

AGOSTINO BURZIO, Student, University of Santa Clara

THOMAS V. BUSCEMI, B.S. (St. Francis Coll.) Teacher, Clayton High School, New Jersey

LORETO M. BUZZEO, Student, School of Education, Fordham University

ELWIN F. CAMMACK, M.Ed. (Brigham Young) Instr., Weber College

J. FRANCIS CAMPBELL, B.S. (Montana State Coll.) Grad. Asst., Montana State College

GERALD D. CHAPMAN, Student, University of California, Los Angeles

HARRY T. CHASE, JR., Student, University of Santa Clara

CHARLES E. CHRISTIAN, Student, Purdue University

CHARLES L. CHUSTZ, B.S. (Louisiana S.U.) Teacher, LaGrange Senior High School, Lake Charles, Louisiana

WILLIAM W. CLARK, B.A. (Reed Coll.) Teacher, Fremont Junior High, Monterey, California

MCLOUIS CLAYTON, Student, Shaw University

MOTHER M. ANCILLA CLEARY, M.S. (Fordham) Asst. Prof., College of New Rochelle

LT. FREDERICK H. CLEVELAND, B.S. (Texas A&M) U. S. Air Force

MAJOR CORNELIUS J. COMBER, U. S. Air Force

RICHARD S. COOKE, B.S. (Northeastern) Programmer, Sylvania Electric

WILLIAM SECORD COOPER, M.S. (M.I.T.) Grad. Student, University of California, Berkeley

WILLIAM E. COPPAGE, M.S. (Texas A&M) Asst. Instr., Ohio State University

HARRY S. COUZINS, B.A. (California, Berkeley) Chairman of Dept., Chaffey High School, Ontario, California

MRS. CAROLYN G. CRANDELL, Student, William Smith College

EDWARD W. CREEKMUR, B.A. (San Diego S.C.) Instr., San Diego Junior College

ROBERT N. CROCKETT, Student, Church College of Hawaii

CHARLES C. DALTON, M.A. (Alabama) Physicist, Army Ballistic Missile Agency

EVMENIOS P. DAMON, M.S. (Drexel Inst. Tech.) Leader, Programming Group, Radio Corporation of America

HERBERT A. DAVID, Ph.D. (London) Prof., Statistics, Virginia Polytechnical Institute

THOMAS N. DELMER, Student, University of Santa Clara

THEODOSIUS L. DEMEN, Ph.D. (St. Louis) Asst. Prof., University of Dallas

BERNARD M. DONAHOE, M.A. (San Jose S.C.) Teacher, Monterey City Schools, California

REV. PEARSE P. DONOVAN, M.A. (San Francisco) Vice Principal, Bishop O'Dowd High School, Oakland, California

STANLEY S. DOTTERER, B.S. (Elizabethtown Coll.) Teacher, Elizabethtown High School, Pennsylvania

ALBERT DOU, D.Sc. (Madrid) Prof., University of Madrid

EDWARD DUBOIS, Student, St. Francis Xavier University

MEYER DWASS, Ph.D. (North Carolina) Asso. Prof., Northwestern University

MERLE H. EASTMAN, Accountant, Euclid 19th Auto Service, Inc., Cleveland, Ohio

GEORGE EILERS, Student, University of Santa Clara

MURRAY EISENBERG, Student, University of Pennsylvania

GEORGE R. ELLISON, B.S.E. (Oklahoma) Physics Research Asst., University of Oklahoma

RONALD J. ENSEY, Student, Hardin-Simmons University

HENRY F. EPSTEIN, Student, Bronx High School of Science, Brooklyn, New York

FLOYD A. ERICKSON, JR., Student, University of Nebraska

DAVID R. FALCONER, B.A. (Texas) Programmer, I.B.M., Bethesda

LAWRENCE FEARNLEY, Ph.D. (Utah) Asst. Prof., Brigham Young University

PAUL I. FEDER, Student, Polytechnic Institute of Brooklyn

BURTON FEIN, Student, Polytechnic Institute of Brooklyn

RAYMOND J. FETZER, Student, St. Ambrose College

RICHARD D. FINLEY, M.S. (Colorado S.U.) Grad. Teaching Asst., Colorado State University

ROGER M. FISCHLER, Student, Polytechnic Institute of Brooklyn

CHARES E. FLAGG, Student, Bates College

DAVID FORSLUND, Student, University of Santa Clara

REV. RAYMOND A. FOURNIER, M.Ed. (Boston) Teacher, Marist College and Seminary

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RAOUL M. FREYRE, Ph.D. (Habana, Cuba) Asst. Prof., Physics Dept., North Carolina State College

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CHARLES J. GIBBONS, Student, St. Ambrose College

FRANK S. GILLESPIE, Student, University of Missouri

FREDERICK J. GILMAN, Student, Michigan State University

TODD GITLIN, Student, Harvard College

MICHAEL C. GODFREY, Student, University of British Columbia

MRS. LUCILLE H. GOODGAME, M.A. (Miss. Southern Coll.) Teacher, George S. Gardiner School, Laurel, Mississippi

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DAVID B. GUSTAVSON, Student, University of Nebraska

MARTIN M. GUTERMAN, Student, Brooklyn College

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EDWARD J. HARTNETT, M.S.E.E. (N.Y.U.) Engineering Leader, Radio Corporation of America
STEPHEN H. HECHLER, Student, California Institute of Technology
E. LEROY HEER, Student, University of Alaska
CHARLES V. HEUER, B.A. (Concordia Coll.) Grad. Asst., University of Nebraska
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EMILIE L. HOPKINS, Student, University of Kansas
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CDR. JAMES A. H. HUNTER, Royal Navy; Author, Toronto, Ontario
THOMAS C. HUTCHINSON, B.A. (Coll. of the Holy Cross) Instr., St. Ambrose College
CARLA J. INGEBO, B.A. (Montana) Asso. Engr., Boeing Airplane Company
JOHN A. JACOBSEN, M.S. (Purdue) Instr., University of Georgia
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DONALD E. JULIE, M.S. (Wisconsin) Applied Science Representative, I.B.M., Minneapolis
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CHARLES L. KELLER, Ph.D. (Illinois) Asst. Prof., University of Dayton
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FATHER A. KIMENAI, M.Ed. (Ottawa) Teacher, St. Francis
DOUGLAS W. KING, Student, University of Buffalo
HOWARD KLEIMAN, M.A. (Teachers College, Columbia) Teacher, Board of Education, Brooklyn
KENNETH F. KLOPFENSTEIN, Student, Iowa Wesleyan College
ANTHONY KOESTLER, Student, Stevens Institute of Technology
RUTH E. KURTZWEIL, B.A. (Rockford Coll.) Grad. Student, University of North Carolina
ERIC S. LANGFORD, B.S. (M.I.T.) Grad. Student, Rutgers—The State University
DAVID G. LASH, Student, Oklahoma State University
DONALD R. LATORRE, Student, Wofford College
DENNIS E. LAURER, Student, University of Kansas
ROBERT LEAF, B.S. (Oregon S.C.) Asso. Engr., Boeing Airplane Company
JOHN J. LEHRBERGER, JR., M.A. (Louisville) Asst. Prof., Madison College
BETTY LEVINE, Student, Hunter College
JOEL E. LEVY, Student, Brooklyn College
C. NELSON LI, B.S. (Taiwan) Grad. Asst., University of Tennessee
BRUCE LITTMAN, Student, Dominguez High School, Compton, California
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JOHN A. LUCASH, JR., Student, Yale University
GEORGE B. MACH, M.S. (S.U. of Iowa) Asst. Prof., California State Polytechnic College
TRUEMAN MACHENRY, M.S. (Manchester) Instr., Rutgers University
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PETER MARKS, Student, College of the City of New York
GARRY S. MARLISS, Student, University of Alberta
JOHN H. MASON, Student, Hillfield College
D. RUTH MASSA, A.B. (Randolph-Macon) Grad. Student, Emory University
MARLENE R. MASSARO, Student, University of Colorado
MRS. ELIZABETH E. MAYO, M.S. (Chicago) Asst. Prof., Morehead State College
RAULLO J. MCCONAHY, M.S. (Chicago) Asso. Math., Applied Physics Lab., Johns Hopkins University
KEVIN M. MCCRIMMON, Student, Reed College
MARY E. MCDERMOTT, M.A. (Stanford) Curriculum Consultant, Mt. Diablo Unified School, Concord, California
JAMES J. MCGINNIS, M.A. (NYU) Chairman of Dept., Binghamton Central High School, New York
FRANCIS S. MCGOWAN, Student, College of the City of New York
JULIA R. MCLAIN, Student, duPont Manual High School, Louisville, Kentucky
WILLIAM B. MEINDERS, Civil Eng. (Cincinnati) Chief Engineer, Husky Products, Inc.
PETER MENEGUS, Student, Fairfield University
R. ADM. WILLIAM L. MESSMER, M.Ed. (Duke) USN Retired; Instr., College of William and Mary, Norfolk
FREDERIC T. METCALF, Student, University of Maryland
EDITH E. MILLER, M.A. (Montana) Head of Dept., Whitefish High School, Montana
RANDAL P. MILLER, Student, Xavier University
ROBERT EUGENE MITCHELL, Student, University of Nebraska
SHASHANKA S. MITRA, M.Sc. (Calcutta) Research Asst., University of Washington
MRS. JEAN B. MOBLEY, M.A. (North Carolina) Head of Dept., Flora Macdonald College
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JOHN C. MUSGRAVE, Student, University of Kansas
LAWRENCE A. MYSAK, Student, University of Alberta
GILBERT W. NELSON, M.S. (N.D. Agric. Coll.) Instr., North Dakota Agricultural College
MRS. LOUISE H. NELSON, M.A. (Georgia) Asst. Prof., Georgia State College for Women
PETER G. NELSON, Student, University of Arizona
EDWARD NEWBERGER, Student, College of the City of New York
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JAMES H. NORTHEY, M.S. (Michigan) Asst. Prof., Eastern Michigan University
WILLIAM P. NOVINGER, B.S. (Northeast Missouri S.T.C.) Asst. Instr., University of Missouri
JAMES NUDELMAN, B.A. (California) Teacher, Mountain View High School, California
UNTO K. NYYSSONEN, Student, San Mateo High School, California
ERNEST D. O'NEIL, Student, St. Bonaventure University
CARL G. OPASKAR, Student, Case Institute of Technology
WARREN PAGE, Student, Hunter College
DANIEL J. PALMER, Student, University of Notre Dame
BARBARA M. PANKOS, Student, Loyola University
DAVID A. PASSMORE, Student, Mississippi Southern College
ROLANDO E. PEINADO, M.A. (Nebraska) Asst. Prof., University of Nebraska
LINDSEY S. PERKINS, Student, Central Michigan University
GIRARD C. PESSIS, Student, Bronx High School of Science, New York
JEAN D. PETERSEN, Student, University of Colorado

TED E. PETRIE, Student, Michigan State University
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SIYA RAM, M.Sc. (Lucknow, India) Grad. & Teaching Asst. Statistical Laboratory, Purdue University
VICTOR L. RANSOM, M.S.E.E. (Case Inst.) Member, Tech. Staff, Bell Telephone Laboratories
ALVAH RAYMOND, M.Ed. (Boston University) Head of Dept., Weymouth High School, Massachusetts
MARY E. REED, M.A. (Nebraska) Teacher, Benton Harbor High School, Michigan
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LEO J. SCHNEIDER, Student, Xavier University
MRS. CERES M. SCHROER, B.A. (California) Grad. Asst., University of Rochester
LAURENCE L. SCHUMAKER, Student, South Dakota School of Mines and Technology
JONATHAN A. SEAMAN, Student, Case Institute of Technology
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SISTER JAMES FRANCIS, M.A. (Catholic Univ.) Instr., D'Youville College
SISTER M. ELEANOR CHRISTINE, B.S. (Villanova) Grad. Student, University of Notre Dame
SISTER MARY AQUIN, M.S. (Notre Dame) Head of Dept., Our Lady of Mercy High School, Rochester, New York
SISTER MARY BRUNO, B.A. (Mt. Mary College) Teacher, Marinette Catholic Central High School, Wisconsin
MRS. DORA SKYPEK, A.B. (Florida State Univ.) Grad. Student, Emory University
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LUCILLE A. SMITH, M.A. (Missouri) Head of Dept., Maplewood-Richmond Heights High School, Missouri
MORTON J. SMITH, Student, College of the City of New York
MITCHELL SNYDER, Student, Yeshiva University
THOMAS T. SPENGLER, Technician, Engineering Experiment Station, Georgia Institute of Technology
ROBERT SPIVACK, Student, Rutgers University
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DANIEL J. STERLING, A.M. (Columbia) Teaching Asst., University of Wisconsin
BRYAN W. STONESTREET, M.A. (West Virginia) Instr., General Motors Institute
JOHN P. STORTO, Student, University of Detroit
EDGAR L. STOUT, Student, Oregon State College
ROBERT S. STRICHARTZ, Student, Bronx High School of Science, New York
H. E. W. NICHOLAS STURM, M.S. (Purdue) Head of Science Dept., Southwest Texas Junior College
ALVIN SWIMMER, M.S. (N.Y.U.) Teaching Asst., University of California, Berkeley
GEORGE H. TAYLOR, M.A. (Ohio S.U.) Teacher, Worthington High School, Ohio
REV. E. J. TEMPLETON, Th.D. (Philadelphia Divinity School) Chairman of Dept., St. John Baptist School, Mendham, New Jersey
GEORGE THOMAS, Sr., M.S. (Oklahoma S.U.) Asst. Prof., Shaw University
ROBERT S. D. THOMAS, Student, Leaside High School, Toronto, Canada
GUY B. TORCHINELLI, M.S. (Illinois) Asst. Prof., State University of New York, College of Education at Buffalo
EILEEN T. TORMEY, A.M. (Boston) Asst. Prof., Rhode Island College of Education
MRS. JOAN U. TREKELL, Student, University of Tulsa
JOSEPH W. TRIGG, JR., B.S. (Lincoln) Grad. Student, University of Missouri
ROBERT J. TRUSTY, Student, St. Ambrose College
JAMES L. TSCHANTZ, B.A. (Coll. of Wooster) Grad. Student, San Jose State College
MRS. JUANITA R. TURNER, M.S. (Illinois) Chairman of Dept., Manassas High School, Memphis, Tennessee
JULIAN J. VANDECAVEYE, B.S. (Michigan) Grad. Student, University of Michigan
LEWIS E. VOGLER, B.S. (Oregon) Mathematician, National Bureau of Standards
RAYMOND E. WALCH, M.A. (Trinity) Director of Mathematics, Town of Westport, Connecticut
BEYRL E. WALROD, A.B. (Columbia) Inventory Control Supervisor, The Budd Company
MRS. HAZEL WALTERS, M.A. (Oklahoma State) Teacher, Cordell High School, Oklahoma
BENJAMIN WEISS, Student, Yeshiva University
SHIRLEY G. WEISSMAN, M.S. (C.C.N.Y.) Teacher, Hackett Junior High School, Albany, New York
WILLES L. WERNER, B.A. (UCLA) Grad. Teaching Asst., University of Utah
MACK L. WHITAKER, M.S. (Appalachian) Grad. Student, Florida State University
MILO E. WHITSON, Ed.D. (Southern Calif.) Head of Dept., California State Polytechnical College
ROBERT E. WILKERSON, B.S. (Colorado) Physicist, National Bureau of Standards
DONALD R. WILLIAMS, B.S. (Oklahoma S.U.) Grad. Asst., Oklahoma State University
RICHARD K. WILLIAMS, Student, Vanderbilt University
CARROLL K. WILSON, M.A. (N. Texas S.T.C.) Acting Chairman, Eastern New Mexico University
JAMES R. WILTS, Ph.D. (Calif. Inst. of Tech.) Asso. Math., IBM, Johnson City
ALBERT K. WONG, B.S. (Nat'l Taiwan) Grad. Student, Atlanta University
ROBERT L. WOOD, Student, St. Bonaventure University
JOHN E. WOODWARD, M.A. (Oklahoma) Mathematician, Dept. of Army, Fort Sill, Oklahoma
RAYMOND A. ZACHARY, JR., M.A. (Texas) Operations Research Analyst, Texas Instruments
RAYMOND S. ZAWADA, B.S. (Indiana Univ.) Computer Programmer, Inland Steel Company
FRANK W. ZURCHER, JR., Student, Stevens Institute of Technology

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1960 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	A. B. CUNNINGHAM, West Virginia University
Indiana	M. E. SHANKS, Purdue University
Kentucky	W. C. ROYSTER, University of Kentucky
Met. New York	C. T. SALKIND, Polytechnic Institute of Brooklyn
Nebraska	EDWIN HALFAR, University of Nebraska
Northern California	C. D. OLDS, San Jose State College
Oklahoma	O. H. HAMILTON, Oklahoma State University
Rocky Mountain	M. L. MADISON, Colorado State University
Wisconsin	R. D. WAGNER, University of Wisconsin

In four sections, more than 50% of the membership voted. Nebraska was high with a very fine 65% response. The next three were Oklahoma—57%, Kentucky—55%, and Wisconsin—51%.

L. J. MONTZINGO, *Associate Secretary*

THE TWENTY-FIRST ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twenty-first annual William Lowell Putnam Mathematical Competition will be held on Saturday, December 3, 1960. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed about October first to the regular mailing list. If an application blank is not received by October 15, you may secure one by writing the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 7, 1960. For further details of the examination and the list of prizes (including the $3000 scholarship to Harvard), see the announcement which will accompany the application blank.

Reports of the previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August–September 1947; December 1948; August–September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August–September (announcement of winners) and November (questions and solutions) 1957; August–September 1958; and August–September 1959.

CONFERENCES FOR MATHEMATICS INSTRUCTORS IN NSF 1960 SUMMER INSTITUTES

The Committee on Institutes of the MAA organized five regional conferences for mathematics instructors in NSF 1960 summer institutes. These conferences were financed by a grant of $49,000 to the MAA from the National Science Foundation, and were held at the following places and times: San Francisco, April 8 and 9; Washington, D. C., April 8 and 9; Chicago, April 9 and 10; New York, April 29 and 30; St. Louis, May 6 and 7.

The programs of the conferences were devoted primarily to a consideration of appropriate courses for institutes for high school mathematics teachers. Experienced lecturers outlined possible courses in algebra, geometry, analysis, probability and statistics, and foundations. These talks served as a basis for small group discussions of courses in the various branches mentioned. Reports on the group discussions were made to the entire conference in each case, with time allowed for general discussion. A talk by a high

school teacher who had attended a summer institute and a discussion of high school mathematics curricula developments by a speaker competent in this area were a part of the program of each conference. About 320 persons attended the conferences.

Summaries of the talks by the speakers and reports of the group discussions for all five of the conferences have been mimeographed for distribution. This material will be of special interest to directors and lecturers in mathematics at summer, inservice, and academic year institutes for high school teachers. Copies can be obtained free by writing to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 26, 1960 at the University of Michigan, Ann Arbor, in conjunction with the meeting of the Michigan Academy of Science, Arts, and Letters. Professor W. D. Baten of Michigan State University presided at both the morning and afternoon meetings and at the luncheon-business meeting. A total of 88 persons attended the meeting, including 67 members of the Association.

A nominating committee consisting of Professors F. A. Beeler, Chairman, E. E. Ingalls and B. M. Stewart proposed the following slate of officers: Professor E. D. Rainville of the University of Michigan as Chairman, Professor Wallace Givens of Wayne State University as Vice-Chairman, and Professor L. E. Mehlenbacher of the University of Detroit as Secretary-Treasurer. The slate was elected unanimously.

Professor R. M. Thrall, member of the Board of Governors representing the Michigan Section, gave a brief report of the state of the Mathematical Association of America.

Professor F. L. Celauro, Secretary of The Committee on the Michigan Mathematics Prize Competition, gave the following report:

The Competition Examination was given on March 3, 1960 to 9200 high school contestants from 432 high schools in Michigan. The Examination consisted of two parts: Part I; objective type, designed to test general proficiency in the four years of mathematics taught in high school, and Part II; written or essay type, designed to reveal mathematical maturity, originality and clarity of expression. The awards will include 94 college scholarships valued from $120.00 to $600.00 each, six medals from the Competition Committee and 15 awards from $100.00 to $200.00.

The following motions pertaining to the Competition Committee were passed at the business meeting: (1) That the Committee for the Michigan Mathematics Prize Competition consist of six members to be appointed by the Chairman of the Michigan Section to serve terms of three years, with two new members being appointed each year; and (2) that members of the Committee be authorized to receive compensation for expenses of attending committee meetings from committee funds, depending upon the judgment of the Committee.

The following papers were presented:

1. *What statistics is, and how statisticians should be trained,* by Professor Herman Rubin, Michigan State University. (By invitation.)

We can define statistics as the theory of decision making under uncertainty. There are two kinds of uncertainty—first, an uncertainty due to probability, and second, due to the lack of knowledge of the probabilities involved. The treatment of the first is merely a problem in analysis; the second is statistics proper. Recommendations concerning the teaching of statistics are given.

2. *Inequalities related to convex curves,* by Professor J. L. Ullman, University of Michigan. The main result of this talk is that

$$\left| \int_0^{2\pi} f(\theta) e^{-i\theta} d\theta \right| \geqq 4,$$

where $f(\theta)$ is a sense-preserving and arc-length preserving homeomorphism of the unit circle and a convex curve. This result can be interpreted to yield an interesting geometrical theorem on convex curves.

3. *Elementary covering problems*, by Professor L. M. Kelly, Michigan State University.

THEOREM. *If each three points of a set in E_n can be covered by a given closed circular disc, then the entire set can be covered by this disc.*

This theorem is due to C. V. Robinson and is a prototype of the problems with which we are concerned. For the most part this is a report of Robinson's work but a few new results are also included.

4. *Bernoulli numbers modulo 27000*, by Professor J. S. Frame, Michigan State University.

This paper will be published in this MONTHLY.

5. *Emphasis on the seventh ratio in trigonometry*, by Professor G. E. Moore, Radiation Laboratory, University of Michigan.

Starting with $\theta = s/r$ and a minimum of ideas from plane geometry, practically all of the topics in trigonometry are developed and correlated. Some of the topics considered are: graphs of the functions, vectors, complex numbers and de Moivre's formula, the wrapping process and ordered number pairs, circular and inverse circular functions, function defined by number pairs, polar coordinates, roots of unity, etc. (This is somewhat the reverse of the order employed in the Report of the Commission on Mathematics—*Appendices*, and has certain advantages).

6. *Summer programs for gifted high school students*, by Professor Isobel Blythe, Michigan State University.

The National Science Foundation has supported or partially supported programs of various sorts for students who have not yet reached the college level. In 1959 there were 112 such programs in 105 institutions. Some were limited to only one subject matter area while others included several areas. The variations in length of time, facilities, purposes, staff, age of students, etc. were presented. Also the results of these programs and reactions to them by students and others were discussed. Finally, the programs for 1960, especially any significant changes, were presented.

7. *Two nonparametric significance tests*, by Professor F. A. Beeler, Western Michigan University.

The object of this talk is to present two nonparametric tests that have proved to be useful. A nonparametric significance test is a test which is valid regardless of the population from which the samples come provided that the observations are random and independent.

8. *The maximum rate in an add-on installment contract*, by Professor H. E. Stelson, Michigan State University.

In an installment contract in which the time price differential is an add-on, the yield return increases as the contract gets longer up to a point. Thereafter the yield declines as the length of the contract increases. The formula for the present value of an annuity can be differentiated implicitly but it is impossible to obtain exact maximum values for the time and rate. Approximate values of the maximum interest rate can be obtained by using approximate values of the interest rate as presented in the December 1953 issue of this MONTHLY by the author.

L. E. MEHLENBACHER, *Secretary*

THE APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 34th meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at Grove City College, Grove City, Pennsylvania, on April 30, 1960. Professor P. N. Carpenter of Grove City College presided at the morning session,

and Dr. B. H. Mount, Chairman of the Section, presided at the afternoon session. Thirty-eight persons were registered for the meeting.

The morning session was devoted to technical papers, as reported below. During the afternoon session, a report was made on the high school examinations in West Virginia and in western Pennsylvania. Income from the contest examinations has exceeded expenses over the past three years. Numerous suggestions were offered for the best use of these funds in improving the program of high school examinations. Final action was deferred until the next meeting.

Dr. B. H. Mount read a paper prepared by Professor J. H. Neelley on the "Joint MAA and NCTM Meeting in Chicago on January 30," in which the author reported his impressions of the meeting.

Professor H. F. Bechtell presented a "Progress Report on the Grove City Plan." Grove City College installed a mathematics curriculum that requires all students taking elementary mathematics through the calculus to follow the same sequence of courses and terminating them at the end of the sophomore year. (Cf. May Meeting of the Allegheny Mountain Section, this MONTHLY, volume 66, number 7.) The plan has very successfully met its objectives. A survey analysis of the prerequisites of the entering students and their eventual success in the first course of algebra and trigonometry strongly supports it as the initial course in the sequence.

Professor J. C. Knipp distributed copies of and discussed a "Report on the Mathematics Program of the Regional Commission on Educational Coordination." Recommendations for coordination of mathematics curricula in high schools and universities were prepared by a panel of mathematics teachers and industrial mathematicians from the western Pennsylvania area.

The following papers were presented:

1. *Maclaurin coefficients for trigonometric and hyperbolic functions*, by Professor F. H. Steen, Allegheny College.

The recursion formula

$$A_n = B_n + 4\binom{n}{4}A_{n-4} - 4^2\binom{n}{8}A_{n-8} + 4^3\binom{n}{12}A_{n-12} - \cdots$$

in which A_n represents the nth derivative of such functions as sec x, sech x, tan x, evaluated at $x=0$, and B_n represents the corresponding values of the product of the function by cos x cosh x, was derived and discussed, together with a similar formula for functions such as x csc x, x csch x, x^2 csc x csch x, x cot x.

2. *Normal approximation to the distribution of two independent binomials, conditional on fixed sum*, by Professor W. L. Harkness, Pennsylvania State University.

Using Feller's superior normal approximation theorem for the binomial distribution, it is shown under suitable restrictions that the distribution of two independent binomials is, for fixed sum, asymptotically normal. The result has applications in the study of asymptotic power functions for the test of independence in 2×2 contingency tables.

3. *A binomial series transformation*, by Mr. H. W. Gould, West Virginia University.

Let $f(k)$ be independent of n and define

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{a+bk}{n} f(k), \qquad f(0) = 1.$$

Then

$$\sum_{k=0}^{\infty}\binom{a+bk}{k} z^k f(k) = x^a \sum_{n=0}^{\infty} (x^{-1}-1)^n F(n),$$

where $z=(x-1)x^{-b}$. This generalizes a procedure used by the author (this MONTHLY, vol. 63, 1956, pp. 84–91) to obtain sums of certain infinite series.

4. *Some relations between alternative rational-choice criteria for consumer behavior theories based on weak orderings*, by Professor E. M. Fels, University of Pittsburgh, introduced by the Secretary.

Starting from formalizations of choice-set functions (as generalized consumer-demand functions), weak (i.e. connected and transitive) orderings, and revealed and indirectly revealed preference relations, five alternative criteria for "rational" choice functions (by Ville, Houthakker, Arrow, Chernoff, and Samuelson) are investigated with a view to their mutual derivability. A conjecture that the so-called Weak Axiom of Revealed Preference (Samuelson) implies the so-called Strong Axiom of Revealed Preference (Houthakker, Ville) is refuted (unless some definitions are changed), utilizing a systematic refutation procedure in accordance with Quine's (1955) proof procedure for quantification theory.

EVAN JOHNSON, JR., *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The 47th regular meeting of the Iowa Section of the Mathematical Association of America was held at the State University of Iowa, Iowa City, on the afternoon of April 22 and the morning of April 23, 1960. Professor J. J. L. Hinrichsen, Chairman of the Section, presided. Total attendance was 115, including 45 members of the Association.

The following officers were elected: Chairman, Professor H. C. Trimble, Iowa State Teachers College; Vice-Chairman, Reverend T. J. Taylor, St. Ambrose College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program:

1. *Approximation on an infinite interval*, by Professor E. N. Oberg, State University of Iowa.

By means of the criteria of least mth power methods, and an integral formulation of Bernstein's theorem, it is shown that a function $f(x)$ that is continuous and which is of the order $c_0x^k+c_1x^{k-1}+\cdots+c_n$ as x becomes infinite can be approximated by sums of orthonormal functions whose derivatives are orthogonal. The rapidity of convergence can be shown to depend upon the properties of continuity of the function $f(x)$.

2. *Explicit formulas for some continuous probability measures*, by Professor C. E. Langenhop, Iowa State University.

For any real x, $0\leqq x\leqq 1$, $x=\sum_{i=1}^{\infty} a_i2^{-i}$, where for each i, $a_i=1$ or 0. Let $S_n=\sum_{i=1}^{n} a_i$ and define $F(x)=\sum_{i=1}^{\infty} a_iq^i(p/q)^{S_{i-1}}$, where $p+q=1$, $p>0$, $q>0$, It can be shown that F is a continuous monotone increasing function on $0\leqq x\leqq 1$ with $F(0)=0$ and $F(1)=1$ and that F is the distribution function on $0\leqq x\leqq 1$ induced by the probabilities of finite sequences of Bernoulli trials with probabilities of success ($a_i=1$) and failure ($a_i=0$) on each trial being p and q, respectively.

3. *The two-color theorem*, by Professor L. E. Pursell, Grinnell College.

The "two-color" theorem asserts that if a sphere is separated into regions by a finite graph all of whose vertices are of even order, then it is possible to partition this collection of regions into two classes (or colors) such that no two regions of the same class have an arc of the graph in common. This theorem has been stated in the literature, but the author does not know of the publication of a proof of the theorem (except for special cases). Two proofs of this theorem are given. With proper restrictions on the graph, these proofs are valid for any surface which satisfies the Jordan Curve Theorem.

4. *A revision of a completion method and its adaptation to ill-conditioned matrices*, by Professor W. L. Waltmann, Wartburg College.

An essentially new method for the inversion of $n\times n$ matrices which is closely related to the method of completion attributed to Sherman, Morrison, and Bartlett has been developed. Their method gives the inverse for a nonsingular matrix that differs in only one column from a matrix whose inverse is known. The revision begins with a diagonal matrix and by successive column

changes produces the inverse of the desired matrix. This technique controls the conditioning of the matrix by controlling both the size of its elements and its determinant. The revised method can be readily adapted to electronic digital computers.

5. *Newtonian and Galilean reference frames (of the special theory of relativity) defined and compared by elementary mathematics*, by Professor C. J. Wolfsa, Grinnell College, introduced by the Chairman.

A survey is made of the available literature regarding the terms, reference frame, reference system, Galilean system, Newtonian system, and inertial systems. Precise mathematical definitions of these terms are constructed culminating in the definition of a Galilean inertial reference system. A survey is made concerning the linearity premise used by Einstein, and many others after him, in deriving the Lorentz transformation equations for special relativity. Lastly, this linearity premise is shown to follow directly from the previous definition of a Galilean inertial reference systems.

6. *On linear combinations of unbiased statistics*, by Professor R. V. Hogg, State University of Iowa.

Let T_1 and T_2 be independent unbiased statistics for the paramenter m. Let σ_1^2 and σ_2^2 be respectively the variances of these statistics. It is well known that $T=(\sigma_2^2T_1+\sigma_1^2T_2)/(\sigma_1^2+\sigma_2^2)$ is the unbiased linear combination of T_1 and T_2 having smallest variance. If σ_1^2 and σ_2^2 are unknown, we define new statistics T' by replacing, in T, σ_1^2 and σ_2^2 by various estimates of them. It is proved that if m is the midpoint of a symmetric distribution, if T_1 and T_2 are unbiased odd location statistics for m, and if the estimates of σ_1^2 and σ_2^2 are even location-free statistics, then each T' is an unbiased statistic for m.

7. *Generalized Riemann integrals in subspaces of the reals*, by Mr. T. J. Robinson, University of North Dakota, and Professor D. E. Sanderson, Iowa State University, presented by Mr. Robinson.

For real-valued functions defined on $E\subset[a, b]$, the integral as a limit of "Riemann" sums, upper sums S, and lower sums s, for partitions of norm Δ, are defined as usual except only subintervals containing points of E are used. The upper (lower) integral is $J=\sup_{\iota>0}\inf_{\Delta<\iota} S$ ($I=\inf_{\iota>0}\sup_{\Delta<\iota} s$). Results: $\lim_{\Delta\to 0} S=J$, $\lim_{\Delta\to 0} s=I$ and the integral exists if and only if (i) $I=J$, (ii) given $\epsilon>0$, $S-s<\epsilon$ for some partition, or (iii) $\lim_{x\to a}\inf f(x)=\lim_{x\to a}\sup f(x)$ for almost all a in E. Functions integrable on E are extendable to integrable functions $\bar{E}([a, b])$ whose integral over all subsets is the same (arbitrarily close—equality not always attainable).

8. *A remark on a certain quadratic form*, by Professor A. T. Craig, State University of Iowa.

Let Q_1 denote a real symmetric quadratic form of rank r, $0<r<n$, in n random values of a variable that is normally distributed with mean μ and variance σ^2; and let the distribution of Q_1 be free of μ. Let Q_2, stochastically independent of Q_1, be another quadratic form in the same random variables. If the rank of Q_2 is $n-r$, the distribution of Q_2 must depend upon μ.

9. *Methods of nomographic disjunction*, by Joanna Enzmann, Sylvania Electronic Systems, introduced by the Chairman.

In order to construct a nomogram from a function of three variables, the function must first be put into the proper form. This form is a three by three determinant, called the disjunction, each row of which contains only one variable, and which, when expanded, yields the original function. To do this, one must find a minimal basis for one variable, and, using this basis as the first row of the determinant, find the other two rows by numerical means. Certain special and degenerated cases may be solved by modifying the general method, and in addition some special cases may be solved by using a partially graphical method.

10. *A theoretical study of a randomly excited mechanical oscillator*, by Professors V. W. Bolie and F. M. Long, Iowa State University, presented by Professor Bolie, introduced by the Chairman.

The paper deals with a mathematical investigation of the feasibility of extracting usable mechano-electric power from a critically damped and randomly excited mechanical oscillator.

Two different analytic functions are used to approximate practical squared-velocity spectra for the motion of the vehicle supporting the oscillator, and the corresponding expressions for extractable power are developed. Numerically computed results are presented graphically to illustrate the effects of varying the oscillator parameters.

11. *Error terms of numerical integration formulas*, by Professor R. J. Lambert, Iowa State University.

This paper gives a proof that the polynomial function $Q_n(s)=\int_0^s t(t-1)\cdots(t-n)dt$ does not change sign on the interval $(0, n)$. Heretofore it was generally believed that $Q_n(s)$ changed sign on the interval $(0, n)$ when n was an odd integer. The technique of proof is to show that when n is odd, $Q_n(s)$ has an upper bound $Q_n(n-1)$ for $(n-1)/2 \leqq s \leqq n$ which is shown to be negative. This result simplifies the treatment of the error terms in certain numerical integration formulas which involve divided differences. The simplified treatment is given here.

12. *A note on dialectics in mathematics*, by Professor W. A. Small, Grinnell College.

By the use of some elementary set-theoretic concepts, a precise formalization and a generalization are given of a result stated by Frederick Engels on the meaning of dialectics in mathematics. A complete dialectical process is defined, and it is shown that such a process is a function, and that every real function is a complete dialectical process. Some general implications of this result are discussed.

13. *Convex functions*, by Professor W. T. Reid, State University of Iowa (by invitation).

This paper is devoted to a survey of basic properties of functions which are convex on an open interval, and extensions of these properties to the case of functions that are convex relative to the totality of solutions of a nonoscillatory linear generalized differential equation $d[r(t)u'(t)] - u(t)dm(t)=0$. Special attention is given to those properties that are variational in nature.

14. *Homomorphisms of extensive systems*, by Mr. M. F. Ruchte, Ames, Iowa, introduced by the Chairman.

15. *Logical and semantical examination of non-Euclidean geometries*, by Professor H. G. Apostle, Grinnell College.

Intuition is necessary in mathematics. If "space" and "dimension" are restricted to physical space, a univocal use of "straight line" leads to logical difficulties, and an ambiguous use does not necessarily refute Kant or render Euclidean geometry a special case of geometry. Further, interpretations of non-Euclidean geometries by recourse to Euclidean models indicate that non-Euclidean geometries may be branches of Euclidean geometry. On the other hand, the treatment of geometry from the general to the particular (Von Staudt, Klein, Veblen, and Young, etc.) is scientifically better from the modern as well as from the ancient (Plato, Aristotle) point of view.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-fifth annual meeting of the Kansas Section of the Mathematical Association of America was held at Kansas State College of Pittsburg, Pittsburg, Kansas, on April 30, 1960, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. Of the 175 people registered, 69 were members of the Association. Professor J. D. Haggard, chairman, presided at the sessions.

The following officers were elected for one-year terms: Chairman, Professor W. D. Bemmels, Ottawa University; Vice-Chairman, Professor L. E. Fuller, Kansas State University; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State College of Pittsburg.

At the joint session, held in the morning, Professor M. R. Hestenes, University of California, Los Angeles, addressed the group on *Some properties of matrices.*

The following short papers were presented at the afternoon session:

1. *Affine spaces*, by Professor T. J. Head, Washburn University.

A relation between the theorem of Cayley in group theory and the concepts of parallelism and translation was discussed.

2. *The teaching of mathematics at British universities*, by Professor Wendy J. Robertson, University of Kansas.

The mathematical training offered in British schools and universities differs in several respects from that available here; some of these differences were discussed. Examples were given of what sort of mathematics a student might have learned in typical English and Scottish schools. A description followed of the methods of teaching and of the examination system in various universities. The contents of courses leading to Pass degrees in subjects including mathematics and to Honours degrees in mathematics, and those given to students in engineering and science, were summarized in detail.

3. *Some uses of matrices in coding*, by Professor L. E. Fuller, Kansas State University.

Messages can be coded by setting up numerical equivalents for the letters of the alphabet, two marks of punctuation, and one for a space. The message is divided into sets with the same number of symbols. Column vectors are formed from the numerical equivalents and multiplied by a square matrix over I_{29}. The resulting vectors are the numerical equivalents of the coded message. The same process using the inverse of the matrix will decode any message. The numerical part of the process is readily adapted to use of a desk calculator. It can also be programmed for a digital computer.

4. *Solvability of factorizable groups*, by Professor L. E. Laird, Kansas State Teachers College.

The classification of the subgroups, factor-groups, and direct products of cyclic, Abelian, nilpotent, supersolvable, and solvable finite groups is well known. The problem of classification of a group $G=HK$, where the classifications of the proper subgroups H and K are known, has received attention in the past ten years. The present status of the problem was discussed by the speaker.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Central Missouri State College, Warrensburg, Missouri, on Saturday, April 30, 1960. Professor L. O. Jones, Vice-Chairman of the Section, presided at the morning session and Professor C. E. Kelley, Chairman of the Section, presided at the afternoon session. Forty-four persons, including 31 association members, attended these meetings.

The officers for 1960–1961 are: Chairman, Professor J. L. Zemmer, University of Missouri; Vice-Chairman, Professor J. J. Andrews, St. Louis University; Secretary-Treasurer, Professor N. A. Haynes, University of Missouri. Mr. R. L. Spreckelmeyer, retiring chairman of the Committee on the High School Mathematics Contest, gave a report on the progress made in this project. As a result of discussion introduced by Professor P. B. Burcham, University of Missouri, a committee is to be appointed to investigate the formation and administration of a test for placement and/or credit to be used by the schools in the state.

The program for the meeting consisted of two parts, one in conjunction with the morning session and one at the afternoon session. The following papers were given:

1. *Euclidean geometry without postulates*, by Professor L. M. Blumenthal, University of Missouri. (By invitation)

Distinguishing between "postulates" (statements in the primitive notions *peculiar* to a subject) and "axioms" (common notions), plane Euclidean geometry is based here on the sole common notion "real number" in a manner to exhibit every Euclidean invariant in terms of the mutual

distances of finite subsets of points. Let $(12)=(21)$, $(23)=(32)$, $(13)=(31)$ denote three positive numbers whose Cayley-Menger determinant $D(1, 2, 3)<0$. A point r is defined as an ordered triple of nonnegative real numbers $(1r, 2r, 3r)$ such that $D(1, 2, 3, r)=0$. Distance of points r, s is the unique nonnegative real root of the equation $D(1, 2, 3, r, s; x)=0$, obtained by replacing elements rs and sr in $D(1, 2, 4, r, s)$ by x.

2. *Reckoning and reasoning*, by Professor R. C. Fisher, The Ohio State University. (By invitation)

Axiomatics enters mathematics in two different ways, for different purposes. A complete set of axioms for a mathematical system such as the real number system or Euclidean geometry is given for the purpose of completely specifying the interpretation of that system, that is, it is restrictive in purpose. On the other hand, certain mathematical systems (such as groups) are given axiomatic treatment for purposes of identifying a type of structure in several mathematical systems. High school teachers might better emphasize the latter role of axiomatics (as it is related to structure) than to engage in too formal an approach to mathematics from logical foundations.

MARIAN A. LESHER, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-sixth annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 22–23, 1960, at the University of Nebraska, Lincoln, Nebraska, in conjunction with the seventieth annual meeting of the Nebraska Academy of Sciences. Professor D. W. Miller, Chairman of the Section, presided. There were 50 persons present at the sessions, including 35 members of the Association.

The following officers were elected for 1960–61: Chairman, Professor R. L. Moenter, Midland College; Vice-Chairman, Professor D. W. Miller, University of Nebraska; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

Professor J. M. Earl was continued as Chairman of the Committee on Mathematics Contests. The Committee consists of representatives of the Nebraska Section of the Mathematical Association of America, the Nebraska Section of the National Council of Teachers of Mathematics, the Nebraska Actuaries Club, and the Nebraska Academy of Sciences.

Professor L. J. Mordell of Cambridge University, Visiting Professor at the University of Colorado, gave the Academy Address on the subject *Reflections of a Mathematician.*

At the Association meetings, the following papers were presented:

1. *Recent results in number theory*, by Professor L. J. Mordell. (By invitation)

2. *Subgroups of division rings*, by Professor William Scott, University of Kansas. (By invitation)

3. *The undergraduate mathematics curriculum in Nebraska colleges and universities*, Panel discussion, by Professor Joseph Wampler, Nebraska Wesleyan University; Professor L. M. Larsen, Nebraska State Teachers College, Kearney; Professor A. K. Bettinger, Creighton University; Professor J. M. Earl, the University of Omaha; Professor Hubert Hunzeker, University of Nebraska, Moderator.

4. *Results of the Nebraska Mathematics Contest*, by Professors J. M. Earl, University of Omaha, and H. M. Cox, University of Nebraska.

The 1960 Mathematics Contest had 2,616 students from 140 Nebraska High Schools enrolled. This compares with 1,635 and 2,428 students from 127 and 133 schools in 1958 and 1959. Of the total of 211 different schools, 63, 62, and 86 were entered in 3, 2, and 1 contests respectively.

5. *On a necessary and sufficient condition that a ring be a field*, by Professor G. H. Meisters, University of Nebraska.

6. *Latin squares*, by Professor Trevor Evans, Emory University and the University of Nebraska. (Invited paper)

7. *Factorizable groups*, by Professor William Scott, University of Kansas. (Invited paper)

8. *Equations that have only trivial models*, by Mr. Robert Schwabauer, University of Nebraska.

As applied to groupoids, the trivial one element algebra is a model of every equation. For some equations it is the only model. Such an equation must be of the form $x=\alpha$, where x is not the first or last variable in α. A sufficient condition for $x=\alpha$ to have no nontrivial models is that no variable appear more than once in α and x is not the first or last variable in α.

9. *An elementary proof of the Schroeder-Bernstein theorem*, by Professor H. H. Schneider, University of Nebraska.

The Schroeder-Bernstein Theorem states: If A, B and C are sets such that A is a subset of B and B is a subset of C, and if there exists a one-to-one correspondence between the elements of A and of C, then there exists also a one-to-one correspondence between the elements of A and of B. The proof of this theorem is given by elementary means, that is, no use is made of the axiom of choice in any form whatsoever. No construction is employed which involves any countable process such as recursive definitions of mappings, etc., or which uses properties of indefinite sets. The proof can easily be formalized within an axiomatic treatment using only the basic set-forming axiom-schemata.

10. *Recent investigation of the Selberg sieve method*, by Professor W. E. Mientka, University of Nebraska.

This paper indicates how the lower bound of the Selberg sieve method was used to obtain two theorems concerning the problem of the existence of at least one prime between consecutive perfect squares. The problem of determining the upper and lower bounds for various sets of numbers satisfying certain conditions is also considered along with possible application to some unsolved problems in number theory.

11. *Semigroups with restricted subsemigroups*, by Mr. C. V. Heuer, University of Nebraska.

Call a semigroup $(S,.)$ a T-semigroup if each proper subsemigroup of $(S,.)$ possesses the property of T. Several results concerning the structure of such semigroups are obtained, of which the following is typical. Theorem: Let $S=(S,.)$ be a semigroup each of whose proper subsemigroups is a group. (i) If S is cyclic, with generator a, then S is finite and either S is a group or $S-(a)$ is a group. (ii) If S is not cyclic then either S is a group or S is the multiplicative semigroup of the field with two elements.

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE SOUTHEASTERN SECTION

The thirty-ninth annual meeting of the Southeastern Section of the Mathematical Association of America was held April 1–2, 1960, at the Wade Hamilton Hotel, Columbia, South Carolina, with the University of South Carolina acting as host institution. Professors T. C. Carson, Chairman of the Section, T. H. Lee, Vice-Chairman, H. A. Robinson, W. L. Williams and E. B. Shanks presided over the general and divisional sessions. There were 176 in attendance, including 140 members of the Association.

The following officers were elected for the coming year: Chairman, Professor H. A. Robinson, Agnes Scott College; Vice-Chairman, Professor E. B. Shanks, Vanderbilt University; Secretary-Treasurer, Professor C. L. Seebeck, Jr., University of Alabama. Professor J. G. Kemeny, Dartmouth College, presented the featured lecture at the invitation of the Section and the University of South Carolina. At the business meeting a committee consisting of Professors H. A. Robinson, Tomlinson Fort, and C. L. Seebeck, Jr. was formed to propose a plan for splitting the Section. It was decided that a mail ballot of all members of the Section be the means of deciding on the committee's proposal.

The Section took no action on a proposal to form a committee to study participation in the High School Mathematics Contest.

The following program was presented:

1. *Bounds for positive solutions of systems of linear diophantine equations*, by Professor B. M. Seelbinder, Wake Forest College.

Consider the system of diophantine equations $\sum_{j=1}^{k} a_{ij}x_j = v_i (i=1, \cdots, n)$, where $n \leqq k$, $a_{ij} \geqq 0$, are given integers such that $\sum_{i=1}^{n} a_{ij} > 0$ for $j=1, \cdots, k$ and the n-rowed minors of the coefficient matrix A have g.c.d. unity. Let D be any positive valued n-rowed minor with elements a_{ij_r}, $r=1, \cdots, n$. The claim is made that for solvability of the system in positive integers it is sufficient that $(v_1, \cdots, v_n)$ lie in the interior of a convex polyhedral cone determined by the rays from the point $(f_1, \cdots, f_n)$ to the points $(f_1+a_{1j_r}, \cdots, f_n+a_{nj_r})$, $r=1, \cdots, n$, where

$$f_i = (D/t) \sum_{j=1, j \neq j_r}^{k} a_{ij},$$

t=g.c.d. of elements of D; r, $i=1, \cdots, n$.

2. *Variation of parameters: an explanation*, by Professor C. G. Phipps, University of Florida.

In solving differential equations, textbooks merely show *how* the method of variation of parameters works. The purpose of this paper is to show *why* it works.

3. *Generalization of two well-known results concerning Hermitian forms*, by Professor Lonnie Cross, Atlanta University, (presented by title).

This paper establishes the following two generalizations of two well-known results concerning Hermitian forms: (1) Let A and B be bounded, Hermitian integral operators defined on $L^2(a, b)$ and be such that A is completely continuous, B is strictly positive definite, $|(A\phi, \phi)| \leqq C(B\phi, \phi)$ $(C>0)$, $\phi \in L^2(a, b)$. Then the equation (*) $A\phi = \rho B\phi$ has a discrete spectrum. Moreover, if A is positive definite, then for any $\phi \in L^2(a, b)$ $(A\phi, \phi) = \sum_{\nu=1}^{\infty} \rho_\nu |(B\phi, \phi_\nu)|^2$, where the ρ_ν are the eigenvalues of (*), ϕ_ν the corresponding eigenfunctions. (2) (Minimum-maximum characterization of the eigenvalues of (*))Let $g_1, \cdots, g_{n-1}$ be elements of $L^2(a, b)$ and denote by $\rho_n(g_1, \cdots, g_{n-1})$ the maximum of $(A\phi, \phi)$ under the conditions $(B\phi, \phi)=1$ and $(B\phi, g_i)=0$ $(i=1, \cdots, n-1)$. Then the nth positive eigenvalue ρ_n^+ is equal to the minimum of $\rho_n(g_1, \cdots, g_{n-1})$ for all possible choices of the functions $g_1, \cdots, g_{n-1}$, and it is attained for $g_i = \phi_i^+$ $(i=1, \cdots, n-1)$.

4. *Matrix slide rules*, by Professor E. P. Miles, Jr., Florida State University.

This paper will appear as a Classroom Note in this MONTHLY.

5. *Research and teaching in college*, by Professor T. C. Carson, East Tennessee State College.

The speaker called attention to a growing tendency of college and university administrations to stress research rather than teaching. This increasing practice has the following very serious results: 1st. To place inexperienced fellows and scholars to do the teaching of young students while the older experienced and higher salaried men are engaged in research is cheating them of their rights as students and their parents of the money they have invested in the school and students' expenses. 2nd. The grading of faculty members by the amount of research published is a very poor standard due to varying subjects and amount of time it takes for each subject. 3rd. The emphasis on research is harmful to research itself. Failure to develop sound backgrounds in the student tends to fail to develop promising candidates for research in the future. Teaching is the most important function of a college faculty. Research should be a secondary part of its work.

6. *Some experiences in teaching advanced mathematical concepts to talented 4th–9th graders*, by Professor Robert Kalin, Florida State University.

Experiences with Florida State University Mathematics Camp, mathematics clubs, and demonstration classes suggest that bright 4th–9th graders can learn abstract, advanced mathematical

ideas, if properly presented. Subject matter taught has included equation and inequality solution, finite arithmetics, etc. Demonstration classes proved excellent for showing teachers new ideas in mathematics and its instruction.

7. *The hyperbolic functions are elementary*, by Professor C. L. Seebeck, Jr., University of Alabama.

The functions cosh ϕ and sinh ϕ are defined as the coordinates (x, y) of the curve $x^2-y^2=1$, where ϕ is the usual area. Starting with three defining postulates for logarithms, the functions $\ln x$ and e^x are developed, and the usual expressions for the hyperbolic functions in terms of e^ϕ are obtained using only high school mathematics.

8. *A matrix problem for college students*, by Professor R. C. Blackwell, Furman University.

The determinant of the square matrix formed by taking the variable in every position and adding a constant to every position of the main diagonal was shown to be the product of the constants plus the variable multiplied by the sum of the products of the constants taken $n-1$ at a time. The proof was given by altering the first two rows and applying a Laplace development and an induction. It was suggested that the evaluation might be a good exercise for students beginning the study of determinants.

9. *An application of skip-sampling to student performance*, by Professor M. C. Palmer, Clemson College.

While most of us who teach mathematics would like to direct our efforts toward our more zealous students, we often find ourselves teaching larger and larger sections in which many of the students are somewhat less than ideal. The purpose of this paper is to focus attention to the problem of producing a maximum effort from the more or less dilatory students while preserving the energy of the instructor for the more pleasant aspects of his occupation. A method of sampling students' assignments is proposed which places a large measure of pressure on the students while requiring a modest amount of paper work by the instructor.

10. *Global stability theory for a particular system of nonlinear differential equations*, by Professor J. A. Nohel and Mr. W. P. Timlake, Georgia Institute of Technology, presented by Mr. Timlake.

For the real nonlinear system (*)$\dot{x}=-a_0(y)y-\sum_{i=1}^{n}a_i(z_i)z_i$, $\dot{y}=b_0(y)f(x)$, $\dot{z}_i=-h_i(t, x, y, z_1, \cdots, z_n)z_i+b_i(z_i)f(x)$, $i=1, \cdots, n$, where a_i, b_i, h_i, f are given functions and $(\cdot=d/dt)$, sufficient conditions are given which insure that every solution of (*) exists on $0\leqq t<\infty$ and approaches zero as $t\to+\infty$. This result, an extension of a theorem of Levin and Nohel (to appear Arch. Rational Mech. Anal., 1960), can also be obtained from a theorem proved independently by Levin.

11. *On criteria for nonlinear stability*, by Professor C. W. McArthur, Florida State University.

For each real number t let $z(t)$ denote an element of Euclidean n-space E_n and $A(t)$ an $n\times n$ real matrix. Let $\lambda(t)$ and $\mu(t)$ denote the least and largest eigenvalues respectively of the matrix $\frac{1}{2}[A(t)+A^T(t)]$ where A^T is the transpose of A. Let f denote a function on E_n to E_n such that $f(0)=0$. Suppose $z(t)$ is a nontrivial solution of the system $dz/dt=A(t)z+f(z)$. It follows that $\lambda(t)-\|f(z)\|/\|z\|\leqq d(\ln\|z(t)\|)/dt\leqq\mu(t)+\|f(z)\|/\|z\|$. Further, if $(\|f(z)\|/\|z\|)\to 0$ as $\|z\|\to 0$, if $\mu(t)$ is integrable, and if there is a $b>0$ such that $\mu(t)+b<0$ for all t, it is shown that the identically zero solution of the above system is asymptotically stable.

12. *A stability theorem for a system of delay differential equations*, by Mr. L. J. Grimm and Professor J. A. Nohel, Georgia Institute of Technology, presented by Mr. Grimm.

Results for a single delay differential equation by the senior author (J. Math. Phys., vol. 38, 1960, pp. 295–311) are obtained for a particular system of such equations.

13. *An analysis of the Rosser rocket function*, by Mr. W. L. Hales, University of Alabama.

The generalized Rosser rocket function and its relationships to other functions are examined. It is found to generate a series of incomplete Beta functions.

14. *Mean value theorems for functions with finite derivates*, by Professor D. B. Goodner, Florida State University.

Since the law of the mean and many of its generalizations require the existence of the derivative, it seems desirable to have expressions which may serve us when there is no derivative. This paper presents extensions of the law of the mean, Taylor's formula with remainder, and Cauchy's generalized law of the mean to functions which have finite one-sided upper and lower derivates.

15. *Differential equations of the first order and first degree*, by Professor Tomlinson Fort, Emory University.

This paper points out the advantages of treating the equation of the first order and first degree by definitions and methods of solution that are symmetric in x and y. Such a treatment avoids difficulties brought in by implicit functions. The paper emphasizes the necessity of discussing domain and range at all times. A definition of a solution is given and it is shown that many functions not usually considered solutions must properly be so described.

16. *On the representation of integers by ternary quadratic forms*, by Professor E. H. Hadlock, University of Florida.

The method employed by L. E. Dickson in finding the progressions of integers which are represented by a genus of positive ternary forms with given invariants is generalized in the form of several theorems. Applications of these theorems involve very little work in finding the progressions of integers represented by a genus of positive forms. The theorems also apply to indefinite forms.

17. *The summation of trigonometric series*, by Professor C. B. Smith, University of Florida.

It is well known that certain special types of trigonometric series may be summed by means of the theory of residues. This summation depends on the integration of an appropriate meromorphic function around the contour $|z| = R$, where R is eventually taken to be infinite. This paper gives a method for finding the sums of more general types of trigonometric series by using contour integration around the large circle $|z| = R$ and the unit circle. This procedure also gives the values of some rather unusual looking definite integrals.

18. *Markov chain theory—expository treatment*, by Professor J. G. Kemeny, Dartmouth College.

Markov chain theory is introduced through a series of simple, concrete examples. Basic theorems on both absorbing and ergodic chains are stated without proof, and illustrated in applications. It is briefly shown how the theory of finite Markov chains can be reduced to matrix problems. Some results and certain unsolved problems for infinite Markov chains are mentioned.

19. *Quasi-isomorphisms of certain algebras*, by Professor B. F. Bryant and Mr. J. F. Kepler, Vanderbilt University.

A one-to-one correspondence $x \rightarrow x'$ of a ring R onto a ring R' is called an anti-isomorphism {quasi-isomorphism} provided (1) $(a+b)' = a' + b'$ and (2) $(ab)' = b'a'$ {for each pair a and b in R, at least one of the equations $(ab)' = a'b'$, $(ab)' = b'a'$ holds}. Hua (*On the Automorphisms of an S-field*, Proc. Nat. Acad. of Sci. U.S.A., vol. 35, 1949, pp. 386–389) has shown that each quasi-isomorphism of a ring R onto a ring R' is either an isomorphism or an anti-isomorphism. For algebras with one binary operation, requirement (1) in the above definitions is dropped. Examples were given to show that (1) there do exist proper quasi-isomorphisms of loops; *i.e.*, quasi-isomorphisms which are neither isomorphisms nor anti-isomorphisms, (2) some loops are not properly quasi-isomorphic to any loop, and (3) a loop may be properly quasi-isomorphic to non-isomorphic loops, which is in contrast to the situation for anti-isomorphisms.

20. *Busche-Ramanujan identities*, by Professor P. J. McCarthy, Florida State University.

The class of arithmetical identities called "Busche-Ramanujan identities" by Vaidyanathaswamy (Trans. Amer. Math. Soc., vol. 33, 1931, pp. 579–662) are discussed, and a generalization of these identities is obtained.

21. *Subspaces spanned by discrete polynomials*, by Professor Andrew Sobczyk, University of Florida.

In any vector space of dimension $N=2^\nu$, ν any positive integer, there exists an orthogonal basis of vectors of the form $\phi^{(1)}=\{\epsilon_t^{(1)}\}, \cdots, \phi^{(N)}=\{\epsilon_t^{(N)}\}$, $t=1, \cdots, N$, where $\epsilon_t^{(k)}=\pm 1$ for all k and t. Denote the orthonormal polynomials on the set $(1, \cdots, N)$ by $p_0(t), \cdots, p_{N-1}(t)$, the degree being indicated by the subscript. The matrix of the orthogonal transformation between $\phi^{(1)}, \cdots, \phi^{(N)}$ and $p_0, \cdots, p_{N-1}$ is determined; in particular if S_k denotes the subspace containing p_k which is spanned by the minimal subset of $\phi^{(1)}, \cdots, \phi^{(N)}$, it is shown that S_0, S_1, S_2 are mutually orthogonal, of respective dimensions 1, ν, and $\frac{1}{2}\nu(\nu-1)$.

22. *On certain functions of a set and a point*, by Professor L. T. Ratner, Vanderbilt University.

This paper deals with real-valued functions $f(x, S)$, where x and S are a point and a subset, respectively, of a given topological space X. Several conditions on X, S and/or $\{f(x, S)\}$ are developed which insure the continuity of $g(S)=_{Df} f(x, S)$ from 2^X to R^X.

23. *A probabilistic inequality for testing two distribution functions*, by Professor H. P. Kuang, Agricultural and Technical College of North Carolina.

Let the discrete marginal distributions of the two stochastic variables X and Y be $F_1=\{p_i\}$ and $F_2=\{q_i\}$ $(i=1, \cdots, r)$ respectively, where $p_i=P$ $(X=s_i)$, $q_i=P$ $(Y=s_i)$; $\sum_{i=1}^r p_i=1$, $\sum_{i=1}^r q_i=1$, and s_i is an arbitrary interval containing a finite number of the mass points. Let further the corresponding observations of X and Y be $\{n_i\}$ and $\{m_i\}$ respectively, where the sample values n_i and m_i belong to the interval s_i, and $\sum_{i=1}^r n_i=n$, $\sum_{i=1}^r m_i=m$. It was shown that if F_1 and F_2 belong to a certain set of distributions, then for any positive ϵ, $P\{\phi(F_1, F_2)\geqq 1-\frac{1}{2}\epsilon^2\}$ $>1-(r^2+r-1)(1/\sqrt{n}+1/\sqrt{m})^4/\epsilon^4$ where $\phi(F_1, F_2)=\sum_{i=1}^r [(n_i/n)(m_i/m)]^{1/2}$.

24. *Estimation in the Poisson distribution when sample values of $c+1$ are sometimes erroneously reported as c*, by Professor A. C. Cohen, Jr., University of Georgia.

The method of maximum likelihood is employed to estimate the Poisson parameter when erroneous observation or reporting sometimes results in reporting as c, sample values of the random variable that are actually $c+1$. The proportion of erroneous observations is also estimated. Asymptotic variances and covariances of the estimates are obtained, and a numerical example is included.

25. *A note on the theory of Lagrange multipliers*, by Professor E. B. Shanks, Vanderbilt University.

In this note a theorem is proved which justifies the ordinary usage of Lagrange multipliers in determining stationary points of a function under a set of constraining relations. This note differs from other treatments in that the theorem states necessary and sufficient conditions, thus establishing the complete adequacy of Lagrange's method. Also, the theorem is proved without the use of differentials.

26. *On the inhomogeneous heat equation*, by Mr. D. G. Herr and Professor J. A. Nohel, Georgia Institute of Technology. (By title)

The equation (*) $u_{xx}-u_t=g(x, t)$, where g is a given function $(-\infty<x<\infty, t\geq 0)$ is studied under various assumptions regarding g. Results of Gevrey (J. Math. Pures Appl., 1914) and some extensions are obtained in a systematic fashion.

27. *A generalization of Hermite's differential equation*, by Professor Russell Cowan, University of Florida.

The differential equation contains two parameters so selected that if one is set equal to zero, the equation reduces to Hermite's differential equation. For certain integral values of the parameters polynomial solutions are obtained. These polynomial solutions are shown to be orthogonal and the integrated square is calculated. Several interesting relations between polynomial solutions are derived.

C. L. SEEBECK, JR., *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of America was held at the Air Force Missile Development Center, Holloman Air Force Base, New Mexico, on April 29–30, 1960. Professor R. B. Crouch, Chairman of the Section, presided at the afternoon session on April 29, and also at the morning session on April 30. There were 50 persons in attendance, including 35 members of the Association.

The following officers were elected: Chairman, Professor Harvey Cohn, University of Arizona; Vice-Chairman, Professor I. I. Kolodner, University of New Mexico; Secretary-Treasurer, Professor George Baldwin, New Mexico State University.

The following papers were presented:

1. *A remark on Pontrjagin duality*, by Professor George Baldwin, New Mexico State University.

Ellis (Trans. Amer. Math. Soc., vol. 83, 1956, pp. 301–312) showed "the complement of a Cartesian product of spaces, with the usual product topology, is the direct sum of their complements; and the complement of a direct sum of spaces, if its topology lies within a certain interval, is the Cartesian product of the complements." The upper bound of these topologies does not make the direct sum a topological group. The purpose of this remark is to replace the upper bound with a topology which makes the direct sum a topological group; specifically the Whitehead topology.

2. *Imbedding a group in a group with irreducible sets of generators*, by Mr. A. B. Gray, Jr., New Mexico State University, introduced by the Secretary.

It is shown that any group G can be imbedded in a group H such that G is a direct summand of H, H has an irreducible set of generators, and G/H is a direct sum of cyclic groups.

3. *On a representation of solutions of linear second order ordinary differential equations*, by Professor I. I. Kolodner, University of New Mexico.

Let $f\in C_p(I)$, $L=D^2+f$. There exists a positive "amplitude" $u\in C_p^2(I)$ and a corresponding increasing "phase" $g\in C_p^3(I)$ such that every solution y of (*) $Lx=0$ may be represented in the form $Y(t)=au(t)\cos(g(t)-\theta)$, a and θ constants. If (y_1, y_2) form a fundamental set of solutions of the differential equation (*) with Wronskian $\Delta>0$, $\Delta=$constant, then $u=\sqrt{(y_1+y_2)}$, $g:g'=u^{-2}$ form such a pair; conversely to each pair (u, g) there exists a corresponding set (y_1, y_2). If $\Delta=1$, the amplitudes form the class of positive solutions of $Lx=x^{-3}$. If $I\supset[0, \infty)$, $f=1+k$ and $\int_0^\infty |k(\tau)|\,d\tau<\infty$, there exists a unique "preferred amplitude" with property $\lim_{t\to\infty} u(t)=1$. Applications are given.

4. *On the splitting of almost locally pure groups*, by Professor E. H. Walker, New Mexico State University.

Boyer and Walker have defined and made a preliminary investigation of almost locally pure groups. (Pacific J. Math., vol. 9, 1959, pp. 409–413.) One question left open is whether an almost locally pure group splits over its torsion subgroup. Using some results of Baer on extensions of Abelian groups, it is shown by using homological methods that there exists an almost locally pure group that does not split over its torsion subgroup.

5. *A further generalization of the Riesz theory of compact operators*, by Professor Seymour Goldberg, New Mexico State University, introduced by the Secretary.

It was shown by Graves (Trans. Amer. Math. Soc. vol 79, 1955, pp. 141–149) that if X and Y are Banach spaces, K a compact operator from X into Y and A a bounded linear operator from X onto Y, then a number of theorems which were proved by Riesz for the case where $X=Y$, $A=I$ could be generalized. In turn, a number of theorems which were proved by Graves are further generalized by replacing the operator A above by a closed linear operator defined on a domain dense in X and range all of Y.

6. *A general chain rule without components for derivatives of functions in vector spaces*, by Professor J. V. Lewis, University of New Mexico.

Ordinarily the chain rule for derivatives of functions in vector spaces must be stated in terms of partial derivatives and components. A general chain rule is developed without these limitations. Let f, g, h be such functions that $h(x)=g(f(x))$ for every x. $D_u h(x)=D_v g(y)D_u f(x)$ where $y=f(x)$ and v is a unit vector in direction $D_u f(x)$. Thus the derivative of h in direction u is the magnitude of the rate of change of f in that direction times the rate of change of g in the direction in which f is changing.

7. *Probability of ordered pairs in permutations*, by Mr. Milton Levy, White Sands Missile Range, Las Cruces, New Mexico.

If the elements $(1, \cdots, n)$ are permuted to $(a_1, \cdots, a_n)$, the resulting permutation contains an ordered pair if for some value of i, $a_i+1=a_{i+1}$. The permutation contains k pairs if there are k such values of i. Letting $c(n, k)$ be the number of n-permutations containing k pairs, then $c(n+1, k)=c(n, k-1)+(n-k)c(n, k)+(k+1)c(n, k+1)$ with boundary conditions: $c(n, k)=0$ if $k<0$ or $k\geqq n$, $c(n, n-1)=1$. If each permutation is equally likely, then the probability of an (n, k)-permutation is $P(n, k)=c(n, k)/n!=[(n-k+1)/nk!]\sum_{j=2}^{n-k+1}[(-1)^j/j!]$.

8. *A remark on spectral theory*, by Professor J. B. Giever, New Mexico State University, introduced by the Secretary.

Let X be a complex Banach space, T a bounded linear operator on X to itself and $\sigma(T)$ its spectrum. Let A_T be the algebra of equivalence classes of complex functions analytic on a neighborhood of $\sigma(T)$ (two functions are equivalent if they agree on the intersection of their domains). This algebra has the additional useful structure of being the algebra of cross-sections of a sheaf F. Here F is the restriction to $\sigma(T)$ of the sheaf of Riemann surfaces over the complex plane.

9. *A theorem on commutators of symmetric groups*, by Mr. Don Cartlidge, New Mexico State University, introduced by the Secretary.

Let S be a set of cardinality A, where $A>\aleph_0$. Let $S(A, A^+)$ denote the permutation group on S and denote by $S(A, B)$ the set of permutations of $S(A, A^+)$ which move less than B elements of S. The author proved that if α is a fixed permutation of $S(A, A^+)$ with the property that, given any $\beta\in S(A, B)(\aleph_0<B<A^+)$ there exists a permutation γ of $S(A, A^+)$ such that $\beta=\alpha^{-1}\gamma^{-1}\alpha\gamma$, then the number of i-cycles of α is greater than or equal to B, for all $i=1, 2, \cdots, \aleph_0$.

10. *Some aspects of functional analysis and classical analysis*, by Professor A. E. Taylor, University of California, Los Angeles.

11. *Selecting beginning freshmen for calculus classes*, by Professor F. C. Gentry, University of New Mexico.

Many freshmen quite well qualified for a beginning course in analytic geometry and calculus on the basis of high school records and entrance tests fail to do satisfactory work. There seems to be a high correlation between first test grades and final grades in the course. Either a better criterion for selection should be found or students failing the first test should be given more preparation for the course.

12. *Changes in the University Mathematics Curriculum*, by Professor R. B. Crouch, New Mexico State University, Professor M. S. Hendrickson, University of New Mexico, and Professor E. D. Nering, University of Arizona.

DEONISIE TRIFAN, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the Mathematical Association of America was held on April 8–9, 1960, at San Antonio College, San Antonio, Texas. Professors W. T. Guy, Jr., University of Texas, and P. R. Culwell, San Antonio College,

presided. The total attendance was 153 including 105 members of the Association.

The officers for the next fiscal year are as follows: Chairman, Professor P. R. Culwell, San Antonio College; Vice-Chairman, Professor W. I. Layton, Stephen F. Austin State College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

The following papers were given:

1. *The mapping function for a class of Riemann surfaces*, by Professor H. B. Curtis, Jr., University of Texas.

Because each member of a class of Riemann surfaces has been shown to be parabolic in type, a representation of the function which maps the plane conformally onto a surface of the class is obtained.

2. *The embedding of a lattice in a disjunctive lattice*, by Professor D. E. Edmondson, Southern Methodist University.

The theorem is proved that every lattice may be embedded in a disjunctive lattice in such a way that it is the homomorphic image of the disjunctive lattice in which it is embedded.

3. *An analysis of rounding errors in a solution of a symmetric tri-diagonal matrix*, by Mr. P. M. Blair, Rice Institute, introduced by the Secretary.

The partial differential equation $[a(x)u_x]_x=u_t$, arising from flow problems in one space dimension, may be solved by finite-difference methods. The tri-diagonal matrix obtained when an implicit difference equation is used has a solution which resists the growth of rounding errors.

4. *Use of a digital computer to build the multiplication table for the symmetric group on n elements*, by Professor H. A. Luther, Agricultural and Mechanical College of Texas.

By use of sequences whose terms are the consecutive numbers from 1 to n, n-factorial sequences are created. These sequences are then "multiplied" to give the usual table.

5. *Curves with bounded turning*, by Dr. Guy Johnson, Rice Institute.

A continuous curve γ with parametrization $z(t)=x(t)+iy(t)$, $a\leqq t\leqq b$, is said to have bounded turning if and only if $z'(t)$ is a function of bounded variation and $|z'(t)|\geqq\delta(\gamma)>0$. The class of such functions is self-contained in the sense that a change of parameter $t=t(\tau)$, where $t'(\tau)$ is a function of bounded variation uniformly bounded from zero, yields again a curve of the same class. The arc length parametrization is always included. The name derives from the fact that $\theta(t)=\arg z'(t)$ is a function of bounded variation.

6. *Test for the rank of a matrix*, by Professor Louis Brand, University of Houston.

The $m\times n$ matrix

$$M=\begin{pmatrix}P & Q\\ R & S\end{pmatrix}$$

is partitioned with $m=r+j$ and $n=r+k$. Then if the $r\times r$ submatrix P is nonsingular, M will be of rank r when and only when $RP^{-1}Q=S$. When this is the case, the system of m linear equations

$$\begin{pmatrix}P & Q\\ R & S\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}a\\ b\end{pmatrix}$$

in n variables $x_1,\cdots,x_r,y_1,\cdots,y_k$ will be consistent when and only when $RP^{-1}a=b$. Here x, y, a, b, are respectively $r\times 1$, $k\times 1$, $r\times 1$, $j\times 1$ vectors. When consistent, the equations have the solution $x=P^{-1}a-P^{-1}Qy$, where y is an arbitrary column vector with k components.

7. *Concerning a generalization of convexity for functions of one variable*, by Mr. Larry Armijo, Rice Institute.

Convexity is defined with respect to the two parameter family of curves $A\phi_1(x)+B\phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ are continuous in the open interval (a, b) and are such that the two equations $y_1=A\phi_1(x_1)+B\phi_2(x_1)$, $y_2=A\phi_1(x_2)+B\phi_2(x_2)$, $x_1\neq x_2$, are solvable uniquely for A and B. If $\phi_3(x)$ is

defined on (a, b), then ϕ_3 is convex if and only if it satisfies the inequality $D^0(\phi_j; x_i) \geqq 0$, $x_1 \leqq x_2 \leqq x_3$, where D^0 is the determinant $|\phi_j(x_i)|$, $i, j = 1, 2, 3$. Presuming that ϕ_j is a C^2-function, two other similar determinantal inequalities involving ϕ_j and its derivatives are obtained, and the relations among these three inequalities are examined.

8. *An interesting identity*, by Professor W. S. McCulley, Agricultural and Mechanical College of Texas.

The identity $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$ and its generalizations are considered in relation to number theory, algebra, vector spaces and analysis. Uses of these identities by Fermat, Euler, Cauchy, Schwarz, Hurwitz, Gram, Schmidt, and others are mentioned, and their interpretations discussed. Proofs of various forms of the identity are given.

9. *Conformal invariants and quasiconformal mapping*, by Professor George Springer, University of Kansas.

10. *Some theorems on sequences*, by Professor R. K. Meany, Texas Christian University.

Three theorems on sequences are cited: 1) Mercer's Theorems; 2) A theorem of Erdös, Feller, and Pollard; 3) Some theorems of the author. In all these theorems the given sequence has the property that each term is influenced by or depends upon an average (in some sense) of the preceding terms, and from this property convergence is deduced.

11. *Backward continuation in time of the solution to the heat equation*, by Professor J. R. Cammon, Rice Institute, introduced by the Secretary.

When one wishes to predict a temperature distribution for preceding times from a presently known distribution by solutions of the heat equation, an improper boundary value problem must be solved. Such a problem was considered for the rectangle. By requiring the solution of the heat equation in the rectangle to be initially bounded, an a priori bound for the solution on compact subsets was derived, and an approximation by a finite linear combination of solutions was determined.

12. *A correction to a theorem on semi-metric topological spaces*, by Professor J. R. Boyd, Arlington State College.

A counterexample is presented which shows that the definition of the distance function as given in the argument for the statement "a Moore space is a semimetric topological space" (Pacific J. Math., vol. 6, 1956, p. 325) is not correct. A correction is given.

13. *Estimation of parameters for linear regression subject to certain restraints*, by Professor P. D. Minton and Mr. A. E. Crofts, Southern Methodist University.

Two cases of estimation of parameters of curves are considered: (1) Two straight lines constrained to meet at a known value of the independent variable; (b) A family of straight lines constrained to have a common slope or intercept; families of curves which can be transformed to a family of straight lines, *e.g.*, a family of exponentials with one common coefficient. Estimation formulas are given and examples are suggested.

14. *Systematic method for building the permutations of N numbers in a digital computer*, by Professor R. E. Kilmer, Agricultural and Mechanical College of Texas, introduced by the Secretary.

Permutations of N consecutive numbers, from 1 to N, may be built using N data storage locations in a digital computer by using N modulus arithmetic operations following a systematic rearrangement of the numbers. Permutations of N nonconsecutive numbers may be built using $3N$ data storage locations by using N consecutive numbers as index registers.

15. *An observed property of primitive roots of primes of the form* $4n-1$, by Professor Roger Osborn, University of Texas.

All primitive roots for odd primes less than 1000 were computed on the IBM 650. Several

properties of the primitive roots were observed and counted for each prime. It was observed that for most primes of the form $4n-1$ more than half of the primitive roots exceed $\frac{1}{2}p$. A parallel with the occurrence of quadratic residues can be observed.

16. *The mathematics program in the small college*, by Professor W. I. Layton, Stephen F. Austin State College.

A discussion of course offerings in mathematics and requirements for majoring and minoring in mathematics at Stephen F. Austin State College are given.

17. *The justification for a new course*, by Professor R. S. Underwood, Texas Technological College.

Assuming that a practical plane analytic geometry for n variables can be developed, the question here discussed is whether or not this theory justifies a separate undergraduate college course. In considering it we should note that the two surfaces used chiefly in the preservation and diffusion of knowledge are the printed page and the blackboard. The high usefulness of the geometry of space does not automatically warrant the exclusion of a parallel development on the plane, with its own peculiar and obvious advantages. In line with this premise, a textbook has been prepared for a course now being taught.

18. *SMSG twelfth-grade course*, by Professor L. K. Durst, Rice Institute.

19. *The national high school mathematics contest*, by Professor W. H. Fagerstrom, Pan American College.

The Committee on the National High School Contest is an outgrowth of the Contest Committee of the Metropolitan New York Section, which was organized on October 30, 1949. Dr. W. H. Fagerstrom of the City College of New York was elected Chairman. The first contest was held May, 1950, with less than 5,000 students from approximately 200 schools. This contest has grown steadily since its organization. At the Seattle meeting of the Association, on the suggestion of the Contest Chairman, the contest became international under the joint sponsorship of the Association and the Society of Actuaries. The 1960 contest has over 150,000 students from about 5,000 schools in the United States and Canada.

20. *What, besides mathematics, should the mathematics teacher be teaching?* by Professor E. R. Heineman, Texas Technological College.

The mathematics teacher, not only in the secondary school but also in the college, should (1) teach the student how to learn by himself, (2) encourage the student to reason and think logically, and (3) insist upon precise language in the description of and the solution of mathematical problems.

21. *Research on ordinary linear differential equation of second order*, by Visiting Professor Ping-Chang Van, Texas Technological College, introduced by the Secretary.

Part I, FACTOR THEOREM AND FORMULAS: A given equation can be factorized and formulated, if one of the two complementary functions is known. Part II, FORMULA AND PROOF FOR THE METHOD BY VARIATION OF PARAMETERS. Solving the two formulas for the particular integral, one of the two unknowns is the required particular integral and the other is the proof. Part III, PROOFS FOR CO-I.F.E. THEOREM: There exists another equation corresponding to the given one, each complementary function of the one being the integrating factor of the other; three proofs. Part IV, CONDITION THEOREM CONCERNING (1) $f^2(x)+F^2(x)=c^2$ and (2) $f(x)F(x)=c^2$, where $f(x)$ and $F(x)$ are the two complementary functions of the given equation and c a constant. The solution can be given by formula.

C. R. SHERER, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-ninth annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois Wesleyan University, Bloomington, Illinois, on

May 13–14, 1960. Professor D. E. Myers, Chairman of the Section, presided at all sessions. There were 80 persons in attendance, including 58 members of the Association.

Reports were given by the Committee on the Strengthening of the Teaching of Mathematics, the Committee on Contests and Awards, and the Committee on the Role of the Section. The Section approved the recommendation to conduct the High School Lecture Program for another year and instructed the Chairman to appoint a committee to do this. The Illinois Section will pay the expenses of this program. The following officers were elected to serve for the coming year: Chairman, Professor Douglas Daly, Illinois Wesleyan University; Vice-Chairman, Professor Toivo Rine, Illinois State Normal University; Secretary-Treasurer, Professor Wayne McGaughey, Bradley University.

Following a brief welcome by Dean John Smith of Illinois Wesleyan University, the following program was presented:

1. *Information theory*, by Professor David Blackwell, University of California, Berkeley (by invitation)

An exposition of some of the work of Shannon, Feinstein, and others.

2. *Probabilistic methods in number theory*, by Professor W. J. LeVeque, University of Michigan (by invitation)

From among the many applications of probability theory to number theory, the following three were discussed and compared: a) The heuristic principle that the property (or "event") of primality of a positive integer is nearly independent of a variety of other interesting properties which the integer may possess. b) The application of the central limit theorem to additive number-theoretic functions, as initiated by P. Erdös and M. Kac. c) The extension of the various limit theorems of probability theory from the case of independent variables to certain dependent sequences, all elements of which are defined in terms of a single random variable uniformly distributed on [0, 1]. The results are metric theorems describing the "usual" behavior of decimal digits, continued fraction denominators, etc.

3. *Relativity and space travel*, by Professor M. C. Moore, Bradley University (by invitation)

The nature of matter does not allow it to move faster than light. However, future man can still travel over enormous distances, for relativity predicts that he will live more slowly at high speeds. With an apparent acceleration never exceeding that of gravity, he could cross our galaxy and return in a time which for him would be only 45 years, although 200,000 years would have elapsed at home. In half the 45 years, if he never decelerated, he could travel beyond the reach of our largest telescopes. Slides showing diagrams, equations, and astronomical scenes illustrated the talk.

4. *Elementary statistics, an experimental course*, by Professor D. R. Bey, Illinois State Normal University.

Over a period of five years, the author has been experimenting with various topics and sequences of topics suitable for a beginning course in statistics open to students without calculus. Many of the topics included in this course are discussed. Although the direction of the course is not entirely determined, its emphasis is on proofs and mathematical structure.

5. *Elementary Russian mathematics*, by Professor G. H. Miller, Western Illinois University.

An analysis of the Russian mathematics program corresponding to our elementary program indicates that their students cover material which is completed one to two years later in our schools. A basic introduction to such concepts as approximation, equalities, inequalities, and the fundamental laws (commutative, distributive, associative) are considered in the fifth and sixth grades. The official texts used in Russia have a greater emphasis on statement problems and oral computation. Their program is oriented so that the students are provided with a background for higher mathematics.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Virginia, Charlottesville, on Saturday, May 7, 1960. Eighty persons were present, including 70 members of the Association. Professor E. E. Floyd, Chairman of the Section, presided. Professor C. H. Frick, Chairman of the Nominating Committee, placed in nomination the following slate of officers for the coming year: Chairman, Professor D. B. Lloyd, District of Columbia Teachers College; Vice-Chairmen, Professor Mary E. Hamstrom, Goucher College, and Professor S. S. Saslaw, Naval Academy; Secretary, Professor Herta T. Freitag, Hollins College; Treasurer, Professor S. B. Jackson, University of Maryland. They were unanimously elected.

The following papers were presented:

1. *Graphical method for the determination of ionic sites with preferential fixation*, by Dr. Claude Marmasse, Hollins College.

A graphical method for the evaluation of a sum of inverses is used. It concerns the physical theory of elementary processes in the case of the ionization of a polyelectrolyte macromolecule.

2. *On generalized K-nions loops and algebras*, by Professor Volodymyr Bohun-Chudyniv, Morgan State College.

The aims of this paper were: (1) to construct methods of determining the orders of all possible types of the loops, groups, and algebras, which differ from one another at least in the multiplication table; and each multiplication table can be represented by triplets, as in case of K-nions type; (2) to determine the all nonisomorphical types of generalized K-nion algebras and groups of the 10th order, and construct examples for 14th, 20th and other orders; (3) to point out an application of the generalized K-nions algebras for the determining of all types of diagonal Latin squares of the corresponding orders.

3. *Fourier series and eigenfunction expansions associated with a non-self-adjoint differential equation*, by Professor L. I. Mishoe, Morgan State College.

Fourier series may be considered as an eigenfunction expansion associated with the self-adjoint equation: $u''+\lambda u=0$. While Fourier series has been studied with great interest by several mathematicians, apparently no studies have been made connecting expansions associated with self-adjoint equations with expansions for non-self-adjoint equations.

In this paper the non-self-adjoint equation (*) $u''+q(x)+\lambda[p(x)u-u']=0$, where $u(0)=u(1)=0$, is considered. It is shown that if $u_n(x)$ are eigenfunctions of (*) and a_n eigencoefficients determined in the usual manner, then: $\sum_{-\infty}^{\infty} a_n u_n(x)$ is convergent, divergent, or uniformly convergent with the Fourier series $\sum_{-\infty}^{\infty} \exp(2n\pi ix)\int_0^1 h(\xi)\exp(-2n\pi i\xi)d\xi$, where $h(x)=f(x)\exp(g(x))$ for $f(x)$ of class $L^2(0, 1)$ and $g(x)$ is a function of p in (*).

4. *Dissection of the nonacute triangle and the square into acute triangles*, by Mr. P. J. Federico, Examiner-in-Chief, Patent Office, Washington, D. C.

5. *Fourier transforms*, by Professor E. J. McShane, University of Virginia.

For many applications, the theory of the Fourier transform should be based on convergence in the mean; for undergraduates only the Riemann integral is available. In this talk it was shown that if f is square-integrable and has a Fourier transform g, then $|f|^2$ and $|g|^2$ have the same (Riemann) integral over the real axis; and if in addition f has only finitely many discontinuities in any finite interval, the conjugate Fourier transform of the g is f.

6. *Geometrical theorems for abscissas and weights of Gauss type*, by Dr. P. J. Davis, Chief, Numerical Analysis Section, National Bureau of Standards, Washington, D. C.

It is commonplace to apply rules of numerical analysis to computation. It is something of a

novelty to see computation applied to rules of numerical analysis. From tables of Gauss, Laguerre, Hermite, and Lobatto quadrature formulas which have recently been computed with great accuracy, and using the "method of inspection," P. Rabinowitz and the lecturer have formulated and proved a number of asymptotic results relating the abscissas to the weights. Other conjectures, including one by G. Szegö, seem plausible numerically, but proofs are still to be supplied.

7. *Nets and calculus*, by Professor B. J. Pettis, University of North Carolina. (By invitation)

D. B. LLOYD, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-third annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the United States Air Force Academy, Colorado Springs, Colorado, May 6 and 7, 1960. On Saturday the Section enjoyed a joint meeting and lecture with the Rocky Mountain Section of the Society for Industrial and Applied Mathematics. The meeting was divided into several sessions with Professors J. W. Ault, R. M. Elrick, J. W. Querry, J. S. Leech, and A. W. Recht presiding. There were 132 persons registered for the meeting, including 102 members of the Association.

Officers elected at the meeting for 1960–1961 were: Chairman, Professor L. W. Rutland, University of Colorado; Vice-Chairman, Professor L. C. Barrett, South Dakota School of Mines and Technology; Secretary-Treasurer, Professor Leota Hayward, Colorado State University; and Director of High School Mathematics Contest, Professor D. C. B. Marsh, Colorado School of Mines.

The following papers were presented:

1. *Cauchy's theory of characteristics*, by Professor R. W. McKelvey, University of Colorado.

This was a discussion of the recent studies of A. Plis and T. Wazewski on the domain of existence of solutions of the first order nonlinear partial differential equation $F(x, u, u_x)=0$, $x=(x_0, \cdots, x_n)$. Wazewski's derivation [Bull. Acad. Polon. Sci. Cl. III No. 4, 1956, pp. 131–135] of Plis' results [same source, pp. 125–129] is in the context of the classical Cauchy theory of characteristics. The method can be applied, by analogy, to the quasi-linear equation

$$\sum a_i(x, u)u_{x_i}, \ -b(x, u) = 0,$$

and is simple enough to find a place in elementary textbooks.

2. *Variation of parameters by vector methods*, by Professors L. C. Barrett and C. A. Grimm, South Dakota School of Mines and Technology, presented by Professor Grimm.

For the equation, $a_0y'''+a_1y''+a_2y'+a_3y=f(x)$, where the a's and $f(x)$ are functions of x continuous on a closed interval over which $a_0\neq 0$, whose reduced equation has the independent solutions y_1, y_2, y_3, we assume the particular solution $y=u_1y_1+u_2y_2+u_3y_3=\vec{U}\cdot\vec{Y}$, $\vec{U}$ to be determined by substitution. From this dot product representation it follows immediately that $y=\int_{x_0}^{x}[\vec{Y}(x)\cdot\vec{w}(t)f(t)][a_0(t)W(t)^{-1}]dt$, $\vec{w}=\vec{Y}\times\vec{Y}'$, $W=\vec{w}\cdot\vec{Y}''$, the Wronskian. It was shown how to generalize the results to other orders of equations.

3. *Existence and stability of periodic solutions of weakly nonlinear differential equations*, by Dr. H. R. Bailey, The Ohio Oil Company, Denver Research Center.

Existence and stability theorems for periodic solutions of weakly nonlinear differential systems have been given recently in a number of papers using a convergent method of successive approximations. This method was originally considered by Lamberto Cesari in 1940 for linear systems.

In the present paper the existence and stability theorems are specialized to the case of a weakly nonlinear differential equation and then applied to a nonlinear Mathieu equation. A summary of the corresponding results in more general situations is given.

4. *On solutions of second order differential equations by changing variables*, by Professor F. M. Stein and Mr. R. D. Finley, Colorado State University, presented by Mr. Finley.

In this paper the authors consider the various methods of changing the dependent and independent variables in the equation $y''+P(x)y'+Q(x)y=0$, where it is assumed that the solutions are known to exist, to reduce it to some classical form or to an equation which may be solved by standard means.

5. *Particular solutions for systems of:* (1) *nonhomogeneous, linear, ordinary differential equations*, (2) *nonhomogeneous, linear, ordinary difference equations*, by Professors L. C. Barrett and R. A. Jacobson, South Dakota School of Mines and Technology, presented by Professor Barrett.

In this paper Lagrange's identity and the bilinear concomitant, so familiar in ordinary differential equation theory, are generalized and then applied, as an alternative to variations of parameters, in solving systems of nonhomogeneous linear ordinary differential equations. A parallel treatment of difference equations is also given.

6. *An integral transform*, by Professor F. M. Hudson, Western State College.

If it is possible to write a differential equation in the form (1) $g(gF')'+gF'+F=0$, where g represents any function of x, then the solution of the equation can usually be found by use of the integral transform $T[F(x)]=\int_0^\infty K(m, x)F(x)dx$. The kernel $K(m, x)$ will depend on g and will be given by the equation $K(m, x)=g^{-1}\exp[\int-(m/g)dx]$. This transform will have as its basic differentiation property $T[gF']=mT[F]$. If an equation is written in the form (1), then the most convenient transform will be suggested by the function g. The Laplace and Mellin Transforms are special cases of the transform $T[F(x)]$.

7. *School Mathematics Study Group experimentation in Colorado*, by Professors W. E. Briggs and L. W. Rutland, Jr., University of Colorado, presented by Professor Rutland.

The objectives and accomplishments of the School Mathematics Study Group and of the Colorado Junior High Center and Colorado Geometry Center were reviewed. An outline was given of the SMSG texts for grades seven through twelve.

8. *The power series coefficients of certain L-functions*, by Professor W. E. Briggs, University of Colorado, and Professor R. G. Bushman, Oregon State College, presented by Professor Briggs.

An example is given to indicate a general method for determining the power series coefficients of functions defined by Dirichlet series. If χ is a principal character modulo k and $h=\phi(k)/k$, then $L(s)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$ can be written as $h/(s-1)+\sum_{r=0}^{\infty}(-1)^rV_r(s-1)^r/r!$. Let $L_1(s)=L(s)-hs/(s-1)$. Using an integral representation of $L(s)$, the authors derive the Theorem: *If* $u<0$, *then* $\sum_{n\leqq x}n^u\chi(n)\log^r n=h\int_1^x t^u\log^r t dt+(-1)^rL_1^{(r)}(-u)+0(1)$. By setting $u=-1$ and letting $x\to\infty$, it follows that $V_0=\lim_{x\to\infty}[\sum_{m\leqq x}n^{-1}\chi(n)-h\log x]+h$ and

$$V_r=\lim_{x\to\infty}\left[\sum_{n\leqq x}n^{-1}\chi(n)\log^r n-\{h/(r+1)\}\log^{r+1}x\right],\qquad r=1, 2, \cdots.$$

9. *New teacher education program in mathematics at Colorado State College*, by Professor D. O. Patterson, Colorado State College.

The program in mathematics at Colorado State College is to become, beginning in September 1960, more extensive in its offering and follow fairly closely the recommendations of the National Commission on Mathematics. New courses for secondary school teachers include such titles as "set theory," "modern algebra," "probability theory," "analysis," and "statistics." For the elementary school teachers new courses in mathematics are "arithmetic for elementary teachers" and "foundations of arithmetic."

10. *Astronautics program at USAF Academy*, by Colonel R. C. Gibson, USAF Academy.

One of the principal values of the astronautics program at the Air Force Academy is pedagogic, in that mathematics, physics and chemistry are brought together in a challenging and timely way during the two senior semesters.

The astronautics courses are using most, if not all of the mathematics taught in the regular cadet mathematics sequence. Both the mathematics department and the astronautics department are finding it extremely challenging to design the courses to maximize the cadet's ability to use and appreciate the magnificence of mathematics.

11. *Are mathematicians playing fair with Uncle Sam?* by Professor A. W. Recht, University of Denver.

Recent Russian advances in science forced a hysterical Uncle Sam into a crash program to support so-called "modern mathematics." NSF millions have been spent, and perhaps squandered, to support mathematical study groups and institutes to indoctrinate high school teachers. College professors have gone on a binge of riding their mathematical hobbies at government expense. Payola in mathematics is not on the grand scale of TV payola and rigged programs, but has had a profound effect on the morale of mathematics education. Have the mathematical touts had Uncle Sam gamble on the wrong horses? Have the mathematicians a real solution or a racket?

12. *Applications of mathematics in technology*, by Professor N. W. McLachlan, Visiting Professor of Applied Mathematics, University of Colorado (Invited address)

13. *Anti-associative systems*, by Professor C. H. Cunkle and Mr. D. R. Rogers, Utah State University, presented by Mr. Rogers.

A binary operation defined on a set S is anti-associative provided that $a(bc)\neq(ab)c$ for each a, b, c in S. Examples of anti-associative systems of n elements ($n>1$) with one binary operation were constructed, and these were characterized for sets of 2 and 3 elements. A generalization of associativity was defined for binary operations mapping a finite set onto itself. These operations fell into classes which formed the elements of a permutation group whose identity element was the class of associative operations.

14. *A new nomographic treatment of quartic equations*, by Professor C. R. Wylie, Jr. and Mr. Elbert Johnson, University of Utah, presented by Professor Wylie.

The equation $\phi_1(t)+a\phi_2(t)+b\phi_3(t)+c=0$ can be written as the pair of equations $\phi_1(t)+a\phi_2(t)-p=0$, $b\phi_3(t)+c+p=0$, for each of which a nomogram can be constructed, provided that in the second equation $c+p$ be treated as one of the variables. The contributions of this paper are (1) a description of a mechanical device to facilitate the simultaneous use of these nomograms in the trial and error case when a, b, and c are given and t is required, and (2) the application of this procedure to the solution of the quartic equations $Ut^4+t^3+Vt^2\pm t+W=0$ which are obtained from the general quartic equation $x^4+a_1x^3+a_2x^2+a_3x+a_4=0$ via the substitutions $x=(\pm a_3/a_1)^{1/2}t$, the plus or minus sign being chosen according as $a_1a_3>0$ or $a_1a_3<0$.

15. *Probability in differential equations*, by Captain R. L. Eisenman, USAF Academy.

The solution $y=a-b\cos(Ct+d)$, where a, b, d are constants and C is a random variable, is eventually distributed as $F(k)=\cos(a-k)/b$ regardless of the distribution of C. The problem was motivated by perturbation of a satellite in circular orbit.

16. *An elementary method of determining initial conditions for missile trajectories*, by Professor C. H. Cunkle, Utah State University.

A missile trajectory can be predicted by solving a set of simultaneous differential equations. Such solutions can be very costly, and, in the process of designing a missile, methods are sought for reducing the number of solutions required. By assuming that the range is a quadratic function and using three-point interpolation, the proper initial conditions for a desired range are approximated in a minimum number of trials. The process involves only elementary analytic geometry.

17. *A matrix application of Newton's identities*, by Professor D. W. Robinson, Brigham Young University.

A novel proof, which is based upon Newton's identities, is given of the following theorem. *Let A be an n-by-n matrix over a field of characteristic zero or prime $p>n$. Then A is nilpotent if and only if trace $A^k=0$, $k=1, \cdots, n$.* Other applications are also suggested.

18. *An application of Markov processes in inventory theory*, by Professor P. W. Zehna, Colorado State College.

In recent investigations in the area of inventory depletion, it was possible to define a sequential model for issuing items from a stockpile. By considering the ages of the items to be random variables, a stochastic process was determined for each of two issuing schemes that were of interest. Further examination revealed imbedded Markov processes in each scheme. It was then possible to find unique stationary absolute probability distributions for each scheme and compare them on the basis of the statistical equilibrium afforded by the stationary distributions.

19. *Reflections of a mathematician*, by Professor L. J. Mordell, Visiting Professor of Mathematics, University of Colorado. (Invited address—SIAM)

The topics discussed were selected from the following: What is mathematics, and what are the difficulties in its study? How are mathematicians made, and how do they work? How do problems arise, and how are they solved? What help is given by the electronic computers? What part is played by memory and luck, and what kind of mistakes and errors do mathematicians make? Finally there are the aesthetic and international aspects of mathematics.

The following papers were presented by title:

20. *Limits of iterated discontinuous functions*, by Professor Emeritus A. J. Kempner, University of Colorado.

21. *On Whitworth's Exercise 667*, by Lt. D. R. Barr, USAF Academy.

At least four solutions of problems equivalent to Exercise 667 in Whitworth's *DCC Exercises in Choice and Chance* have appeared in the literature since the publication of the latter in 1897 (see references in J. O. Irwin, J. Roy Statist. Soc., vol. 118, pp. 393–396). A new solution, using only basic definitions and the formula for the simultaneous occurrence of exactly m among N events, is presented.

22. *Green functions for systems of differential equations*, by Professors L. C. Barrett and R. A. Jacobson, South Dakota School of Mines and Technology.

The present note is concerned with extending the familiar concept and usage of Green's function to nth order systems of ordinary linear differential equations with two-point boundary conditions.

23. *Methods for calculating principal idempotents*, by Mr. J. C. Higgins, Brigham Young University.

24. *On a generalization of Pythagorean theorem*, by Professor Aboulghassem Zirankzade, University of Colorado.

The generalized Pythagorean theorem, in Euclidean n-space, is well known and a simple proof, using methods of vector analysis, exists. A more elementary proof, using only plane and solid Euclidean geometry, is given for the case $n=3$ or $n=4$. It is also shown how this method could be modified to prove the generalized theorem for the case $n>4$. Because of the existence of this simple proof, the generalized theorem becomes a suitable topic to be offered in secondary school.

F. M. CARPENTER, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-eighth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Mount Mary College, Milwaukee, Wisconsin, on May 7, 1960. Professor C. B. Hanneken, Chairman of the Section, presided. This meeting was held jointly with the May meeting of the Wisconsin Mathematics Council and there were 129 present, including 52 members of the Association and 62 members of the Wisconsin Mathematics Council.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor Henry Van Engen, University of Wisconsin; Vice-Chairman, Professor Earl Swokowski, Marquette University; Secretary-Treasurer, Dr. E. F. Wilde, Beloit College.

The following papers were presented:

Morning Session for members of the Association

1. *A combinatorial problem on integers*, by Professor Frank Wagner, Marquette University.

Let $0 \leqq \alpha \leqq 9$ and $x > 1$ be integers. Let $p(\alpha, x)$ be the number of occurrences of the digit α in the list of integers from 1 up to but not including x. If $x = \sum_{i=0}^{n} \beta_i 10^i$, $0 \leqq \beta_i \leqq 9$, let $B_k = \sum_{i=k}^{n} \beta_i 10^i$. Then $B_n = \beta_n 10^n$, $B_0 = x$. We obtain the relation

$$p(\alpha, B_k) - p(\alpha, B_{k+1}) = p(\alpha, \beta_k 10^k) + \beta_k 10^k \sum_{j=k+1}^{n} \delta_{\alpha\beta_j},$$

where δ is the Kronecker δ. From this we get

$$p(\alpha, x) = \sum_{1} j\beta_j 10^j + \sum_{2} 10^j + \sum_{3} \sum_{i=0}^{j-1} \beta_i 10^i,$$

where the summation ranges in $\sum_1$, $\sum_2$, $\sum_3$, are $0 \leqq j \leqq n$; $\beta_j > \alpha$, $0 \leqq j \leqq n$; $\beta_j = \alpha$, $1 \leqq j \leqq n$; respectively. If $\alpha = 0$, from this last we must subtract $(10^{n+2} - 9n - 19)/81$.

2. *Analytic mappings of a finite area*, by Mr. Daniel Waterman, University of Wisconsin, Milwaukee.

The condition that an analytic function map a domain in the plane onto a region of finite area on its Riemann surface, was the problem under consideration. The Tauberian nature of this condition was discussed and the domains considered were subsets of the circle, part of whose boundary is an arc of the circle and regions in the half-plane of convergence of a Laplace transform with intervals on the abscissa of convergence contained in the boundary.

3. *Some common problems in college mathematics programs*, by Dr. Lawrence Wahlstrom, Wisconsin State College, Eau Claire.

The author listed ten questions for discussion by a proposed Wisconsin Committee for Mathematics in the State Colleges. By meeting regularly for a sufficient length of time, the committee could come to grips with such problems as (1) desirable changes in the sequence of mathematics courses for prospective mathematics teachers; (2) a common proficiency examination; (3) courses and course-content requirements; (4) developing effective liaison with public school teachers; (5) the graduate program in mathematics; (6) pooling staff resources for an effective research program in the teaching of mathematics. This "Wisconsin-idea" would be advisory and not obligatory in actual operation.

4. *Geometry and group theory*, by Dr. G. O. Losey, University of Wisconsin.

The purpose of this exposition was two-fold: (1) to illustrate how the fundamental ideas of group theory may be introduced by the consideration of the group of allowable transformations of a geometry; (2) to indicate the nature of recent studies on the ways that abstract group theorists are using the language and techniques of geometry to learn more about groups. The principal example for part (1) was the group of collineations of a projective plane. In part (2) the work of N. S. Mendelsohn, G. Bachman, and D. G. Higman and J. B. McLaughlin was discussed.

Morning Session—Wisconsin Mathematics Council

5. *The ninth-grade course of the School Mathematics Study Group*, by Mr. Henry Swain, New Trier Township High School, Winnetka, Illinois.

"A First Course in Algebra," one of the sample textbooks produced by the School Mathematics Study Group, puts meaning and good mathematics into algebra by building the course

about a central theme, the structure of the real number system. All the usual manipulations of algebra are found to depend on a few basic properties of numbers. Some proving is done to give some simple experience with proofs, not to make the course completely rigorous. The language of sets, the use of the number line, and absolute values, and the treatment of inequalities all contribute to a course which interests the students, is teachable and is mathematically sound.

Afternoon Session—joint session of the Wisconsin Section M. A. A. and the Wisconsin Mathematics Council.

6. *Geometric algebra*, by Professor Ernst Snapper, Indiana University.

Geometric algebra is a term coined by Artin who wrote a book by this title. This term conveys the fact that the modern algebraist replaces as much as possible concrete algebraic objects like polynomials, matrices, etc., by the abstract geometric concepts which they represent. As an example of this geometric attitude it was discussed how the theory of quadratic forms should be developed in terms of "metric vector spaces." Witt's theorem was discussed as an example of a theorem which could not even be discovered until after the geometric attitude had been adopted.

E. F. WILDE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–31, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA

IOWA, State University of Iowa, Iowa City, October 10–11, 1960.

KANSAS

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, December 3, 1960.

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA, University of North Dakota, Grand Forks, Fall, 1960.

MISSOURI, University of Missouri, Columbia, Spring, 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 5, 1960.

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA, Oklahoma City University, October 21–22, 1960.

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, University of Colorado, Boulder, May, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

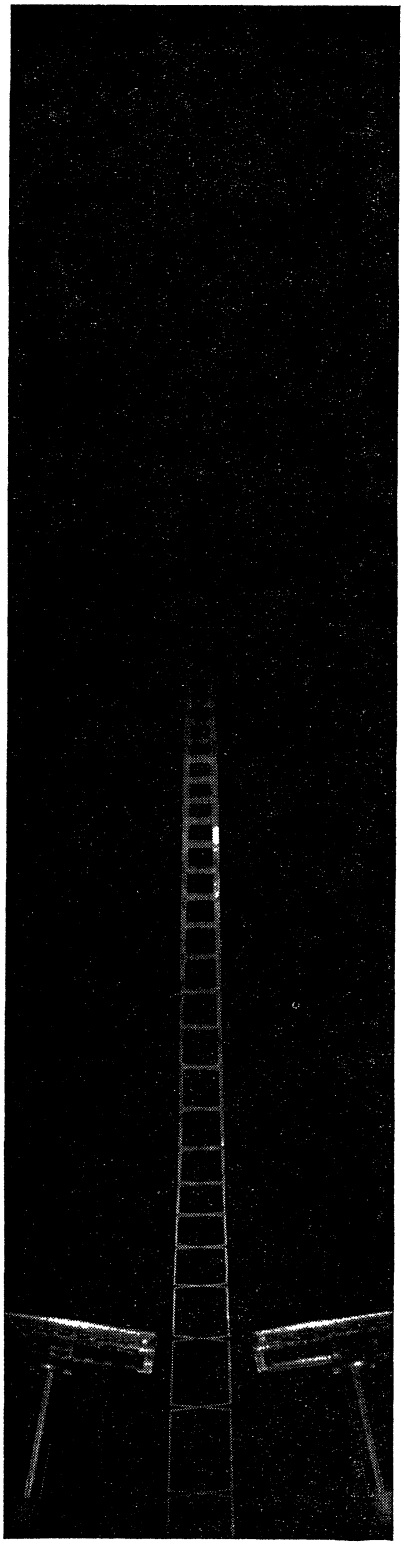
SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, April, 1961.

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Endicott, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May, 1961.



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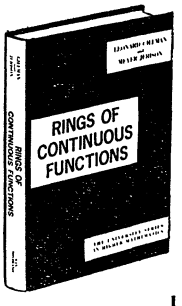
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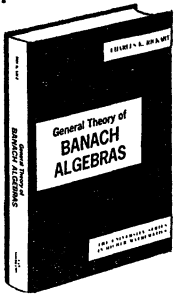
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The Eighth

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R. H. BING

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1. INTRODUCTION

Topology as an offshoot of geometry. In many respects, topology may be considered an offshoot of geometry. It is a study of properties that are somewhat geometric. Just as it would be difficult to define geometry to a person unacquainted with such terms as points, lines, triangles, and planes, it is difficult to give an understandable definition of topology without using some of the terminology of topology. We delay giving a definition of topology until later.

A person may approach the study of topology in somewhat the same spirit that a student begins the study of synthetic Euclidean plane geometry. The student brings into his geometry class a rich background of information about the Euclidean plane. There is a listing of certain properties of the plane. This list is called a set of *postulates* or *axioms*. An effort is made to make the list so complete that the theorems about the plane can be proved on the basis of the postulates alone without making any recourse whatever to our intuitive knowledge about the plane. We strive for efficiency in our list of postulates by eliminating any postulate that can easily be proved from the remaining postulates.

The study of topology frequently starts with a set of postulates or axioms. In a later section we list some axioms that can start us on a study of topology. In plane geometry we use the plane as a model to suggest the postulates. The plane also acts as a model to suggest our beginning axioms for topology. However, a line also makes a good model for these axioms. There are other models as suggested in a later section *Some examples of topological spaces.*

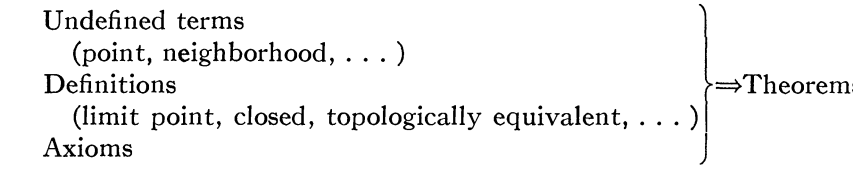
In geometry we try to give explicit definitions of the terms we use. This is in addition to the descriptive discussion accompanied by examples that convey additional information. Explicit *definitions* describe the thing being defined in terms of things whose meaning is already assumed to be understood. This is different from the definitions as given in dictionaries where a "mesh-type" of definition is given and there is no clear understanding as to which term is defined first, but enough synonyms are included so that a person who knows the meaning of many other words is likely to have enough information to decipher the meaning of the word under consideration. In mathematics, definitions are given in a "tree-like" fashion where a term is defined by using previously ac-

cepted terms. We try to avoid the "circular" lines of definitions found in dictionaries.

The basic terms that we start with are not defined in terms of previously defined terms. These are our *undefined terms*. If there is to be a beginning, there must be some term used first. Perhaps a more compelling reason for having undefined terms is that although we may have an intuitive feeling as to the meaning of a basic term under consideration, any meaningful formulation of its meaning involves more complicated terms. For example, the fuzzy definition "A point is that which has position but no dimension" may be criticized on the basis that "position" and "dimension" were used. A similar criticism may be made of the definition "A point is the intersection of two nonparallel lines." In topology we use "point" as an undefined term. In the last section of this article we mention another advantage of undefined terms in topology—namely, if a term is left undefined, any meaning can be assigned to it and as long as the axioms are still satisfied, the theorems we have proved on the basis of the axioms hold true with this meaning. This results in richer interpretations of our theorems.

The set of all points under consideration is called *space*. If we are studying plane geometry, we take the Euclidean plane E^2 to be space. In studying solid geometry, we take Euclidean 3-space E^3 to be space. In topology it is customary to take a set of axioms that will hold in many different spaces and the theorems we prove will hold true in each of the spaces. We suppose that the reader started this chapter with a knowledge of the Euclidean line E^1, the Euclidean plane E^2, and Euclidean 3-space E^3. One section of this article is devoted to a listing of various other spaces.

In geometry we prove theorems on the basis of some axioms, definitions, and undefined terms. We do the same thing in topology—although we use different axioms, undefined terms, and definitions. This schematic diagram shows the axiomatic approach.



Before considering a set of axioms for topology, we shall indulge in an intuitive treatment of several aspects of topology. Our treatment contains some rubber-sheet geometry that may be of interest in spite of the fact that it does not add much to our knowledge of topology. We hope that by mentioning the axiomatic approach in this introduction and by returning to it after a brief sojourn to enrich our topological background in which some topological oddities are mentioned, we leave the impression that the true spirit of topology lies in proving theorems rather than in the oddities.

To point out things that are studied in topology but not in geometry we consider the simple closed curves shown in Figure 1. Geometrically they are



FIG. 1

different. However they are topologically equivalent. In a course in topology we study properties common to such figures. We list three topological properties of a simple closed curve.

1. A simple closed curve is connected. (We give a definition later of "connected" but our intuitive notion of "being in one piece" may suffice here.)

2. If a point is removed from a simple closed curve, the remaining part is connected. (Although a person could not physically remove a point from an object, we can do the operation mentally.)

3. If two points are removed from a simple closed curve, the remaining part is not connected but is in two pieces. (These pieces are called "components" of the remainder.)

Exercise 1. If three points are removed from a simple closed curve, how many components does the remainder have? Into how many pieces does n points separate a simple closed curve?

In some respects there are as many, if not more, simple closed curves in the plane of the fourth type shown in Figure 1 as there are triangles, rectangles, and circles. In fact, if there were some way of selecting a simple closed curve at random from the plane, one might consider it odd indeed if the one chosen even contained a straight line segment or the arc of a circle. It does not require much imagination to see how a person who was seeking to learn about all kinds of plane figures would be led to a study of simple closed curves.

Exercises are scattered throughout the article so that the reader who wishes to do more than skim can exercise. Answers are found at the end of the article. Much of this article resembles the chapter entitled *Point Set Topology* from *Insights into Modern Mathematics*, 23rd yearbook of the National Council of Teachers of Mathematics, and is used with permission.

Perhaps you have already felt a need for better explanations for the meaning of such terms as simple closed curve, topologically equivalent, connected, components, separate. It is hoped that the following sections will partially fill this need. The list of terms found at the end of this article may help the reader in locating definitions.

2. TOPOLOGICAL EQUIVALENCE

We mentioned in the first section of this article that a triangle and a "wiggly" simple closed curve are alike in some respects. We say that they are topologically equivalent. A circle, a square, an ellipse, and a triangle are all called simple closed curves.

Topological equivalence. Two sets A, B are topologically equivalent if there is a 1-1 correspondence between them that is continuous both ways.

The curved arc A and the straight line segment B shown in Figure 2 are topologically equivalent. Although there are many 1-1 correspondences that could be chosen, the projection of A onto B provides a convenient one—that is, a point p of A corresponds to the point $f(p)$ of B directly beneath it.

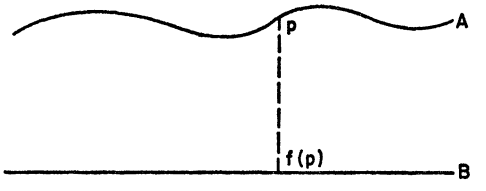


FIG. 2

The 1-1 correspondence from A to B above is continuous because points close together in A go into points that are close together in B. The 1-1 correspondence is continuous the other way because points close together in B correspond to points close together in A. The expression "close together" is not precise and we give a better explanation of continuity in the section *Transformations*.

A 1-1 correspondence of a set A onto a set B that is continuous both ways is called a *homeomorphism* of A onto B. The projection of A onto B is an example of a homeomorphism of A onto B. A homeomorphism of one set onto another set is a special type of transformation and will be discussed further in later sections.

Two sets are *topologically equivalent* if there is a homeomorphism of one onto the other. Accordingly, instead of saying that the two sets are topologically

equivalent, we may say that they are *homeomorphic.* Each set is topologically equivalent or homeomorphic to itself because each point of the set may be made to correspond to itself.

It is possible to get a homeomorphism of a short segment I_1 onto a long segment I_2. (See Fig. 3.) The homeomorphism this time is given by a projection from a point. This homeomorphism shows that, in a certain sense, there are as many points on a short segment as on a long segment.

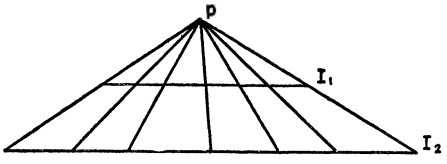


FIG. 3

We mention some other objects that are topologically equivalent or homeomorphic—the surface of a tetrahedron and the surface of a ball; a teacup and a doughnut; the x-axis and the graph of $y=x^2$; the half-open segment* $(0, 1]$ and the graph of $y=1/x$ $(0<x\leqq 1)$. (We suppose that a segment contains its endpoints unless we state otherwise.)

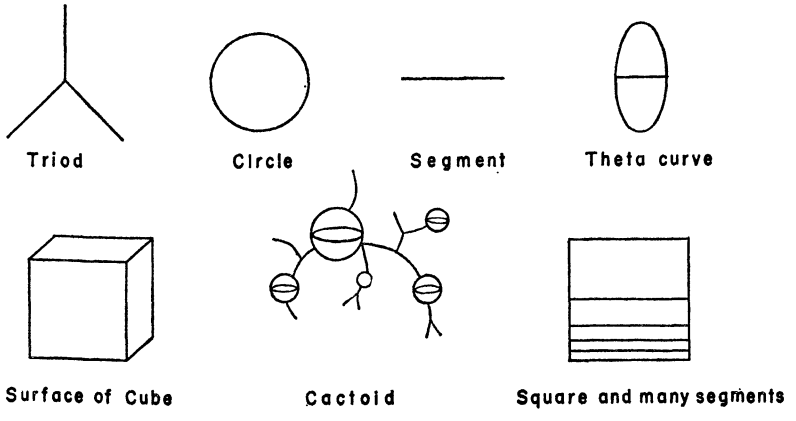


FIG. 4

In Figure 4 we show some objects no two of which are homeomorphic. For example, the circle is not topologically equivalent to the segment because no point of it could be made to correspond (in the proper way) to the end of the segment.

A misconception about topological equivalence. One might suppose that if two sets are topologically equivalent it is possible to deform one onto the other

* By "half-open segment" is meant a segment which contains only one of its endpoints. For example, the half-open segment (0, 1] contains the point 1 but not 0.

by "pulling and stretching but without breaking and tearing." We give some counterexamples to show that this is not always possible.

In E^3 (Euclidean 3-space—the familiar three-dimensional space of everyday experience) consider two sets each consisting of two tangent spheres; in the first set the spheres are tangent externally and in the second set one sphere lies on the interior (except for the point of tangency) of the other. The two sets are topologically equivalent because there is a 1-1 correspondence of the proper sort between the two sets. However, it is not possible to deform one set onto the other in E^3 by "pulling and stretching but without breaking and tearing."

In the next section *Examples of surfaces*, we give other counterexamples to the faulty "pulling and stretching" definition. If one figure can be deformed onto another by pulling and stretching but without breaking and tearing, the figures are topologically equivalent; the converse of this statement is not true. We repeat that two sets are topologically equivalent if there is a 1-1 correspondence between them that is continuous both ways. The definition makes no mention of pulling stretching, and deforming.

Exercise 2. Which pairs of the objects in Figure 5 are topologically equivalent?

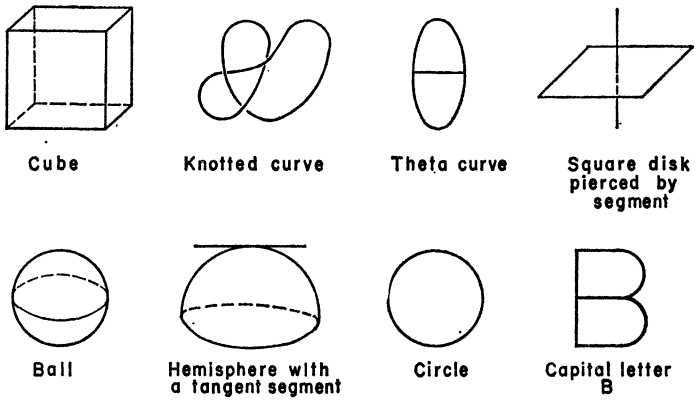


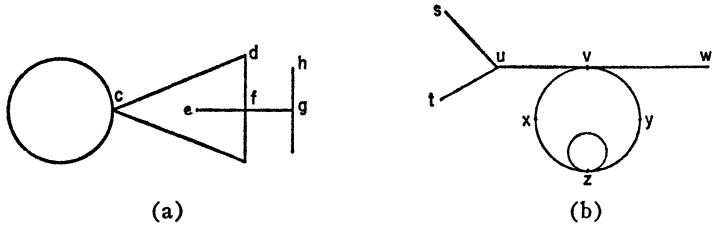
FIG. 5

Exercise 3. Figure 6 shows the capital letters as they may appear without serifs. Group the letters so that letters in the same group are topologically equivalent. Is the grouping different if the letters have width?

A B C D E F G H I
J K L M N O P Q R
S T U V W X Y Z

FIG. 6

Exercise 4. In setting up a 1-1 correspondence between the objects in Figures 7a and 7b to show they are topologically equivalent there are four of the points c, d, e, f, g, h that would necessarily correspond to particular points in Figure 7b, there is one of the six points for which there would be two choices of points in Figure 7b, and there would be another of the six for which there would be many choices. Which points correspond to points c, d, e, f, g, h?



(a) (b)

FIG. 7

3. EXAMPLES OF SURFACES

In this section we give examples of surfaces, some of which are familiar. These examples provide more counterexamples to the faulty "pulling and stretching" definition of topologically equivalent discussed in the preceding section.

Bands. Consider rectangular bands as shown in Figure 8. The arrows in Figure 8a indicate that the side AB of the rectangle is to be "sewed" to CD

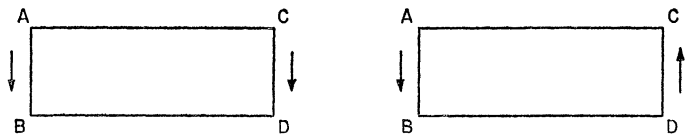


FIG. 8

with A coinciding with C and B with D. Perhaps a cylinder results as shown in Figure 9a but there is no assurance that before the sewing is done, several twists are not made as, say, in Figure 9b, where a full twist is first made. Although the bands in Figure 9a and 9b are topologically equivalent, neither can be obtained from the other by "pulling and stretching without breaking and tearing."

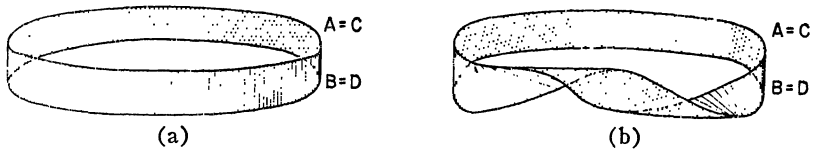


FIG. 9

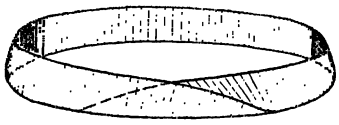


FIG. 10

The arrows in Figure 8b indicate that AB is attached to CD with A coinciding with D and B with C. The resulting band is called a Moebius band. The Moebius band shown in Figure 10 could not be deformed into one with more twists even though the two would be topologically equivalent.

The Moebius band is an example of a one-sided surface. If a fly were to walk around the band an odd number of times, he would be on the "other side" (if there were two sides). Machinery belts are sometimes made into Moebius bands so that they will not wear out on only one side and remain good on the other—they only have one side.

If Moebius bands are split, the results may be unexpected. If it is split down the middle, there results a two-sided band with twists. If this twisted two sided band is split down the middle, there results linking twisted two-sided bands.

The Moebius band has been the butt of jokes and stories. We list several.

1. There are stories of people being given the job of painting one side of a Moebius band and removing all the paint off the other side. See "Alexander Botts and the Moebius Strip," a story by Wm. H. Upson in the December 22, 1945, issue of the *Saturday Evening Post.*

2. One lecturer told the story of how a certain witch doctor kept a supply of bands in the hut in which he gave advice to couples wondering if they were suited to each other. If he wished to prophesy a break-up in their proposed union, he would split an untwisted two-sided band and the two pieces would fall apart. If he wished to prophesy that they would stay together, he would split a two-sided band with a full twist. If he wished to prophesy a really solid marriage, he would split a Moebius band where even after splitting it remained one band. Questioned after his lecture as to the practical value of bands, the lecturer replied, "You might use them if you wanted to be a witch doctor."

3. A three-dimensional poster was used to advertise a certain topology lecture. The poster showed a burlesque dancer whose figure was partially covered by a Moebius band or Moebius strip. The caption was "Would you like to see a Moebius strip? Come to the Auditorium at 8 o'clock tonight."

4. The July, 1949, issue of *Ford Times* carried a story by Wm. H. Upson entitled "Paul Bunyan Versus the Conveyor Belt," telling how Paul Bunyan lengthened a belt in the shape of a Moebius band by splitting it down the middle. Bunyan and his mechanic Fordson won two bets from Loud Mouth Johnson by knowing properties of bands.

5. "No-sided Professor" is the title of an article by Martin Gardner which appeared in the January, 1947 issue of *Esquire.* It tells of the mishaps of a research mathematician who, along with one of his colleagues, disappeared into one of the higher-dimensional spaces while experimenting with odd surfaces,

We mention a more practical use of bands. Suppose a person wishes to study the function $z=y\cos(\pi x/5)$ defined in the strip between the parallel lines $y=1$ and $y=-1$ in the plane. If he divides the strip into rectangles by taking vertical segments on the lines $x=0$, $x=\pm 10$, $x=\pm 20\cdots$, it may be shown that z takes on the same values on one rectangle as on any other rectangle. Hence the function need be studied on only one rectangle. Also if two points on vertical sides of one of the rectangles have the same y coordinates, z assumes the same values on these two points since

$$y\cos(\pi x/5) = y\cos[\pi(x+10)/5].$$

Hence we may consider the opposite vertical sides of the rectangle joined to make a cylinder. (See Fig. 11.)

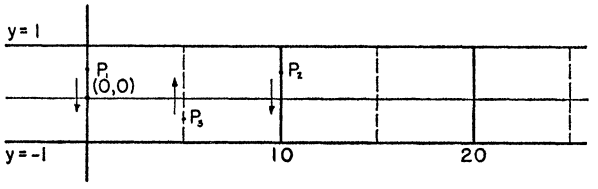


FIG. 11

If one divides the strip into rectangles by taking vertical segments every 5 units rather than every 10 units, two points on opposite vertical sides of one of these smaller rectangles have corresponding z values assigned to them if they are symmetric with respect to the center of the rectangle since

$$y\cos(\pi x/5) = -y\cos[\pi(x+5)/5].$$

Exercise 5. If the vertical sides of this smaller rectangle are joined so that z takes on the same value on joined points, what figure results?

The large and small rectangles with the two vertical sides joined so that joined points take on the same z values showed why Figure 8a and 8b are useful representations of cylindrical bands and Moebius bands.

Spheres, tori, Klein bottles, and projective planes. If a square as shown in Figure 12a were sewn up as indicated by the arrows, AB would be sewn to AD

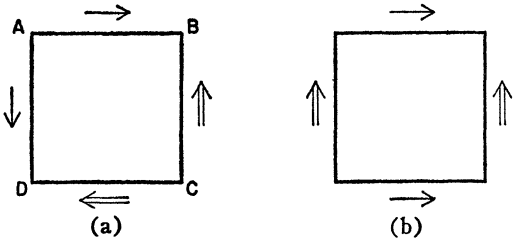


FIG. 12

and *CB* to *CD*. There would result a figure that could be distorted into the surface of a ball. A set topologically equivalent to the surface of a ball is called a 2-sphere.

If only the top and bottom of the square shown in Figure 12b were attached, there would result a cylinder. In order to make the rest of the sewing possible in Euclidean 3-space E^3, we suppose that there was no twisting before the first sewing was done. Now suppose the right and left sides are joined as indicated. There are a number of ways that this may be done, three of which are as indicated in Figure 13.

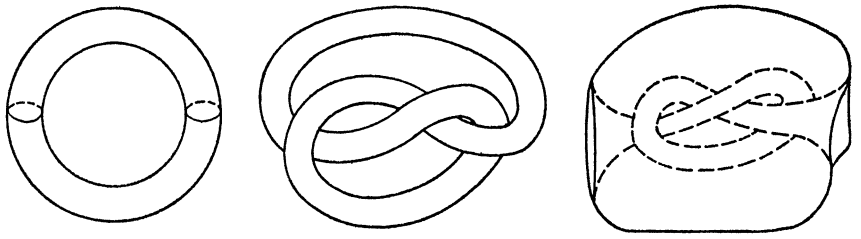


FIG. 13

In Figure 13a, there is no "funny business" and an ordinary *torus* results. In 13b, the cylinder was stretched, a knot was tied in it, and then the final sewing was done. The third is the hardest of the three to visualize. Not only is a knot tied in the cylinder but one end of the cylinder is pulled entirely over the knot before the final sewing is done. If a hole were bored in a cubical block of wood, the surface of the remaining wood would probably be like Figure 13a after certain deformation. If a worm bored a knotted hole in a block of wood, the surface of the remaining wood might resemble Figure 13c. The three surfaces shown in Figure 13 are topologically equivalent to each other but not one of them can be stretched to make it fit on any other one of them. We thus have another counterexample to the faulty pulling and stretching definition.

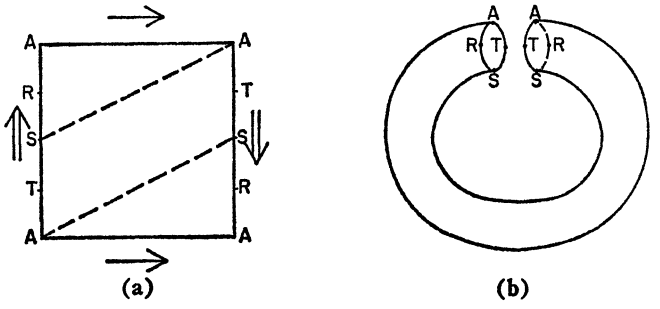


FIG. 14

Not all the objects studied in topology can actually be constructed in E^3. Figure 14 shows such an object. When the first sewing is done a cylinder results as shown in Figure 14b. However, the sewing cannot be completed since the front left arc ARS is to be sewed onto the back right arc ARS and the back left arc ATS is to be sewed onto the right front arc ATS. However we can study objects that we cannot actually construct. If the joining is abstractly made, the resulting surface is called a Klein bottle. If such a surface could be constructed in E^3, it would be a one-sided surface and would have no inside nor outside. However, no such bottle exists in E^3.

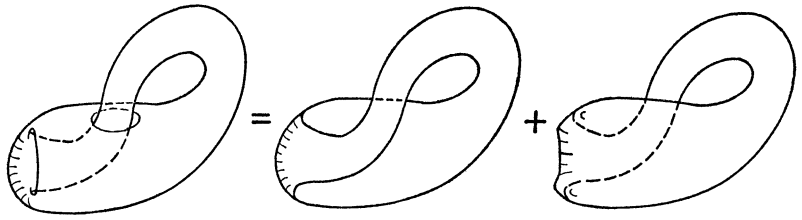


FIG. 15

Glass models of Klein bottles appear much as shown in Figure 15a. The left end of a glass tube is run inside the tube, the ends of the tube are joined together on opposite sides of the middle of the tube, and the viewer is to use his imagination and suppose that they miss the middle of the tube where they join. The tube may be split down the middle to make two Moebius bands as shown in Figure 15b. The cut is indicated in Figure 14a by dotted lines.

If a sphere is subdivided into spherical triangles, an order may be chosen on the boundary of each of these triangular disks so that if two of them share an edge in common, the order on this edge is different when viewed from the different disks. For example, we may order the boundaries of the disks clockwise when viewed from the outside of the sphere. In Figure 16, the orientation of

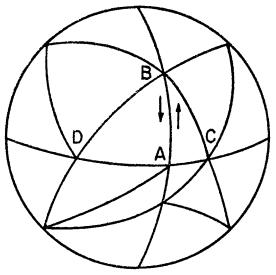


FIG. 16

segment AB is from A to B when considered from spherical triangle ABC but

it is from B to A when considered from disk ABD. Since such an orientation may be chosen, we say that a sphere is *orientable*. Since the rectangle in Figure 14a can be subdivided into small triangles, a Klein bottle can be triangulated into curvilinear disks. However, it is not orientable.

Exercise 6. Suppose the square of Figure 14a is subdivided into 9 squares, each of these squares is divided into two isosceles right triangles, and a clockwise orientation is chosen on the boundary of each triangle. When the sewing is done to get a Klein bottle, where will there be two curvilinear triangles sharing an edge so that the order on this edge is the same from both triangles?

Suppose a top is spinning on the surface of the earth and the top is moved in a path so that it returns to its original position. If the direction of spinning does not change, the top will be spinning in the same direction when it returns. Suppose that instead of having the top reaching out from the surface, we consider a direction of turning (say clockwise) about each point. It is possible to select this direction of turning or orientation so that as a point moves on the surface of a ball the orientation does not change.

The condition is somewhat different on a Klein bottle. Suppose an orientation about a point is chosen and the point is moved the long way around the Klein bottle but so that it returns to its original position. If the orientation is

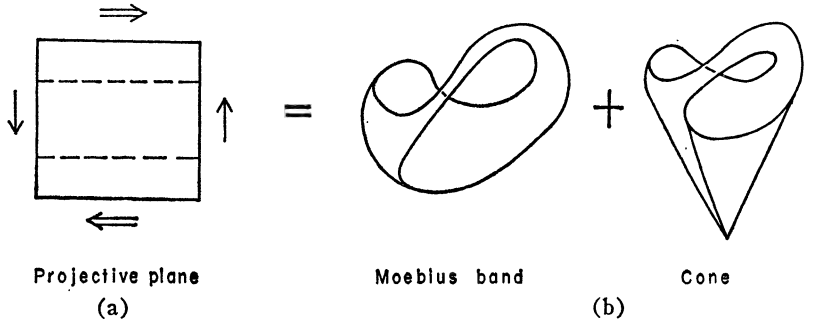


FIG. 17

not changed as the point moves, the orientation will be different at the end from at the beginning. The Klein bottle is not orientable.

The object shown in Figure 17 is a *projective plane*. It is the sum of a Moebius band and a disk. The fitting seems best if the disk is shown as a cone. If it is cut along the dotted lines shown in Figure 17a, the middle portion makes a Moebius band and the outside pieces make a disk. A projective plane is not orientable.

A projective plane is sometimes defined as a circular disk with diametrically opposite points of the boundary connected. The three-dimensional counterpart of a projective plane is projective three space which may be thought of as a solid ball with diametrically opposite points of the boundary joined.

The projective plane is studied in projective geometry. Yet not everyone who studies projective geometry knows of any geometric model for the projective plane.

Exercise 7. Consider a circular disk with a circular hole in the middle as shown in Figure 18. If diametrically opposite points of the outer boundary are joined, what figure results?

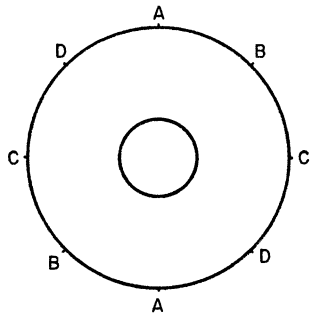


FIG. 18

Exercise 8. Suppose the function $z=\cos x \cos y$ is defined on the horizontally based square in the plane with opposite vertices (0, 0) and (π, π). What figure is formed if opposite vertical sides are sewed together so that z takes on the same value on joined points? If the top and bottom are also joined in this fashion, what figure is formed?

Exercise 9. Describe a function $z=f(x, y)$ that can be defined on a horizontally based square with opposite vertices (0, 0) and (1, 1) so that studying z on the square is equivalent to studying it on a Klein bottle after points of the boundary are joined.

The *complement* of a figure is the set of points not in the figure. (Complement of A in space S is $S-A$.) For example, on the surface of the earth the surface of the oceans, seas, lakes, etc., is the complement of the surface of the land. In the plane the complement of a circle is the sum of its interior and its exterior. The complement of a circle in E^3 would be all in one piece.

We now show that even though two figures in the same space are topologically equivalent, their complements may be topologically different.

In Figure 13, each of the surfaces has an interior and an exterior. It could be shown that the interior of the first is topologically equivalent to the interior of the second but not to the interior of the third. A small creature who knew about topological equivalence but not about straightness, length, etc., could not tell the difference between the inside of the first tube and the interior of the second. However, the exterior of the first tube in Figure 13 is like the exterior of the third but not like the exterior of the second. No two of the three figures have complements that are topologically equivalent.

The knotted curve shown in Figure 5 is topologically equivalent to a circle but their complements are topologically different.

Examples and counterexamples. Although examples are not the main forte of topology, they do serve a useful purpose. A wealth of examples helps our intuition in picking out certain theorems that are likely to be true and labeling others as false.

Examples may be used as counterexamples to certain conjectures and show

that it is useless to try to prove them. A counterexample is a convincing way of showing that a "theorem" is, in fact, false. We know in this way that each of the following conjectures is a false statement.

FALSE STATEMENT A. *If two objects in E^3 are topologically equivalent, it is possible to deform one onto the other with pulling and stretching without breaking and tearing.*

Exercise 10. Name 3 pairs of topologically equivalent objects that serve as counterexamples to False Statement A.

FALSE STATEMENT B. *Each surface is orientable.*

Exercise 11. Name three counterexamples to False Statement B.

FALSE STATEMENT C. *Each surface is two-sided.*

Exercise 12. Name a counterexample to False Statement C.

FALSE STATEMENT D. *If two surfaces in E^3 are topologically equivalent, their complements are topologically equivalent also.*

Exercise 13. Give a counterexample to False Statement D.

4. LIMIT POINTS

The notion of a limit is a very important one in mathematics. A student may do well in arithmetic, algebra, and even geometry without understanding limits, but he must learn this concept in order to go far in mathematics. Although limits are used extensively in calculus, some students do the mechanical parts of this subject without gaining an insight into the concept of limits. Limits are used so extensively in point set topology that it is inconceivable that a person could make much headway here without learning a considerable amount about limits. Learning about limit points is a good starting point.

We use the notion of a limit to tell us whether a point is "infinitely close" to a set—whether it "sticks onto" a set. In such a nice space as the Euclidean plane we use the idea of distance to define limit points. In the more general spaces such as those discussed in the section *Some examples of abstract spaces* we use neighborhoods to define limit points.

NEIGHBORHOOD. *A neighborhood in the Euclidean plane E^2 is the interior of a circle. If a point lies on the interior of a circle, this interior is called a neighborhood of the point.*

Each point has many neighborhoods,—some small and some large, some with the point in the middle and some asymmetric with respect to the point. Each neighborhood is a neighborhood of many points—in fact, of each point in it.

A neighborhood in Euclidean 3-space E^3 is the interior of a sphere; in Euclidean 1-space E^1 (or a line) it is an open segment (segment without its end points)—in fact, in any space with a distance, it is the interior of a generalized sphere. In more general spaces such as those discussed in the section *Some examples of abstract spaces*, there may be no distance function and the neighborhoods are certain designated point sets that satisfy certain conditions or axioms for these abstract spaces. A distance function is defined in the section *Some examples of topological spaces* but our intuitive notion of how far things are apart will suffice here.

NEIGHBORHOOD DEFINITION OF LIMIT POINT. *The point p is a limit point of the set X if each neighborhood of p contains a point of X other than p.*

(Note that a limit point of X is not necessarily a point of X.)

DISTANCE DEFINITION OF LIMIT POINT. *The point p is a limit point of the set X if for each positive number ϵ there is a point x of X such that $0<\rho(p, x)<\epsilon$.*

(We use $\rho(p, x)$ to denote the distance from p to x.) This definition is not applicable in abstract spaces without distance functions.

Exercise 14. Suppose that, in the plane, X is the set of all points satisfying the condition $0<x^2+y^2<1$ or $x=y=1$. Which of the following points are points or limit points of X? $p=(0, 0)$, $q=(0, 1)$, $r=(1, 1)$, $s=(2, 0)$, $t=(0, \frac{1}{2})$.

Exercise 15. In an abstract space whose points are the points of the plane and whose neighborhoods are horizontal lines, which of the points p, q, r, s, t are limit points of the set X of Exercise 14?

Sets. A *point set* is a collection, each of whose members (or elements) is a point. The collection of all points with a particular property R is designated by $\{p \mid p \text{ has property } R\}$. A point set with only one point p is designated by $\{p\}$. We use the following terminology in discussing sets.

SUBSET. *We say that A is a subset of B if each element of A is an element of B.*

We denote this by $A \subset B$. If p is an element of A, we write $p \in A$. We may say that p belongs to A.

SUM OR UNION. *The sum of A and B (denoted by $A+B$ or $A \cup B$) is the set of all elements in either A or B—that is $A+B=\{p \mid p \in A \text{ or } p \in B\}$.*

INTERSECTION, PRODUCT, OR COMMON PART. *The intersection of A and B* (denoted by $A \cdot B$, $A \cap B$, *or* AB) *is the set of all elements in both A and B—that is, $A \cdot B=\{p \mid p \in A \text{ and } p \in B\}$.*

If there is no element which is in both A and B, we say that A does not intersect B and may write $A \cdot B=0$.

Some people use the artificial convention that a collection need have no elements. This may lead to complications in having to make exceptions for the null set, as this artificial thing is called. For example, instead of saying that two parallel lines do not intersect, some of those who use this "null set" convention say that two parallel lines meet in the "empty set." Instead of saying that each bounded collection of numbers has a least upper bound, they say that each nonnull bounded collection of numbers has a least upper bound. The definition of "connected" as given in the section *Properties preserved by mappings* would contain an extra condition if the "null set" convention were used. On the other hand, the convention may have some advantage in the algebra of sets (Boolean algebra) for then two sets always have a product and a difference. We believe the disadvantages of introducing the null set in our treatment of point set topology outweighs the advantages so we do not use it here. Anytime we say that X is a set, we imply that there is at least one element in X.

Not only do we add and multiply two sets but we do the same for three sets, four sets . . . or even an infinite collection of sets. Suppose $\{A_\alpha\}$ is a collection (perhaps infinite) of point sets and A_α is a typical set in $\{A_\alpha\}$. The sum (or union) of the elements of $\{A_\alpha\}$ is the set of all points p such that each belongs to some A_α. We use $\sum A_\alpha$ or $\cup A_\alpha$ to denote this sum. Then $\sum A_\alpha = \{p \mid p \in \text{some } A_\alpha\}$. The product, intersection, or common part of the elements of $\{A_\alpha\}$ may be written $\prod A_\alpha$ or $\cap A_\alpha$. Then $\prod A_\alpha = \{p \mid p \in \text{each } A_\alpha\}$. If there is no point common to all elements of $\{A_\alpha\}$ we say that this intersection does exist and write $\prod A_\alpha = 0$.

DIFFERENCE. *We use $A-B$ to denote the collection of elements of A that do not belong to B—that is, $A-B=\{p \mid p \in A$ and $p \notin B\}$. If $A \subset B$, we write $A-B=0$.*

An example. If a person were to squirt some black ink on the plane, the set A of points in the dark spot is an example of a point set. Suppose a set B is determined by squirting some red ink on the plane. Then $A+B$ designates the set of points covered by ink, $A \cdot B$ designates the set covered by both kinds of ink, and $A-B$ designates those covered by black but not red ink. (See Fig. 19.)

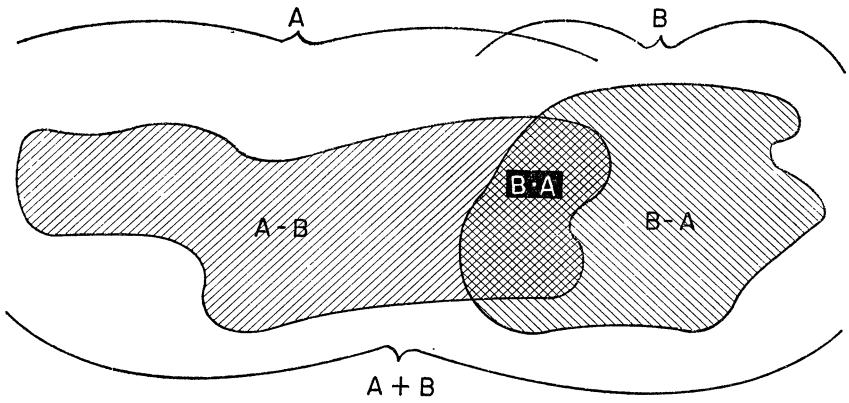


FIG. 19

OPEN SET OR DOMAIN. *A point set D is an open set if for each point p of D there is a neighborhood of p that lies in D. An open set is called a domain. The intersection of a set X and an open set is called an open subset of X.*

(Note that an open subset of X may not be an open set.)

An open set is the sum of neighborhoods, although they may be infinite in number. For the Euclidean plane, the interior of a square is a domain but a disk (circle plus its interior) is not.

Exercise 16. In showing that the sum of two open sets A, B is open, one shows that if p is a point of $A+B$, p belongs in a neighborhood which lies in $A+B$. There are two cases in the proof—either p belongs to A or it does not. If it does not belong to A, show that there is a neighborhood N of p in $A+B$ containing p.

Exercise 17. Suppose S is the interior of a square in the plane. For each point p of S describe a radius r_p such that if $V(p, r_p)$ is the interior of a circle with center at p and radius r_p, then $\sum V(p, r_p) = S$.

CLOSED SET. *A closed set is one which contains all its limit points. Each of its limit points belongs to it. A line is closed in the plane.*

The *closure* of a set X is the sum of X and all its limit points. This closure is designated by $\overline{X}$. The closure of the set X of Exercise 14 is the sum of a disk and $\{r\}$. The closure of the set X of Exercise 15 is the set of all points above the line $y = -1$ and on or below the line $y = 1$.

Exercise 18. For the Euclidean plane give an example of (a) a closed set, (b) an open set, (c) a set neither open nor closed, and (d) a set that is both open and closed.

Exercise 19. What would need to be proved to show that the closure of each set is closed?

Exercise 20. If, in the plane, G is the graph of $y = \sin 1/x$ $(0 < x < 1/\pi)$, what is $\overline{G}$?

It is possible to prove each of the following theorems from the definitions alone without even knowing what a neighborhood looks like:

1. *The sum of two open sets is open.*
2. *The product of two closed sets is closed.*
3. *The complement of a closed set is open.*
4. *The complement of an open set is closed.*
5. *The closure of each set is closed.*

The reader may gain experience by proving these from the definitions. Exercises 16, 21, 22, 23, 24 point in the direction of these proofs. The first two theorems may be generalized to say that the sum of any collection (perhaps infinite) of open sets is open and the product of any collection of closed sets is closed.

Exercise 21. In showing that the product of two closed sets A, B is closed, one considers a limit point p of $A \cdot B$ and shows that p is a point of $A \cdot B$. Show that if p is a limit point of $A \cdot B$, it is a point of A.

Exercise 22. If A is a closed set and p is a point of its complement, show that the assumption that there is no neighborhood of p that misses A leads to the contradiction that $p \in A$.

Exercise 23. If A is an open set and A' is its complement, show that if $p \notin A'$, then p lies in a neighborhood N that misses A'.

Exercise 24. Show that if A is a set and $p \notin \bar{A}$, then p is not a limit point of $\bar{A}$.

5. TRANSFORMATIONS

Ordinarily, a person regards a transformation as a change. A seed becomes a tree, baby becomes an adult, wood turns to smoke and ashes, a bomb changes to fragments, a wad of gum is squeezed, one object is transformed into another.

In mathematics a person may come to regard a transformation as a rule or law associating points of one set X with points of another set Y. If f denotes the transformation and $x \in X$, we use $f(x)$ to denote the point of Y associated with x. (See Fig. 20.)

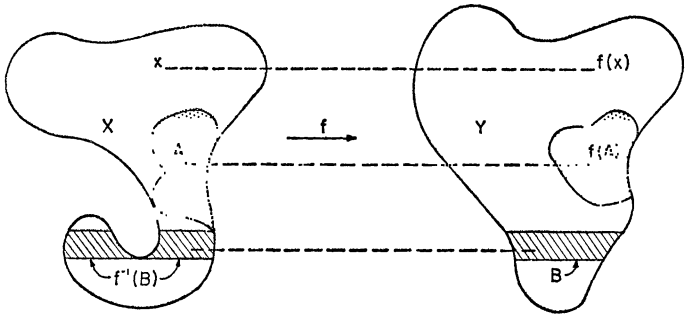


FIG. 20

We call $f(x)$ the *image* of x under f. If A is a subset of X, $f(A)$ denotes the collection of images of points of A. If B is a subset of Y, $f^{-1}(B)$ is the set of all points of X that go under f into a point of B; $f^{-1}(B)$ is called the *inverse image* of B.

A useful and more sophisticated approach is to define a transformation of X into Y to be a collection of ordered pairs that satisfy certain conditions. For example, if one considers the equation $y=x^2$, the corresponding transformation might be regarded as the set of all ordered pairs of the form (a, a^2). The pairs would be the coordinates of points on the parabola. A definition of a transformation based on ordered pairs may seem somewhat artificial and perhaps not exactly describing the intuitive notion that we have in mind, but it does have the advantage of being a precise definition given in terms of more elementary concepts.

ORDERED PAIRS DEFINITION OF TRANSFORMATION. *A transformation of X into Y is a collection of ordered pairs (x, y) such that $x \in X$, $y \in Y$, and each element of X is the first element of one and only one of the ordered pairs.*

FUNCTION DEFINITION OF TRANSFORMATION. *We may regard a transformation of X into Y as a single-valued function that assigns one and only one element of Y to each element of X.*

This definition leaves it to those who define function to decide if it is a change, a rule, a collection of ordered pairs, or something else.

If each element of Y is the image of an element of X under f, we say that f takes X *onto* Y instead of merely into Y. Hence, of the two transformations of the reals into the reals given by $y=x^2$ and $y=x^3$, the second is an onto transformation but the other one is not.

If no element of Y is the image of two elements of X, we say that f is *one-to-one* or 1-1. Hence $y=\sin x$ is 1-1 for $0<x<\frac{1}{2}\pi$ but not for $0<x<\pi$.

Intuitively, we say that a transformation f is continuous if points "close together" in X go into points "close together" in Y. The notion "close together" is not precise so we give the following definition:

NEIGHBORHOOD DEFINITION OF CONTINUOUS. *A transformation f of X into Y is continuous at the point x of X if for each neighborhood U of $f(x)$ there is a neighborhood V of x such that $f(V \cdot X) \subset U$.* (See Fig. 21.)

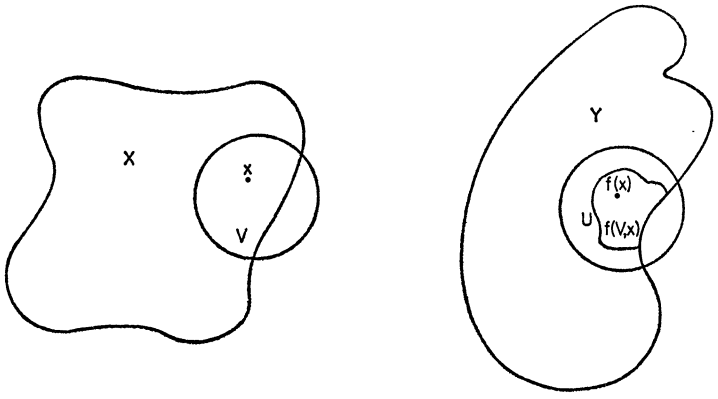


FIG. 21

The size of V in the above definition is a function of x and U—if U is taken to be small, V may need to be chosen small. However, Example 8 below shows that V may not need to be small even if U is.

If the set X on which f is defined is all of space, the definition may be simplified by replacing "$f(V \cdot X) \subset U$" by "$f(V) \subset U$." We could make this change anyway if we called V an open subset of X containing x rather than a neighborhood of x.

The set on which a function is defined is called the *domain of definition* of

the function. Note that a domain of definition may not be an open set or a domain as defined earlier.

A *function is continuous* if it is continuous at each point of its domain of definition X. A continuous function is called a *map*.

For metric spaces (Euclidean spaces and other spaces with distance functions), we may use the following definition:

$\epsilon-\delta$ DEFINITION OF CONTINUITY. *The transformation f of X into Y is continuous at the point x of X if for each positive number ϵ there is a positive number δ such that for each point x' of X within a distance δ of x, $f(x')$ is within ϵ of $f(x)$.*

Let us consider some examples of transformations of the reals into the reals described by equations.

1. $f(x)=x^3-1$.
2. $f(x)=x^3-x$.
3. $f(x)=x^2$.
4. $f(x)=\sin x$.
5. $f(x)=\text{Arc tan } x$.
6. $f(x)=0$ if x is irrational or $0, f(p/q)=1/q$ if $p/q\neq 0$ is rational and in lowest terms.
7. $f(x)=x$ or x^3 according as x is algebraic or nonalgebraic.
8. $f(x)=1-x^2$, 0, or x^2-1 according as $x<-1$, $-1\leqq x\leqq 1$, $x>1$.

Exercise 25. Which of the above examples is (a) onto, (b) 1-1, (c) continuous, (d) continuous at $x=0$?

Here is an example of a homeomorphism f of the interior of a circle of radius 1 onto the plane. The center c of the circle goes into itself and if p is another point of the interior of the circle, $f(p)$ is on the ray from c through p and $\rho(c, f(p))$ is $\rho(c, p)/(1-\rho(c, p))$.

Now let us consider a transformation of one segment AC onto a segment $C'B'$ as shown in Figure 22. The points between A and B go into points of $A'B'$ through parallel projections and the points of BC go into points of $B'C'$ by projections through a point.

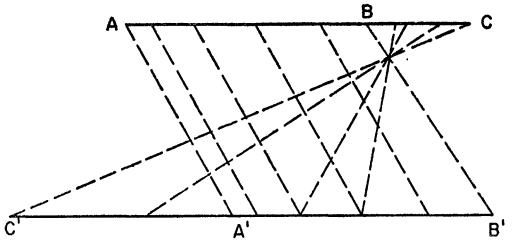


FIG. 22

If P is a point of the segment AB, the ratio in which P divides AB is the

quotient obtained by dividing the distance from A to P by the distance from P to B. The map from AB to $A'B'$ is called *linear* if for each point P of AB, P' divides $A'B'$ in the same ratio that P divides AB. We note that while the map shown in Figure 22 is linear on AB and is linear on BC, it is not linear on AC. We make use of linear maps in the section *The Jordan curve theorem.*

This transformation indicated by Figure 22 may be represented by the graph shown in Figure 23. The interval AC is placed on the x-axis and the interval $C'B'$ on the y-axis. To find the image of a point of AC, project the point vertically onto the graph and then project this point horizontally onto the y-axis. The equation in this case is $f(x)=x+2$ or $17-4x$ according as $1\leqq x\leqq 3$ or $3\leqq x\leqq 4$.

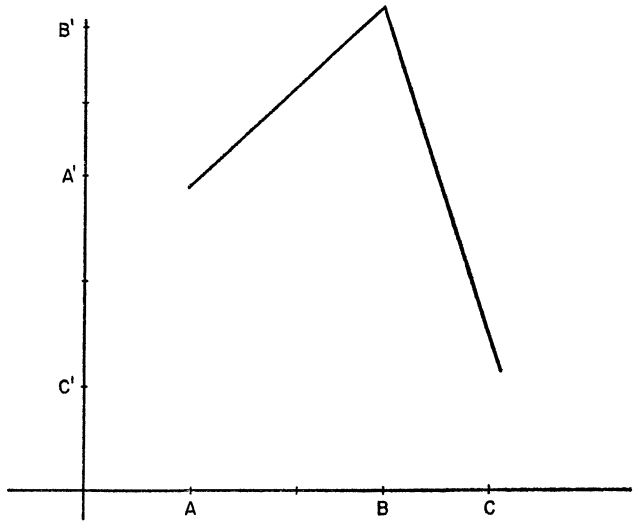


FIG. 23

If X is a square plus its interior and Y is a segment, a transformation of X into Y may be represented by a graph in 3-space which may be a surface. However, if Y were also a square plus its interior, it would require a space of four dimensions to draw the graph—two dimensions for X and two others for Y.

The $\epsilon-\delta$ definitions of continuity of functions given in analytics and calculus texts are puzzling to most students when they are encountered for the first time. In some cases, even an expert could not tell from the definitions given whether or not a function such as $f(x)=\sqrt{(1-x^2)}$, $-1\leqq x\leqq 1$, is continuous at $x=1$ since mention of the domain of definition is sometimes omitted—in other cases the function would be called discontinuous because it is not defined for all values of x. From the definitions we have given, the corresponding transformation would be continuous.

Here is a geometric definition of the continuity of a graph which may be

more meaningful to many students than the corresponding $\epsilon-\delta$ definition. It is understood that the graph is in the plane.

DEFINITION OF CONTINUITY OF A GRAPH. *The graph G is continuous at the point p of G if for each pair of horizontal lines, one above and one below p, there is a pair of vertical lines, one to the right and one to the left of p, such that each point of G between the vertical lines is between the horizontal lines.* (See Fig. 24.)

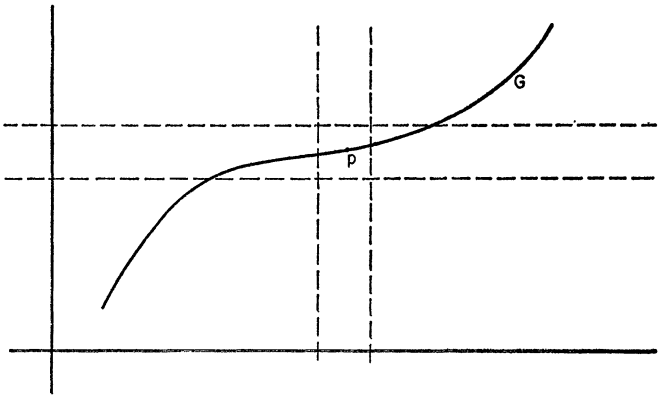


FIG. 24

Exercise 26. What in the above definition corresponds in the $\epsilon-\delta$ definition to (a) x, (b) $f(x)$, (c) ϵ, (d) δ?

Exercise 27. Under what conditions would a graph in 3-space be continuous at one of its points p?

The definition of topologically equivalent given in the section *Topological equivalence* may have been unsatisfactory since no precise definition of continuous had been given. This flaw can now be removed.

A 1-1 transformation f of A onto B is said to be *reversibly continuous* if f^{-1} is continuous at each point of B.

A 1-1 transformation f of A onto B is called a *homeomorphism* of A onto B if it is continuous and reversibly continuous.

Two sets are said to be *topologically equivalent* (or *homeomorphic*) if there is a homeomorphism of one onto the other.

6. PROPERTIES PRESERVED BY MAPPINGS

A continuous transformation is called a *mapping*. In this section we point out that certain properties are preserved by mappings. The following exericse shows that in a certain sense, limit points are preserved.

Exercise 28. Show that if p is a limit point of X and if f is a map of $\bar{X}$ into Y, $f(p)$ is either a point or a limit point of $f(X)$.

Suppose $x_1, x_2, \cdots$, is an infinite sequence of points—we do not suppose that different elements of the sequences are different points. The sequence is said to *converge* to a point x_0 if for each neighborhood U of x_0 there is an integer n such that each point of $x_n, x_{n+1}, \cdots$ lies in U. A proof similar to that used to answer Exercise 28 shows that if f is a mapping of X, X contains $x_0, x_1, x_2, \cdots$, and $x_1, x_2, \cdots$ converges to x_0, then $f(x_1), f(x_2), \cdots$ converges to $f(x_0)$.

The preceding result may remind us of the definition of continuity of a function given in some calculus books. The function $f(x)$ is sometimes said to be continuous at $x=a$ provided the limit, as x approaches a, of $f(x)$ is $f(a)$.

Connectedness is another property that is preserved by mappings. Intuitively, we think that a thing is connected if it is "all in one piece." For example, the disk D shown in Figure 25 is connected but the Cantor set C is not. However, D is the sum of its interior and its boundary, so in a certain sense it is in

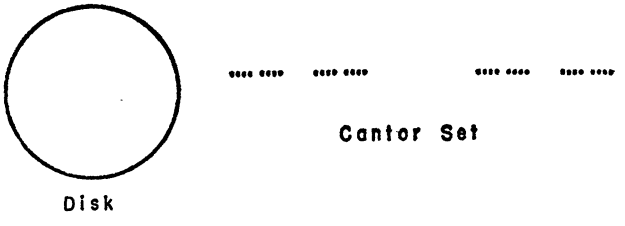


FIG. 25

two pieces. (The *boundary* of M is the intersection of $\overline{M}$ and the closure of the complement of M.) We shall give a more precise definition of connected. The

Cantor set shown in Figure 25 is obtained by starting with a segment, removing the open middle third of it, removing each of the open middle thirds of the remaining pieces, etc.

Two sets are *mutually separated* if neither contains either a point or a limit point of the other. (A, B mutually separated means $\overline{A}\cdot B = A\cdot\overline{B} = 0$.) The interior of a disk and its boundary are not mutually separated since a point of the boundary is a limit point of the interior.

A set is *not connected* if it can be expressed as the sum of two mutually separated sets. The Cantor set is not connected because its right-hand half and left-hand half are mutually separated.

DEFINITION OF CONNECTED. *A set is connected if it is not the sum of any two mutually separated sets.*

Exercise 29. Show that if f is a mapping of a connected set X, then $f(X)$ is connected also.

A closed and connected point set is called a continuum. Being a continuum is not preserved under a mapping—we showed in the last section a homeomorphism of the interior of a circle (not a continuum) onto the plane (a continuum). The inverse of this homeomorphism is a mapping of a continuum onto something that is not closed.

A *component* of a set X is a maximal connected subset of X. Hence, the graph of a hyperbola has two components but the graph of a parabola has only one. The complement of a hyperbola in the plane has three components while the complement of a parabola has only two. A continuum has only one component but a Cantor set has infinitely many.

Being open and closed is preserved in a certain sense by the inverses of mappings. In fact, rather than say that a transformation f of X into Y is continuous if it is continuous at each point of X, it is sometimes defined to be continuous if the inverse of each open subset in Y is an open subset of X.

Exercise 30. Show that if f is a continuous transformation of X onto Y, then the inverse of each open subset of Y is an open subset of X.

Exercise 31. Show that the transformation f of X into Y is continuous if the inverse of each open subset of Y is an open subset of X.

Exercises 30, 31 show that a transformation f of X into Y is continuous if and only if the inverse of each open subset of Y is an open subset of X. Similarly it might have been shown that f is continuous if and only if the inverse of each closed subset of Y is a closed subset of X. (A closed subset of X may not be closed but is merely the intersection of X with a closed set.)

As the population of the earth increases, it is inevitable that there be centers of dense population. In contrast, it is felt that if space travel becomes feasible and space is habitable, the population of the universe can increase without us having crowding. Even in the plane, it is possible to have infinitely many points such that no point is within 100 units of any two of them. (Put the points at (0, 0), (300, 0), (600, 0), $\cdots$.) The limited property of the earth we have men-

tioned reminds us of a topological property called compactness. Compactness is preserved under a mapping if we use the following definition.

DEFINITION OF COMPACTNESS. *A set M is compact if each infinite subset of M has a limit point in M.*

The plane is not compact but, as shown in the next section, a closed segment is. (Some definitions do not impose the condition that the limit point of M need belong to M. If this condition were omitted, the interior of a circle would be compact and Exercise 32 could not be done. Other definitions impose even more stringent conditions.)

Exercise 32. Show that if f is a mapping of X and X is compact, then $f(X)$ is compact also.

A set of tarpaulins is said to cover a football field if each spot of the field is under one of them. Similarly, a collection G of point sets is said to *cover* a point set X if each point of X belongs to an element of G. If each element of G is a domain, G is called an *open covering.*

The collection of all neighborhoods covers the plane but no finite subcollection of them does. However, if a collection of neighborhoods covers the surface of a sphere or a segment, a finite number of them does also. This reminds us of the following topological property that is preserved under a mapping.

DEFINITION OF BICOMPACTNESS. *A set X is bicompact if each open covering G of X contains a finite subcovering—that is, a finite subcollection of G which covers X.*

Exercise 33. Show that bicompactness is preserved by a mapping.

The two properties, compactness and bicompactness, are not equivalent because there are topological spaces that are compact but not bicompact. However, any bicompact space is compact and any compact space with a distance function is bicompact. Therefore, bicompactness is a stronger condition and a person who uses it in the hypothesis of a theorem is not proving as strong a result as one who uses mere compactness instead. Because of lack of agreement on terminology, some people use the term compact to mean bicompact as we have defined it here. Not everyone that mentions the word compact is using the weaker condition.

Exercise 34. Show that bicompactness implies compactness.

A set X is *homogeneous* if for each pair of points p, q of X there is a homeomorphism of X onto itself that takes p onto q. A circle is homogeneous because a rotation will take any point into any other point. A closed segment is not homogeneous because there is no homeomorphism of the segment onto itself that takes an end point onto a non-end point.

A definition of topology. Topology may be defined as a study of properties that remain invariant under homeomorphisms. Such a definition might be meaningless to persons unacquainted with such properties as connectedness,

compactness, bicompactness, homogeneity, etc. If one were to give such a definition to an unenlightened gathering, the question might be asked as to what is a homeomorphism. If a definition of "homeomorphism" were given such as appears in this article, the next question might be, what is the meaning of the word "continuous"? Neither the neighborhood nor the $\epsilon-\delta$ definition of continuity is likely to be of much help. Perhaps it is best not to define topology to people who do not already know what it is.

Unsolved problems. A difficulty in discussing unsolved problems in topology is that most of these problems involve complicated concepts meaningful only to the advanced worker. Most people would have neither the technical vocabulary nor the experience to appreciate the discussion. However, here is a problem which we have at least developed the vocabulary to present.

A *simple closed curve* is a set topologically equivalent to a circle. Is a simple closed curve the only homogeneous compact plane continuum other than a point? For many years it was conjectured that the answer was in the affirmative—in fact some papers appeared in scientific journals giving faulty proofs to this effect. However, in 1948, an example was given of a bounded plane continuum which was neither a simple closed curve nor a point but which was homogeneous nevertheless. In 1954 another example was announced. Are there other homogeneous bounded plane continua?

7. THE JORDAN CURVE THEOREM

The interior of circles and triangles appear to be somewhat alike. However, if one has a complicated simple closed curve such as shown in Figure 26, one might have difficulty in deciding if its interior was like the interior of a circle.

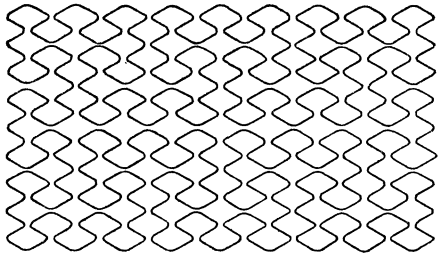


FIG. 26

If the curve had been even more complicated, it might have been impossible to tell by merely looking whether or not it even had an interior.

A subset X of a set Y is said to *separate* Y if Y is connected but $Y-X$ is not connected. It may seem intuitively obvious that a simple closed curve in the plane separates the plane but it is not easy to prove this.

The *Jordan curve theorem* states that the complement of each simple closed curve J in the plane has exactly two components and is the boundary of each of them. One extension of the Jordan curve theorem due to Schoenflies may be stated as follows:

AN EXTENSION OF THE JORDAN CURVE THEOREM. *If J_1 and J_2 are simple closed curves in the plane, there is a homeomorphism of the plane onto itself that takes J_1 onto J_2.*

In plane analytical geometry one studies translations and rotations of the plane onto itself—and perhaps even reflections through a line. These are examples of homeomorphisms of the plane onto itself. Under each of these particular homeomorphisms, a figure would go into a figure congruent to itself. However, the types of homeomorphisms used to show the truth of the above extension of the Jordan curve theorem would frequently change the shapes of objects.

To those who think the theorem needs no proof since it is intuitively obvious, it may come as a surprise that the analogous theorem is not true in 3-space. Although the horned sphere described at the end of this section is topologically like the surface of a ball, there is no homeomorphism of 3-space onto itself that will take one onto the other.

The Jordan curve theorem is one of the more frequently used theorems of plane topology; it is used in such subjects as analysis or complex variables. Although several proofs of this theorem have been given—and some only recently—these proofs are still complicated.

In this section, we only treat the Jordan curve theorem where the simple closed curves involved are uncomplicated. However, if the reader works through even the first of the exercises, he will get some understanding of the meaning of the theorem.

Exercise 35. Describe a homeomorphism of the plane onto itself that takes a given triangle onto a given circle. (See Fig. 27 for a hint as to how to proceed.)

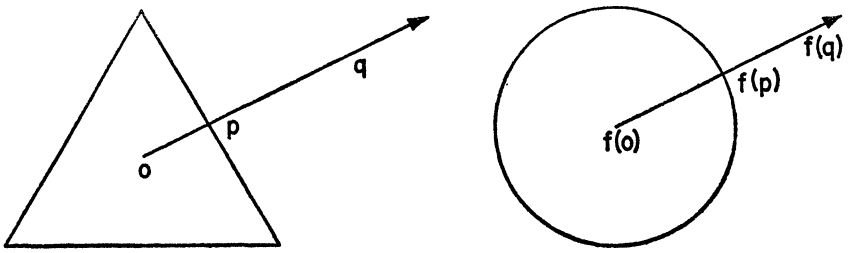


FIG. 27

Exercise 36. Describe a homeomorphism of the plane onto itself that takes a given triangle onto the graph of $\{y-\sqrt{(1-x^2)}\}\{2y-\sqrt{(1-x^2)}\}=0, (-1\leqq x\leqq 1)$.

By using mathematical induction on the number of sides, it can be shown that for any polygon there is a homeomorphism of the plane onto itself that takes the polygon onto a triangle. One without training in topology who obtained an airtight proof of such a theorem could be proud of his accomplishment. Indeed, it is no easy task for the beginner to get even Exercise 37. Some of the methods used in working the following exercise can be used to get the more general result.

Exercise 37. Describe a homeomorphism of the plane onto itself that takes polygon *ABCDEF* of Figure 28 onto a triangle. (See Fig. 29 for a hint as to how to proceed.)

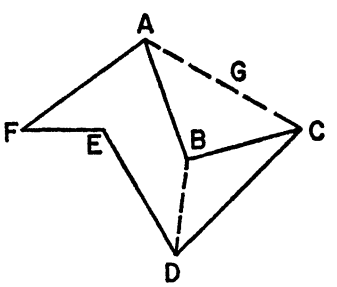


FIG. 28

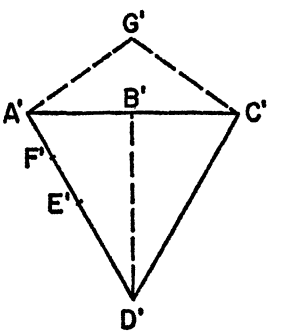


FIG. 29

A set in the plane is *bounded* if it lies on the interior of some large circle. The interior of a triangle in the plane is bounded but its exterior is not.

It is to be noted that we regard a triangle as the sum of its three sides. The complement of a triangle in the plane has two components. If ABC is a plane triangle we use Int ABC to denote the bounded component of the complement of ABC. Either the Jordan curve theorem or the Schoenflies theorem shows that each simple closed curve in the plane has a complement with a bounded component.

The reader who is more interested in getting an overall view of this chapter than in going into details may skip the rest of this section and proceed with the next section *Properties of the real numbers*.

We now give three theorems relating to the Schoenflies theorem. Each is of interest in its own right.

RETRACTION THEOREM. *Suppose P is a polygon and ABC is a triangle such that $P\cdot(ABC+\text{Int } ABC)$ is the sum of two sides AB, BC of ABC. If AC is the third side of ABC, there is a homeomorphism h of the plane onto itself that takes P onto $P-(AB+BC)+AC$.*

Let M be the midpoint of AC and R, S be the points on the line MB as shown in Figure 30 so that P intersects the disk bounded by the polygon $ARCS$

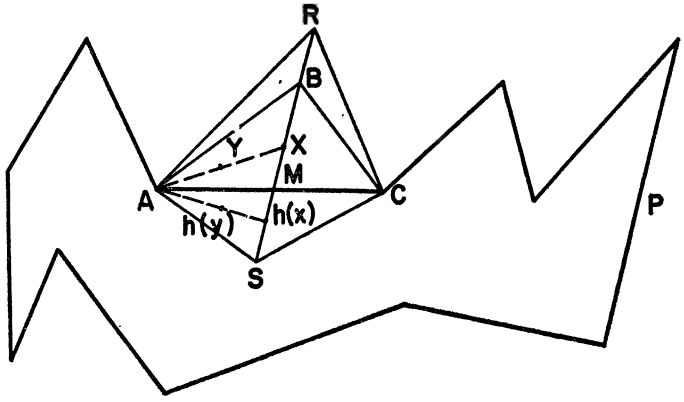


FIG. 30

only in $AB+BC$. The homeomorphism h is the identity outside the disk, takes RB linearly onto RM and BS linearly onto MS.

Exercise 38. If X is a point of RS and Y is a point of Int ARS between A and X, how might h be defined on Y?

SCHOENFLIES THEOREM FOR POLYGONS. *For each polygon P in the plane there is a homeomorphism h of the plane onto itself that takes P onto a triangle.*

SHELLING THEOREM. *If P is a polygon with more than three sides and UV is a side of P, there is a triangle ABC such that AB, BC are sides of P different from UV, and (ABC+Int ABC) lies, except for $AB+BC$, on* Int P.

These two theorems may be proved by induction on the number $n(P)$ of sides of P by making use of the retraction theorem. Each holds for $n(P)=3$ or 4 so we assume $n(P)=k$ is the first integer for which there is a polygon P for which one of the two results does not hold.

Let L be a line of support for this P (P intersects L but not both components of the complement of L) such that L contains two points of P. Since the theorems would be easier to prove if P were assumed to be a convex polygon, we suppose that L contains a segment AB that intersects P exactly in its end points. (See Fig. 31.)

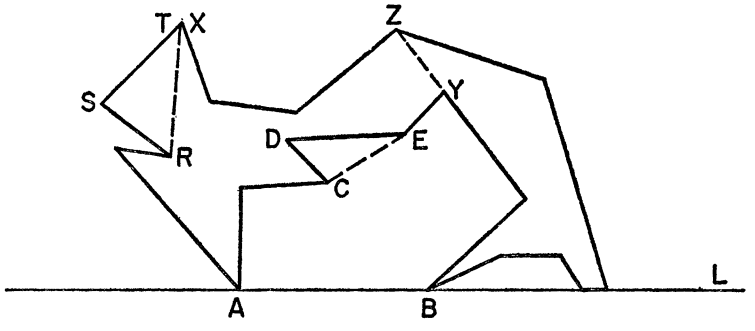


FIG. 31

Let AXB and AYB be the two broken lines in P with ends A and B. With no loss of generality we suppose that X and Y are vertices of P chosen to be at a maximum distance from L and such that X is as far if not farther from L than Y. If X is farther than Y from L, we find that there is a vertex Z of AXB such that Z is farther from L than Y and YZ intersects P only at its end points. If it should happen (which it does not, but we are not yet ready to prove that it does not) that X is the same distance from L as Y, we would select Y so that there is a vertex Z of AXB such that YZ is parallel to L and intersects P only at Y and Z.

We now show that the Schoenflies theorem is satisfied on P. Consider the polygon $AB+AYB$. If it is a triangle, it follows from the retraction theorem that the Schoenflies theorem is satisfied on P. If $AB+AYB$ has more than three sides, the following five exercises indicate why the Schoenflies theorem holds on P.

Exercise 39. Why is the shelling theorem satisfied on $AB+AYB$?

Exercise 40. In applying the shelling theorem to $AB+AYB$ to show that there is a triangle CDE such that CD and DE are sides of P other than UV, and CDE lies, except for $CD+DE$, on Int ($AB+AYB$), what should be used for UV to insure that CD, DE are sides of P?

Exercise 41. Why will not the segment CE of Exercise 40 intersect P at a point other than C and E?

Exercise 42. Name a polygon P' with one fewer sides than P such that the retraction theorem says that there is a homeomorphism of the plane onto itself taking P onto P'.

Exercise 43. If h is a homeomorphism of the plane onto itself taking P onto P' and g is a homeomorphism of the plane onto itself taking P' onto a triangle, name a homeomorphism of the plane onto itself that takes P onto a triangle.

Finally we consider why the shelling theorem is satisfied on P. Let $ZA\,Y$ and $ZB\,Y$ be the two arcs whose sum is P. If UV is a given arc of P, we choose the one of $ZA\,Y$, $ZB\,Y$ that does not contain it. With no loss of generality we suppose it is $ZA\,Y$. The shelling theorem is satisfied on polygon $ZA\,Y+YZ$ since this polygon has fewer sides than P. It follows from this shelling theorem that there is a triangle RST such that RS, ST are sides of $ZA\,Y+YZ$ other than YZ and such that $(RST+\text{Int } RST)$ lies, except for $RS+ST$, on Int $(ZA\,Y+YZ)$. If we show that Int $(ZA\,Y+YZ)\subset$Int P, the triangle RST shows that the shelling theorem is satisfied for P. Consider

$$I(P) = \text{Int}(ZA\,Y + YZ) + \text{Int}(ZB\,Y + YZ) + \text{open segment } (YZ).$$

If we show that Int $P=I(P)$, it will follow that Int$(ZA\,Y+YZ)\subset$Int P. Recall that this was the last link in establishing the shelling theorem for P. It follows from the Schoenflies theorem applied to P that the complement of P has only one bounded component. Hence, we show that $I(P)=\text{Int } P$ by showing that $I(P)$ and the complement $E(P)$ of $P+I(P)$ are mutually separated.

Since $P+I(P)$ is closed, $E(P)$ contains no limit point of $I(P)$. We show that $I(P)$ and $E(P)$ are mutually separated by showing that no point of $I(P)$ is a limit point of $E(P)$. Points of the open segment (YZ) are the only ones in question. No point of this open segment is a limit point of $E(P)$ if Int$(ZA\,Y+YZ)$ contains points on one side of (YZ) and Int$(ZB\,Y+YZ)$ contains points on the other side. This will be the case if Int$(ZA\,Y+YZ)$ does not intersect Int$(ZB\,Y+YZ)$ since each of the open sets has YZ on its boundary. The following two exercises show why these two open sets do not intersect.

Exercise 44. Why does $ZB\,Y$ not intersect Int $(ZA\,Y+YZ)$? Why does $ZA\,Y$ not intersect Int $(ZB\,Y+YZ)$?

Exercise 45. Show that there is no point common to Int $(ZA\,Y+YZ)$ and Int $(ZB\,Y+YZ)$.

J. W. Alexander described the *horned sphere* illustrated in Figure 32. It is topologically equivalent to the surface of a ball but its exterior is not like the exterior of a ball. Recall that a set topologically equivalent to the surface of a ball is called a 2-sphere but we sometimes use the word sphere alone for brevity.

We might describe the horned sphere as follows. A long cylinder closed at both ends is folded until the ends are near each other and parallel. Then tubes are pushed out each end until they hook as shown. The process is continued by pushing out additional tubes, etc. The resulting set has the property that,

although it is topologically equivalent to the surface of a sphere, there is a circle in the exterior of the horned sphere that cannot be shrunk in the exterior of the horned sphere to a point.

Although a horned sphere is topologically equivalent to a sphere, the complement of a horned sphere is not like the complement of a sphere because each simple closed curve in the complement of a sphere can be shrunk to a point without touching the sphere. The simple closed curve J shown in Figure 32 cannot be shrunk to a point in the complement of the horned sphere.

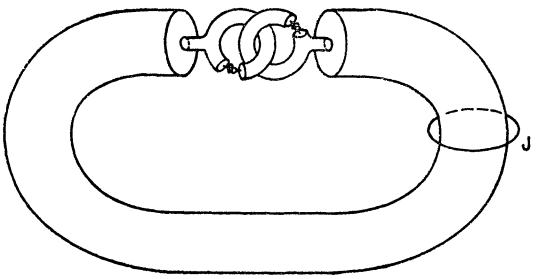


FIG. 32

The four-color problem. There are many unsolved problems in mathematics. It was not known for many years whether or not the Jordan curve theorem was true—though during part of this time many people suspected it to be true. The four color problem is still unsolved. It asks if each map on the surface of a sphere can be colored with four colors if the boundaries of the states are simple closed curves. States whose boundaries meet only at a point are allowed to have the same color, but states whose boundaries meet along an arc are not allowed to have the same color. If some of the states had boundaries that contained part of the graph of $y=\sin 1/x$, we might have a situation such as depicted in Figure 33 so it is essential to impose the condition that the boundaries of the states be simple closed curves.

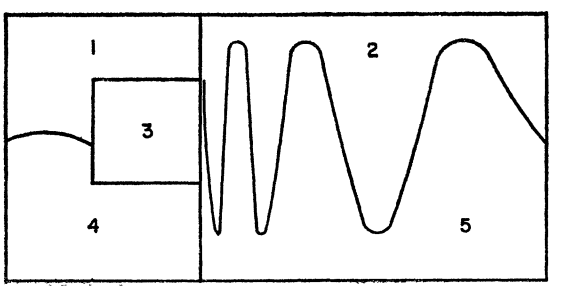


FIG. 33

8. PROPERTIES OF THE REAL NUMBERS

Much of mathematics is devoted to a study of the real numbers. In analytics much use is made of the fact that there is a 1-1 correspondence between the real numbers and the points on a line. Certain of the topological properties of a line can be developed from a study of the real numbers. With the view of getting a set of axioms that will characterize the line, we list certain properties of the set of real numbers E^1.

PROPERTY I. *E^1 is linearly ordered—that is, its elements a, b, c are ordered so that*:

1. *If $a \neq b$, either $a < b$ or $b < a$ (each pair of elements is related)*;
2. *$a \not< a$ (no element is less than itself)*;
3. *If $a < b$ and $b < c$, then $a < c$ (transitive property)*.

When we say that $a = b$, we mean that a is the same as b. Ordinarily we use the plural to denote more than one and, if we say that a, b are elements, we imply that they are distinct.

Any set (whether it is E^1 or not) is said to be *linearly ordered* if it has an order relation satisfying the above conditions.

Exercise 46. Show that if a, b are elements of a linearly ordered set, then not both $a < b$ and $b < a$.

An element a of a linearly ordered set X is said to be a *maximum* of X if for each other element b of X, $b < a$. Also, a is a *minimum* if for each other element b of X, $a < b$.

PROPERTY II. *E^1 has no maximum or minimum.*

Exercise 47. Give an example of a linearly ordered set failing to satisfy Property II.

If A, B are two (nonnull) subsets of a linearly ordered set X, then $[A, B]$ is called a *cut* of X if (1) each point of X belongs to either A or B, and (2) each point of A is less than each point of B.

Exercise 48. (a) Give an example of a cut $[A, B]$ of the reals E^1 so that A has no maximum. (b) Give an example of a cut $[D, E]$ of E^1 so that D has a maximum. (c) In which of the above cases does B or E have a minimum?

PROPERTY III. *E^1 satisfies the Dedekind cut condition—that is, if $[A, B]$ is a cut of E^1, either A has a maximum or B has a minimum.*

Exercise 49. Give an example of set X satisfying Properties I, II, but not III.

Exercise 50. Give an example of a set X different from E^1 satisfying Properties I, II, III.

A set is called *countable* if either it is finite or its elements can be put in a one-to-one correspondence with the positive integers—that is they can be arranged in a sequence. It may be shown that each subset of a countable space is countable.

A set is called *uncountable* if it is not countable. It is known that the rational numbers are countable while the real numbers are uncountable.

a is between b and c if either $b<a<c$ or $c<a<b$.

PROPERTY IV. *E^1 is separable—that is, it contains a countable set of points $p_1, p_2, \cdots$ such that if a, b are any two points of E^1 there is some p_i between a and b.*

Exercise 51. What is a countable set of numbers that shows the real numbers are separable?

Two elements of a linearly ordered set are *adjacent* if there is no element between them.

Exercise 52. Show that no two points of a set satisfying Property IV are adjacent.

If a, b are two points, the set consisting of a, b and all points between a and b is called a *closed interval.* If a, b are adjacent, the corresponding closed interval contains only two points.

Two closed intervals *intersect* if some point belongs to both.

Exercise 53. Show that if G is a collection of closed intervals of the real numbers and no two elements of G intersect, then G is countable.

Example 1. Consider the points of a horizontally based square linearly ordered as follows: $a<b$ means that either a, b are on the same vertical line and b is above a or a lies on a vertical line to the left of b.

Exercise 54. Show that above example does not satisfy Properties II, IV. How may it be modified so as to satisfy Property II?

Example 2. Suppose that each finite ordered set of letters is a *word.* Hence CAT, AAAC, ZZRNT, A, NAACP, KKK, etc., are words. Suppose the words are linearly ordered as would be done in an alphabetical ordering.

Exercise 55. Example 2 satisfies Property I. Show that it has the following properties: a. It has a minimum but no maximum. b. It does not satisfy Property III. c. There is no word preceding CAT which is adjacent to CAT but there is a word following CAT which is adjacent to CAT. d. There are only a countable number of words in the set.

Unsolved problem. Mathematicians have shown that any set L satisfying Properties I, II, III, IV is like the set of reals E^1 in the sense that there is a 1-1 correspondence between the elements of L and the numbers of E^1 that preserves the order relation. It is an unsolved problem as to whether or not any set satisfying Properties I, II, III, IV′, IV″ is like a line where IV′, IV″ are as follows:

PROPERTY IV′. *No two elements of L are adjacent.*

PROPERTY IV″. *Any collection of nonintersecting closed intervals of L is countable.*

This question was raised (in French) in a Polish mathematical journal in 1920 by a Russian mathematician named Souslin. It is called the Souslin problem. Despite many attempts to solve it, it remains unsolved today. One could solve it by either showing that Properties I, II, III, IV′, IV″ imply Property IV or by constructing an example satisfying Properties I, II, III, IV′, IV″ but not Property IV. Of course, one should not do both.

9. A TOPOLOGICAL CHARACTERIZATION OF A LINE

We use the properties of the real numbers considered in the preceding section to give a topological characterization of a line. This characterization is a set of axioms satisfied by the line such that anything that satisfies the axioms is topologically equivalent to a line.

A space is a linear topological space if its points are ordered and its neighborhoods are the sets of the following three sorts—(1) the set of points between two fixed points, (2) the set of points before some fixed point, or (3) the set of points following some fixed point. In studying a line, we only need the first kind of neighborhood but if we study a segment (which is also a linear topological space) the other types of neighborhoods are needed. (Sometimes other kinds of neighborhoods are needed but we do not find this convenient for our present purposes.)

A topological space is said to be *separable* if it contains a countable set of points such that each neighborhood contains one of these points.

The following is a topological characterization of a line L.

AXIOM 1. *L is a linear topological space.*

AXIOM 2. *L has no maximum or minimum.*

AXIOM 3. *L is connected.*

AXIOM 4. *L is separable.*

It may be shown that any space satisfying the above axioms is topologically equivalent to a line. Of course, there is no assurance that this space is "straight."

We recall some examples from the preceding section to see why all these axioms are necessary. To see that Axiom 2 is necessary (*i.e.*, Axioms 1, 3, 4 are not sufficient), consider a segment. To see that Axiom 3 is necessary, one could consider the set obtained by removing one point from a line. To see that Axiom 4 is needed, one could consider the points of the plane between and on two horizontal lines ordered as were the points in Example 1 of the preceding section.

Can Axiom 4 be replaced by the following?

AXIOM 4′. *L does not contain uncountably many mutually exclusive neighborhoods.*

There is no known proof that Axioms 1, 2, 3, 4′ characterize the line. Neither is there a known example of a space satisfying Axioms 1, 2, 3, 4′ that is not topologically equivalent to a line. This again is the Souslin problem.

Since there is 1-1 correspondence between points of a line and the real numbers that preserves order, the above axioms may be used to prove some useful theorems about real numbers. We illustrate with examples.

THEOREM. *A bounded collection of numbers has a least upper bound.*

We shall give a proof of this theorem based on the axioms. First we elaborate on its meaning.

A number x is an *upper bound* for a collection N of numbers if x is greater than or equal to each value of N. The *least upper bound* is the smallest of all the upper bounds. Similarly, a *lower bound* and a *greatest lower bound* are defined. A set of numbers with both an upper and a lower bound is called *bounded.*

Now for a proof of the theorem. Suppose N is a bounded collection of numbers. Let R be the set of all points p of the x-axis such that p is associated with an upper bound of N, and L be the rest of the x-axis. (See Fig. 34.)

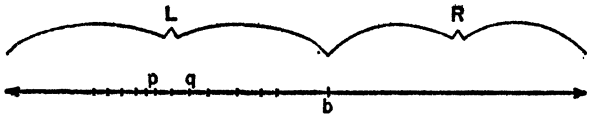


FIG. 34

There is a point in R because N was given to be bounded. There is a point in L because any number less than a number in N is not an upper bound of N.

No point p of L is a limit point of R because if p is in L, there is a point q corresponding to a number of N such that q is to the right of p—no point to the left of q is a point of R.

Axiom 3 tells us that L, R cannot be mutually separated, so some point b of R is a limit point of L. This point will correspond to an upper bound x_b of N since b is in R. Since b is a limit point of L, for each positive number ϵ there is a point of N within ϵ of x_b. Then x_b is the least of all the upper bounds of N.

We now mention some theorems of a related nature frequently used in calculus.

1. *A collection of numbers with a lower bound has a greatest lower bound.*

2. *A bounded monotone increasing sequence of numbers converges to the least upper bound of the sequence.*

3. *The Dedekind cut proposition holds.*

(The proposition, sometimes taken as an alternative axiom, states that if $[A, B]$ is a cut of the reals, either A has a maximum or B has a minimum.)

4. *A bounded infinite collection of numbers has a greatest limit.*

(This limit is sometimes called the upper limit. If the numbers are regarded as points, it will be a limit point that is larger than any other limit point.)

Exercise 56. How would the x-axis be broken into two sets L, R to show that the above Theorem 4 holds?

Exercise 57. Show from the axioms that a segment is connected.

A subset of the plane is said to be bounded if it lies on the interior of some circle. It can be shown that each closed and bounded subset of the plane is bicompact and hence compact. Also, each compact subset of the plane is closed and bounded.

Exercise 58. Show that each compact subset of a line is closed and bounded.

Exercise 59. In showing that each closed and bounded subset of the x-axis is bicompact, one must show that if G is an open covering of a closed and bounded subset S of the x-axis, then a finite number of elements of G cover S. How would the x-axis be broken into two sets L, R to show that the assumption that no finite subcollection of G covers S leads to the contradiction that the x-axis is not connected?

The following theorems used in calculus and concerning a continuous function $f(x)$ defined on a segment $(a \leqq x \leqq b)$ are special cases of more general theorems we have considered.

1. $f(x)$ *is bounded.*

2. $f(x)$ *has a maximum and a minimum.*

3. *If* $f(x)$ *takes on two values, it takes on each value between them.*

The first of these results follows from the facts that a segment is bicompact (Exercise 59), its image under a continuous transformation is bicompact (Exercise 33) and hence compact (Exercise 34), and a compact set of numbers is closed and bounded (Exercise 58).

The second of these results follows from the further facts that a closed and bounded set of numbers has a least upper bound and a greatest lower bound since it is bounded. It takes on these values since it is closed.

The third result follows from the facts that a segment is connected (Exercise 57) and therefore its image is connected (Exercise 29).

A set X is said to have the *fixed-point property* if for each mapping f of X into itself, for some point p of X, $f(p) = p$. A circle does not have the fixed point property because a rotation can move each of its points. It can be shown that a segment, a disk, and a solid cube do have the fixed point property.

Exercise 60. Show that a segment has the fixed point property.

Unsolved problems. It is conjectured that if C is a bounded plane continuum whose complement is connected in the plane, then C has the fixed-point property. Partial solutions have been given of two types—one of these says that if the continuum has certain properties (such as being like a segment or a disk), it has the fixed-point property; the other type says that if certain restrictions are placed on the mapping, some point is left fixed. In spite of the vast amount of time spent on this problem by some talented workers, the answer still eludes us. Perhaps some student coming along will succeed in verifying the conjecture that has baffled so many of his predecessors—or maybe someone will construct a counterexample showing that the conjecture is false.

An arc is a set topologically equivalent to a segment. If one were asked to name a plane continuum with more than one point which was topologically equivalent to each continuum in it with more than a point, one might think of an arc (See Fig. 4). For many years no one knew of any example other than an arc, although the question was raised in mathematical journals as to whether or not there were any. In 1948 such an example was found. The problem as to whether or not there are still other examples is unsolved.

10. SOME EXAMPLES OF TOPOLOGICAL SPACES

What is a point?

If one is studying the real number system or a real line one might answer that a point is a real number. For analysis, the answer might be—a complex number. In plane analytics, a point is an ordered pair of real numbers—in solid analytics it is an ordered triple of real numbers.

In synthetic geometry and in topology, a point is an undefined term. If we have a collection of elements (which we call points), and certain subcollections of these elements (which we call neighborhoods), we may regard the original collections as a topological space if the neighborhoods satisfy certain conditions called axioms.

In this section we mention some abstract spaces somewhat out of the ordinary. We describe them by telling what the points and neighborhoods of these spaces are.

Euclidean 4-space. Here each point has four coordinates instead of three. (No mention need be made of time. In fact, any student who has worked problems with four variables may consider himself as having worked with 4-space.) A point is an ordered quadruple of numbers and a neighborhood with center at (x_0, y_0, z_0, w_0) and radius ϵ is the set of all points $(x, y, z\ w)$ such that $((x_0-x)^2+(y_0-y)^2+(z_0-z)^2+(w_0-w)^2)^{1/2}<\epsilon$.

Hilbert space. The points of this space are sequences $(a_1, a_2, \cdots)$ of numbers such that $\sum a_n^2$ exists. The distance between two points $(x_1, x_2, \cdots)$, $(y_1, y_2, \cdots)$ is $(\sum(x_n-y_n)^2)^{1/2}$. Neighborhoods are the interiors of generalized spheres.

A function space. A point in this abstract space is a continuous function $y=f(x)(0\leqq x\leqq 1)$. A neighborhood with center at $f_0(x)$ and radius ϵ is the collection of all functions $f(x)$ such that $|f(x)-f_0(x)|<\epsilon$ for each x between 0 and 1 inclusive.

Metric spaces. Many of the spaces we deal with in topology are metric spaces. *Metric spaces* are topological spaces with distance functions $\rho(p, q)$ (called the distance between p and q). *Distance functions* satisfy the following conditions.

1. $\rho(x, y)\geqq 0$, *the equality holding if and only if* $x=y$.
2. $\rho(x, y)=\rho(y, x)$ (*symmetry*).
3. $\rho(x, y)+\rho(y, z)\geqq\rho(x, z)$ (*triangle condition*).
4. $\rho(x, y)$ *preserves limit points.*

This fourth condition means that a point x belongs to the closure of a set M if and only if for each positive number ϵ, there is a point m of M such that $\rho(x, m) < \epsilon$.

Exercise 61. Describe a distance function for the above function space.

We shall now describe a topological space that has no distance function.

Another function space. This is a function space used extensively in studying linear spaces. Again a point in this abstract space is a function $y=f(x)$ $(0 \leqq x \leqq 1)$ but this time we do not suppose that it is continuous. To get a neighborhood in this function space, we select a finite collection of numbers $x_1, \cdots, x_n$ between 0 and 1 inclusive and a positive number ϵ. The neighborhood $N(f_0(x); x_1, \cdots, x_n; \epsilon)$ is the set of all functions $f(x)$ defined for $0 \leqq x \leqq 1$ and such that $|f_0(x_i) - f(x_i)| < \epsilon$ $(i=1, \cdots, n)$. To get another neighborhood, we select another function, another finite set of points, and another positive number.

If p is a limit point of a set X in a metric space, there is a sequence of distinct points of X converging to p.

In the function space we have just described, let $f_0(x)$ be the function such that $f_0(x) = 0$ for $0 \leqq x \leqq 1$. Let M be the collection of all functions that take on the value 0 at a finite collection of numbers and on the value 1 everywhere else. Although $f_0(x)$ is a limit point of M, no sequence of points of M converges to $f_0(x)$. This shows that the function space we have described is not a metric space.

Exercise 62. If $f_1(x), f_2(x), \cdots$, is a sequence of points of M in the above function space, what is a neighborhood of $f_0(x)$ that contains no point of $f_1(x), f_2(x), \cdots$?

Exercise 63. If p is a limit point of a set M in a metric space, how might one select a sequence of distinct points of M converging to p?

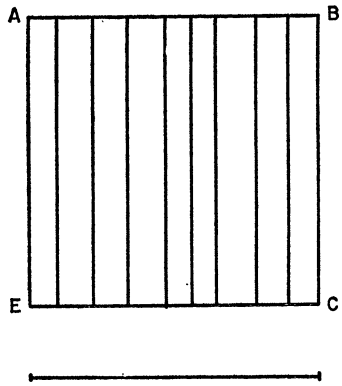


FIG. 35

Decomposition spaces. Suppose a square $ABCE$ plus its interior is decomposed into vertical straight line segments as shown in Figure 35. Let D be the

space whose " points" are these vertical segments and whose "neighborhoods" are the sum of such collections of " points" which are open subsets of the square disk. The decomposition space D is topologically equivalent to a segment.

Exercise 64. What "points" of D correspond to the ends of the segment equivalent to D?

If the square disk were decomposed in the manner illustrated in Figure 36, it is possible to define "points" and "neighborhoods" so that the decomposition space is topologically equivalent to the sum of two spherical surfaces and two segments.

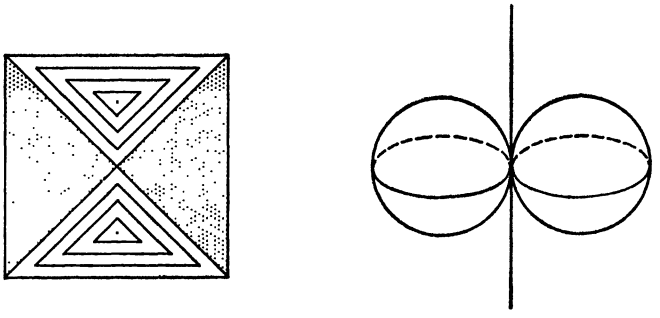


FIG. 36

Exercise 65. Which element of the decomposition space corresponds to the point of tangency of the two spheres? Which elements correspond to the outer ends of the two segments?

These two examples illustrate how topology may be used to study collections of figures rather than ordinary point sets.

The topologist is acquainted with dozens of different spaces—he may invent new spaces as the needs arise. Some of these spaces are much simpler than the ones we have described and others are more complex.

Queer spaces play an important role in topology. A wealth of examples of spaces with certain properties helps our understanding of these properties. It suggests to us that theorems are likely to follow from these properties.

Also, queer spaces are useful as counterexamples. There is no more convincing way of showing that one set of properties do not imply a second set than by exhibiting a space with the first set of properties but without the second set. Of the problems solved during recent years which had been regarded as difficult unsolved problems for many years, a goodly number of them were solved by the use of counterexamples.

11. SOME AXIOMS FOR TOPOLOGY

In geometry we may prove certain theorems on the basis of only part of our axioms—for example, we may omit the parallel axiom and study what theorems can be proved with the remaining set of axioms. While the axioms we then use do not completely describe the plane, we find that we can prove a restricted set of theorems on the basis of them. The beginning axioms we list are satisfied by a line, a plane, and E^3, but do not completely describe any of them. An advantage of working with such a restricted set of axioms is that the theorems we prove on the basis of these axioms will not only hold true on a line, in the plane, and in E^3, but in any space whatsoever that satisfies these axioms.

Suppose we have a collection S (which we call space) of things (which we call points) and certain subcollections of S (which we call neighborhoods). The points and neighborhoods may or may not be like the things we ordinarily think of as points and neighborhoods. We suppose however that they satisfy the following axioms.

AXIOM 1. *If a point p belongs to each of two neighborhoods N_1, N_2, then p lies in a neighborhood N_3 such that $N_3 \subset N_1 \cdot N_2$.*

AXIOM 2. *If p, q are two points, there are neighborhoods N_p, N_q containing p, q, respectively, such that $N_p \cdot N_q = 0$.*

We take "point" and "neighborhood" as undefined terms with the restriction that each neighborhood is a point set. Then without even knowing what these points and neighborhoods are, we can prove certain theorems on the basis of these axioms. Suppose that we have the neighborhood definition of limit point and the definitions of open, closed, sum, intersection, as given earlier. Then the following theorems can be proved.

THEOREM 1. *If p is a limit point of the sum of two sets, it is a limit point of one of them.*

THEOREM 2. *The intersection of two open sets is open.*

THEOREM 3. *The sum of two closed sets is closed.*

THEOREM 4. *A point set with only one point in it is closed.*

The spaces considered in the preceding section *Some examples of topological spaces* satisfied the two axioms we listed. These axioms are called the *Hausdorff axioms* and a space satisfying them is called a *Hausdorff space.* Hence, the four theorems are true in these spaces. Many additional theorems can be proved on the basis of them.

Exercise 66. An equivalent statement of Theorem 1 says that if p is not a limit point of either A or B, it is not a limit point of $A+B$. Prove this alternative statement. Which of the other three theorems is implied by Theorem 1?

Exercise 67. Theorem 2 can be proved from Theorem 3 by resorting to the results that the complements of closed sets are open and the complements of open sets are closed. However, prove it directly from Axiom 1.

Exercise 68. Which of the axioms is used to prove Theorem 4?

Exercise 69. Let $V(p, \epsilon)$ denote the set of all points whose distance from p is less than ϵ. To show that any metric space with the $V(p, \epsilon)$'s as neighborhoods satisfies Axiom 2, we might demonstrate that if p, q are two points, then $V(p, \rho(p, q)/2) \cdot V(q, \rho(p, q)/2) = 0$. How do we know that there is no point common to $V(p, \rho(p, q)/2)$ and $V(q, \rho(p, q)/2)$?

Exercise 70. To prove that a metric space satisfies Axiom 1 we might show that if r were a point in $V(p, \epsilon) \cdot V(q, \delta)$, then there is a positive number γ such that $V(r, \gamma) \subset V(p, \epsilon) \cdot V(q, \delta)$. What could we use for γ?

There are many theorems that could not be proved on the basis of Axioms 1 and 2 alone. For example, it could not be proved that a simple closed curve separates the plane. This result holds on the surface of a sphere or in a plane and is a consequence of the Jordan or Schoenflies theorems. However, additional axioms need to be assumed to prove these results from axioms. We frequently introduce additional axioms in studying topology. Such axioms as those given in the section *A topological characterization of a line* might be used. We might add an axiom saying that space is metric. We might axiomatize that neighborhoods are connected, that space is compact, that simple closed curves separate space, etc.

The following two exercises show why the space L of the section *A topological characterization of a line* is a Hausdorff space. If p and q are two points with $p<q$, we use (p, q) to denote the set of all points between p and q.

Exercise 71. To prove that L satisfies Axiom 1 of the present section, we show that if $r \epsilon (p_1, q_1) \cdot (p_2, q_2)$, then there is a (p_3, q_3) such that $r \epsilon (p_3, q_3) \subset (p_1, q_2) \cdot (p_2, q_2)$. What might we take for (p_3, q_3)?

Exercise 72. To show that L satisfies Axiom 2 of this section, we would consider two points p and q of L. Let r be a point between them, and let N_p, N_q be the sets of points preceding and following r respectively. Which of the axioms for L would not be satisfied if there were no point between p and q?

13. TRENDS TOWARD ABSTRACTNESS

One of the aspects of modern mathematics is the trend toward abstractness. This is not only true in topology but also in algebra, analysis, and other branches of mathematics. Let us examine how this trend has affected topology.

In proving a theorem in plane geometry, the student does not use all the properties of the plane but merely uses certain of the axioms. Similarly, in proving a topological theorem in the plane, one might note that his proof depends only on certain properties of the plane. The theorem would then hold true in any space with these essential properties. Let us illustrate this with an example.

THEOREM. *The sum of two closed plane sets is closed.*

This theorem can be proved without mentioning many properties of the plane. There is no need to mention the x-axis, y-axis, or straight lines. The theorem can be proved by using the definitions of closed sets, sum, limit point, along with the fact that if N_1, N_2 are two neighborhoods containing the same point, there is a neighborhood N_3 of the point in $N_1 \cdot N_2$ (Axiom 2 of the preceding section).

Exercise 74. Prove the above theorem on the basis of the definitions and the fact that if a point of the plane lies in each of two neighborhoods $N_1 \cdot N_2$, there is a neighborhood N_3 of the point in $N_1 \cdot N_2$.

As pointed out in the preceding section, we can prove the theorem about the sum of two closed sets being closed in any of the spaces mentioned in the section *Some examples of topological spaces* since all of these spaces satisfy Axiom 2 of the preceding section. One method of studying topology is to consider what theorems can be proved in a space with certain properties. One may not even know what space is under consideration as long as it is known that the space has the properties used in proving the theorem. Naturally the mathematician is

interested in getting the maximum in the way of results or conclusions from a minimum in the way of hypotheses and axioms.

If one is able to prove that certain theorems are true on the basis of certain properties of a space, at some time in the future a use may be found for a particular space with these properties, and the theorems proved before the space was invented will be known to hold.

An advantage (other than giving the imagination and mind ample working space) in dealing with abstract spaces is that our theorems tell us more than things about ordinary points and also reveal truths about collections of other sorts—for example, about collections of functions, such as described in Examples 3 and 4 in the section *Some examples of topological spaces.*

There are other abstract spaces which have for points other sorts of unexpected things—closed sets, continua, matrices.

Types of unsolved problems. We finish this article by mentioning some types of unsolved problems.

As we have indicated, some theorems go about as follows: Each space with a certain set of properties has some other property. Quite a few of the many mathematical papers published each year are devoted to proving this kind of theorem. We shall not discuss here the properties of interest but they include such things as being compact, bicompact, and having distance functions.

A more concrete type of theorem tells us that certain theorems (such as the Jordan curve theorem) hold in the plane or in some other familiar space. During the past forty years we have made much headway in proving theorems for the plane. In more recent years some headway has been made at hacking away at some of the complicated theorems in 3-space.

Intermediate between the abstract and concrete types of theorems are the characterization theorems. Show that a well-known space (such as a line, plane, or 3-space) satisfies a certain minimum set of conditions—then show that any space that satisfies these same conditions must be topologically equivalent to the well-known space.

We have some good topological characterizations of the line. However, one of the interesting unsolved problems (Souslin problem) is in this area. It gives a certain set of conditions satisfied by the line and asks if any space which satisfies them must be homeomorphic with a line.

Topological characterizations of the plane have been discovered during the past 40 years. An unsolved problem for many years in this area may be stated as follows: Is the connected compact metric space S topologically equivalent to the surface of a ball if it has more than one point and satisfies the following conditions?

1. *The neighborhoods of S are connected.*
2. *The complement of no simple closed curve is connected.*
3. *The complement of each pair of points is connected.*

The answer was given in the affirmative in 1946. The resulting characterization of a 2-sphere is called the Kline sphere characterization.

Although topological characterizations of 3-space have been given, these are still complicated. In view of the recent spurt toward increasing our knowledge of the topological properties of 3-space, there is hope for improvements here.

It is difficult to specify the types of unsolved problems that are of most interest to research topologists. Researchers vary in their interests and the field is wide. Things in which one worker is intensely interested may not even concern someone else at the moment, because he may be too busy working on something else. Although there are many unsolved problems in topology, those of interest are not likely to have easy solutions. However, we have talented young people coming along each year who are taking their places as ingenious researchers and making contributions that help push the frontier of mathematics farther ahead.

LIST OF TOPOLOGICAL TERMS

The following list showing where topological terms are defined in this article may be of convenience as a reference.

adjacent, 36
arc, 4, 41
band, 8
between, 36
bicompact, 27
bound—greatest lower, least upper, lower, upper, 34
boundary, 25
bounded in plane, 31
Cantor set, 26
closed, 19
closed interval, 36
closure, 19
compact, 27
common part, 17
complement, 14
component, 3, 26
connected, 3, 26
continuous, 4, 21, 22, 24
continuum, 26
converge, 25
countable, 36
cover, 27
cut, 36
decomposition spaces, 43
Dedekind cut condition, 36
difference, 18
distance function, 16, 42
domain, 18
domain of definition, 21
Euclidean spaces, 2, 42
fixed-point property, 40
four-color problem, 34
function spaces, 42
half-open segment, 5
Hausdorff axioms, 46
Hausdorff space, 46
Hilbert space, 42
homeomorphic, 5, 24
homeomorphism, 4, 24
homogeneous, 27
horned sphere, 33
image, 20
intersect, 36
intersection, 17
into, 21
inverse image, 20
Jordan curve theorem, 29
Klein bottle, 12
Kline sphere characterization, 51
limit point, 17
linear map, 22
linear topological space, 38
linearly ordered, 35
map, 22
mapping, 25
maximum, 35
metric space, 42
minimum, 35
Moebius band, 9
mutually exclusive, 26
mutually separated, 26
neighborhood, 16, 45
null set (empty set), 17
one-to-one transformation, 21
onto, 21
open covering, 27
open set, 18
open subset, 18
orientable, 13
point, 2, 45
point set, 17
product, 17
projective plane, 13
retraction theorem, 31
reversibly continuous, 24
Schoenflies theorem, 29, 31
segment, 5, 6
separable, 36, 38
separate, 3, 29
shelling theorem, 32
simple closed curve, 3, 28
Souslin problem, 37, 39
space, 2, 45
sphere, 10, 34
subset, 17
sum, 17
topologically equivalent, 3, 4, 5, 24
topology, 1, 27
torus, 11
transformation, 21
transitive property, 35
triangle condition, 43
uncountable, 36
union, 17

ANSWERS TO EXERCISES

1. Three. n.

2. Cube and ball, knotted curve and circle, theta curve and capital letter **B**, square disk pierced by segment and hemisphere with a tangent segment.

3. (A, R), (B), $(C, G, I, J, L, M, N, S, U, V, W, Z)$, (D, O), (E, F, T, Y), (H, K), (P, Q), (X). Yes—if the letters have width, each letter is equivalent to a disk or a disk with one or two holes in it.

4. The following pairs correspond: (c, z), (e, w), (f, v), (g, u). Either s or t might correspond to h. Any point on the circle $vxzy$ other than v and z might correspond to d.

5. A Moebius band.

6. The triangle with the upper left vertical edge and the one with the lower right vertical edge; or the triangle with the lower left vertical edge and the one with the upper right vertical edge; or the triangles with the leftmost and rightmost middle vertical sides.

7. A Moebius band.

8. A Moebius band. A projective plane.

9. Any function $f(x, y)$ satisfying the conditions $f(x, 0)=f(x, 1)$ and $f(0, y)=f(1, 1-y)$ will suffice. Examples of such functions are $f(x, y)=\cos 2\pi x \cos 2\pi y$ and $f(x, y)=x(x-1)(x-3)+y(y-1)(x+y)(x+y+1)^2(x+y-2)$.

10. A disk pierced by a segment and a hemisphere with a tangent segment, two homeomorphic bands with a different number of twists, some pair of the tori shown in Figure 13.

11. A Moebius band, a projective plane, a Klein bottle.

12. A Moebius band. We avoided mentioning a projective plane and a Klein bottle because they do not lie in E^3 and the sides of a surface refer to parts of the space in which they lie.

13. Any pair of the tori shown in Figure 13.

14. r and t are points of X; p, q, t are limit points.

15. p, q, s, t are limit points of X.

16. If $p \notin A$, $p \in B$ by the definition of $A+B$. Since B is open, p belongs to a neighborhood N in B by the definition of open. Then N lies in $A+B$.

17. Let r_p be the distance from p to the boundary of S. It is also permissible for it to be a fractional part of this distance.

18. (a) A line, (b) the interior of a square, (c) a segment without its end points, (d) the plane.

19. Show that each limit point of the closure belongs to the closure.

20. $\overline{G}$ is the sum of G, the segment from (0, 1) to (0, −1) and the set whose only element is the point $(1/\pi, 0)$.

21. Since p is a limit point of $A \cdot B$, each neighborhood of p contains a point of $A \cdot B - \{p\}$ by the definition of limit point. Then each neighborhood of p contains a point of $A - \{p\}$ since each point of $A \cdot B$ is a point of A. Then p is a limit point of A by the definition of limit point. Since A is closed, $p \in A$.

22. If each neighborhood of p contains a point of A, it contains a point of $A - \{p\}$ since $A = A - \{p\}$. Then p would be a limit point of A by the definition of limit point and $p \in A$ since A is closed.

23. If $p \notin A'$, $p \in A$ by the definition of A. Then p lies in a neighborhood N in A since A is open. No point of N lies in A' since each of its points lies in A.

24. Since $p \notin \bar{A}$, there is a neighborhood N of p that contains no point of A. Then N contains no point of $\bar{A}$ and p is not a limit point of $\bar{A}$.

25. (a) 1, 2, 7, 8. (b) 1, 5, 7. (c) 1, 2, 3, 4, 5, 8. (d) All.

26. (a) Projection of p on x-axis. (b) Projection of p on y-axis. (c) Distance from p to horizontal lines if they are equidistant from p. (d) Distance from p to vertical lines if they are equidistant from p.

27. If for each pair of horizontal planes, one above and one below p, there is a neighborhood of p such that each point of the graph in, above, or below the neighborhood is between the horizontal planes.

28. We show that each neighborhood U of $f(p)$ contains a point of $f(X)$. This insures that $f(p)$ is either a point or a limit point of $f(X)$. Let V be a neighborhood of p such that $f(V \cdot X) \subset U$. Since p is a limit point of X, V contains a point x of X and $f(x)$ is in U.

29. If $f(X)$ were the sum of two mutually separated (nonnull) sets A, B, then X would be the sum of the sets $f^{-1}(A), f^{-1}(B)$. These two sets are mutually exclusive because f is single valued, and each is an open subset of X by Exercise 30. Hence if $f(X)$ were the sum of two mutually separated (nonnull) sets A, B, X would be the sum of two mutually separated (nonnull) sets $f^{-1}(A), f^{-1}(B)$.

30. Suppose p is a point of $f^{-1}(D)$, where D is an open subset of Y. If U is a neighborhood of $f(p)$ with $U \cdot Y \subset D$, the continuity of f implies that there is a neighborhood V of p such that $f(V \cdot X) \subset U$. Then $V \cdot X \subset f^{-1}(D)$. The sum of all such V's for all points p of $f^{-1}(D)$ is an open set W and $W \cdot X = f^{-1}(D)$ is an open subset of X.

31. If U is an open subset of Y containing $f(p)$, we have that $f^{-1}(U)$ is an open subset of X, *i.e.*, $f^{-1}(U) = D \cdot X$, where D is open. We can use a neighborhood V of p in D to show that the definition of continuity is satisfied at p. Since f is continuous at each point p of X, it is continuous.

32. We show that each infinite collection $y_1, y_2, \cdots$ of points ($y_i \neq y_j$ if $i < j$) of $f(X)$ has a limit point in $f(X)$. Let x_i be a point of $f^{-1}(y_i)$. Then $x_1, x_2, \cdots$ has a limit point x_0 in X. Let K be infinite set formed by subtracting $f^{-1}f(x_0)$ from $x_0, x_1, x_2, \cdots$. Since x_0 is a limit point of K but $f(x_0)$ is not a point of $f(K)$ then, by Exercise 28, $f(x_0)$ is a limit point of $f(K)$ and hence of $y_1, y_2 \cdots$.

33. If G is a collection of domains covering $f(X)$ the inverse of each of these is an open subset of X by Exercise 30. If X is bicompact, a finite number of these inverses cover X and the collection of corresponding elements of G covers $f(X)$.

34. We show that if K is an infinite collection of points in a bicompact space X, then K has a limit point in X. For each point p of K let G_p be the sum of p and the complement of $\overline{K}$. No finite subcollection of the G_p's covers X so either the G_p's do not cover X or some G_p is not open. If some point q of X does not lie in any G_p, this point q is a limit point of K because it belongs to $\overline{K}$ but not to K. If some G_p is not open the corresponding point p is a limit point of K because if it were not, G_p would be the sum of two open sets—the complement of $\overline{K}$ and a neighborhood containing p but no other point of K.

35. We use functional notation to describe the homeomorphism—that is, the point corresponding to x is called $f(x)$. Let o be a point of the interior of the triangle and $f(o)$ be the center of the circle. For each point p on the triangle, let $f(p)$ be the point on the circle such that the ray from o through p points in the same direction as the ray from $f(o)$ through $f(p)$. Each point q of the first ray corresponds to the point $f(q)$ of the second ray such that $f(q)$ divides $f(o), f(p)$ in the same ratio (internally or externally) that q divides o, p. (See Fig. 27.)

36. First we describe a homeomorphism g which takes the plane onto itself and the circle with equation $x^2+y^2=1$ onto the given graph. If p is a point of the interior of the circle, $g(p)$ is on the same vertical line with p and divides the points where the vertical line hits the graph in the same ratio that p divides the points where the line hits the circle. If p is beneath the circle, $g(p)$ is on the same vertical line with p and the same distance below the graph as p is below the circle. If p is a point of the plane that is neither interior to, below, or on the lower semicircle of the circle, $g(p)=p$.

Suppose f is a homeomorphism such as given in Exercise 35 of the plane onto itself that takes the given triangle onto the circle with equation $x^2+y^2=1$. Then gf is the required homeomorphism —to find the point that corresponds to p, find the point $f(p)$ that corresponds to p under the homeomorphism f and then find the point $gf(p)$ which corresponds to this point $f(p)$ under g.

37. Construct segment AGC and segment BD as shown in Figure 28. Also construct the "kite-shaped" figure shown in Figure 29. We describe the homeomorphism f of the polygon $ABCDEF$ onto the triangle $A'C'D'$ by steps.

(i) Let f be a homeomorphism that takes AB linearly onto $A'B'$—if p is a point of AB, $f(p)$ divides A', B' in the same ratio that p divides A, B. Extend f to take BC, CD, DE, EF, FA, AG, GC, BD linearly onto $B'C'$, $C'D'$, $D'E'$, $E'F'$. $F'A'$, $A'G'$, $G'C'$, $B'D'$, respectively.

(ii) We now extend f so that it takes the interior of $ABCG$ onto the interior of $A'B'C'G'$. Let

p and $f(p)$ be points of the interiors respectively of these two triangles. If x is a point other than p of the interior of $ABCG$, extend the segment from p through x until it intersects triangle $ABCG$ at a point y. Now $f(y)$ has already been defined in the first step. The point $f(x)$ is the point that divides $f(p), f(y)$ in the same ratio that x divides p, y.

(iii) Similarly extend f to take the interior of triangle BCD onto the interior of triangle $B'C'D'$.

(iv) To extend f to the interior of $ABDEF$, take a point p of this interior such that no ray from p intersects $ABDEF$ in two points (we might choose p on the line which bisects angle E) and let $f(p)$ be any point on the interior of $A'B'D'E'F'$. For each point x other than p of the interior of $ABDEF$, let y be the point where the ray from p through x intersects $ABDEF$ and $f(p)$ be the point that divides $f(p), f(y)$ in the same ratio that x divides p, y.

(v) To extend the homeomorphism f to the exterior of $AGCDEF$, consider the same points $p, f(p)$ used in Step (iv). For each point x of the exterior of $AGCDEF$ let y be the point where the segment from p to x hits $AGCDEF$. Then $f(x)$ is the point that divides $f(p), f(y)$ externally in the same ratio that x divides p, y.

38. So that $h(Y)$ lies on the segment from A to $h(X)$ and divides this segment in the same ratio that Y divides AX.

39. This follows by mathematical induction on $n(P)$ since $AB+AYB$ has fewer sides than P.

40. Let $UV=AB$.

41. Since X lies in Ext $(AB+AYB)$ (X is at a maximum distance from L) AXB lies except for its ends in this exterior. Then CE does not intersect AXB except possibly at C or E. Also, CE was chosen so as not to intersect AYB except at C and E since it intersects $AB+AYB$ only there.

42. $P-(CD+DE)+CE$.

43. gh.

44. B lies in Ext $(ZAY+YZ)$ as does each non-endpoint of ZBY. A lies in Ext $(ZBY+YZ)$ as does each non-endpoint of ZAY.

45. If there were such a point P, consider a segment PQ that misses YZ and has only the point Q on ZAY. We would have a contradiction of the first or second part of Exercise 44 according as PQ does or does not intersect ZBY.

46. If $a<b$ and $b<a$, then $a<a$ by the transitive property. This contradicts the fact that no element is less than itself.

47. The numbers between 0 and 1 inclusive.

48. (a) Let A be the set of negative numbers and B the others. (b) Let D be the set of non-positive numbers and E the others. (c) B has a minimum but E does not.

49. The set of rational numbers or the reals with 0 removed.

50. The integers. Another interesting example is the set of points in the plane on or between two horizontal lines ordered so that $p<q$ if p is below or to the left of q.

51. The rationals.

52. If some two points a, b were adjacent, there would be no point of the countable set $p_1, p_2, \cdots$ between them since there is no point whatsoever between them.

53. Let each of the intervals correspond to a rational number in it. This sets up a 1–1 correspondence between the intervals and a subset of the rationals. The set of rationals is countable so each subset of it is also. Hence, the collection of intervals is countable.

54. Example 1 does not satisfy Property II because the point of the square in the upper right hand corner is a maximum and the point in the lower left hand corner is a minimum. If these two points were removed, the remainder would satisfy Property II. Each vertical segment in the square is a closed interval in the linearly ordered set. The square contains uncountably many such vertical segments, no two of which intersect. Example 1 could not satisfy Property IV because it can be shown by techniques similar to those used in solving Exercise 53 that no space satisfying Property IV can have uncountably many mutually exclusive closed intervals.

55. a. Example 2 has the word A as a minimum. For each given word we can obtain a bigger word by adding another letter.

b. To check Property III, consider an expression such as CCC $\cdots$ (there are infinitely many

c's in this expression). It is not a word since only finite ordered sets of letters are words. If A is the set of words preceding CCC $\cdots$ and B is the set following, the cut $[A, B]$ does not satisfy the Dedekind cut condition.

c. The word CATA follows CAT and is adjacent to CAT. However, if a word precedes CAT, we obtain a new word between it and CAT by adding an A to the end of this original word.

d. There are 26 one-letter words, $(26)^2$ two-letter words, $(26)^3$ three-letter words, $\cdots$. We put these words in a 1-1 correspondence with the positive integers as follows: The one-letter words correspond to the integers between 1 and 26 inclusive, the two-letter words correspond to the integers between 27 and $26+(26)^2$ inclusively, the three-letter words correspond to the integers between $27+(26)^2$ and $26+(26)^2+(26)^3$ inclusively, $\cdots$.

56. Let L be the set of points of the x-axis associated with numbers n that have infinitely many numbers of the collection which are bigger than n while R is the rest of the x-axis.

57. Suppose pq is the segment with left end p and right end q, L_p is the part of the line to the left of p and L_q is the part of the line to the right of q. The segment pq is connected because the assumption that it is the sum of two mutually separated sets A, B leads to the contradiction that the line is not connected since it is the sum of two mutually separated sets A', B'; A' is A, $A+L_p$, $A+L_q$, $A+L_p+L_q$ according as A contains neither p nor q, A contains p but not q, A contains q but not p, A contains both p and q; B' is defined in a similar fashion.

58. Suppose that S is a compact (nonnull) subset of the line. Denote a fixed point of S by s_0. If S were not bounded, for each integer i there would be a point s_i in S which would not belong to the neighborhood with center at s_0 and radius i. This is impossible since S is compact but $s_1, s_2, \cdots$ would have no limit point. If S were not closed there would be a point x_0 not of S which is a limit point of S. For each integer i there would be a point s_i of S in the neighborhood with center at x_0 and radius $1/_i$. This is impossible since S is compact and $s_1, s_2, \cdots$ would not have a limit point in S.

59. Let L be the collection of all points p of the x-axis such that a finite subcollection of G covers each point of S to the left of p. Let R be the rest of the x-axis.

60. Suppose f is a mapping of a horizontal segment S into itself, L is the set of points of S that move to the right under f, and R is the set of points of S that move to the left. Unless f leaves some point fixed, the left end of S belongs to L and the right end belongs to R. If a point p of S moves to the right, it follows from the continuity of f that there is a neighborhood of p such that each point of S in the neighborhood moves to the right. Hence, L is an open subset of S. Also, R is an open subset of S. Since S is connected (Exercise 56), it is not the sum of two mutually separated sets L and R. Any point of $S-(L+R)$ is left fixed under f.

61. $\rho(f(x), g(x))$ is the maximum value of $|f(x)-g(x)|$.

62. $N(f_0(x); y; \frac{1}{2})$ where y is a value such that $1=f_1(y)=f_2(y)=\cdots$.

63. For each integer i let m_i be a point of M whose distance from p is less than $1/i$ and such that m_i is different from any of $m_1, \cdots, m_{i-1}$. Then $m_1, m_2, \cdots$ converges to p.

64. The segments AE and BC.

65. The sum of the perimeter of the square and its diagonals. The centers of the upper and lower nested collections of triangles.

66. If p is not a limit point of A, it lies in a neighborhood N_1 which contains no point of $A-\{p\}$; also, if p is not a limit point of B it lies in a neighborhood N_2 which contains no point of $B-\{p\}$. It follows from Axiom 1 that p lies in a neighborhood $N_3 \subset N_1 \cdot N_2$. Then N_3 contains no point of $A+B-\{p\}$ and p is not a limit point of $A+B$. Theorem 3 is implied by Theorem 1.

67. Let p be a point in the intersection of two open sets U_1, U_2. We show that p lies in a neighborhood $N \subset U_1 \cdot U_2$. It follows from the definition of open sets that p lies in a neighborhood N_1 with $N_1 \subset U_1$ and p lies in a neighborhood N_2 with $N_2 \subset U_2$. It follows from Axiom 1 that p lies in a neighborhood $N_3 \subset N_1 \cdot N_2$. Then $N=N_3 \subset U_1 \cdot U_2$.

68. Axiom 2.

69. If r were a point common to $V(p, \rho(p, q)/2)$ and $V(q, \rho(p, q)/2)$, $\rho(p, r)+\rho(r, q)<\rho(p,q)$. This violates the triangle condition for a metric space.

70. Let γ be the minimum of $\epsilon-\rho(p, r)$ and $\delta-\rho(q, r)$.

71. Let p_3 be the larger of p_1, p_2 and q_3 be the smaller of q_1, q_2.

72. L would not be connected. Axiom 2.

73. (a) An ordered pair of numbers. (b) An ordered triple of numbers. (c) An ordered quadruple of numbers. (d) A sequence of numbers the sum of whose squares exists (is a number). (e) A continuous function $y=f(x)$ defined for $0\leqq x\leqq 1$. (f) A vertical segment.

74. To show that the sum $A+B$ of two closed sets A, B is closed, we need to show that each limit point of $A+B$ belongs to $A+B$.

Suppose p is not a point of $A+B$. Then it is not a point of A (from the definition of sum). Since A is given to be closed, p is not a limit point of A (definition of closed). Hence, there is a neighborhood N_1 of p which contains no point of A (definition of limit point). Likewise, p is not a point or limit point of B and there is a neighborhood N_2 of p which contains no point of B.

There is a neighborhood N_3 of p which lies in both N_1 and N_2. (We cannot justify this from the definitions.) No point of A is in N_3 since no point of A is in N_2—also, no point of B is in N_3. Hence N_3 contains no point of $A+B$ since it contains no point of A and no point of $A+B$. Hence, $A+B$ is closed.

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GENERALIZED FIBONACCI NUMBERS AND ASSOCIATED MATRICES

E. P. MILES, JR., The Florida State University

Introduction. We define k-generalized Fibonacci numbers ($k \geqq 2$) in such a way that for $k=2$ we get the ordinary Fibonacci numbers. We prove several interesting facts about these k-generalized numbers which reduce for $k=2$ to well-known properties of the ordinary Fibonacci numbers. We study a sequence of 2 by 2 nonsingular matrices with elements consecutive Fibonacci numbers whose members become arbitrarily ill-conditioned if we progress far enough in the sequence. This result is later generalized to obtain a sequence of k by k matrices with k-generalized Fibonacci numbers for elements and comparably ill-conditioned members.

The k-generalized Fibonacci numbers $f_{j,k}$ are defined as follows:

$$(1) \qquad f_{j,k} = 0, \quad 0 \leqq j \leqq k-2, \quad f_{k-1,k} = 1, \quad f_{jk} = \sum_{n=1}^{k} f_{j-n,k}, \quad j \geqq k.$$

When $k=2$ the numbers $f_{j,2}$, or simply f_j, which satisfy

$$(1)' \qquad f_0 = 0, \quad f_1 = 1, \quad f_j = f_{j-1} + f_{j-2}, \quad j > 1,$$

are the ordinary Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, $\cdots$. We first state without proof three well-known (see for instance [1]) properties (2), (3), (4) of these f_j. However, these properties are just special cases of their counterparts (2)″, (3)″, and (4)″ for the $f_{j,k}$ which we prove later. The first properties to be generalized are

$$(2) \qquad f_n = [(1+\sqrt{5})^n - (1-\sqrt{5})^n]/2^n\sqrt{5}, \quad n = 0, 1, \cdots,$$

$$(3) \qquad \lim_{n\to\infty} f_{n+1}/f_n = (1+\sqrt{5})/2,$$

$$(4) \qquad f_n f_{n+2} - f_{n+1}^2 = (-1)^{n+1}, \quad n = 0, 1, \cdots.$$

It is convenient for recognizing the form which our generalizations will take to rewrite (2), (3), and (4) in terms of the concepts introduced in (5), (6), and (7) which follow. We consider the equation,

$$(5) \qquad E_2(x) = x^2 - x - 1 = 0,$$

having roots

$$(6) \qquad r_1 = (1-\sqrt{5})/2, \quad r_2 = (1+\sqrt{5})/2, \quad r_1 < r_2,$$

and the family of 2 by 2 matrices,

$$A_n = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} \qquad n = 0, 1, \cdots. \tag{7}$$

In terms of quantities just defined, (2), (3), and (4) become

$$f_n = r_1^n/(r_1 - r_2) + r_2^n/(r_2 - r_1), \tag{2'}$$

$$\lim_{n\to\infty} f_{n+1}/f_n = r_2, \tag{3'}$$

$$|A_n| = (-1)^{n+1}. \tag{4'}$$

Since the family of matrices A_n defined by (7) has some interesting properties not immediately apparent, we proceed with a development and discussion of these properties before passing to our generalizations. The matrix A_n is the nonsingular coefficient matrix for the system

$$f_n x_1 + f_{n+1}x_2 = f_{n+2}, \qquad f_{n+1}x_1 + f_{n+2}x_2 = f_{n+3}, \tag{8}$$

with the obvious unique solution (1, 1).

In view of (3) it is obvious that the system (8) becomes "nearly" dependent with increasing n, since the corresponding coefficients become almost proportional. This leads to excellent classroom illustrations of systems with integral coefficients and integral exact solutions which are highly unstable if the coefficients are known only approximately. For instance, consider the system (8) for $n=10$ under the assumption that all coefficients are exactly known except that of x_2 in the second equation which may have an error no greater than .02%. The system may be indicated as follows:

$$\begin{aligned} 55x_1 + 89x_2 &= 144, \\ 89x_1 + (144 \pm \epsilon)x_2 &= 233, \qquad 0 \leqq \epsilon \leqq .0288. \end{aligned} \tag{9}$$

For $\epsilon=0$ the correct solution is (1, 1), for $\epsilon=.02$ it becomes (18.8, -10), for $\epsilon=.018$ it is (-159.2, 100). This wide variation in the solutions for small changes in ϵ is to be expected since the system for $\epsilon=1/55$ is inconsistent and hence has no solution at all.

By using the P-condition number of von Neumann (absolute value of the ratio of larger eigenvalue of A_n to the smaller), we can make a precise statement about the ill-condition of A_n. The characteristic equation for A_n,

$$(f_n - \lambda)(f_{n+2} - \lambda) - f_{n+1}^2 = 0, \tag{10}$$

may, with the aid of (4), be written as

$$\lambda^2 - (f_n + f_{n+2})\lambda + (-1)^{n+1} = 0. \tag{11}$$

It is easily verified that A_0 has eigenvalues $\lambda_{1,0}=r_1=(1-\sqrt{5})/2$ and $\lambda_{2,0}=r_2=(1+\sqrt{5})/2$ and P-condition number $=r_2/r_1=r_2^2=(3+\sqrt{5})/2$. Likewise A_1 has

eigenvalues $\lambda_{1,1}=(3-\sqrt{5})/2=r_1^2$ and $\lambda_{2,1}=(3+\sqrt{5})/2=r_2^2$ and P-condition number r_2^4. This suggests the general result which we now prove* that A_n has eigenvalues $\lambda_{1,n}=r_1^{n+1}$, $\lambda_{2,n}=r_2^{n+1}$ and P-condition number r_2^{2n+2}. From (5) we see that $r_1r_2=-1$. Then by (2)′ we have

$$
\begin{aligned}
f_n+f_{n+2} &= \frac{r_1^n}{r_1-r_2}+\frac{r_2^n}{r_2-r_1}+\frac{r_1^{n+2}}{r_1-r_2}+\frac{r_2^{n+2}}{r_2-r_1} \\
&= \frac{r_1^{n+1}}{r_1-r_2}\left(\frac{1}{r_1}+r_1\right)+\frac{r_2^{n+1}}{r_2-r_1}\left(\frac{1}{r_2}+r_2\right) \qquad (12) \\
&= \frac{r_1^{n+1}}{r_1-r_2}(-r_2+r_1)+\frac{r_2^{n+1}}{r_2-r_1}(-r_1+r_2) \\
&= r_1^{n+1}+r_2^{n+1}.
\end{aligned}
$$

Using (12), (11) becomes $\lambda^2-(r_1^{n+1}+r_2^{n+1})\lambda+r_1^{n+1}r_2^{n+1}=0$, which obviously has the roots specified. The P-condition number of A_n is thus seen to be r_2^{2n+2} which becomes arbitrarily large as n increases.

The conditions (1) completely determine the value of $f_{n,k}$. This is readily ascertainable from the general solution of the kth-order difference equation with constant coefficients,

$$a_0f(n+k)+a_1f(n+k-1)+\cdots+a_kf(n)=0, \qquad (13)$$

which has been known since Bernoulli (1728) (see, for instance, Aitken [2]). The general solution of (13) is of the form

$$f(n)=c_1z_1^n+\cdots+c_kz_k^n \qquad (14)$$

whenever the algebraic equation

$$a_0z^k+a_1z^{k-1}+\cdots+a_k=0 \qquad (15)$$

has k distinct roots $z_1, \cdots, z_k$. Substituting for n the values $0, 1, \cdots, k-1$ in (14) imposes k independent and consistent conditions on the constants $c_1, \cdots, c_k$ which determine them uniquely.

Thus we turn now to a study of the roots of

$$E_k''(x)=x^k-x^{k-1}-\cdots-x-1=0, \qquad (5)''$$

showing them to be distinct roots $r_{1,k}, \cdots, r_{k,k}$ which may be ordered so that

* An alternative proof involves showing that $A_0^n=A_{n-1}$, an interesting property noted by the author only after the first submission of the paper. This property, which for $k\geqq 2$ does not apply to the $A_{n,k}$ of this paper, has been generalized in the author's Classroom Note, On matrix slide rules, pages 788–791 of this issue of the MONTHLY.

(6)″ $\quad |r_{j,k}| < 1, \quad 1 \leqq j \leqq k-1 \quad \text{and} \quad 1 < r_{k,k} < 2.$

Since $E_k''(1) = 1-k < 0$ and $E_k''(2) = 1$, there is at least one root of $E_k''(x) = 0$ on the open interval (1, 2). Let the largest such root be called $r_{k,k}$. We now show that $E_k''(z)$ has no roots on the unit circle. In particular it does not have the root $z = 1$, so we may write (5)″ in equivalent form

$$z^k = (z^k - 1)/(z - 1), \qquad z \neq 1. \tag{16}$$

Assuming that (16), and hence (5)″, has a root $\cos\theta + i\sin\theta = z_1$ we obtain, by substituting z_1 in (16), equating the square of the absolute values of each side, and simplifying, the condition $\cos k\theta = \cos\theta$, which is satisfied only for $\theta = (2n\pi)/(k-1)$ or $\theta = (2n\pi)/(k+1)$ and integral n. The first value for θ makes $z_1^k = z_1 = 1$ which is already ruled out; the second value for θ makes z_1^k equal to $1/z_1$ and, by (16), to $-1/z_1$, which is impossible. Thus (5) has no roots on the unit circle. It is now helpful to introduce

$$E_k'(z) = z^{k+1} - 2z^k + 1 = 0, \tag{5′}$$

formed by multiplying (16) by $z-1$ and collecting terms. Obviously $E_k' = 0$ has k roots in common with $E_k'' = 0$. It has one additional root, $z = 1$, on the unit circle and at least one root $r_{k,k}$ outside the unit circle. We show that its roots inside the unit circle are distinct. Any repeated root of $E_k' = 0$ would also be a root of $(k+1)z^k - 2kx^{k-1} = 0$ and accordingly either $z = 0$ or $z = (2k)/(k+1)$. Of these possibilities $z = 0$, the only one inside the unit circle, is not a root of $E_k' = 0$. We complete our analysis of the roots of $E_k'' = 0$ by showing that $E_k' = 0$ has exactly $k-1$ roots interior to the unit circle. We do this by showing that $E_k'(z)$ and $-2z^k + 1 = 0$ have the same number of zeros interior to the circle Γ_ϵ, Γ_ϵ: $z = (1+\epsilon)^{1/k}$, for sufficiently small positive ϵ. Let $f(z) = z^{k+1}$ and $g(z) = -2z^k + 1$. On Γ_ϵ we have $|f(z)| = (1+\epsilon)^{(k+1)/k}$ and $|g(z)| = |-2(1+\epsilon)e^{i\theta} + 1| > 2(1+\epsilon) - 1 = 1 + 2\epsilon$. Thus on Γ_ϵ,

$$|g(z)| / |f(z)| > [(1 + 2\epsilon)/[(1 + \epsilon)^{(k+1)/k}] = F(\epsilon).$$

Now, $F(0) = 1$ and

$$F'(0) = \frac{(1 + \epsilon)^{(k+1)/k}[2] - [1 + 2\epsilon][(k + 1)/k](1 + \epsilon)^{1/k}}{(1 + \epsilon)^{(2k+2)/k}}\Bigg|_{\epsilon=0},$$

so that $F'(0) = (k-1)/k > 0$. Thus for some positive ϵ^* sufficiently small we have $F(\epsilon) > 1$ for $0 < \epsilon \leqq \epsilon^*$.

Applying Rouche's theorem we see that, for all sufficiently small positive ϵ, $g(z) = -2z^k + 1$ and $f(z) + g(z) = E_k'(z)$ have the same number of zeros in Γ_ϵ. Since ϵ may be chosen as near zero as we wish, the k zeros of $f(z) + g(z)$ must be in or on the unit circle. Only one of these zeros ($z = 1$) is on the unit circle, so the other $k-1$ distinct zeros must be interior to the unit circle.

As noted above we may assume that the jth term of our k-generalized

Fibonacci sequence is expressible as a linear combination of the jth powers of the distinct roots $r_{1,k}, \cdots, r_{k,k}$ of $E_k' = 0$, that is, that

$$f_{j,k} = \sum_{i=1}^{k} B_i(r_{i,k})^j, \qquad j = 0, 1, \cdots, k-1, k, \cdots. \tag{17}$$

Using (1) we see that the coefficients B_i are determined by the linear system

$$\begin{aligned} \sum_{i=1}^{k} B_i(r_{i,k})^m &= 0, \qquad m = 0, 1, \cdots, k-2, \\ \sum_{i=1}^{k} B_i(r_{i,k})^{k-1} &= 1, \end{aligned} \tag{18}$$

of k equations in k unknowns. In solving (18) by Cramer's rule we see that

$$B_i = N_i/D_k, \tag{19}$$

where D_k is the Vandermonde determinant

$$D_k = \begin{vmatrix} 1 & \cdots & 1 \\ r_{1,k} & \cdots & r_{k,k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ r_{1,k}^{k-1} & \cdots & r_{k,k}^{k-1} \end{vmatrix} = \prod (r_{m,k} - r_{n,k}), \tag{20}$$

where, in the product, $m > n$, $2 \leqq m \leqq k$, and N_i is obtained from D_k by replacing its ith column by a column of $k-1$ zeros followed by a 1. Expanding D_k by its ith column we see that it is $(-1)^{k+i}$ times a Vandermonde determinant of order $k-1$ involving all the $r_{n,k}$ except $r_{i,k}$. Thus we have

$$N_i = (-1)^{k+i} \prod (r_{m,k} - r_{n,k}), \tag{21}$$

where, in the product, $i \neq m > n \neq i$, $2 \leqq m \leqq k$. Substituting (21) and (20) in (19) we have

$$B_i = (-1)^{k+i} / \prod\nolimits_1 (r_{m,k} - r_{n,k}) = \prod\nolimits_2 (r_{i,k} - r_{n,k})^{-1}, \tag{22}$$

where, in $\prod_1$, $m = i$ or $n = i$, $m > n$, $2 \leqq m \leqq k$, and in $\prod_2$, $i \neq n$, $1 \leqq n \leqq k$. The last step in (22) follows on changing the sign of the $k-i$ factors $(r_{m,k} - r_{i,k})$, $m > i$, and introducing the compensating factor $(-1)^{k-i}$ to preserve equality. Thus we have

$$f_{j,k} = \sum_{i=1}^{k} [\prod\nolimits_2 (r_{i,k} - r_{n,k})^{-1}](r_{i,k})^j. \tag{2}''$$

We note that (2)′ and (2) follow readily from (2)″ in the special case $k = 2$.

From (2)″ and the facts that (a) $|r_{i,k}| < 1$, $i < k$, and (b) $|r_{k,k}| > 1$, we conclude that $\lim_{j \to \infty} (f_{j,k} - B_k r_k^j) = 0$, whence we obtain

$$\lim_{j\to\infty} (f_{j+1,k})/(f_{j,k}) - r_{k,k} = 0, \tag{3}''$$

which includes (3)′ and (3) as special cases.

The matrix $A_{j,k}$ which occurs in the generalization we make of (4)′ is defined as follows. $A_{j,k}$ is the matrix with general element $a_{mn}=f_{j+m+n-2,k}$, $1\leqq m, n \leqq k$; *i.e.*,

$$A_{j,k} = \begin{pmatrix} f_{j,k} & \cdots & f_{j+k-1,k} \\ f_{j+1,k} & \cdots & f_{j+k,k} \\ \vdots & & \vdots \\ f_{j+k-1,k} & \cdots & f_{j+2k-2,k} \end{pmatrix}. \tag{7}''$$

We now show that

$$|A_{j,k}| = (-1)^{(2j+k)(k-1)/2}, \tag{4}''$$

which reduces to (4)′ or (4) for $k=2$.

We observe that the matrix $A_{0,k}$ is lower triangular with determinant

$$|A_{0,k}| = (-1)^{k-1}(-1)^{k-2}\cdots(-1)^{1} = (-1)^{k(k-1)/2}. \tag{23}$$

We also note that

$$|A_{j,k}| = (-1)^{k-1}|A_{j-1,k}| \tag{24}$$

because $|A_{j-1,k}|$ is equivalent to the determinant $|A''_{j,k}|$ obtained from $|A_{j-1,k}|$ by replacing each element of its first column by its row sum, and $|A''_{j,k}|$ has identical columns with those of $|A_{j,k}|$ permuted cyclically so that the last column of $|A_{j,k}|$ is the first column of $|A''_{j,k}|$. Repeated application of (24) yields

$$|A_{j,k}| = (-1)^{j(k-1)}|A_{0,k}|. \tag{25}$$

By substitution of $A_{0,k}$ from (23) in (25) we may complete the proof of (4)″.

From (3)″ it follows that the rows of $A_{j,k}$ become almost dependent for large j. We obtain a crude measure of the ill-condition of these matrices from the following considerations.

The eigenvalues of the real symmetric matrix $A_{j,k}$ are the k real roots of an equation of form

$$\lambda^k - \sum_{n=0}^{k-1} f_{j+2n,k}\lambda^{k-1} + \cdots + (-1)^k(-1)^{(2j+k)(k-1)/2} = 0.$$

The largest eigenvalue of $A_{j,k}$ exceeds $f_{j,k}$ which is the smallest trace element. Thus for $j>k$ we have $\lambda_{k,k}>f_{j,k}>1$. Since the product of the eigenvalues has absolute value unity and one of them exceeds unity the smallest must be such that $|\lambda_{1,k}|<1$. This means that for fixed k and $j>k$ the P-condition number of $A_{j,k}$ exceeds $f_{j,k}$ which we have seen to be of the order of magnitude of $C(r_{k,k})^j$.

We consider a relatively unstable system of three equations associated with

the matrix $A_{5,3}$ under the assumption that all of its elements except a_{33} are exactly known. We may write the system in the following form.

$$4x + 7y + 13z = 24, \qquad 7x + 13y + 24z = 44, \qquad 13x + 24y + (44 + \epsilon)z = 81.$$

For certain values of ϵ corresponding to errors in a_{33} of less than 1% we note the following:

$$\begin{aligned} \epsilon &= 0: && (x, y, z) = (1, 1, 1), \\ \epsilon &= .3: && (x, y, z) = (-2, 13, 10), \\ \epsilon &= .4: && (x, y, z) = (3, 11, -5), \\ \epsilon &= 1/3: && \text{no solution.} \end{aligned}$$

We conclude with a generalization of the well-known fact that the ordinary Fibonacci numbers $f_{n,2}$ or f_n may be obtained by diagonal summing of the binomial coefficients arranged in a Pascal triangle. This familiar result, for which an inductive proof was given by Ganis [3], may be expressed analytically as follows:

$$f_n = \sum_{m=0}^{[(n-1)/2]} \binom{n-1-m}{m}. \tag{26}$$

In order to write the generalized form of (26) which the author has obtained, the following notation is introduced. The symbol $(a_1, \cdots, a_k)^*$ is defined to have the value

$$\left(\sum_{j=1}^{k} a_j\right)! \Big/ \prod_{j=1}^{k} (a_j!)$$

for a_j integral and nonnegative and the value zero otherwise (in particular if one of the a_j is negative). Consider ordered k-tuples $(a_1, \cdots, a_k)$; the set $S_{n,k}$ is defined as the set of all such k-tuples whose elements are nonnegative integers such that

$$\sum_{j=1}^{k} j \cdot a_j = n. \tag{27}$$

With the above notation (26) may be rewritten as

$$f_{n,2} = \sum_{S_{n-1,2}} (a_1, a_2)^*, \qquad n \geqq 1, \tag{28}$$

a special case of our general result

$$f_{n,k} = \sum_{S_{-k+1,k}} (a_1, \cdots, a_k)^*, \qquad n \geqq k - 1, \tag{28}''$$

which we now establish. The proof depends upon the following extension to multinomial coefficients $(a_1, \cdots, a_k)^*$ of the familiar Pascal relation for bi-

nomial coefficients. We accept as our starting point this easily proved relation,

$$(29)\qquad (a_1, a_2, \cdots, a_k)^* = (a_1 - 1, a_2, \cdots, a_k)^* + (a_1, a_2 - 1, \cdots, a_k)^* + \cdots + (a_1, a_2, \cdots, a_k - 1)^*,$$

which holds for any set of k nonnegative integers $a_1, \cdots, a_k$, at least one of which is positive.

Now suppose that for a given k, and n replaced by each of the k consecutive integers $n-k, \cdots, n-1$, the relation (28)'' holds, we can show using (1) that (28)'' holds in general. Consider the result of replacing each term on the right hand side of (28)'' by its expansion in terms of (29). The replacement for a single term will consist of k terms, some of which may have the value zero, with the property that the jth such term belongs to the right-hand side of the assumed version of (28)'' in which $f_{n-j,k}$ is expressed as a sum over $S_{n-j-k+1,k}$. When the complete breakdown of the right-hand side of (28)'' for $f_{n,k}$ is accomplished in this fashion we note that the totality of terms on the right in this equation coincides with the totality of the corresponding terms in the expansions for $f_{n-1,k}, \cdots, f_{n-k,k}$. Furthermore these last quantities which occur on the left in our assumed expansions total $f_{n,k}$ by (1) and the desired result follows. There remains only the problem of demonstrating k consecutive values of n for which (28)'' holds. This may be observed to hold for $n=k-1, k, k+1, \cdots, 2k-2$ for which the values $f_{n,k}$ are successively $1, 1, 2, 4, \cdots, 2^{k-2}$. By the above arguments and the principle of induction, (28)'' holds for a fixed $k \geqq 2$ and all $n \geqq k-1$. We check the result for the case $n=9$ and $k=4$. The sequence $f_{n,4}$ goes 0, 0, 0, 1, 1, 2, 4, 8, 15, 29, $\cdots$, so that $f_{9,4}=29$. On the other hand by (28)'' we get

$$\begin{aligned} f_{9,4} &= \sum_{S_{6,4}} (a_1, a_2, a_3, a_4)^* \\ &= (6,0,0,0)^* + (4,1,0,0)^* + (3,0,1,0)^* + (2,2,0,0)^* \\ &\quad + (2,0,0,1)^* + (1,1,1,0)^* + (0,3,0,0)^* + (0,1,0,1)^* + (0,0,2,0)^* \\ &= 1+5+4+6+3+6+1+2+1 = 29, \end{aligned}$$

which is what it should be.

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REMARKS ON A GENERALIZATION OF THE TRIGONOMETRIC FUNCTIONS

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Shelupsky [1] has recently investigated generalizations of the trigonometric functions defined by the differential equations

$$(d/dx)\alpha_s(x) = \beta_s^{s-1}(x), \qquad (d/dx)\beta_s(x) = -\alpha_s^{s-1}(x).$$

The purpose of this note is to point out prior work on this topic. Günther ([2], pp. 350–353) introduced these functions in terms of the geometry of the curves $x^m+y^m=1$ and defined, similarly, generalized hyperbolic functions in terms of the curves $x^m-y^m=1$.

The most exhaustive treatment appears to be that of Grammel [3]. He first defines the inverse circular functions

$$\mathrm{ArCos}\,(n)x = \int_x^1 (1-t^n)^{(1-n)/n}dt,$$

$$\mathrm{ArSin}\,(n)x = \int_0^x (1-t^n)^{(1-n)/n}dt$$

(the same as (5) and (6) of [1]), thence the direct functions, and proceeds to derive in considerable detail the properties of these functions, their series representations, and integral formulas. A similar treatment is given of the generalized hyperbolic functions. The results are extended to nonintegral values of n. Generalizations to curves defined by the more general relation $f(x, y)=1$ are also indicated. In another paper [4], Grammel has tabulated the values of the generalized circular functions for $n=4$, 6, 8.

It should be noted that the correct value for $\pi(3)$ is $3.533\cdots$, not $3.595\cdots$ as given in [1].

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ON THE "CLOCK PARADOX" IN RELATIVITY THEORY*

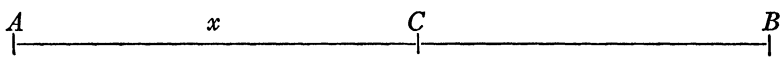
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If of two newborn babies one were put on a space ship traveling with a speed v equal to 99% of the speed of light c to the star Alpha Centauri, which is about four light-years from the earth, then turned around and came back to the earth, he would be a little more than one year old while his twin brother would have grown to be about eight years old. The reason for this is that the boy on the space ship lives, grows, and ages according to his own time, the time of a coordinate system with respect to which he is at rest. In short one says: Clocks on moving coordinate systems are slower.

The "apparent" paradox arises from the question: What if we assume a frame of reference fixed to the space ship so that relative to this system the boy who stayed on earth is moving? Should not then the boy who stayed home turn out to be younger than his space-traveling twin?

We give the answer to this question in three parts.

1. To make sure we do not have to worry about the effect of the velocity-reversal of the space-ship after arrival at Alpha Centauri we make the following arrangement:



We reflect the distance x from earth (A) to Alpha Centauri (C) on C which leads to B with $\overline{AC}=\overline{BC}$. Assume a colony of astronauts at B who are able to send newborn babies towards C. Have an astronaut at C release a light-flash at a certain moment. At the moment of the arrival of this light-flash a baby is sent out from earth (A) and from the space station (B) towards C. We first concentrate our attention on the earth baby traveling from A to C; when he meets the other baby at C we switch our attention to this other baby traveling from C to A. From the symmetry of this arrangement it is clear that the time necessary for the round trip $A\rightarrow C\rightarrow A$ is equal to the time required for the trip from A to B, or twice the time necessary for traveling from A to C.

2. We determine this time necessary to travel from A to C in both coordinate systems.

In the system resting with our solar system, the space ship traveling with velocity v obviously needs the time $t=x/v$ to cover the distance x according to

$$x = vt. \tag{1}$$

In the language of relativity theory, x and t are the world coordinates of the event (x, t), which is the arrival of the space ship at C. In the coordinate system in which the space ship is at rest, this same event of the ship's arrival at C has

* Cf. Alfred Schild, The clock paradox in relativity theory, this MONTHLY, vol. 66, 1959, pp. 1-18.

coordinates (x', t'). These coordinates are related to x and t by the Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{\{1 - (v^2/c^2)\}}}; \qquad t' = \frac{t - (v/c^2)x}{\sqrt{\{1 - (v^2/c^2)\}}}. \tag{2}$$

Substituting from (1) into (2) leads to

$$x' = 0; \qquad t' = t\sqrt{\{1 - (v^2/c^2)\}}. \tag{3}$$

The first of equations (3) confirms what we already know: the arrival of the ship at C takes place at a distance zero from the ship and nowhere else. The second of equations (3) tells us that, according to ship's time, this event will occur at an earlier time. If, according to a terrestrial observer, the ship needs $t=4$ years to cover the distance of four light-years from earth to Alpha Centauri, then the ship's clocks will read $t'=4\sqrt{\{1-(99/100)^2\}}$ or approximately half a year.

3. Now for the "paradox." Taking the coordinate system that moves along with the ship as the primary system, we should say: At the beginning of the voyage from A to C the earth was at the origin of this coordinate system $(x'=0)$; during the voyage, it moves backward with velocity $-v$ until C is at the origin and A is a distance $-x'$ behind the ship and

$$-x' = -vt'. \tag{4}$$

Since A and C are receding with velocity $-v$ relative to the space ship, this distance $\overline{AC}$ appears Lorentz-contracted for physicists on the space ship. Therefore

$$x' = x\sqrt{\{1 - (v^2/c^2)\}}. \tag{5}$$

Because of the Lorentz contraction (5), equation (4) becomes $-x\sqrt{\{1-(v^2/c^2)\}} = -vt'$ or $t'=(x/v)\sqrt{\{1-(v^2/c^2)\}}$. Since from (1) we know that $x/v=t$, we again have the second of equations (3). Thus, no matter which coordinate system is used, the twin who did the traveling is always younger by a factor $\sqrt{\{1-(v^2/c^2)\}}$.

MATHEMATICAL COMPETITIONS IN CHINA*

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Widespread American interest in the state of mathematics education in the Soviet Union was not aroused until the launching of the first Sputnik. Lacking any comparable incentive there has as yet been little concern with this subject as it applies to China. Yet it would seem advisable to make some beginning now in looking into this question, as there are several reasons for expecting a significant increase in the number and quality of mathematicians in China.†

China's population of some half to two-thirds of a billion people provides a vast reservoir of potential talent. Throughout its history this populous society has placed great store on intellectual activity, as is attested, among other things, by the fact that the Chinese equivalent of our Horatio Alger was the poor but bright young man who rose to high position by success in the rigorous competitive examinations for the imperial civil service. Today under the new regime on the mainland the rewards for academic success, both in terms of prestige and material advantage, are perhaps greater than ever before. This is particularly true in such officially preferred areas as mathematics.

One of the ways in which this official preference manifests itself is in the promotion of mathematical competitions as a means of identifying and encouraging mathematically talented students. Such contests were held for the first time in 1956 in four major cities in China. The Chinese Mathematical Society played a leading role in the actual conduct of the competitions. As a guide in the anticipated expansion of this activity throughout the country the Middle School Mathematics Study Group of the Shanghai Branch of the Chinese Mathematical Society published a report‡ on the competitions in China's largest city. I present herewith some notes based on this report together with a full translation of the contest questions used in Shanghai.

In broad perspective the mathematical competitions were undertaken in

* I am indebted to the American Council of Learned Societies and the Social Science Research Council for a joint grant in support of a project on Chinese mathematics of which this study is a part.

† The use of the phrase "mathematicians in China" rather than "Chinese mathematicians" is intended to exclude from consideration those of the latter who are even now serving with distinction in our own institutions of higher learning.

‡ Shang-hai shih 1956–57 nien chung hsüeh-sheng shu-hsüeh ching-sai his-t'i pien-hui [Compilation of problems from the 1956–57 Mathematical Competitions for Middle-School Students in Shanghai Municipality], Shanghai, Hsin chih-shih, 1958.

response to the official summons to the Chinese people to catch up with the advanced industrial nations of the world by 1970. (Great Britain is the rival most frequently mentioned in connection with this goal.) The report stresses the need for a rapid rise in the level of mathematical competence so as to satisfy the demands of the "great leap forward," particularly in such areas as agriculture, engineering, and the natural sciences.

The four major cities of Shanghai, Peking, Tientsin, and Hankow were the first to undertake mathematical competitions in 1956, their example being followed by many more cities in the following year. It was expected that eventually the contests would spread so widely that they would be comparable in scope to the Mathematical Olympiads of the Soviet Union and other countries of Eastern Europe.§

As to the contests in Shanghai, at the beginning of 1956 the local branch of the Chinese Mathematical Society, acting at the behest of the parent organization, proceeded to organize a Committee on Mathematical Competitions for Middle School Students. The 17 members of the committee were drawn from the Mathematical Society, the Shanghai Office of Education, and the Shanghai Branch of the All-China United Association of Natural Science Professional Societies. Support was also obtained from teachers and various social organizations. In addition the press and radio lent a hand in providing city-wide publicity to supplement the efforts made directly to the students in school to arouse interest in the contests.

The Shanghai competitions of 1956 and 1957 were held at two levels, one for second-year and another for third-year upper middle school students. (The Chinese lower- and upper-middle schools are of three years each and include grades 7 through 12. Hence the contestants corresponded approximately to our high school juniors and seniors.) Each competition consisted of three rounds. The first or preliminary round was conducted by the schools with their own examinations. Of the students enrolled in the last year of upper middle school, more than 4000 participated in the preliminary competition in 1956, and many more did so in 1957, but this was still less than 50 per cent of the total. The schools selected the best students to participate in the second round of the contest, the number chosen being limited to 3 per cent of the students enrolled in each class. There were 732 second-round contestants in 1956, and 1063 in 1957. The second and third rounds of the competitions were conducted by the Committee on Mathematical Competitions, which selected the top 6 to 10 per cent of the second-round contestants for participation in the third round, and then designated 10 students from each class as winners of the final round.

At the conclusion of each competition an awards assembly was held so as to increase the enthusiasm of the students who had already participated and to

§ R. Creighton Buck, A look at mathematical competitions, this MONTHLY, vol. 66, 1959, pp. 201–212; J. Aczél, A look at mathematical competitions in Hungary, this MONTHLY, vol. 67, 1960, pp. 435–437; Izaak Wirszup, The seventh mathematical olympiad for secondary-school students in Poland, The Mathematics Teacher, vol. 51, 1958, pp. 585–589.

encourage other students to participate in the future. Those attending the assembly included winners in each of the three rounds, student representatives of the first year of upper middle school, teachers, principals, parents of the third-round winners, and representatives of social organizations. The best students in the second- and third-round competitions were given certificates of merit and awards. In addition a representative of the Central Ministry of Higher Education made the announcement that the three top students in the final competition for the third-year class would be permitted to study any speciality in mathematics, astronomy, physics, and allied fields in any specified local comprehensive university without being subjected to further examination.*

There was one reservation as to the qualifications of these winners. This was that the student had to be in good health and politically acceptable. The feeling was expressed that inadequate attention had been given to this last point and that it should be taken into consideration in the future in selecting contestants for the second and third rounds. Ideally the end product of the educational process was a person who was, as the slogan put it, "both a Communist and a specialist."

As to the specifically mathematical aspects of the competitions, both positive and negative results were reported. On the positive side was the fact that the top ten names in the final round of the 1957 competition for third-year middle school students were distributed among nine schools, and the top 44 names in the second round were distributed among 26 schools, a picture which was said to be true also of the competitions for second-year students. This was considered evidence of the high quality of mathematics teaching throughout the middle schools of Shanghai. Nevertheless the judgment was expressed that the level of mathematical competence among students was still quite inadequate. Only fragmentary data were reported on student performance on the individual test problems, but some specific comments were made with respect to general shortcomings. These were as follows:

1. Many students are deficient in analyzing a problem on the basis of hypothesis and conclusions, getting at the heart of the problem, and then applying logical reasoning to its solution. Students should be given more opportunity for independent analysis and synthesis. It would be well for them to think about some problems for several days and not to be given the solutions too quickly.

2. Students should be given more practice in handling problems which involve a synthesis of algebra, geometry, and trigonometry.

3. More attention should be devoted in middle school to the study of inequalities so as to facilitate the transition from middle school to college mathematics.

4. Students pay insufficient attention to restating problems and are inadequately skilled in performing operations. The latter was especially apparent in complicated problems. Hence students should be given comparatively complicated problems for homework so as to develop their operational skills.

* Although students attending institutions of higher learning do not pay tuition fees and are entitled to receive grants in aid which provide for basic living expenses, the number of students attending such institutions is limited by competitive examinations.

It was further recommended that in order to enable students to go beyond the basic concepts and theorems contained in their textbooks, they should be encouraged to read additional books and journals, to form study groups for discussion of problem-solving and extracurricular reading, and to participate in regular, small-scale mathematical competitions.*

The emphasis in the preceding remarks on the importance of mathematical reasoning is reflected in the choice of problems for the competitions. Although these undoubtedly call for factual knowledge, they are primarily of the perceptive or aptitude type. Only five or six problems were assigned in any one competition, and 150 minutes, or about half an hour per problem, was allotted for their solution.

The problems for the second and third rounds of the 1956 and 1957 competitions are given below.

Problems of the second round, 1956

1. (a) Given a positive integral number n, prove that $13^{2n}-1$ is a multiple of 168.

(b) What is the nature of the natural number n that will make $1\cdot 2\cdot 3\cdot \cdots \cdot n$ divisible by $1+2+3+\cdots+n$?

2. Solve the inequality $10^{2\log x}+4x-\log_2 32>0$.

3. Between two low-lying fields is an angled ditch (Fig. 1). It is desired to straighten both banks of the ditch without changing the area of either field.

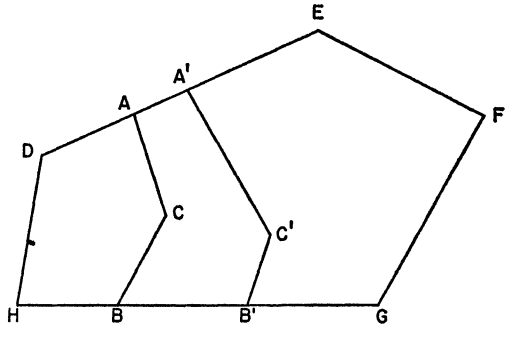


FIG. 1

(a) If the points A, A' are not changed, what should be done to the ditch?

(b) If the point A is not changed and the two banks of the ditch are to be parallel, what should be done to the ditch?

Explain the method of procedure in each case.

4. Find the six-digit number $1abcde$ which when multiplied by 3 becomes $abcde1$.

5. Construct a circle and, using only compasses (not a straight edge), divide the circle into 4 equal parts.

6. From a point A due west of an air-borne balloon the angle of elevation is 45°. From a point

* An interesting application of this emphasis on collective work and mathematical competitions is the combination of the two in the form of mathematical relay races. These are described by a middle school teacher, Ting Han-wen, in an article entitled "Shu-hsüeh chieh-li sai [Mathematical Relay Races]", Chung-hsüeh shu hsüeh [Middle School Mathematics], No. 7, 1959, 19–20.

B due southeast the angle of elevation is 67°30′. The points A and B are 266 meters apart and are both one meter above the ground. How high above the ground is the balloon?

Problems of the final round, 1956

1. From among the natural numbers from 1 to 100, in how many ways is it possible to select two numbers such that their sum exceeds 100?

2. (a) Given the arithmetic progression $a_1, a_2, a_3, \cdots, a_{n+1}$, prove that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\cdots+\frac{1}{\sqrt{a_n}+\sqrt{a_{n+1}}}=\frac{n}{\sqrt{a_1}+\sqrt{a_{n+1}}}.$$

(b) Given the geometric progression $a_1, a_2, a_3, \cdots, a_{n+1}$, prove that

$$\sqrt{(a_1a_2)}+\sqrt{(a_2a_3)}+\cdots+\sqrt{(a_na_{n+1})}=\frac{\sqrt[2n]{(a_1a_{n+1})}}{\sqrt[n]{a_1}-\sqrt[n]{a_{n+1}}}(a_1-a_{n+1}).$$

3. Given the polynomial $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$ with integral coefficients, and an odd number α and an even number β such that $f(\alpha)$ and $f(\beta)$ are both odd. Prove that the equation $f(x)=0$ has no integral roots.

4. Given any $\triangle ABC$, prove that

(a) $tg^2\frac{1}{2}A+tg^2\frac{1}{2}B+tg^2\frac{1}{2}C\geqq 1$; (b) $\cos A+\cos B+\cos C>1$.

5. In a given circle construct an inscribed equilateral triangle such that the sum of its base and its height is a maximum.

6. Figure 2 represents a flattened solid. That is to say, if properly put together it will form a solid.

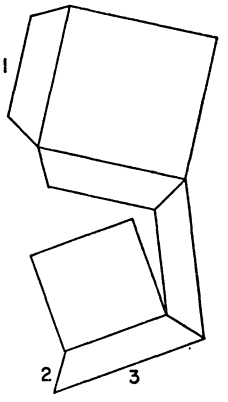


FIG. 2

(a) When the figure is assembled, which line segments meet line segments 1, 2, 3 respectively? Write in the corresponding numbers.

(b) What is this figure? Why?

Problems of the second round, 1957

A. Problems for second-year students

1. Find 10 consecutive natural numbers each of which is a composite number.

2. If the equations $x^2+ax+b=0$ and $x^2+px+q=0$ have one root in common, find the quadratic equation whose roots are their dissimilar roots.

3. For what values of x is the following meaningful:

$$\sqrt{\lg\sqrt{\lg\sqrt{\lg\sqrt{\lg\sqrt{\lg x}}}}}.$$

4. About a given equilateral triangle circumscribe another equilateral triangle whose area is a maximum.

5. If A is an acute angle, prove that

$$\begin{aligned}&\sec A + \sec(A/2) + \sec(A/3) + \cdots + \sec(A/n)\\ &+ \csc A + \csc(A/2) + \csc(A/3) + \cdots + \csc(A/n)\\ &> \sec A \csc A + \sec(A/2)\csc(A/2) + \sec(A/3)\csc(A/3) + \cdots + \sec(A/n)\csc(A/n).\end{aligned}$$

B. Problems for third-year students

1. Same as Problem 1 for second-year students.

2. Find the positive root of the following equation, and prove that there is only one positive root:

$$x = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+x}}}}.$$

3. Find the real values of x which will satisfy the following relationship: $|3x-2|+|3x+1|=3$.

4. D, C are two points on a semicircle with diameter AB, and x is any point on AB. Prove that $\operatorname{tg} ACX\cdot\operatorname{tg} BDX=\operatorname{tg} BAC\cdot\operatorname{tg} ABD$.

5. $V-ABC$ is a trihedral angle and VD is a line bisecting the face angle BVC. If $\angle AVD\gtreqless\frac{1}{2}\pi$, prove that $\frac{1}{2}(\angle AVB+\angle AVC)\lesseqgtr\angle AVD$.

6. Prove that in any triangle the continued product of the lines bisecting the three angles is less than the continued product of the three sides.

Problems of the final round, 1957

A. Problems for second-year students

1. If the natural number $62ab427$ is a multiple of 99, find a, b.

2. Given a rectangular plot of land $ABCD$ which is divided into three segments by the straight lines EF and GH. (Fig. 3). How can these three segments be transformed into rectangles of respectively equal area such as in Figure 4?

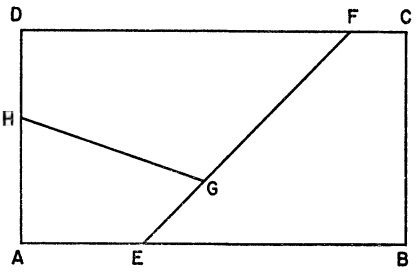


FIG. 3

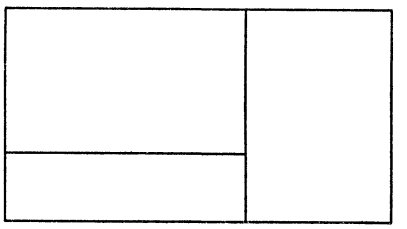


FIG. 4

3. If $x+y+z=0$, prove that

$$\left(\frac{x^2+y^2+z^2}{2}\right)\left(\frac{x^5+y^5+z^5}{5}\right)=\frac{x^7+y^7+z^7}{7}.$$

4. Given four concentric circles whose radii are respectively 1, 2, 3, 4 and the square $ABCD$ circumscribed about the largest circle.

(a) Make a square contained in $ABCD$ whose four sides are respectively tangent to the four circles* and are parallel to or identical with the four sides of $ABCD$.

(b) How many such squares can be made?

5. (a) In any given triangle ABC with sides a, b, c (Fig. 5), prove that $a^2=b^2+c^2-2bc\cos A$.

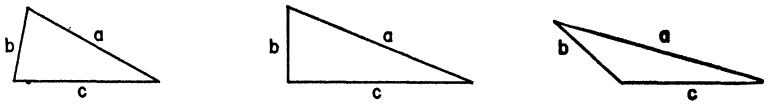


FIG. 5

(b) Given a point P on the circumference of a unit circle and the vertices $A_1, A_2, \cdots, A_n$ of an inscribed regular polygon of n sides (Fig. 6), prove that $\overline{PA}_1^2+\overline{PA}_2^2+\cdots+\overline{PA}_n^2$ is a constant.

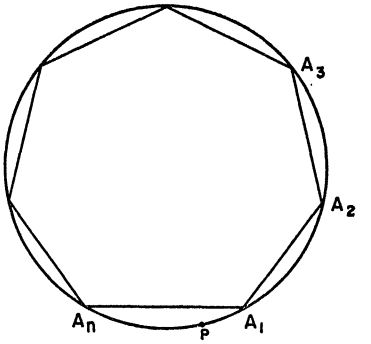


FIG. 6

B. Problems for third-year students

1. Same as Problem 1 for second-year students.

2. Same as Problem 2 for second-year students.

3. Given the whole number a larger than 2 and the composite number b ($b>0$), if b can be divided evenly by r different positive integral numbers, prove that a^b-1 can be divided evenly by at least r different positive integral numbers.

4. Prove that the area of any polygon of n sides inscribed in a unit circle does not exceed $\frac{1}{2}n\sin(2\pi/n)$.

5. Given the points $A_1, A_2, A_3, \cdots$, on a straight line, with $\overline{A_1A_2}$ having length 1; A_{n+2} a point on the segment $\overline{A_nA_{n+1}}$; and $\overline{A_nA_{n+2}}$: $\overline{A_{n+2}A_{n+1}}=\overline{A_{n+2}A_{n+1}}$: $\overline{A_{n+1}A_n}$, in which $n=1, 2, 3, \cdots$. Prove that the sequence formed by the lengths $\overline{A_1A_2}$, $\overline{A_1A_3}$, $\overline{A_1A_4}$ has a limit, and that this limit is $\frac{1}{2}(\sqrt{5}-1)$ (Fig. 7).

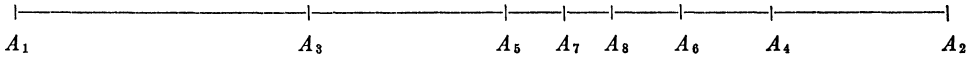


FIG. 7

* It is not clear from the wording of the problem that in some cases the sides of the square must be extended if they are actually to touch the circles.

ON GENERALIZATIONS OF THE EXPONENTIAL FUNCTION

S. TAUBER, Portland State College

1. Introduction. This paper deals with functions that are defined as infinite series of the nth power (n fixed) of the terms of the Maclaurin series of the exponential function. Such functions are primarily met in the study of generating functions of sets of orthogonal functions. To our knowledge such functions have neither been studied nor tabulated although they seem useful (aside from the above-mentioned application) as a generalization of the exponential function and contain, for their second order, a certain kind of Bessel functions.

It will be shown that there is a linear relation between the function of order n and its derivatives up to order n; thus the function satisfies a linear differential equation of order n. The n linearly independent solutions of the differential equation are found as logarithmic series. It follows that the function can be expanded in logarithmic series.

2. Notations. The following notations are currently used in finite differences (see, for example, [1]), a being any complex number and b any positive integer.

Factorial monomial:

$$(a)_b = a(a-1)\cdots(a-b+1). \tag{1}$$

Stirling numbers of the first kind: St_b^m $(1 \leqq m \leqq b)$ defined by

$$(a)_b = \sum_{m=1}^{b} St_b^m a^m. \tag{2}$$

Stirling numbers of the second kind: st_b^m $(1 \leqq m \leqq b)$ defined by

$$a^b = \sum_{m=1}^{b} st_b^m (a)_m. \tag{3}$$

Differential operators:

$$D_t = d/dt, \qquad D_x = d/dx, \qquad D_s = d/ds. \tag{4}$$

3. Definition of the function T_n. Differential equation. Let n be a positive integer; we define the function T_n by:

$$T_n(x) = \sum_{m=0}^{\infty} x^{mn}/(m!)^n. \tag{1}$$

The general term of the series is the nth power of the general term of the exponential series. The series converges, just as the exponential series does, for all values of the variable. We denote by T_n^p the pth derivative of T_n. Hence for any positive integer $p \leqq n$,

$$T_n^p(x) = D^p T_n(x) = \sum_{m=1}^{\infty} (nm)_p x^{nm-p}/(m!)^n. \tag{2}$$

If $n<p$, the expressions are the same, but the summation will start with 2 for $n<p\leqq 2n$; with 3 for $2n<2p\leqq 3n$; etc. For the special case $p=n$, we have

$$D_x^n T_n(x) = T_n^n(x) = \sum_{m=1}^{\infty} (nm)_n x^{nm-n}/(m!)^n. \tag{3}$$

T_n^n can be expressed in terms of T_n^{n-1}, T_n^{n-2}, $\cdots$, T_n^1, and T_n with the help of Stirling numbers of the first and second kind. Direct calculations are long and complicated. It is simpler to make the substitution $x=e^{t/n}$, $t=n \ln x$. We thus obtain

$$T_n(e^{t/n}) = \sum_{m=0}^{\infty} e^{mt}/(m!)^n, \tag{4}$$

$$D_t^n T_n(e^{t/n}) = e^t T_n(e^{t/n}), \tag{5}$$

so that

$$(D_t^n - e^t) T_n(e^{t/n}) = 0, \tag{6}$$

which proves that $T_n(e^{t/n})$ is a particular solution of the differential equation

$$(D_t^n - e^t)y = 0. \tag{7}$$

It is then easy to return to x, since $dx/dt=x/n$, and $D_t^2 y=(xD_x y+x^2 D_x^2 y)/n^2$. More generally it is found that

$$D_t^p y = n^{-p} \sum_{q=1}^{p} A_p^q x^q D_x^q y, \tag{8}$$

where the A_p^q are numerical constants.

It is easily seen that the A_p^q-coefficients satisfy the relation $A_{p+1}^q = A_p^{q-1} + qA_p^q$ if we define $A_p^0 = A_p^{p+1} = 0$. This is the same relation as the one satisfied by Stirling numbers of the second kind (cf. [1], p. 169). Since $A_1^1 = A_2^1 = A_2^2 = 1$, it follows that the A_p^q-coefficients are no other than the Stirling numbers of the second kind, $A_p^q = st_p^q$, and (8) becomes for $p=n$

$$D_t^n y = \sum_{q=1}^{n} st_n^q x^q D_x^q y/n^n. \tag{8a}$$

Then (7) becomes

$$\left[\sum_{q=1}^{n} st_n^q x^{q-1} D_x^q - n^n x^{n-1} \right] y = 0. \tag{9}$$

Equation (9) has the following special cases:

$$y' - y = 0 \qquad (n = 1),$$
$$xy'' + y' - 4xy = 0 \qquad (n = 2),$$
$$x^2y''' + 3xy'' + y' - 27x^2y = 0 \qquad (n = 3),$$
$$x^3y''' + 6x^2y''' + 7xy'' + y' - 256x^3y = 0 \qquad (n = 4).$$

For $n=1$ the solution is $y=Ce^x$. For $n=2$ the solution is given in [2], p. 440; there equation 2.162 (1a) is $x^2y''+axy'+(bx^m+c)y=0$, which for $a=1$, $b=-4$, $c=0$, $m=2$, has the solution $y=C_1J_0(2ix)+C_2Y_0(2ix)=C_1I_0(2x)+C_2K_0(2x)$. For a tabulation of Stirling numbers of the second kind see [1], p. 170.

From (9) we obtain $T_n^n=n^nx^{n-1}-\sum_{q=1}^{n-1} st_n^q x^{q-1}T_n^q$ which expresses T_n^n in terms of T_n^k, $k<n$, and Stirling numbers. For example, we have

$$T_1^1 = T_1, \qquad T_2^2 = -(T_2^1 - 4xT_2)/x, \qquad T_3^3 = -(3xT_3^2 + T_3^1 - 27x^2T_3)/x^2,$$
$$T_4^4 = -(6x^2T_4^3 + 7xT_4^2 + T_4^1 - 256x^3T_4)/x^3.$$

4. General solution of the differential equation. Equation (9), Section 3, is of Fuchsian type (Cf., [4], vol. II). The point $x=0$ is a regular singular point of the differential equation. We obtain the indicial equation by looking for solutions of the form

$$(1) \qquad y = \sum_{m=0}^{\infty} a_m x^{c+m}, \qquad a_0 \neq 0.$$

Since $D_x^q y=\sum_{m=0}^{\infty} a_m(c+m)_q x^{c+m-q}$, we obtain by substituting into (9), Section 3,

$$\sum_{q=1}^{n}\sum_{m=0}^{\infty} a_m(c+m)_q st_n^q x^{c+m-n} - n^n\sum_{m=0}^{\infty} a_m x^{c+m} = 0.$$

Equating to zero the coefficient of x^{c+m} and using (3), Section 2, we obtain $a_{m+n}(c+m+n)^n-n^na_m=0$ or $a_m(c+m)^n-n^na_{m-n}=0$, if we let $a_j=0$ for $j<0$. For $m=0$ this last expression reduces to $a_0c^n=0$, and since $a_0\neq 0$, this equation has n roots in c equal to zero. According to Fuchs' theory ([4] vol. II) it follows that (9), Section 3, has one single particular solution in power series, while all the other particular solutions contain logarithmic terms or can be expressed in power series of logarithms. This last form is easily obtained from (7), Section 3 by searching for solutions of the form $y=\sum_{m=0}^{\infty} a_m t^{c+m}$ to the differential equation $(D_t^n-e^t)y=0$. By direct substitution we obtain, after division by t^c,

$$\sum_{m=0}^{\infty} a_m(m)_n t^{m-n} - \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_q t^{p+q}/p! = 0.$$

Taking $a_0, a_1, \cdots, a_{n-1}$, as indeterminate coefficients, we obtain by equating equal powers of t, for $m>n-1$,

$$(n + p)_n a_{n+p} = \sum_{q=0}^{p} a_q/(p - q)! \qquad p = 0, 1, 2, \cdots.$$

The coefficients $a_0, a_1, \cdots, a_{n-1}$ can be chosen arbitrarily and the remaining coefficients a_{n+p} determined from them. In particular, we can obtain n independent solutions

$$y_{q,n}(t) = \sum_{m=0}^{\infty} a_m^q t^m, \qquad q = 0, 1, \cdots, n - 1$$

by taking

$$a_m^q = \begin{cases} 1 \text{ if } m = q \\ 0 \text{ if } m \neq q, \end{cases} \qquad 0 \leqq m \leqq n - 1.$$

It is convenient to write $N(n, q, m) = a_m^q m!$. The following table gives the values of $N(n, q, m)$ for $n = 2, 3, 4$, $m \leqq 10$:

m	0	1	2	3	4	5	6	7	8	9	10
$N(2, 0, m)$	1	0	1	1	2	5	13	36	109	359	1266
$N(2, 1, m)$	0	1	0	1	2	4	10	29	90	295	1030
$N(3, 0, m)$	1	0	0	1	1	1	2	6	17	54	98
$N(3, 1, m)$	0	1	0	0	1	2	3	5	12	48	89
$N(3, 2, m)$	0	0	1	0	0	2	6	12	22	48	96
$N(4, 0, m)$	1	0	0	0	1	1	1	1	2	7	23
$N(4, 1, m)$	0	1	0	0	0	1	2	3	4	6	14
$N(4, 2, m)$	0	0	1	0	0	0	2	6	12	20	32
$N(4, 3, m)$	0	0	0	1	0	0	0	6	24	60	120

The particular solutions of (7), Section 3, can now be written

$$y_{q,n}(t) = \sum_{m=0}^{\infty} N(n, q, m) t^m/m!, \qquad q = 0, 1, \cdots, n - 1, \tag{2}$$

and the solutions of (9), Section 3, as

$$y_{q,n}(n \ln x) = \sum_{m=0}^{\infty} N(n, q, m) n^m (\ln x)^m/m!. \tag{3}$$

The solution $T_n(x)$ is a linear combination of the $y_{q,n}(x)$,

$$T_n(x) = \sum_{q=0}^{n-1} K_q y_{q,n}(x) = \sum_{m=0}^{\infty} \sum_{q=0}^{n-1} K_q N(n, q, m) n^m (\ln x)^m/m!.$$

Thus we have expressed $T_n(x)$ as a power series in $\ln x$. In the case $n = 2$, for example, we obtain

$$T_2(e^{t/2}) = \sum_{m=0}^{\infty} e^{mt}/(m!)^2 = K_0 y_{0,2}(t) + K_1 y_{1,2}(t),$$

$$D_t T_2(e^{t/2}) = \sum_{m=0}^{\infty} me^{mt}/(m!)^2 = K_0 y'_{0,2}(t) + K_1 y'_{1,2}(t).$$

Taking $t=0$, we obtain $y_{0,2}(0)=1$ and $y_{1,2}(0)=0$, $y'_{0,2}(0)=0$, and $y'_{1,2}(0)=1$, so that

$$K_0 = T_2(1) = 2.279\ 585\ 518\cdots, K_1 = \sum_{m=1}^{\infty} m/(m!)^2 = 1.590\ 638\cdots.$$

For other values of n the calculations are similar.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE TWIN PRIME CONSTANT

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Let $T(x)$ be the number of prime twins with first member not exceeding x. Brun [1] showed that $T(x)=O(x/\log^2 x)$, and Hardy and Littlewood [2] proved that if $T(x)\sim C(x/\log^2 x)$ for some constant C (which is a plausible conjecture with excellent empirical support from the tables of prime numbers) then

$$C = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) = 1.32032\cdots. \tag{1}$$

In 1954, E. M. Wright [3], in his *Postscript on prime pairs*, attributed a non-rigorous probabilistic derivation of (1) to Lord Cherwell, citing a 1946 paper [4]. This reference was repeated recently in this MONTHLY [5] by Pólya. Actually, the first probabilistic derivation of (1) can be found explicitly as early as Selmer's paper [6] which appeared in 1942, and perhaps implicitly in earlier writings of Brun (cf. [7]).

The probabilistic derivation of (1) is actually simpler than the presentations of it in [3], [5], and [6]. It may be done in this manner:

By the prime number theorem, $\Pr(x \text{ is prime})\sim 1/\log x$. However, the proba-

bility that $x+2$ is prime, *given* that x is prime, departs from $1/\log(x+2)$ as follows: Since x is prime, x is odd with probability 1, and $x+2$ is odd, doubling the chance of $x+2$ being prime. For any odd prime p, $x \neq p$ is *not* divisible by p with probability 1, which *decreases* the probability that $x+2$ is relatively prime to p from $(p-1)/p$ to $(p-2)/(p-1)$. Thus the probability that $x+2$ is prime departs from $1/\log(x+2)$ by the factor

$$2\prod_{p>2}\left(\frac{p-2}{p-1}\right)\Big/\left(\frac{p-1}{p}\right) = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

Thus

$$T(x)/x \sim \frac{1}{\log^2 x}\cdot 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

The question of how many primes to include in the product need never arise (as it does in the references cited), because the infinite product is clearly convergent. Thus the product can be extended over all odd primes, rather than starting with $2<p<x^{\mu}$ and letting $x\to\infty$, where x^{μ} can only be justified on seemingly mystical grounds [5] at best. (In fact, different authors use different values of μ.)

In fairness to the references cited, the question of how many primes to include in the product *is* significant in estimating $\pi(x)$, the number of primes up to x, but as shown in this note (which follows the treatment in [8]), the question can be sidestepped entirely in the probabilistic estimation of $T(x)$.

It is also interesting to observe that

$$C = 2\sum_{\text{odd } n=1}^{\infty}\frac{\mu(n)}{\phi^2(n)}, \tag{2}$$

where μ and ϕ are the Möbius and Euler functions, respectively. This formula arises in the "singular series" approach of Hardy and Littlewood. Actually, (1) may be regarded as an Euler product expansion for (2).

Also,

$$C = \frac{1}{4}\sum_{\text{odd } n=1}^{\infty}\frac{\mu(n)2^{\nu(n)}\log^2 n}{n}, \tag{3}$$

where $\nu(n)$ is the number of distinct prime factors of n. The expression (3) for C arises in the approach to twin primes followed in [8].

'Another interesting constant connected with the twin primes is "Brun's sum,"

$$(\tfrac{1}{3}+\tfrac{1}{5})+(\tfrac{1}{5}+\tfrac{1}{7})+(\tfrac{1}{11}+\tfrac{1}{13})+(\tfrac{1}{17}+\tfrac{1}{19})+\cdots,$$

the sum of the reciprocals of the twin primes, which Brun showed is either convergent or finite. Selmer [6] evaluated this sum to three decimal places,

omitting the pair (1/3+1/5), and obtaining the value 1.368 · · · for the rest, guaranteed to within 0.1 per cent.

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THE QUADRATIC SUBFIELD OF THE FIELD GENERATED BY THE p-th ROOT OF UNITY

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It is well known that the field $R(\zeta_p)$ generated by the pth root of unity ζ_p (p an odd prime) contains the field generated by $\sqrt{p}$ as its only quadratic subfield if $p \equiv 1$ (4) and the field generated by $\sqrt{(-p)}$ if $p \equiv 3(4)$. An elementary proof for this discrepancy follows from the Gauss sums. Another proof can be obtained from the known expression for the discriminant of the cyclotomic field which contains p as its only prime factor. This implies that the quadratic subfield must be generated by the square root of an integer $n \equiv 1(4)$. The following alternative proof uses no algebraic number theory.

Let $p = 4n+3$. Then $p-1 = 4n+2 = 2m$ where m is odd. Since $(2, m) = 1$ the field $R(\zeta_p)$, which is cyclic and of degree $p-1$, is the product of a quadratic field and of a field of odd degree which is necessarily real. Since $R(\zeta_p)$ contains complex quantities the quadratic field cannot be real.

For $p = 4n+1$ we can argue, *e.g.*, as follows: Since $p-1 = 4n$ the field $R(\zeta_p)$ contains a biquadratic cyclic field and the quadratic subfield of such a field is known to be real.*

In a vague way, a transition between the two cases is given by the case of the integers of the form $4n$. The field $R(\zeta_{4n})$, $n > 1$, is no longer cyclic and contains a real and an (actually several) imaginary quadratic field. For, the field of the 8th root of unity contains $\sqrt{(-1)}$ and $\sqrt{2}$, while any $R(\zeta_{4p})$, p an odd prime, contains $\sqrt{(-1)}$ and $\sqrt{p}$ or $\sqrt{(-p)}$, hence both.

* See, *e.g.*, H. Weber, Algebra, vol. 1 (2nd ed.), 1898, p. 699.

THE BASIS THEOREM FOR FINITELY GENERATED ABELIAN GROUPS

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The following proof of the basis theorem for finitely generated Abelian groups may be of interest because of its simplicity and because of the theorem presented here on which it is based.

If a_1 and a_2 are integers with greatest common divisor 1 then there are numbers r_1 and r_2 so that $a_1r_1+a_2r_2=1$. This fact is extended in the lemma below.

LEMMA. *If $a_1, \cdots, a_n (n>1)$ are integers whose greatest common divisor is 1, then there is an $n\times n$ matrix with integral elements whose determinant is 1 in which the $a_1, \cdots, a_n$ appear as the elements of the first row.*

Proof. For $n=2$ this follows from the statement of the paragraph preceding the lemma. We suppose that for $i=2, \cdots, n-1$, $a_i=b_id$ where d is the greatest common divisor of the a_i and hence the b_i have greatest common divisor 1; then by the induction assumption the b_i are elements of the first row of an $n-1$ square matrix of determinant 1.

Since a_n and d have greatest common divisor 1, there are integers s and t so that $sa_n+dt=1$. If e is the proper choice of ± 1 and if the $n-1$ square matrix of the previous paragraph is bordered as follows:

$$\begin{pmatrix} b_1d & \cdots & b_{n-1}d & a_n \\ * & \cdots & * & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ * & \cdots & * & 0 \\ esb_1 & \cdots & esb_{n-1} & t \end{pmatrix},$$

the resulting matrix has determinant 1 and the lemma is proved.

THEOREM. *If $x_1, \cdots, x_n$ are generators of an additively written Abelian group and $a_1, \cdots, a_n$ are integers with greatest common divisor 1, then $a_1x_1+\cdots+a_nx_n$ may be chosen as one of a set of n generators of the group.*

Proof. Let A be the matrix of the lemma. Then $A^{-1}=$Adjoint of A has integral entries also. If X is the column vector on the symbols $x_1, \cdots, x_n$ then the elements of the group corresponding to the rows of the column vector AX are a set of generators for the group. For it is clear that $X=A^{-1}AX$; hence the generators $x_1, \cdots, x_n$ may be expressed as integral combinations of the new generators.

BASIS THEOREM. *If G is an Abelian group generated by a finite number of elements, then G is the direct product of cyclic groups.*

Proof. Suppose every set of generators of G contains n or more elements. Then choose a set of n generators so that one of them, x_n, has minimal order k in the sense that no other set of n generators has an element of smaller order.

The other $n-1$ generators, $x_1, \cdots, x_{n-1}$, generate a proper subgroup H of G. By an induction argument on n we may assert that H is a direct product of $n-1$ cyclic subgroups. The theorem will be proved when it is shown that the intersection of H with the cyclic group C generated by x_n is zero. If this were not so then for some integers $a_n < k$, and $a_1, \cdots, a_{n-1}$, $a_n x_n = a_1 x_1 + \cdots + a_{n-1} x_{n-1}$ or $a_1 x_1 + \cdots + a_{n-1} x_{n-1} - a_n x_n = 0$. If the greatest common divisor of $a_1, \cdots, a_n$ is d, then by the above theorem the element $g = (a_1/d)x_1 + \cdots + (a_{n-1}/d)x_{n-1} - (a_n/d)x_n$ is a member of a set of n generators of G of order a divisor of $d < a_n < k$. This is contrary to the choice of x_n. Accordingly we conclude that $H \cap C = 0$ as desired.

ON DIFFERENTIATION OF SERIES TERM-BY-TERM

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It is of course common knowledge that the convergence of a sequence of continuously differentiable functions on the unit interval does not imply the differentiability of the limit function nor the convergence of the derivatives of the functions in the sequence. We happened to notice that if the second derivatives of the functions in the sequence exist, and are uniformly bounded, then term-by-term differentiation is indeed valid. Since this simple fact does not seem to be widely known, nor in the literature, we offer this note.

The following lemma, adequate for our purposes, is a weakened version of results of Landau [3, 4] whose results in turn were stimulated by Littlewood's rediscovery [2, 5] of an observation of Moore ([6], p. 316.)

LEMMA 1. *Let g be a twice-differentiable real-valued function defined on an interval I of length l. Suppose that $|g(x)| \leqq \epsilon$ and $|g''(x)| \leqq k$ for all x in I, where ϵ and k are fixed positive numbers. Lastly, suppose $4(\epsilon/k)^{1/2} \leqq l$. Then $|g'(x)| \leqq 2(\epsilon k)^{1/2}$.*

Proof. Taylor's theorem gives $g(x+t) = g(x) + tg'(x) + t^2 \int_0^1 (1-y)g''(ty+x)dy$ for all x and $x+t$ in I. Therefore

$$|t|\,|g'(x)| \leqq |g(x+t) - g(x)| + t^2 \int_0^1 (1-y)\,|g''(ty+x)|\,dy$$

$$\leqq 2\epsilon + t^2 \int_0^1 k(1-y)dy = 2\epsilon + kt^2/2.$$

Consequently, $|g'(x)| \leqq 2\epsilon/|t| + k|t|/2$ for all t such that x and $x+t$ are in I. For x in the first half of I, let $t = 2(\epsilon/k)^{1/2}$; for x in the second half of I, let $t = -2(\epsilon/k)^{1/2}$. Since $2(\epsilon/k)^{1/2} \leqq l/2$, $x+t$ is in I. Using this t we get $|g'(x)| \leqq 2(\epsilon k)^{1/2}$.

THEOREM 1. *Let f_n be a sequence of real-valued functions defined on a closed interval and satisfying* (i) *f_n converges uniformly; and* (ii) *f_n is twice differentiable, and for some constant k, $|f_n''(x)| \leqq k$ uniformly in n and x. Then f_n' converges uniformly.*

Proof. Let ϵ be a sufficiently small positive number. Then for all i and j sufficiently large, $|f_i - f_j| \leqq 2\epsilon$. Also, of course, $|f_i' - f_j'| \leqq 2k$. We apply Lemma 1 to conclude that $|f_i' - f_j'| \leqq 4(\epsilon k)^{1/2}$.

COROLLARY. *The limit f of the sequence f_n is differentiable and f_n' converges to f' uniformly. Furthermore f' satisfies the Lipschitz condition $|f'(x) - f'(y)| \leqq k|x-y|$.*

The hypotheses of the theorem can be relaxed in three ways: first, one may replace (i) by the assumption that f_n converges at each point of a dense subset of the interval; second, one may replace (ii) by the hypothesis that f_n' exists everywhere and satisfies the Lipschitz condition $|f_n'(x) - f_n'(y)| \leqq k|x-y|$ uniformly in n, x and y; third and last, one need only assume that for some positive integer r, the $(r+1)$st derivatives exist and satisfy $|f_n^{(r+1)}| \leqq k$, or more generally that the rth derivatives exist and satisfy a uniform Lipschitz condition. That is:

THEOREM 2. *Let f_n be a sequence of real-valued functions defined on a closed interval and satisfying* (i) *f_n converges at each point of a dense subset of the interval; and* (ii) *for some $r \geqq 1$, the rth derivatives of the f_n exist and satisfy $|f_n^{(r)}(x) - f_n^{(r)}(y)| \leqq k|x-y|$ for all n, x, and y. Then for each j, $0 \leqq j \leqq r$, $f_n^{(j)}$ converges uniformly. (Here $f_n^{(0)}$ means f_n.)*

COROLLARY. *The limit f of the sequence f_n has r continuous derivatives, and for each j, $1 \leqq j \leqq r$, $f_n^{(j)}$ converges uniformly to $f^{(j)}$. Furthermore $f^{(r)}$ satisfies the Lipschitz condition $|f^{(r)}(x) - f^{(r)}(y)| \leqq k|x-y|$.*

Since the ideas needed for the proof of Theorem 2 are well known to analysts, we do not give a proof, but are content to suggest that a proof can be based on the circle of ideas involved in Ascoli's theorem which may be found, for example, in [1].

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THE MATRIX CONGRUENCE $X^2 \equiv I \pmod{p^a}$

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For p an odd prime, we determine all matrix solutions of order m of $X^2 \equiv I \pmod{p^a}$.* It is known ([1] p. 518) that the solutions of $X^2 \equiv I \pmod{p}$ are exactly the matrices similar modulo p to $J_t = \operatorname{diag}(I_t, -I_{m-t})$ with $0 \leqq t \leqq m$.

1. Any solution of

$$X^2 \equiv I \pmod{p^{a+1}} \tag{1}$$

is congruent modulo p^a to some root R of

$$X^2 \equiv I \pmod{p^a}, \tag{2}$$

where R in turn is congruent modulo p to some root Q of

$$X^2 \equiv I \pmod{p}. \tag{3}$$

To find all solutions of (1) which are congruent to R modulo p^a, we let $X = R + p^a T$ and seek all T modulo p such that $(R + p^a T)^2 \equiv I \pmod{p^{a+1}}$, or, equivalently, all T such that

$$\frac{R^2 - I}{p^a} + TQ + QT \equiv (0) \pmod{p}. \tag{4}$$

Now we choose $Q = J_t$ and we suppose R known. In solving (4) for

$$T \equiv \begin{bmatrix} A^{(t)} & B \\ C & D^{(m-t)} \end{bmatrix} \pmod{p},$$

we get equivalently

$$\begin{bmatrix} 2A & (0) \\ (0) & -2D \end{bmatrix} \equiv \frac{I - R^2}{p^a} \pmod{p}. \tag{5}$$

Now (5) is solvable for A and D modulo p and uniquely so if and only if null matrices of appropriate sizes occur in the upper right corner and the lower left corner of $p^{-a}(I - R^2)$ modulo p. This can be accomplished by replacing R by $R + p^a V$ with suitable V, if need be. Then, since (5) is independent of B and C, we see that there are exactly p^N solutions for T, where $N = 2t(m-t)$ is the number of entries in B and C. Since, moreover, for $Q = J_t$ and R fixed, incongruent solutions T_1, T_2 modulo p of (4) lead to incongruent solutions X_1, X_2 modulo p^{a+1} of (1), and incongruent solutions R_1, R_2 modulo p^a of (2) lead to incongruent solutions X_1, X_2 modulo p^{a+1} of (1), an induction argument shows us that there are exactly $p^{N \cdot a}$ solutions X of (1) for which $X \equiv J_t \pmod{p}$. In addition, (5) enables us to write down, albeit recursively, exactly what these solutions are.

* The referee has kindly pointed out that the number of solutions given in the present note is a special case of a result announced in [2].

2. Here we use the binomial theorem to obtain explicit, nonrecursive forms of the solutions discussed in Section 1.

Let

$$X \equiv \begin{bmatrix} E^{(t)} & F \\ G & H^{(m-t)} \end{bmatrix} \pmod{p^{a+1}},$$

where F and G are chosen arbitrarily modulo p^{a+1}, except that we take $F \equiv (0) \pmod{p}$ and $G \equiv (0) \pmod{p}$; and where we choose

$$(6) \quad E \equiv I - \tfrac{1}{2}FG - \tfrac{1}{8}FGFG - \tfrac{1}{16}FGFGFG - \tfrac{5}{128}FGFGFGFG - \cdots \pmod{p^{a+1}}$$

and

$$(7) \quad H \equiv -I + \tfrac{1}{2}GF + \tfrac{1}{8}GFGF + \tfrac{1}{16}GFGFGF + \tfrac{5}{128}GFGFGFGF + \cdots \pmod{p^{a+1}},$$

the coefficients in (6) and (7) being those in the expansion of

$$(1-x)^{1/2} = 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 - \cdots.$$

We note first that there are only a finite number of terms in (6) and (7) since $F \equiv (0) \pmod{p}$ and $G \equiv (0) \pmod{p}$. We also observe from the arbitrary character of F and G that we have $p^{N \cdot a}$ choices for X modulo p^{a+1}. Clearly, each such X is congruent to J_t modulo p. Finally,

$$X^2 \equiv \begin{bmatrix} E^2 + FG & EF + FH \\ GE + HG & GF + H^2 \end{bmatrix} \pmod{p^{a+1}},$$

and this reduces by our choices above to $X^2 \equiv I \pmod{p^{a+1}}$.

3. The solutions of (1) which are yet to be considered are those built up from the matrices similar modulo p to some J_t, $0 \leqq t \leqq m$. We view them as transforms of the solutions we already have as follows: Let us fix t and take P an integral matrix with determinant prime to p. If now X is any solution of (1) which is congruent to J_t modulo p, then $P^{-1}XP$ modulo p^{a+1} is a solution of (1) which is congruent to $P^{-1}J_tP$ modulo p. Moreover, $X_1 \not\equiv X_2 \pmod{p^{a+1}}$ is equivalent to $P^{-1}X_1P \not\equiv P^{-1}X_2P \pmod{p^{a+1}}$. Thus there are exactly as many solutions of (1) which are congruent to $P^{-1}J_tP$ modulo p as there are solutions of (1) which are congruent to J_t modulo p, and we have just determined what they are.

4. The number of solutions of (1) is thus given by

$$\sum_{t=0}^{m} S_0(m, t)\, p^{2t(m-t)a},$$

where $S_0(m, t)$ is the number of matrices of order m which are similar modulo p to J_t. Finally, Hodges' paper ([1] pp. 518–519) gives $S_0(m, t) = g_m/g_t g_{m-t}$ where $g_t = \prod_{i=0}^{t-1} (p^t - p^i)$.

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ON THE EVALUATION OF FINITE SUMS BY RESIDUES

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The evaluation of finite sums is of considerable importance in many branches of pure and applied mathematics. This note presents two theorems describing transformations of certain sums into residues expressions. These finite summation transformations are quite analogous to the Watson transformations [1] used for the evaluation of double-ended infinite series, and are related to the contour-integral summation schemes of Milne-Thomson [2].

THEOREM 1. *Suppose $a(z)$ has only a finite number of poles in the complex z-plane, all of which are simple and none of which are located at integers or at infinity. If these are the only singularities, then*

$$\sum_{n=0}^{N} a(n) = -\sum_{k=0}^{K} \zeta_N^{(1)}(z_k)\ \text{Res.}\ \{a(z)\}_{z=z_k} + (N+1)\,a(\infty),$$

where the residues are evaluated at the poles of $a(z)$, and $\zeta_N^{(1)}(z) = \sum_{n=0}^{N} (z-n)^{-1}$.

The proof follows immediately from the contour integral

$$\oint_C a(z)\zeta_N^{(1)}(z)dz$$

taken around an infinitely large circle centered at the origin of the complex z-plane, since $\zeta_N^{(1)}$ is constructed with poles at the integers $0, 1, \cdots, N$ and residues equal to unity. For alternating series, a similar function with alternating residues leads to the following theorem.

THEOREM 2. *If $a(z)$ once again has only simple poles, not at integers or at ∞, then*

$$\sum_{n=0}^{N} (-1)^n a(n) = -\sum_{k=0}^{K} \zeta_N^{(2)}(z_k)\ \text{Res.}\ \{a(z)\}_{z=z_k} + \tfrac{1}{2}[1+(-1)^N]a(\infty),$$

where the residues are evaluated at $a(z_k)=\infty$, and $\zeta_N^{(2)}(z) = \sum_{n=0}^{N} (-1)^n(z-n)^{-1}$.

The practical utility of these theorems depends on the number of poles of $a(z)$ being much less than the number of terms of the summation (*i.e.*, $K \ll N$), and is also dependent on the ease of finding these poles. These transformations then reduce the summation task to that of looking up a small number of values of the functions $\zeta_N^{(1)}$ and $\zeta_N^{(2)}$, which can be easily tabulated for the desired ranges

of z and N. We have denoted these functions by the Greek zeta because of their resemblance to incomplete Riemann zeta-functions of the first order, recalling that the complete Riemann function is defined by $\zeta(s, z) = \sum_{n=0}^{\infty} (z-n)^{-s}$. Some sketches of such incomplete zeta-functions are presented in Figure 1, and reveal the similarity to the common functions $\pi \cot \pi z$ and $\pi \csc \pi z$, respectively, due to the locations of the poles.

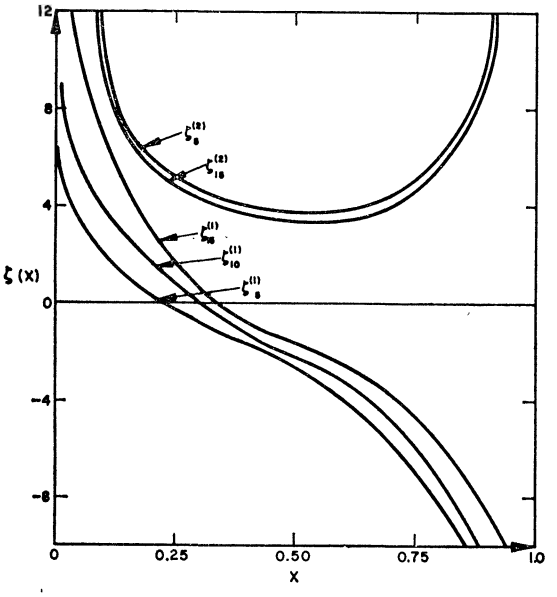


FIG. 1. Incomplete zeta-functions of first and second kinds.

As an application of the residues transformation, consider the following example:

$$\sum_{n=0}^{N} \frac{(n-a_1)(n-a_2)}{(n-b_1)(n-b_2)(n-b_3)} = -\frac{(b_1-a_1)(b_1-a_2)}{(b_1-b_2)(b_1-b_3)}\zeta_N^{(1)}(b_1) - \frac{(b_2-a_1)(b_2-a_2)}{(b_2-b_1)(b_2-b_3)}\zeta_N^{(1)}(b_2) - \frac{(b_3-a_1)(b_3-a_2)}{(b_3-b_1)(b_3-b_2)}\zeta_N^{(1)}(b_3),$$

where the a's and b's are distinct and nonintegral. The variation of this sum with increasing N is a function of the ζ-curves alone, and could be read directly from a zeta-table.

In conclusion, I should like to acknowledge gratefully the advice and encouragement of Dr. Aharon Ksienski.

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A GENERALIZATION OF THE FACTOR THEOREM

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A simple property of finite differences is used below to develop a necessary and sufficient condition for a polynomial $p(x)$ to be divisible by $(x-r)^m$. Since $p(x)$ is divisible by $(x-r)^m$ if and only if $p(rx)$ is divisible by $(x-1)^m$, only the case $r=1$ will be considered.

Let $F[x]$ consist of all polynomials in x with coefficients in a field F. If F has characteristic p, let m be a positive integer less than $p+1$.

THEOREM. *A polynomial* $p(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$ *in* $F[x]$ *is divisible by* $(x-1)^m$ *if and only if*

$$a_0(n+1)^s+a_1n^s+a_2(n-1)^s+\cdots+a_{n-1}2^s+a_n=0 \quad \text{for } s=0,1,\cdots,m-1. \tag{1}$$

Necessity. Let $p(x)=q(x)(x-1)^m$ with $q(x)=\sum_{i=0}^{n-m}q_ix^{n-m-i}$ in $F[x]$. Since $(x-1)^m=\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{m-j}$, one has

$$p(x)=\left(\sum_{i=0}^{n-m}q_ix^{n-m-i}\right)\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{m-j}\right)$$
$$=\sum_{i=0}^{n-m}q_i\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{n-i-j}.$$

The condition (1) for $p(x)$ now becomes

$$\sum_{i=0}^{n-m}q_i\left[\sum_{j=0}^{m}(-1)^j\binom{m}{j}(n-i-j+1)^s\right]=0 \quad \text{for } s=0,1,\cdots,m-1. \tag{2}$$

For each i, the expression in brackets in (2) vanishes since it is an mth difference of x^s with $m>s$. Hence (2) holds and the necessity is established.

Sufficiency. The hypothesis now is that $p(x)$ satisfies (1) for $s=0,1,\cdots,m-1$. We use induction on m. The case $m=1$ follows from the Factor Theorem. Assume the theorem true for $m=k$. We wish to show that $p(x)$ is divisible by $(x-1)^{k+1}$ assuming that $p(x)$ satisfies (1) for $s=0,1,\cdots,k$ and (from the hypothesis of the induction) that $p(x)$ is divisible by $(x-1)^k$.

Divide $p(x)$ by $(x-1)^{k+1}$ obtaining

$$p(x)=q(x)(x-1)^{k+1}+r(x), \tag{3}$$

with $r(x)$ of degree not greater than k. Since $p(x)$ is divisible by $(x-1)^k$, $r(x)=a(x-1)^k$ with a in F.

The necessity part of the theorem shows that $q(x)(x-1)^{k+1}$ satisfies a condition such as (1) for $s=0,\cdots,k$. Since the condition (1) is linear and homogeneous in the coefficients a_i, the difference $r(x)$ of two polynomials which satisfy such a condition also does. The condition analogous to (1) with $s=k$ for

$r(x)=a\sum_{j=0}^{k}(-1)^j\binom{k}{j}x^{k-j}$ is

$$a\left[(k+1)^k-k\cdot k^k+\binom{k}{2}(k-1)^k-\cdots+(-1)^k\cdot 1^k\right]=0. \tag{4}$$

The expression within the brackets in (4) is $k!$ since it is a kth difference of x^k. If F has characteristic p, $k=m-1<p$. Hence $k!\neq 0$. Therefore $a=0$. Hence $r(x)=0$ and (3) shows that $p(x)$ is divisible by $(x-1)^{k+1}$. This completes the proof.

Finally, one may note that the theorem still holds if one replaces the powers $(n+1)^s, n^s, \cdots, 2^s, 1^s$ in (1) by the sth powers of any $n+1$ elements of F which are in arithmetic progression.

NORMS OF POLYNOMIALS

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Many investigations in number theory and Fourier series require integral estimations which are just somewhat beyond what the Schwarz inequality supplies. In this note we will break through the Schwarz barrier! This breakthrough, however, is so puny that it should be quite apparent that our main purpose is really to entice further research in this interesting and difficult subject.

Let $P(z)=z^n\pm z^{n-1}\pm\cdots\pm 1$, $n>0$ and let $\int|P|$ denote $(2\pi)^{-1}\int_0^{2\pi}|P(e^{i\theta})|\,d\theta$. The Schwartz inequality gives $\int|P|<\sqrt{(n+1)}$. We shall prove the

THEOREM. $\int|P|<\sqrt{(n+.97)}$.

Proof. Case 1: $\operatorname{Max}_{|z|=1}|P(z)|\leqq 2\sqrt{(\pi n)}$. Writing $P(e^{i\theta})=\sum_{k=0}^{n}\epsilon_k e^{ik\theta}$, we have

$$|P(e^{i\theta})|^2=P(e^{i\theta})P(e^{-i\theta})=\sum_{j=-n}^{n}\delta_j e^{ij\theta},$$

where $\delta_j=\epsilon_0\epsilon_{|j|}+\cdots+\epsilon_{n-|j|}\epsilon_n$. Now note that $\delta_0=n+1$ and that δ_j, being an integer and odd if $n-|j|$ is even, satisfies $|\delta_j|\geqq 1$ for $n-|j|$ even. Hence

$$\int|P|^4=\sum\delta_j^2\geqq(n+1)^2+2[(n+1)/2]\geqq(n+1)^2+n$$

and so, since $\int|P|^2=n+1$, we obtain $\int(|P|^2-(n+1))^2\geqq n$.

Now $|P|+\sqrt{(n+1)}\leqq 2\sqrt{(\pi n)}+\sqrt{(n+1)}\leqq(2\sqrt{\pi}+\sqrt{2})\sqrt{n}<5\sqrt{n}$ so that

$$\int(|P|-\sqrt{(n+1)})^2\geqq n/(5\sqrt{n})^2=\tfrac{1}{25};$$

again, since $\int|P|^2=n+1$, this shows that $\int|P|\leqq\sqrt{(n+1)}-1/\{50\sqrt{(n+1)}\}$. Hence $(\int|P|)^2\leqq n+1-\frac{1}{25}+\frac{1}{5000}<n+.97$.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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A NOTE ON δ AND ϵ

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Our freshmen, at least some of them, can solve the problem: $\lim_{x\to 2} 3x = 6$, by showing that if $\delta = \frac{1}{3}\epsilon$, then $0 < |x-2| < \delta$ implies $|3x-6| < \epsilon$, for any positive number ϵ. But confront them with $\lim_{x\to 2} x^2 = 4$ and they haven't the remotest idea how to proceed. Of course one can write $|x^2-4| = |x-2| \cdot |x+2|$ and start by restricting x to values between 1 and 3 so that $|x+2| < 5$, but this leaves me cold. The purpose of this note is to exhibit a systematic technique for attacking this and similar problems, applicable in fact to all polynomials.

Considering the problem at hand, *i.e.*, $\lim_{x\to 2} x^2 = 4$, we wish to exhibit a positive number δ such that if $0 < |x-2| < \delta$, then $|x^2-4| < \epsilon$ for every positive ϵ. The whole trick is to expand x^2-4 in powers of $x-2$. Thus $x^2-4 = (x-2)^2 + 4(x-2)$. Then

$$|x^2 - 4| = |(x-2)^2 + 4(x-2)| \leqq |x-2|^2 + 4|x-2|.$$

The latter will be less than ϵ provided that (completing the square)

$$|x-2|^2 + 4|x-2| + 4 < \epsilon + 4$$

or

$$|x-2| + 2 < \sqrt{(\epsilon+4)} \text{ or } |x-2| < \sqrt{(\epsilon+4)} - 2.$$

Thus, choosing $\delta = \sqrt{(\epsilon+4)} - 2$, the problem is solved.

Similarly, to prove that $\lim_{x\to 2} x^3 = 8$, one chooses $\delta = \sqrt[3]{(\epsilon+2^3)} - 2$. In general, for $a > 0$, to prove $\lim_{x\to a} x^n = a^n$, one chooses $\delta = \sqrt[n]{(\epsilon+a^n)} - a$. Obviously, such a δ is always positive.

If $a < 0$ for the case $\lim_{x\to a} x^n = a^n$, we choose $\delta = \sqrt[n]{(\epsilon + |a|^n)} - |a|$; this choice of δ will, of course, work for $a > 0$ as well.

For a problem such as $\lim_{x\to 2} 5x^2 = 20$, since $|5x^2-20| < \epsilon$ if $|x^2-4| < \frac{1}{5}\epsilon$, we choose $\delta = \sqrt{(\frac{1}{5}\epsilon+4)} - 2$, and so on. In the case of a general polynomial this technique is applied to the individual terms.

As a simple illustration consider $\lim_{x\to 1} (x^3+5x^2) = 6$. We wish to exhibit a positive number δ such that if $0 < |x-1| < \delta$, then $|x^3+5x^2-6| < \epsilon$. Now

$$|x^3 + 5x^2 - 6| = |x^3 - 1 + 5x^2 - 5| \leqq |x^3 - 1| + 5|x^2 - 1|.$$

Also, $|x^3-1| < \frac{1}{2}\epsilon$ if $0 < |x-1| < \delta_1$, where $\delta_1 = \sqrt[3]{(\frac{1}{2}\epsilon+1)} - 1$; and $5|x^2-1| < \frac{1}{2}\epsilon$ if $0 < |x-1| < \delta_2$, where $\delta_2 = \sqrt{(\frac{1}{10}\epsilon+1)} - 1$. Hence, if we choose δ to be the smaller of δ_1 and δ_2, the problem is solved.

FINDING THE CHANGE OF VARIABLE CARRYING ONE FINITE INTERVAL ONTO ANOTHER

F. A. FICKEN, New York University

It is often required to find a linear change of variable, $t=px+q$ (*), in such a way that a given interval $a<x<b$ is mapped onto a prescribed interval $\alpha<t<\beta$. One way is to substitute into(*) and solve for p and q. Another way, motivated by the fact that the ratio of the lengths of the intervals is $(\beta-\alpha)/(b-a)$, is displayed in the following inequalities:

$$a < x < b, \qquad 0 < x - a < b - a,$$

$$0 < \frac{\beta - \alpha}{b - a}(x - a) < \beta - \alpha,$$

$$\alpha < \frac{\beta - \alpha}{b - a}(x - a) + \alpha < \beta.$$

If $\alpha=-\beta<0$ (for example, we often wish $-\pi<t<\pi$ in Fourier analysis), a still simpler process exploits the fact that the midpoint of the given interval is to be mapped onto the origin:

$$a < x < b,$$

$$-\frac{b - a}{2} < x - \frac{a + b}{2} < \frac{b - a}{2},$$

$$-\beta < \frac{2\beta}{b - a}\left(x - \frac{a + b}{2}\right) < \beta.$$

Suppose, for illustration, that we are given the interval $-1<x<4$. The inequalities

$$-\frac{5}{2} < x - \frac{3}{2} < \frac{5}{2}, \qquad -\pi < \frac{2\pi}{5}\left(x - \frac{3}{2}\right) < \pi$$

lead us to put $t=2\pi(x-3/2)/5$. Alternatively, the inequalities

$$0 < x + 1 < 5, \qquad 0 < \frac{2\pi}{5}(x + 1) < 2\pi$$

suggest a variable $u=2\pi(x+1)/5$ running over the interval from 0 to 2π.

Experiments indicate that these extremely easy manipulations can be performed quite rapidly in numerical examples. The explicit translation of the origin and change of scale seem enlightening to students.

THE TRIANGULAR INEQUALITY IN THE PROJECTIVE MODEL OF A HYPERBOLIC GEOMETRY

C. F. MOPPERT, University of Melbourne

The triangular inequality is the main obstacle if one wishes to present Klein's model of a hyperbolic geometry in a synthetic way. Once this inequality is established, the rest of the way is smooth.

As I could not find this in any text book, I give here the proof of a theorem (Theorem 3) from which the triangular inequality follows in a well-known way. As was to be expected, the relation to be proved has to be found equivalent to a relation of separation. This separation is the one stated in Theorem 2 and makes use of the fact that a point is assumed to be an interior point.

I adopt the notations used in H. S. M. Coxeter, *The Real Projective Plane*, Cambridge, 1955, and assume the knowledge of the following definitions and theorems:

1. *Given two projectively related ranges a and b,* $A \in a$, $B \in b$, $A \barwedge B$† *then the lines AB envelop a conic touching a and b.*

2. *Given a conic by two projectively related ranges a and b and a point P on the line joining the points of contact of a and b with the conic. Then the conic associates an involution to P in such a way that if* $A_\mu A_\nu \barwedge B_\mu B_\nu$ *in the projectivity* $a \barwedge b$ *then* PA_μ, PB_μ, PA_ν, $PB_\nu \barwedge PB_\mu$, PA_μ, PB_ν, PA_ν.

3. *The point P is interior point of the conic if and only if this involution is elliptic.*

4. *An involution* $ABCD \barwedge BADC$ *is elliptic if and only if* $AB \| CD$.

5. *Two chords of a conic are perpendicular if they are on lines conjugate with respect to the conic.*

6. *If A, B are two points inside a conic c and if PQ is the chord containing them we put* $\|AB\| = |\log R_I(PQ, AB)|$, *where* R_I *denotes the cross ratio.*

7. *A harmonic homology with center A and axis a is a* 1-1 *mapping of the projective plane such that if* $P \to P'$ *by this homology then A, P, P' are collinear and* $H(AA_0, PP')$ *where* $A = PP_0' \cdot a$. *By a harmonic homology, lines are mapped into lines.*

8. *If center and axis of a harmonic homology are pole and polar with respect to a conic c then this homology has the conic as an invariant in such a way that interior points are mapped into interior points. If A, B are interior points,* $A \to A'$, $B \to B'$ *by this homology then* $\|AB\| = \|A'B'\|$.

† The symbol $\barwedge$ denotes a projectivity.

THEOREM 1. *Let $ABCD$ be four points on a conic c. Put $AB\cdot CD=Q$, $AD\cdot BC=R$, $QR=p$. Denote by a, b, c, d the tangents on c in A, B, C, D, put $bd=E$, $ac=E^*$. Then $EE^*=QR$ and $H(QR, EE^*)$.*

Proof. p is the polar of $P=AC\cdot BD$. This proves $EE^*=QR$. The harmonic homology with Q as center, PR as axis has c as an invariant and $E\to E^*$ by this homology. This proves the theorem.

THEOREM 2. *Let A, B, C, D be four points on a conic c, let $P=AC\cdot BD$, $Q=AB\cdot DC$, $R=AD\cdot BC$ be the vertices of the diagonal triangle of the quadrilateral $ABCD$. Put $p=QR$; $S=AC\cdot p$, $T=BD\cdot p$. If b is tangent on c in B and $E=bp$ then P is interior point of c if and only if $QR\|ET$. If $E=Q$ viz. $E=R$ then the conic degenerates.*

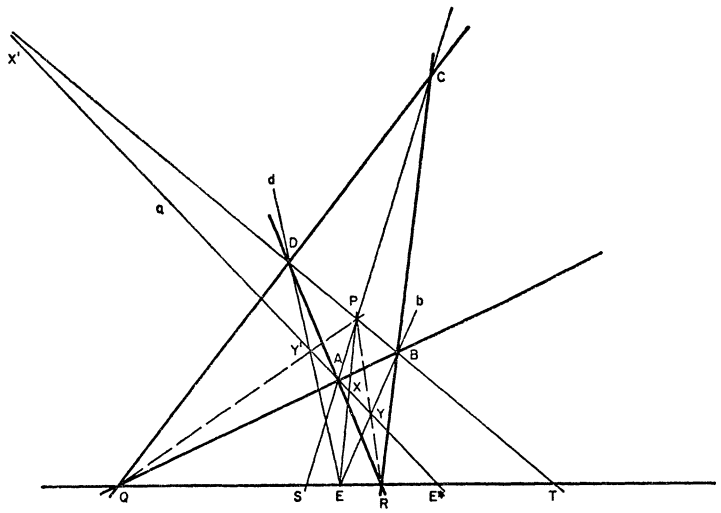


FIG. 1

Proof. (Fig. 1) Let a, d be the tangents on c in A viz D. It is $bd=E$. Furthermore, if $ap=E^*$ then $H(QR, EE^*)$ (Theorem 1). Put $F=PR\cdot AB$, $Y=ab$. As $H(QR, ST)$ and $QRST\overset{P}{\underset{\wedge}{=}}QFAB$§ it is $H(QF, AB)$. Therefore $QREE^*\overset{\frown}{\wedge}QFBA$. This being a perspectivity it follows F, Y, R *viz.* P, Y, R are collinear points. In the same way P, Y', R are collinear where $Y'=ad$. Put $X=PE\cdot a$, $X'=BD\cdot a$. Then $X'Y'XY\overset{P}{\underset{\wedge}{=}}TQER$.

Consider c being generated by the two projective ranges b and d. The pairs of points, namely YY' and ED are corresponding points. The involution associated to P by c is therefore determined by the pairs PY, PY' and $PE=PX$, $PD=PX'$. It is elliptic if and only if $XX'\|YY'$ and as $XX'YY'\overset{\frown}{\wedge}ETRQ$ it is elliptic if and only if $ET\|RQ$. On the other hand a point is interior point of a conic if and only if the involution associated to it is elliptic.

§ The symbol $\overset{P}{\underset{\wedge}{=}}$ denotes a perspectivity from P.

If $E=R$ or $E=Q$ then the tangent b has two distinct points in common with c and thus the theorem is proved.

THEOREM 3. *If ABC is a triangle inside a conic c, rectangular at C then $\|AB\|>\|AC\|$.*

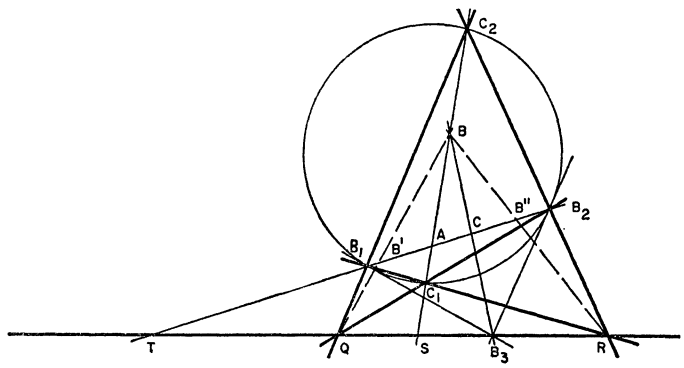


FIG. 2

Proof. (Fig. 2) Let AC be on the chord B_1B_2, AB on the chord C_1C_2. Put $B_1C_2\cdot B_2C_1=Q$, $B_1C_1\cdot B_2C_2=R$, $QR=p$, $B_1B_2\cdot p=T$, $C_1C_2\cdot p=S$. If $B'=B_1B_2\cdot BQ$, then $B\to B'$, $A\to A$ by the harmonic homology with Q as center, AR as axis, therefore $\|AB\|=\|AB'\|$. In the same way, $\|AB\|=\|AB''\|$ with $B''=B_1B_2\cdot BR$. Denote b_1, b_2 the tangents in B_1, B_2. If $B_3=b_1b_2$ then $B_3\in p$ and $B_3T\|QR$ according to Theorem 2. It is $C=BB_3\cdot B_1B_2$ as ABC is rectangular at C. If $\|AC\|>\|AB\|$ then $AC\|B'B''$. As $B'ACB''\overset{B}{\underset{\wedge}{=}}QSB_3R$ this is equivalent to $SB_3\|QR$. This contradicts $B_3T\|QR$.

EVALUATION OF DIRICHLET'S INTEGRALS

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The following method of evaluating the well-known multiple integral

$$I_n=\int\cdots\int_{x_h\geq 0,\,\Sigma x_h\leq 1}(1-x_1-\cdots-x_n)^{\alpha_0-1}x_1^{\alpha_1-1}\cdots x_n^{\alpha_n-1}dx_1\cdots dx_n$$

for constant $\alpha_k>0$ in terms of the gamma function may be interesting. In the case $n=1$, it gives the evaluation of the beta function [cf. **1**, **2**, **3**].

Consider the integral

$$g_{n+1}=\int\int\cdots\int_{\xi_k\geq 0,\,\Sigma\xi_k\leq\omega}\xi_0^{\alpha_0-1}\xi_1^{\alpha_1-1}\cdots\xi_n^{\alpha_n-1}f(\xi_0+\xi_1+\cdots+\xi_n)d\xi_0d\xi_1\cdots d\xi_n$$

with constant $\alpha_k > 0$ and $\omega > 0$, the function f being such that $\int_0^\omega x^{\alpha-1} f(x)dx$ exists for every $\alpha > 0$. Let us change the variables of integration by putting

$$\xi_0 = x_0(1 - x_1 - \cdots - x_n), \qquad \xi_1 = x_0 x_1, \cdots, \xi_n = x_0 x_n.$$

The Jacobian of this transformation is easily computed. Observing that $\sum \xi_k = x_0$, one gets, by just an addition of rows,

$$\frac{\partial(\xi_0, \xi_1, \cdots, \xi_n)}{\partial(x_0, x_1, \cdots, x_n)} = \frac{\partial(x_0, \xi_1, \cdots, \xi_n)}{\partial(x_0, x_1, \cdots, x_n)} = x_0^n.$$

Hence we find that

$$\mathcal{I}_{n+1} = I_n \int_0^\omega x_0^{\alpha_0+\alpha_1+\cdots+\alpha_n-1} f(x_0)dx_0. \tag{1}$$

Here if we take $f(x) = e^{-x}$ and let $\omega \to \infty$, we at once obtain the desired expression of I_n in terms of the gamma function:

$$I_n = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)}. \tag{2}$$

At the same time, on inserting (2) into (1) and changing the notation slightly, we have another familiar formula:

$$\int\cdots\int_{x_k \geqq 0,\, \Sigma x_k \leqq \omega} x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} f(x_1 + \cdots + x_n)dx_1 \cdots dx_n$$

$$= \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n)} \int_0^\omega x^{\alpha_1+\cdots+\alpha_n-1} f(x)dx. \tag{3}$$

The above procedure has been motivated by the simple observation that a check of (3) can be obtained by choosing $f(x) = e^{-x}$ and letting $\omega \to \infty$.

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1. E. Goursat, A Course in Mathematical Analysis, vol. I, 1904 (English ed.), pp. 312–313.
2. S. K. Lakshmana Rao, On the evaluation of Dirichlet's integral, this MONTHLY, vol. 61, 1954, pp. 411–413.
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LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

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It is well known that the homogeneity of an expression depends upon the many ways that the dimensions or degrees are assigned to its variables; consequently, there are several ways in which an expression can be homogeneous. Subsequent to the assignment of degrees, an expression is said to be homogeneous if each term of its sum has the same degree.

Professor H. B. Phillips points out in his text* that a differential equation which is not homogeneous, say in x and y, can sometimes be made so by assigning dimension 1 to x and its differential dx and at the same time assigning dimension n to y and its differential dy. When an equation is made homogeneous by assigning dimensions in this manner the substitution $y=vx^n$ will render the new equation in v and x separable. It is also pointed out that the only important aspect of the substitution $y=vx^n$ is that the variable v is of dimension zero. Many types of equations usually solved by other methods can also be solved by this "homogeneous" method. In order to determine n in the substitution $y=vx^n$, let x and its differential dx be of dimension 1 in the original equation, and replace y by u^n and dy by u^n, where u is of dimension 1. Some examples follow.

Exact differential equation: $(x^2+3xy^2)dx+(3x^2y-2y^5)dy=0$. The above consideration yields

$$(x^2 + 3x^1u^{2n})dx^1 + (3x^2u^n - 2u^{5n})u^n = 0.$$

The terms are respectively of dimensions 3, $2+2n$, $2+2n$, $6n$. For each term to be of the same dimension, we must have $3=2+2n$, whence $n=1/2$. Therefore $y=vx^{1/2}$ will separate variables.

An integrating factor, $1/xy$: $(x^2y+x)dy+(xy^2-3y)dx=0$.

$$2 + 2n,\ n + 1,\ 2n + 2,\ n + 1.$$

Then $2n+2=n+1$. Here $n=-1$ and the substitution $y=v/x$ will separate variables.

Linear equation in y: $3dy/dx+5y/x=x^2$.

$$n + 1,\ n + 1,\ 4; \qquad n + 1 = 4, \quad \text{whence} \quad y = vx^3.$$

Bernoulli's equation: $xdy/dx+2y=3x^2y^{4/3}$.

$$n + 1,\ n + 1,\ 4n/3 + 4.$$

Then $4n/3+4=n+1$, $n=-9$; $y=vx^{-9}$.

Another type: $(x^{1/3}+x^{5/3}y^{1/5})dx+(x^{32/9}y^{-2/3}+x^{22/9}y^{-5/6})dy=0$.

$$4/3, \quad 8/3 + n/5, \quad 32/9 + n/3, \quad 22/9 + n/6;$$

$$8/3 + n/5 = 4/3, \quad n = -20/3, \quad \text{whence} \quad y = vx^{-20/3}.$$

It is observed that the dimension of each term must be obtained before determining whether a unique n can be found.

* Differential Equations, New York, 1951.

THE CHAIN RULE

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The average calculus student finds difficulty with problems requiring use of the chain rule for differentiation of functions of several variables because he is unable to produce with confidence the correct formula which should be applied. The proof of the chain rule formula, using the continuity of the first partials to obtain an alternative form of Δf (the change in the function f), is too complicated and deep for most students at this level to use as an aid in writing the proper formula. The purpose of this note is to present a scheme which has the properties of a successful mnemonic: sufficiently "natural" and simple to be easily recalled, and quickly applied.

First, it is necessary to be confident of what derivatives make sense, and whether these be partial or ordinary derivatives. A function-variable diagram (or function diagram)* is quite adequate for this task. Assuming that we are concerned with functions $w=w(x_1, \cdots, x_r)$, $x_i=x_i(y_1, \cdots, y_s)$, $i=1, \cdots, r$ and $y_j=y_j(z_1, \cdots, z_t)$, $j=1, \cdots, s$, we construct the following diagram:

$$w$$

$$x_1, \cdots, x_r$$

$$y_1, \cdots, y_s$$

$$z_1, \cdots, z_t$$

The significance of the diagram is not only that w is a function of the x's, and that w is a function of the y's, and also of the z's, but that each variable is a function of the variables in any row lower than that in which the given variable occurs. Thus it makes sense to talk of the derivative of any variable with respect to any variable which appears lower on the function diagram.

A diagram

$$f$$

$$x \quad u \quad v$$

$$x \quad y$$

has special dangers. We can differentiate with respect to x, where f is thought of as a function of x, u and v, or where f is thought of as a function of x and y. It seems preferable, at first, not to use the language of the last sentence but rather to speak of differentiating with respect to the x in the second row or the x in the third row. Of the many possible ways of distinguishing between these derivatives; the usual one $(\partial f/\partial x)_{u,v}$ and $(\partial f/\partial x)_y$ is easily accepted and understood. Finally, the derivative with respect to any variable is ordinary or partial

* Such diagrams are presented in many calculus texts, *e.g.*, G. B. Thomas, Jr., Calculus and Analytic Geometry, Reading, Mass. 1953, p. 503. However it seems preferable to emphasize by the terminology the functions which are implied by such a diagram.

according to whether the variable with respect to which we differentiate is alone in the row, or occurs with other variables.

All of the preceding is rather standard, but the following method for obtaining the chain rule seems to be novel. In the first function diagram, suppose we wish to find $\partial w/\partial z_1$. We define a path starting at w and ending at z_1 to be a sequence w, x_i, y_j, z_1 where $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$, and we associate with such a path the product $(\partial w/\partial x_i)(\partial x_i/\partial y_j)(\partial y_j/\partial z_1)$, assuming r, s, t, are all >1. The chain rule states that $\partial w/\partial z_1$ is equal to the sum of the products associated with all possible paths which start at w and end at z_1.

Thus, to obtain the chain rule formula, we first construct a function diagram, and then enumerate all possible paths* which start at the dependent variable and end at the independent variable. The chain rule states that the derivative is equal to the sum of the products associated with all the paths just enumerated.†

The path concept of the chain rule not only assures us of writing the correct formula but also suggests, in some cases, when substitutions are required. If we are to show that $f(x-at)$ satisfies the one-dimensional wave equation $\partial^2 f/\partial t^2 = a^2(\partial^2 f/\partial x^2)$ we note that the paths of the diagram $\begin{matrix} f \\ x\ t \end{matrix}$ starting at f and ending at x and t have nothing in common, and thus it is not surprising that the derivatives cannot be related without putting another row between f and x, t. (Indeed, it is clear that the only additional rows which can possibly be of aid are those *between* f and x, t.) Although there are many other profitable returns which this visual picture of the chain rule yields, merely its success as a mnemonic is sufficient to warrant its consideration.

In conclusion, we find that given the function diagram, everyone is able to write easily and quickly the chain rule formulas, and that most students are able to produce the appropriate diagram when all their thoughts are concentrated on this effort alone.

MATRIX SLIDE RULES

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We consider a k by ∞ matrix A with columns the vectors C_n, $-\infty < n < +\infty$ whose k elements $c_{1,n}, \cdots, c_{k,n}$ are defined as follows: The k adjacent vectors $C_0, C_1, \cdots, C_{k-1}$ are the successive columns of the identity matrix of order k, and all other $c_{j,n}$ are defined recursively by the relation

$$c_{j,n} = \sum_{m=1}^{k} a_m c_{j,n-m}, \qquad a_k \neq 0. \tag{1}$$

This means in particular that the vector C_k has elements $c_{j,k} = a_{k-j+1}, j = 1, \cdots, k$.

* For a general function diagram, we define a path as a sequence of variables, one and only one from each row in turn, starting with the dependent and ending with the independent variable.

† The product associated with a path is the product of the derivative of each variable in the sequence with respect to its successor.

Let A_j, $-\infty < j < \infty$, be the k by k submatrix of A consisting of the consecutive column vectors C_j, C_{j+1}, $\cdots$, C_{j+k-1}. We note readily that $A_0 = I_k$ and A_1 is a nonsingular matrix with characteristic equation

$$x^k = \sum_{m=1}^{k} a_m x^{k-m}. \tag{2}$$

It may be easily established by induction that $A_j = (A_1)^j$, $-\infty < j < \infty$. In the special case with all the a_j integers and $a_k = 1$ it is obvious that all elements of A are integers.

We now see how finite submatrices of A may be used to construct matrix slide rules which are helpful in suggesting numerous problems involving the matrices A_j and providing a quick check for the answers to these problems. We may, for instance, display in equally spaced columns on our slide rule the $2n+k$ adjacent columns C_{-n}, C_{-n+1}, $\cdots$, C_0, $\cdots$, C_{n+k-1} of A. By sliding across this rule a cover concealing all but a consecutive grouping of k columns we can display in turn the matrices $(A_1)^{-n}$, $(A_1)^{-n+1}$, $\cdots$, $(A_1)^0$, A_1, $\cdots$, $(A_1)^n$. The names of the various matrices thus displayed can be written in a row along the upper part of the rule so that an auxiliary opening will show which matrix is being displayed at a given time.

If one wants the inverse of a given matrix $A_j = (A_1)^j$, $-n \leqq j \leqq n$, one has only to locate the matrix A_{-j} on the slide rule. Should one wish to know the product of A_j and A_m with $-n \leqq j, m, j+m, \leqq n$ one needs only to locate A_{j+m}. Multiplication of any A_j and A_m is commutative, which suggests another type problem, that is, verifying this fact for particular values of j and m. We have excluded the case $a_k = 0$ in order that our base matrix A_1 might have an inverse, although with $a_k = 0$ it would still be possible to display the positive powers of A_1 in a semi-infinite matrix A corresponding to the columns C_0, C_1, $\cdots$, C_k, $\cdots$.

We see from (2) that A_1 may be constructed to have any desired eigenvalue set λ_1, $\cdots$, λ_k where $0 < |\lambda_1| \leqq \cdots \leqq |\lambda_k|$. By extending C_j with a space followed by the array $(\lambda_1)^j$, $\cdots$, $(\lambda_k)^j$ and making an appropriately labeled auxiliary opening in our cover we can display the eigenvalues of the matrix with the matrix itself. When $|\lambda_k/\lambda_1| > 1$ we can pick j such that the P-condition number of A_j is as large as we please. Applying this technique when $k = 2$ and $a_1 = a_2 = 1$ leads to the 2 by 2 matrices of the author's paper, *Generalized Fibonacci numbers and associated matrices*, pages 745–752 in this issue of the MONTHLY. However for $k > 2$ the ill conditioned matrices of the other paper differ from those obtained here. We also note that verification that a particular matrix A_1 satisfies its characteristic equation (2) is immediately evident from the fact that all elements of A_k are defined recursively by (1).

We illustrate the above remarks by considering the matrices displayed below and commenting upon them. If $k = 4$, $a_j = 1$, $j = 1, 2, 3, 4$, and n is taken as 7, the 18 columns from C_{-7} through C_{10} suffice to display the matrix B (enclosed with

its label in the rectangular polygon representing the cover opening of the rule) and its integral powers B^i, $|j| \leqq 7$. We note that the elements $c_{1,j}$ in this case are the generalized Fibonacci numbers $f_{j,4}$ of the author's paper mentioned above, and that $c_{4,j}=f_{j+1,4}$. However the $c_{2,j}$ and $c_{3,j}$, although determined by the same recursion relation, are not numbers of the same set because the initial values are ordered differently.

B^{-7}	B^{-6}	B^{-5}	B^{-4}	B^{-3}	B^{-2}	B^{-1}	I_4	B	B^2	B^3	B^4	B^5	B^6	B^7			
0	1	−3	2	0	0	−1	1	0	0	0	1	1	2	4	8	15	29
4	1	−2	−1	2	0	−1	0	1	0	0	1	2	3	6	12	23	44
−4	5	−2	0	−1	2	−1	0	0	1	0	1	2	4	7	14	27	52
1	−3	2	0	0	−1	1	0	0	0	1	1	2	4	8	15	29	56

By using a permutation matrix such as D below we could easily construct a cylindrical slide rule from which any integral power of D, say D^i, might be obtained by rotating j columns of the cylinder past the opening in the cover, when $D^0=I$ was there initially. Since the eigenvalues of D and its powers are readily expressed algebraically in terms of the cube roots of unity, we have indicated in this case how the eigenvalues for a particular power of D may be displayed along with the matrix itself. One needs only to mount the matrix below on a cylinder of circumference nine column widths and fit it with a cover opening as indicated.

	D^{-3}	D^{-2}	D^{-1}	D^0	D	D^2	D^3	
1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0
0	0	1	0	0	1	0	0	1
	1	1	1	1	1	1	1	
	1	w	w^2	1	w	w^2	1	
	1	w^2	w	1	w^2	w	1	

As our last example we display matrices T^i, $0 \leqq j \leqq 4$, where T^i has P-condition number 10^i, and simultaneously display the eigenvalues for T^i.

	I_3	T	T^2	T^3	T^4	
1	0	0	−10	−100	−1010	−10100
0	1	0	1	0	1	0
0	0	1	10	101	1010	10101
	1	−1	1	−1	1	
	1	1	1	1	1	
	1	10	100	1000	10000	

We note that T^n has eigenvalues $(-1)^n$, 1, and 10^n so that its P-condition number (ratio of the largest to smallest absolute values for its eigenvalues),

10^n, is also displayed as the lowest framed number. To illustrate the instability of a system with coefficient matrix of high condition number we consider the following system with its coefficient matrix T_3 having P-condition number 1,000. We suppose all coefficients in this system are exactly known except that of y in the third equation whose error does not exceed 1%. Thus we are really concerned with possible solutions of $s(e)$ for $|e| \leqq 1.01$ where

$$s(e) \sim \begin{aligned} -10x - 100y - 1010z &= -10{,}100 \\ x \qquad\quad + \qquad z &= 0 \\ 10x + (101 + e)y + 1010z &= 10101 \end{aligned}$$

We note that the exact solutions of $s(e)$ within the 1% error range include the widely scattered vectors

$$\begin{aligned} e = 0: &\quad (x, y, z) = (-10, 1, 10), \\ e = -.99: &\quad (x, y, z) = (-.1, 100, .1), \\ e = -1.01: &\quad (x, y, z) = (-20.1, -100, 20.1), \\ e = -1: &\quad \text{no solution.} \end{aligned}$$

We conclude with a few suggestions that simplify the construction of matrices of the above type, should anyone wish to make one for his own amusement or edification. After choosing k and the set $\{a_k\}$ occurring in column C_k, one fills in columns $C_0, C_1, \cdots, C_k$ accordingly. It is now convenient to write on a separate thin strip of paper spaced according to the column width used the symbols

$$\overset{\cdot}{a}_k + \overset{\cdot}{a}_{k-1} + \cdots + \overset{\cdot}{a}_1. \tag{3}$$

Each element of C_{k+1} is now readily computable by placing the strip (3) above its k predecessors and carrying out the indicated operations. Other columns to the right of C_{k+1} can be filled in by successive applications of this technique. Since, when $a_k \neq 0$, the recursion relation (1) is solvable for a given $c_{j,n}$ in terms of $c_{j,n+p}$, $p = 1, \cdots, k$, entries for columns C_{-1} may be computed with the aid of a strip like (3) which takes the form

$$(-\overset{\cdot}{a}_{k-1}/a_k) + (-\overset{\cdot}{a}_{k-2}/a_k) + \cdots + (-\overset{\cdot}{a}_1/a_k) + (\overset{\cdot}{1}); \tag{4}$$

entries for columns $C_{-2}, C_{-3}, \cdots$, etc., may then be obtained in the same manner. A final check of a matrix such as the first illustration above, would be to show that the left-most k by k submatrix, B_{-7}, is the inverse of the rightmost, B_7, by showing that their product is the identity.

Epilog. The general term of any sequence defined recursively by (1) may be expressed in terms of the row vector R: $(r_0, r_1, \cdots, r_{k-1})$ and the column vectors C_j as follows: $r_n = R \cdot C_n$.

Postscript. For an amplification of the theory underlying the results of this note, see Ralph W. Pfouts and C. E. Ferguson, *A matric general solution of linear difference equations with constant coefficients*, Math. Mag., vol. 33, 1960, pp. 119–127.

MATHEMATICAL EDUCATION NOTES

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THE NASDTEC-AAAS TEACHER PREPARATION AND CERTIFICATION STUDY

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The remarkable array of letters in the title stands for the National Association of State Directors of Teacher Education and Certification, and for the American Association for the Advancement of Science. The various state departments of education, through their directors of certification, are attempting to work out a new approach to certification, to meet the problems stated below. Since there is great urgency about mathematics and the sciences, they are beginning with these fields, with the help of the AAAS, and with a grant from the Carnegie Corporation of New York.

The problems. Given our present school and university system, training programs for the secondary teacher must satisfy several boundary conditions. First, the teacher should be equipped to begin teaching after four years of college. This condition is imposed by the beginning teacher, or perhaps rather by the beginning salaries in the schools. Thus there are only 120–130 semester hours available. Since some of this time will be taken up with requirements peculiar to the individual colleges—courses in religion, unusually large requirements in language or some other field—room must be left for these in any general requirements for certification.

Second, the teacher must be prepared to teach two or three subjects. The primary reason for this is the very large number of high schools that are too small for proper specialization. Conant* estimates that 17,000 out of the 21,000 high schools in the country are too small for this. By and large, there is nothing a state department of education can do about the too-small high school except try exhortation. Amalgamation is, generally, up to the voters in the individual school districts. Thus, in one state there are two towns five miles apart, each with its own rather small high school. One combined school half-way between could provide a vastly better program, but amalgamation would end the annual football game between the two schools, and nothing can make the voters give that up. Since small high schools will continue to exist for some time, most states are forced into requiring a minimal preparation in other fields than the major.

Third, the teacher must be trained in teaching methods, by courses in education. I must admit certain prejudices here, but I must also admit to very little

* James B. Conant, The Child, the Parent, and the State, Harvard University Press, 1959, Notes, p. 161 *et seq.*

experience. Certainly, some training is called for, and as of 1957, minima* ranged from 12 to 27 semester hours, with a median of 18 hours and an average of 19−. The requirements in this area are often given in great detail.

Fourth, an increasing number of states†—28 in 1959—have a "general education" requirement. To illustrate this, Kansas requires 10 hours of social science; 10 hours of literature, language, art, philosophy, or general religion; 10 hours of physical and biological sciences; and 15 hours of electives in these areas, making 45 semester hours. A student taking a normal bachelor's degree in a good arts college would almost certainly satisfy these requirements in satisfying his college's graduation requirements. The students who would not meet it are the ones from weak colleges with a minimal subject-matter preparation and with a vast number of courses in education, typing, and bookkeeping, or the ones with degrees in hotel management or mobile homes. To prevent these students from becoming teachers is the intent of the requirement, but it and arithmetic show that there is no easy solution.

Fifth, many schools have special requirements that have been imposed by the legislatures—Washington State history, or the Constitution of Texas, or the harmful effects of alcohol, for example.

With all these conditions, there is not much room to move around in. What has actually grown up is a pattern of requiring a major and a minor field, or a major and two minors, or two majors, and putting the requirements for these entirely in terms of numbers of semester hours. In mathematics, in 1957,§ the range for a major was between 12 and 40 semester hours, the median 18, and the average 20 hours. Very little attempt has been made to insure a uniform level of minimal competency in subject matter,‡ or to see that material needed in the high school is taught. It would be perfectly possible, for example, for a teacher whose certification major is mathematics never to have had a course in geometry after the tenth grade. Whatever control has been exerted has been by the individual colleges, and frequently the responsibility for subject-matter requirements has not been in the hands of the department concerned, even for a certification major. As for minors, often very little attention is paid by anybody to what a student takes to complete a minor.

Each state has made its own solution to the boundary-value problem. These differ widely, so that a teacher well prepared for one state may not be able to meet the neighboring state's requirements.

One can see obvious weaknesses in such a system. (One can also see major difficulties facing the reformer.) A less obvious weakness, that the NASDTEC members take very seriously, is that many teachers with minimum certifica-

* Conant, *loc. cit.*

† Tabular Summary of Certification Requirements for School Personnel in the United States, 1959. National Education Association, Washington, D. C.

§ Conant, *loc. cit.*

‡ Several states give some sort of examination for certification, but very few states or cities test subject-matter proficiency.

tion majors, or who have ended up teaching in a minor, are not prepared to go on to a master's degree in the subject they are teaching. Since a master's degree has cash benefits, such teachers are apt to end up with a M.Ed., with no further subject matter training, or with as little as they can manage. This should be of special concern to us, because it is particularly in the tight fields of mathematics and the sciences that one finds people teaching with only a minor.

The NASDTEC study. The certification directors are working towards an "approved-program" approach. To quote their description,

"In this approach state departments of education and their advisory committees define guide lines for the preparation of teachers and leave to each college the responsibility of planning a program of courses and experiences. These programs are then submitted to the state department of education for approval. After approval is granted, an application for a certificate is considered on the basis of the applicant's having completed the approved program.

"Further, an effort is being made by state departments of education to accept out-of-state candidates by means of the 'approved program approach'. Therefore, the National Association of State Directors of Teacher Education and Certification at its June 1958 meeting agreed to support a national plan of reciprocity based upon accreditation of the preparation program by the National Commission for Accreditation of Teacher Education."

Clearly, the important question is, what will be the nature of these "guide lines" and how will they be determined?

Following Young's corollary to Parkinson's Law—the number of committees increases exponentially with time—the first step taken by the two organizations was to form a board of eight, five from NASDTEC, two from AAAS, including E. G. Begle of Yale and SMSG, and one from the Council of Chief State School Officers. John Mayor was appointed Study Director with Dr. William P. Viall, Chief, Bureau of Teacher Certification, New York State, as Associate Director and chairman of the board.

The first major step was the holding in April, 1960, of four regional meetings, covering all the states, to make recommendations for these guide lines. At each of these there were about 40 people. The meeting I attended was perhaps typical; there were present certification directors from the states of the region, eleven other educators in administrative positions including two state superintendents, and one principal; four high school teachers; one college president, and nine deans; four physical scientists, and five mathematicians, including two of the deans. Most of the actual work was done in small sessions of about ten people, each group taking a field and preparing a list of recommendations. The four sets of recommendations for each field are being circulated to a wider group for criticism. The revised versions will furnish the starting point for studies in each state.

Then finally in June, 1961, a national conference will be held

"with representation from each of the states, persons involved in the several other related studies, specialists in curriculum with particular attention to the fields of science and mathematics, and recognized national leaders in teacher education and in the sciences.

"The purposes of such a conference would be to review the studies and the recommendations

critically, to make suggestions for state and regional adaptation, to lay plans for the development of reciprocity agreements on a national basis, to inaugurate studies in other areas, and to bring to the attention of the public the purpose and progress of this work."

Following this conference, there will be a final report.

The nature of the mathematics recommendations. I can summarize rather quickly the reasons why there should be changes not only in the amount but in the nature of the training of the secondary teacher of mathematics. First, improvements in grades 1 through 8 through such projects as the SMSG program will mean better-prepared high school students, capable of more difficult work. Second, pressure from engineering schools and industry—and from some, though apparently not all, mathematicians—will push down into the high schools more and more work that is now usually done in college. Third, the current changes in the high school curriculum call for a different training. Thus the teacher of the next decade will be taking better-prepared students through new material well into college work. Many people would guess that in a decade 15% of our high school seniors will be taking calculus, a figure not unreasonable if compared with European experience.

In curriculum planning, mathematics appears to be ahead of the other sciences, and there seems to be general agreement on the nature of the changes required. Among the groups in mathematics that have considered the associated problem of teacher training, there is also agreement. With the benefit of all this work, it is not surprising that the reports of the regional groups are very much alike. I will try to construct a sort of average.

The center of the teacher's training should be three year courses. These are: (1) Calculus, or analytic geometry and calculus, with most of the emphasis on calculus, stressing basic concepts and intuitive ideas as well as or rather than manipulative techniques. (2) Modern algebra—the structure of the number system, groups, rings, fields, polynomials, matrices. (3) Geometry—a critical study of the foundations of geometry. Several other topics, such as projective or non-Euclidean geometry or analytic geometry by vectors.

Note that a poorly prepared student may very well have to take a year's work to begin this sequence, work of a nature that the groups did not believe should count as part of a teacher's training. (Indeed, several groups stated explicitly their dissatisfaction with such courses as solid geometry, college algebra, trigonometry, theory of equations, advanced Euclidean or college geometry, and mathematics of finance, courses which have been a large part of many teachers' majors.)

Beyond these three courses the student should have a year's work in courses such as probability and statistics, set theory and elementary logic, or additional work in one of the three main divisions.

I think there was probably general agreement that a minor should include the three main courses.

Fifth-year programs should be advanced courses of a somewhat similar na-

ture, and should prepare the teacher to teach the analytic geometry and calculus for the Advanced Placement Program of the College Entrance Examination Board. If the teacher has only a minor, the fifth year should complete the major program.

In most colleges and universities, the certification to the state that a student has completed the necessary work for a certificate and is competent to be in teaching is in the hands of the Education School or Department. So is the supervision of the student's practice teaching. These, of course, are purely internal matters for the college or university, and not determined by the state. There appeared general agreement that the subject-matter department play more of a part in both of these activities, taking at least the responsibility of attesting the student's mathematical competency.

Some personal observations. The four-year program is weaker than I had hoped for. This is, however, not the result of compromise between educators bent on keeping school standards low and high-principled university scholars nobly fighting for scholarship. The NASDTEC members seemed to me to be as anxious to improve the training of the teacher as I, and to be much better informed as to the obstacles, and perfectly willing to fight through them. If the University people had said that a year of Banach algebra was vital for the teacher, and explained what was involved, I felt the directors would have looked worried, and then gone out and worked to get it. It was the facts of the situation that forced us into the above recommendations as a minimal program, not merely a desire for agreement.

I should like to emphasize that this is a minimal program. I now believe that any competent department of mathematics, in almost all states, can draw up its own program for the teacher, and if it really is sensible, the state department of education will approve it.

I should like to emphasize also that this is a program for a particular point in history. In another ten years, if all goes well, the prospective teacher will enter college with a good bit stronger background in mathematics, and this program will be unduly weak.

The fifth-year program still has many problems. There seems almost universal agreement that it should be a year of further subject-matter study. Yet equating advanced work to a master's degree of the traditional sort, as I have said, almost forces most teachers into an education degree. There seems to me to be an urgent need of special master's degrees aimed at the teacher. As I interpret the position of the curriculum directors on requiring the degree, the degree means that someone besides a local school board has approved a program of study and certified satisfactory completion. Without the degree, entirely too much power is given to local school boards to judge the qualifications of the teacher.

Conclusion. The NASDTEC study is the first occasion that I know for mathematicians to influence directly the policy of state departments of educa-

tion and educational administrators on a nationwide scale. The state studies of the next year offer a chance, not only to help determine policy on an important question, but to gain a voice in educational policy at the highest state level. I believe that the opportunity is opening as a consequence of the evidence we have given in recent years of deep interest in the problems of the high school and of our willingness to work cooperatively with the secondary school people. Such a chance should be seized. In my opinion, the strongest mathematics departments in each state, the ones that should be providing guidance and leadership, should write their state certification officer, and express their willingness to help in the state study. I think you will be welcomed.

WHAT MATHEMATICS IS TAUGHT IN EUROPEAN SCHOOLS

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The Organization of European Economic Cooperation has recently sponsored two events of importance for mathematics instruction in the western world. The first of these was an International Seminar in mathematics held at Royaumont, France, from November 21 to December 5, 1959, which was attended by fifty-one mathematicians and teachers from twenty-one countries of Europe and America. The other is a Survey of European Mathematics Education in which the Mathematical Association of America participated with funds supplied by the National Science Foundation. The complete reports of both these events will be published in the near future. The following article, indicative of some of the findings, is made from a preliminary examination of the Survey Questionnaires.

In making any comparison of the subject matter taught in European schools with that taught in the schools of the United States, one must consider the structure of the European schools. Generally, the school organization, the purposes of instruction, and selection or guidance of pupils is quite distinct from country to country in Europe and in all cases quite different from that held in the United States. In several countries of Europe, children begin formal schooling at ages 5, 6, or 7 years, and the elementary and secondary school program lasts for 11, 12, or 13 years of study. In order to make a description as valid as possible, I shall speak therefore only of that mathematics taught from ages 6 to 17 years.

At another place I have reported on the mathematics taught to the youth of the world, ages 6 to 15 years.* Here, then, I shall speak only briefly on arithmetic instruction. The arithmetic taught in the United States is exactly the same arithmetic as taught in Europe to children 6 to 12 years of age. The grade placement of the subject matter, however, varies considerably and generally, the subject matter covered in the books and classes in the United States is

* The mathematics education of youth—a comparative study, Enseignement Math., vol. 5, 1959, pp. 61–78.

placed about one-third academic year later than the average placement for all European countries. This means that some countries delay longer than the United States in introducing topics, *e.g.*, common fractions, while they also introduce other topics a year earlier than we do. The variation in placement is indicated in the following three problems:

(1) $68+25$ is taught in:

1 country at age 6		4 countries at grade 1
13 countries at age 7	or	15 countries at grade 2
5 countries at age 8		1 country at grade 3
1 country at age 9		

Thus this problem is taught on the average at age $7\frac{1}{2}$ years in grade 2, which is exactly the same time and place of instruction in the United States.

(2) Add: 784.927 is taught in:

27.384
63.673
591.595

3 countries at age 8		1 country at grade 2
6 countries at age 9	or	4 countries at grade 3
8 countries at age 10		9 countries at grade 4
3 countries at age 11		4 countries at grade 5
		2 countries at grade 6

There is great divergence in the placement of this topic. The average placement is in grade 4 toward the age 10 years. In the United States it is generally placed in grade 6 at age $11\frac{1}{2}$ years. The use of the metric system may be one factor in establishing the earlier placement of this subject in Continental Europe.

(3) $2\frac{3}{5}+5\frac{7}{12}$ is taught in:

9 countries at age 10		6 countries in grade 5
5 countries at age 11	or	10 countries in grade 6
5 countries at age 12		3 countries in grade 7

The average placement is at the end of the 5th grade or beginning of the 6th grade at age $11\frac{1}{3}$ years, while in the United States it occurs in the middle of the 5th grade at age $10\frac{1}{2}$ years. The early placement of fractions in England and the United States may be accounted for in the use of our systems of measures, most of which are non-metric.

The teaching of arithmetic as a meaningful mathematical structure is more prevalent in the United States than in any other country of the world. This accounts in part for the delay in placement of some topics. Generally in Europe, while there is a trend to shift to meaningful teaching, the teaching still consists of rote, repetitive, manipulative drill on facts and processes. However, there is a more powerful factor at work in European schools to make for earlier placement of topics. This is the segregation of pupils into intellectual streams at the end of the 4th, 5th, or 6th grade or at ages ten to twelve years. The segregation is usually based on an examination consisting of two subjects—the native language and arithmetic. The whole teaching of arithmetic in grades 1 to 5 or 6 is aimed at preparing those pupils, who can, to pass this examination. For Europe

as a whole, almost one-half (50%) of the children do not take the examinations because (a) they, by class, relegate themselves to a lower working level than the professions which demand higher study and greater means of financing, or (b) they know that it would be totally impossible for them to pass the examination.

Of those pupils who take the examination, about 60 to 70 per cent pass it. Thus in Europe of all children age 12 years, only 30 out of every 100 enter the gymnasium, lycee, gymnas, grammar school, or other college preparatory school. The seventy per cent of the children who are barred from such study, continue in other schools until age 14 or 15 years and then enter either into the world of work or into vocational schools for further training. The arithmetic taught in European schools to children age 6 to 12 years is geared to the learning capacity of the upper 30 to 50 per cent in cognitive intelligence. The lower half learn very little. The caliber of instruction in the United States is known, but nevertheless, in the writer's judgment, it would be a great step backward if the United States were to adopt any of the present European program in place of the program evolving in our own schools. There I am sure we shall develop administrative and curriculum procedures to take care of those highly capable children we have so long neglected, and at the same time continue to teach arithmetic to all the children, adapting the instruction to their capacity and rate of learning.

Turning to the high school (grades 7 through 12), we will here consider only the mathematics taught in college preparatory programs. We have heard of the fine teaching and the thoroughness in learning of the subjects taught in European schools. Generally this is true, and it is primarily due to the excellent mathematical and pedagogical training of the teachers in these schools, as well as to the limitation for entrance to pupils with high abilities to learn. How would we, in the United States, enjoy teaching only the upper 30 per cent of intelligence in grades 7 and 8, the upper 15 per cent of intelligence in grades 9 and 10, and the upper 8 per cent of intelligence in grade 12! For these percentages show the average "student mortality rate" of secondary-school students in Europe.

Yet the subject matter covered is not vastly different from that in our schools and in no case is it modern in the sense of the fine materials written by the School Mathematics Study Group and the University of Illinois School Mathematics Committee, or the program advocated by the Commission on Mathematics. Let us look first at what is not taught in European schools.

1. There is no use of sets, set notations, or set language. Unions, intersections, Cartesian products, function as a mapping or a restricted set of ordered pairs, is nowhere present in the program. Function in all the European schools is merely $f(x)$ where x is a numerical variable.

2. The study of permutations, combinations, and probability, has been dropped in most schools as too difficult and designated as university study.

Statistics, as an organized study, both descriptive and inferential is not generally taught in the secondary school. There are three experiments in the teaching of this subject (Netherlands, Switzerland, and Sweden), but it is not a part of the official program. In England it is an optional subject in the sixth form which is really junior-college-level instruction.

3. The trigonometry of real numbers is nowhere to be found in the secondary school. The treatment of trigonometry is entirely based on angles (including the radian measure of angles). The wrapping function is completely unknown.

4. Formalism and axiomatic development are not stressed. To be sure, in the teaching of geometry there are definitions, axioms, theorems and proof, but not in so rigorous a manner as we use in our teaching. Proof in algebra (*i.e.*, using the usual laws of a group, ring, or field to prove further properties) is never mentioned. The study of formal algebra is strictly a university study, and the secondary school algebra can still be classified as essentially that of Hall and Knight.

5. The theory of determinants (except for 2nd- and 3rd-order determinants) and matrices is not taught in European secondary schools.

6. Of course complex numbers are taught and used, in the form $a+bi$, and in trigonometry in the form $(\cos\theta+i\sin\theta)$ but never in a formalized structure as ordered pairs of real numbers.

7. Projective geometry as a separate subject, (*i.e.*, more than a mention of a few properties, such as dualism) and non-Euclidean geometries do not exist in the instructional program. The study of groups is reserved for special students in the 13th year of study.

Turning to what is taught in Europe and generally not taught in the United States, we see some striking contrasts. A few of these differences demand careful study by American educators, for they give promise of valuable additions to our instruction in mathematics.

1. The first of these is the teaching of the calculus. This is a universal subject in European secondary school instruction and includes the differentiation and integration of the usual functions, with applications. The treatment is informal, with geometrical and physical explanations of the theory, and no attempt is made to give rigorous delta-epsilon proofs of theorems. The students learn how to do and how to use the calculus and they apply it to their study of physics. This study of calculus is to the formal calculus taught to them later in the university as our study of intuitive geometry of junior high school is to the study of deductive geometry in the high school. It gives a basic wide knowledge and experience upon which a formal mathematical system can be constructed.

2. Most European countries continue to teach descriptive geometry as a separate discipline (only The Netherlands has recently dropped this subject). This instruction is more than mechanical drawing, for it involves not only the making of drawings, but the theory of projections, representations, parallelism, perpendicularity, rotations, intersections and development of surfaces. This subject is never taught in the secondary school in the United States and is

found under the title of Engineering Drawing in some of our colleges and universities.

3. Cosmography is a subject quite common in continental European secondary schools. In this course the teaching of trigonometry is extended to the solution of spherical triangles. The celestial sphere and its coordinate systems are studied, and problems are solved in the determination of time, latitude, longitude, and orbits of planets. Recent interest in outer space, satellites, and missiles suggest that this elementary study of cosmography may be a good basis for the more extended and deeper mathematical treatment of space and orbit problems to be studied later in the university.

4. A treatment of Geometry as the set of transformations of a plane into itself (or space into self), called motion geometry (or Bewegungsgéometrie) is gaining considerable attention in Germany where 70 per cent of the gymnasium classes are using this approach to the subject. Here rotations, translations, and reflections (Spiegelungen) are developed intuitively, then the properties of these transformations are used in developing the usual geometry of the plane. Shearing, stretching, and contracting lead to the study of similarity and the introduction of vectors. This program is in harmony with the geometry program proposed by Felix Klein and called the Erlanger program, but it also leads to a better insight into the properties that distinguish one type of geometry from another.

5. I also mention the fundamental ideas of the theory of numbers. While in Europe the arithmetic of the elementary school is very mechanical, the attempt to mathematicize this arithmetic and uncover some of its basic properties is done in the secondary school, after the pupil has studied some algebra. Thus, division $a \div b$ becomes $a = bq + r$, where $r < b$, and from this understanding the Euclidean algorism for finding the greatest common divisor, theorems on divisibility, least common multiples, prime numbers, unique factorization, and the solution of simple Diophantine equations are developed.

6. Lastly, the properties of the conics, developed either synthetically or analytically, occupy a large part of the senior secondary school instruction. In our country this mathematics is fast disappearing, even at the university level, for it represents a treatment that can be greatly simplified by the use of the calculus. Further, most of the properties studied have little to do with either practical applications or the subsequent study of mathematics. However, since the synthetic study of the properties of conics requires a type of mathematical insight and patience for complex thinking, it makes this study an excellent sieve in the selection of university students who will major in mathematics.

Thus we see that, in large measure, Europe teaches the same mathematics we do, albeit with greater depth and complexity, and usually to pupils at a slightly younger age than ours, because their pupils are more highly selected. They also teach mathematics we do not include in our program, not because we feel this mathematics is too difficult, but because through the traditional and cultural patterns we have acquired, we feel it is either not of value or of need to our pupils. More important, however, is the fact that whereas the

European program is still relatively static there are emerging in the United States a number of mathematics programs that include present-day concepts and symbolism, that are in line with technological and scientific needs of our country, and which are capable of study and grasp by a much larger percentage of the youth than that which could successfully pursue the present classical program of European schools.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1431. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Advanced Development*

If $a_{n+1}=(1+a_n a_{n-1})/a_{n-2}$ and $a_1=a_2=a_3=1$, show that a_n is an integer.

E 1432. *Proposed by O. Lowenschuss and A. Rosenfeld, Budd Lewyt Electronics, Inc., Long Island City, N. Y.*

How many distinct products appear in the multiplication table through $N\times N$?

E 1433. *Proposed by Alexander Oppenheim, University of Malaya*

Let P be a point in the interior of a triangle and let the distances of P from the vertices of the triangle be x, y, z and from the sides of the triangle be p, q, r. Show that $xyz \geqq (q+r)(r+p)(p+q)$.

E 1434. *Proposed by Anatole Beck, University of Wisconsin*

Show that every open set in the plane can be represented as a disjoint union of closed straight line segments.

E 1435. *Proposed by J. F. Leetch, Ohio State University*

Find the limit superior and the limit inferior for each of the following sequences of sets of real numbers:

(a) $\{A_n\}_{n=1}^{\infty}$ where A_n is the set of integers mod n,

(b) $\{A_n\}_{n=1}^{\infty}$ where $A_n=\{a/n \mid a\in J \text{ (integers)}\}$.

SOLUTIONS

The Derivatives of $1/(e^u-1)$

E 1401 [1960, 182]. *Proposed by Emil Grosswald, Institute for Advanced Study*

Prove that for every integer $k \geqq 1$,

$$\frac{d^{k-1}}{du^{k-1}}\left(\frac{1}{e^u-1}\right) = (-1)^{k-1}\sum_{r=1}^{k}\frac{\gamma_k(r)}{(e^u-1)^r},$$

where the constants $\gamma_k(r)$ are all positive and satisfy

$$\gamma_k(r) = \sum_{\nu=0}^{r-1}(-1)^\nu\binom{r-1}{\nu}(r-\nu)^{k-1}.$$

In particular, show that $\gamma_k(k) = (k-1)!$.

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. Since

$$\gamma_n(r) = \left[\frac{d^{n-1}}{dx^{n-1}}\left\{e^x(e^x-1)^{r-1}\right\}\right]_{x=0}, \qquad 1 \leqq r \leqq n,$$

we find that $\gamma_{n+1}(r) = r\gamma_n(r) + (r-1)\gamma_n(r-1)$. Since $e^x(e^x-1)^{r-1} = (e^x-1)^r - (e^x-1)^{r-1}$, we see that $\gamma_n(n) = (n-1)!$ and $\gamma_n(n+1) = 0$. The desired formula is true for $k=1$. Suppose now the formula is true for $k = 1, 2, \cdots, n$. Then

$$\begin{aligned}(-1)^n\sum_{r=1}^{n+1}\frac{\gamma_{n+1}(r)}{(e^u-1)^r} &= (-1)^n\left[\sum_{r=1}^{n}\frac{r\gamma_n(r)}{(e^u-1)^r} + \sum_{r=1}^{n}\frac{r\gamma_n(r)}{(e^u-1)^{r+1}}\right] \\ &= (-1)^n\sum_{r=1}^{n}\frac{e^u r\gamma_n(r)}{(e^u-1)^{r+1}} \\ &= \frac{d}{du}\left[(-1)^{n-1}\sum_{r=1}^{n}\frac{\gamma_n(r)}{(e^u-1)^r}\right] = \frac{d^n}{du^n}\left(\frac{1}{e^u-1}\right).\end{aligned}$$

Thus, by mathematical induction, the formula is true for integer $n \geqq 1$.

Also solved by A. N. Aheart, D. W. Bailey, Leonard Carlitz, G. DiAntonio, Jane Evans, S. H. Greene, A. F. Kaupe, Jr., M. S. Klamkin, Donald Knuth, D. C. B. Marsh, C. F. Pinzka, B. E. Rhoades, Herbert Ruderfer, Vencil Skarda, and the proposer.

DiAntonio pointed out that this problem is essentially Prob. 8 on p. 87 of Gibson's *Advanced Calculus*. Pinzka pointed out that the concluding particular result follows from Problem 4183 [1947, 235].

An Inequality Connecting the Inradius and Exradii of a Tetrahedron

E 1402 [1960, 182]. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let r denote the radius of the inscribed sphere of a tetrahedron T and let r_i $(i=1, 2, 3, 4)$ denote the radii of the exspheres of T which touch one face of T and the other three faces of T produced. Show that $\sum_{i=1}^{4} r_i \geqq 8r$, with equality if and only if T is isosceles.

Solution by Joe Lipman, Harvard University. If $3S=\sum_{i=1}^{4} a_i$, where a_i is the area of the ith face, then $r_i=V/(S-2a_i/3)$ and $r=V/S$, where V is the volume of the tetrahedron. This gives $\sum_{i=1}^{4}(1/r_i)=2/r$. But, by elementary algebra, $\sum_{i=1}^{4}(1/r_i)\sum_{i=1}^{4} r_i \geqq 16$, with equality if and only if the r_i, that is the a_i, are equal. The conclusion follows.

Also solved by John Burr, J. W. Clawson, M. S. Klamkin, D. C. B. Marsh, and the proposer.

A Property of a Polynomial

E 1403 [1960, 182]. *Proposed by T. J. Rivlin, IBM Corp., Yorktown Heights, N. Y.*

Let $p(x)=a_0+a_1x+\cdots+a_nx^n$, with the sequence of coefficients a_3, a_5, a_7, $\cdots$, a_m ($m=n$, n odd; $m=n-1$, n even) either all nonnegative or all nonpositive. Show that:

(a) there exists a number y_n such that $\{p(1)-p(-1)\}/2=p'(y_n)$,
(b) if n is odd and >1, $|y_n| \leqq (1/n)^{1/(n-1)}$,
(c) if n is even and >2, $|y_n| \leqq \{1/(n-1)\}^{1/(n-2)}$,
(d) the bounds on $|y_n|$ are the best possible,
(e) when $n=1$, 2 we may take $y_n=0$.

Solution by D. C. B. Marsh, Colorado School of Mines. Write

$$p'(x) - \{p(1) - p(-1)\}/2 \equiv \phi(x) \equiv f(x) + xg(x),$$

where, for $n \geqq 3$,

$$f(x) = \sum_{j=1}^{(m-1)/2} \{(2j+1)x^{2j} - 1\}a_{2j+1}, \qquad g(x) = \sum_{j=1}^{[n/2]} 2ja_{2j}x^{2j-2},$$

and [] denotes the greatest integer function. For $m>1$, let $x^*=(1/m)^{1/(m-1)}$. It is easy to show that, for $j \leqq (m-1)/2$, $(2j+1)(x^*)^{2j} \geqq 1$, so that $f(x^*)$, $f(-x^*)$, and $-f(0)$ have the same sign as the numbers of the set a_3, a_5, $\cdots$, a_m. Now $g(x)$ is an even function so that $x^*g(x^*)$ and $-x^*g(-x^*)$ have opposite signs, one being the same as the sign of the numbers a_3, a_5, $\cdots$, a_m. Thus one of $\phi(x^*)$ and $\phi(-x^*)$ has the same sign as the numbers a_3, a_5, $\cdots$, a_m, whereas $\phi(0)$ has the opposite sign. With $\phi(x)$ continuous there must therefore exist at least one value of x (say y_n) between x^* and $-x^*$ for which $\phi(y_n)=0$, and thus a value for which $p'(y_n)=\{p(1)-p(-1)\}/2$, proving (a), (b), and (c). For odd $n>1$, when $a_j=0$ ($j \leqq n-1$) and $a_n=1$, $\phi(x)=0$ is satisfied by only $x=\pm x^*$ in the interval $(-x^*, x^*)$. For even $n>2$, when $a_j=0$ ($j<n-2$), $a_{n-2}=1$, $a_{n-1}=2\{1/(n-1)\}^{(n-3)/(n-2)}$, $a_n=\{(2-n)/n\}(n-1)^{2/(n-2)}$, we have $\phi(x)=0$ is satisfied only for $x=\pm x^*$ in the interval $(-x^*, x^*)$, proving (d). For $n=1$, the equation becomes $a_1=a_1$, satisfied identically, so we *may* choose $y_n=0$, while for $n=2$, one has $a_1=a_1+2a_2y_n$, permitting $y_n=0$. Thence (e).

Also solved by Jane Evans, S. H. Greene, and the proposer.

The proposer wondered if there is some simple bound for y_n when the coefficients of the polynomial are completely arbitrary.

A Property of the Two-Dimensional Point Lattice

E 1404 [1960, 183]. *Proposed by H. N. Ward, Harvard University*

(a) In Euclidean 2-dimensional space R^2, let L be the lattice of points with integral coordinates, and T any nondegenerate triangle (including its boundary) whose vertices belong to L. Let T' be any translation of $-T(=\{-x \mid x\in T\})$ which has members of L as vertices; that is, $T'=p-T$, where $p\in L$. Show that if $T\cap T'\neq\phi$, then $T\cap T'$ contains a member of L.

(b) Show that an analogous statement in Euclidean n-dimensional space R^n, $n>2$, where simplexes are used for triangles, fails.

Solution by the proposer. Let R^n be Euclidean n-space; let $e_1, \cdots, e_n$ be a fixed basis; and let L^n be all points $p=a_1e_1+\cdots+a_ne_n$ for which the a_i are integers. Then L^n is the integral lattice. For any two subsets A and B of R^n let $A+B$ be all $a+b$ with a in A, b in B. Let A^m stand for $A+\cdots+A$ with m summands. Let A^* be $A\cap L^n$. For any scalar r, let rA be all ra for a in A.

Let A be a convex set in R^n. Then for any positive integer m, $A^m=mA$. For an element of A^m is $p_1+\cdots+p_m$ where each p_i is in A. But then $(1/m)p_1+(1/m)p_2+\cdots+(1/m)p_m=p$ is in A from the convexity, so that $p_1+\cdots+p_m=mp$ is in mA. Then $A^m\subseteq mA$. But if p is in A, $mp=p+\cdots+p$ (m summands) is in A^m, so $mA\subseteq A^m$; that is then, $A^m=mA$.

Now suppose A is convex. Suppose also A is the union of a family $\{A_i\}$ (with a given index set I) of convex sets such that for each A_i one has that $(A_i^*)^m=((A_i)^m)^*$ for all m. Then in fact $(A^*)^m=(A^m)^*$. For, $A^m=mA=\cup mA_i$, and since the A_i are convex, this last is $\cup(A_i)^m$. Intersecting the sets with L^n, one has $(A^m)^*=\cup((A_i)^m)^*=\cup(A_i^*)^m$. Here all unions are over the set I. Say now that $p_1+\cdots+p_m$ is in $(\cup A_i^*)^m$. Suppose p_j is given to be in A_{ij}^*. Then, since A is convex and each p_j is in A, one has that $(1/m)p_1+\cdots+(1/m)p_m$ is in A. Then this last element is in A_i for some i, so that $p_1+\cdots+p_m$ is in $mA_i=(A_i)^m$. Since it is also in L^n because each p_j is, it is then in $((A_i)^m)^*=(A_i^*)^m$. Then $p_1+\cdots+p_m$ is in $\cup(A_i^*)^m$. This last union is a subset of $(\cup A_i^*)^m$, which is $(A^*)^m$. So one has $(\cup A_i^*)^m\subseteq\cup(A_i^*)^m\subseteq(\cup A_i^*)^m=(A^*)^m$. Since the second set in the string is $(A^m)^*$, one has the desired result.

In R^2 let C be a closed convex polygon whose vertices are in L^2. Then (calling such a polygon a lattice polygon) one has that C is a union of lattice triangles $\{T_i\}$ (which are taken to be closed). If one can show that for a lattice triangle T one has $(T^*)^m=(T^m)^*$, the preceding will give $(C^*)^m=(C^m)^*$. Let T be a lattice triangle whose vertices are $a_ie_1+b_ie_2$, $i=1, 2, 3$. Let the parallelogram $0, e_1, e_2, e_1+e_2$ be given unit area. Then the area of T is $|D/2|$, where D is the determinant whose ith row is $a_i, b_i, 1$. The area is a half-integer. Then if one can show that any lattice triangle is the union of lattice triangles of area $\frac{1}{2}$, and that

for any lattice triangle of area $\frac{1}{2}$ one has the relation $(T^*)^m=(T^m)^*$, then again the earlier result will give the desired conclusion.

Let (x, y) be written for xe_1+ye_2. Say T is a lattice triangle of area $\frac{1}{2}$. There is no loss in generality in taking one vertex as 0. Say (a, b) and (c, d) are the other vertices, ordered to give $ad-bc=1$, the value of $|D|$. Suppose (x, y) is a lattice point of T^m. Then $(x/m, y/m)$ is in T. Then $(x/m, y/m)=r(a, b)+s(c, d)+t0$, where r, s, and t are nonnegative and $r+s+t=1$. That is, $x/m=ra+sc$, $y/m=rb+sd$. Solving, one has $r=dx/m-cy/m$, $s=ay/m-bx/m$; that is, $r=u/m$, $s=v/m$, where the u, v must be nonnegative integers for which $u+v$ is at most m. Then $(x, y)=u(a, b)+v(c, d)+(m-u-v)0$, which is a point of $(T^*)^m$. So $(T^m)^*\subseteq(T^*)^m$. Since the reverse is always true, one has $(T^m)^*=(T^*)^m$.

Now let T be a lattice triangle of area equal to or greater than $\frac{1}{2}$; one must show that it is the union of lattice triangles of area $\frac{1}{2}$. One can prove this inductively since the area is always a half-integer. In fact, it is sufficient to show that such a T contains a point of L^2 other than its vertices; for using such an additional point as a vertex, one can decompose T into two or three lattice triangles according as the point is on an edge or in the interior of T. Each such triangle has area strictly less than that of T, and the induction hypothesis works; the statement is true at area $\frac{1}{2}$.

Let the area be S; one may take $S\geqq 1$. Assume one vertex is 0, and say p and q are the others. The parallelogram whose vertices are $p+q$, $p-q$, $-p+q$, $-p-q$ has area $8S\geqq 8$. Call P the parallelogram and form $\sqrt{\frac{1}{2}}P$, whose area is $\geqq 4$. Then by Minkowski's theorem on compact convex sets symmetric about 0 with area greater than or equal to 4, $\sqrt{\frac{1}{2}}P$ contains a lattice point other than 0. Then P contains a lattice point other than 0, which, because of the reducing factor, is also different from any of the vertices. Because P is a union of translations by members of L^2 of T and $-T$, T has a point of L^2 in it other than its vertices, as needed; for, in fact, these translated triangles can be taken as having vertices among the points 0 and the four vertices of P.

One immediate conclusion is a slight generalization of the result asked for in E 1404: let C be any convex lattice polygon and suppose for a given p in L^2 that C meets $p+(-C)$. Then, in fact, there is a member of L^2 in the intersection. For, one must have points a and b in C for which $a=p-b$. Then $p=a+b$; so p is in $(C^2)^*$. But then p is in $(C^*)^2$; so that there are points c and d of L^2 in C^* with $p=c+d$. But then the lattice point $c=p-d$ is a point of L^2 common to C and $p+(-C)$ as wished.

Another corollary is a sort of figurate number result. Let C be a convex lattice polygon; let C_e denote the set formed by the edges, and C_i the interior. Let $S(C)$ denote the area; and for any set A let $N(A)$ be the number of points of L^2 in A. One has the area formula $S(C)=N(C_i)+N(C_e)/2-1$. But for an integer m, $S(C^m)=m^2S(C)$, and $N((C^m)_e)=mN(C_e)$. Then one has that the number of lattice points in $(C^*)^m$ is a polynomial in m of the second degree; for this number is $N((C^*)^m) = N((C^m)^*) = N(C^m) = N((C^m)_e) + N((C^m)_i) = mN(C_e)$

$+S(C^m)-N((C^m)_e)/2+1=mN(C_e)+m^2S(C)-mN(C_e)/2+1$; that is, one has that $N((C^*)^m)=m^2S(C)+mN(C_e)/2+1$.

The Series $1+22+333+\cdots$

E 1405 [1960, 183]. *Proposed by L. L. Garner, University of North Carolina*

Find (a) an expression for the general term, (b) the sum of the first n terms of the series $1+22+333+\cdots$.

Solution by W. B. Carver, Cornell University. The problem is indefinite. The three given terms are the first three terms of infinitely many different series, as, for instance, the series whose general term is $u_k=145k^2-414k+270$, for which the sum of n terms would be $S_n=(290n^3-807n^2+523n)/6$.

However, I *guess* that the proposer intended that the general term of the series should be $u_k=k(10^k-1)/9$. In that case the sum of n terms would be

$$S_n=\sum_{k=1}^{n}(k\cdot 10^k-k)/9.$$

Then

$$9S_n=(1\cdot 10+2\cdot 10^2+3\cdot 10^3+\cdots+n\cdot 10^n)-(1+2+3+\cdots+n).$$

The series in the first parentheses is a combination arithmetic-geometric series which can be summed by the same method as that used for summing the geometric series. Let

$$\begin{aligned} P &= 1\cdot 10+2\cdot 10^2+3\cdot 10^3+\cdots+n\cdot 10^n,\\ 10P &= \qquad 1\cdot 10^2+2\cdot 10^3+\cdots+(n+1)\cdot 10^n+10^{n+1}. \end{aligned}$$

Hence, subtracting,

$$\begin{aligned} 9P &= -(10+10^2+10^3+\cdots+10^n)+n\cdot 10^{n+1}\\ &= -10(10^n-1)/9+n\cdot 10^{n+1}, \end{aligned}$$

and

$$\begin{aligned} S_n &= -10(10^n-1)/729+n\cdot 10^{n+1}/81-n(n+1)/18\\ &= [(9n-1)\cdot 10^{n+1}+10]/729-n(n+1)/18. \end{aligned}$$

It is an interesting exercise to show directly that this expression for S_n gives an *integer* for all integral values of n.

Also solved by A. N. Aheart, R. H. Anglin, D. W. Bailey, J. W. Baldwin, Edward Barbeau, Merrill Barnebey, Alan Beal, W. R. Becker, Roy Bertoldo, A. P. Boblétt, Julian Braun, D. A. Breault, A. M. Broshi, Brother Christopher Beaulieu, J. L. Brown, Jr., A. W. Brunson, R. G. Buschman, J. W. Clawson, F. B. Correia, A. E. Crofts, Jr., Thomas Curry, T. M. Delmer and L. F. Klosinski (jointly), G. DiAntonio, E. S. Eby, Walter Ehrenpreis, L. F. Epstein, Jane Evans, Joyce Farrell, A. E. Fleck, Charles Franti, Michael Goldberg, L. D. Goldstone, H. W. Gould, Mrs. R. H. Gramann, S. H. Greene, Emil Grosswald, Margaret Herzog, H. W. Hickey, Geoffrey

Horrocks, A. R. Hyde, J. M. Kingston, M. S. Klamkin, Donald Knuth, Harry Langman, Joe Lipman, Stephen Lowe, D. C. B. Marsh, J. W. Mettler, N. G. Mouck, Jr., C. S. Ogilvy, M. J. Pascual, R. Q. Petersen, D. J. Peterson, J. L. Pietenpol, C. F. Pinzka, Thomas Porsching, J. H. Rascoff, B. E. Rhoades, D. A. Robinson, J. V. Ryff, R. S. Sabharwal, Barbara Sakitt, E. M. Scheuer, Y. P. Sharma, C. M. Sidlo, Stephen Stein, M. F. Stilwell, R. P. Tapscott, R. C. Weger, T. T. West, R. J. Wisner, C. C. Yalavigi, Walter Zayachkowski, and the proposer.

Editorial Note. A couple of other elementary methods of obtaining $T_n(x) = \sum_{k=1}^{n} kx^k$, $x \neq 1$, are:

(1)
$$\begin{aligned} T_n(x) &= x + 2x^2 + 3x^3 + \cdots + nx^n \\ &= (x + x^2 + x^3 + \cdots + x^n) + (x^2 + x^3 + \cdots + x^n) + \cdots + x^n \\ &= x(x^n - 1)/(x - 1) + x^2(x^{n-1} - 1)/(x - 1) + \cdots + x^n(x - 1)/(x - 1) \\ &= [nx^{n+1} - (x + x^2 + \cdots + x^n)]/(x - 1) \\ &= [nx^{n+1}(x - 1) - x(x^n - 1)]/(x - 1)^2 \\ &= x[nx^{n+1} - (n + 1)x^n + 1]/(x - 1)^2. \end{aligned}$$

(2)
$$\begin{aligned} T_n(x) &= x(d/dx)\left(\sum_{k=1}^{n} x^k\right) \\ &= x(d/dx)[(x^{n+1} - x)/(x - 1)] \\ &= x[nx^{n+1} - (n + 1)x^n + 1]/(x - 1)^2. \end{aligned}$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4923. *Proposed by D. J. Newman, Brown University*

For any set of real numbers, S, we define its "divisor set," $\hat{S}$, to be the set of all numbers in (0, 1) for which some integral multiple lies in S. Are there sets S of arbitrarily large measure for which $\hat{S}$ has arbitrarily small measure?

4924. *Proposed by J. H. Blau, Antioch College and Stanford University*

Let P denote a partition of $[0, 1]$ into n disjoint measurable sets E_i. Denote $\sup m(E_i)$ by $|P|$, and let $x_i \in E_i$. For which functions F on $[0, 1]$ does the following pseudo-integral exist:

$$\lim_{|P| \to 0} \sum_{i=1}^{n} F(x_i)m(E_i) \ ?$$

4925. *Proposed by C. Kacser, Clarendon Laboratory, Oxford, England*

There is a well-known result that

$$\sum_{n=0}^{\infty} (2n + 1)P_n(x)Q_n(z) = \frac{1}{z - x},$$

where P_n and Q_n are the regular and irregular Legendre polynomials of order n. A suggested generalization is

$$\sum_{n=0}^{\infty} (2n + 1)P_n(x)[Q_n(z)]^3 = \left[\log \frac{z - x}{z - 1}\right]^2 \frac{1}{z - x},$$

at least in the limit as z approaches 1 from above. Prove or disprove.

4926. *Proposed by S. Berberian, State University of Iowa*

If $\mathfrak{H}$ is infinite-dimensional (say separable) Hilbert space, construct a densely defined linear transformation T for which the domain of the adjoint transformation T^* is the subspace $\{0\}$.

4927. *Proposed by David Gale, Brown University*

Suppose f is a continuous transformation of n-space into itself, and suppose there is a positive number λ such that $|f(x)-f(y)| \geqq \lambda|x-y|$ for all points x and y, where $|x-y|$ is the Euclidean distance. Prove that f maps onto all of n-space.

4928. *Proposed by C. C. Faith, Pennsylvania State University*

Show by a concrete example that there exists a normal equation $\phi(x) = 0$ of degree n over a field F of characteristic 0 having two conjugates v and v' whose difference $v-v'$ has degree less than n over F, when $n=4$, or when $n=6$.

SOLUTIONS

A Defining Property of the Circle

4869 [1959, 816]. *Proposed by M. S. Klamkin, A VCO Research, Wilmington, Mass.*

A smooth centro-symmetric curve has the property that the centroid of any half-area which is formed by a chord through the center is equidistant from the center. Show that the curve is a circle.

Solution by Harley Flanders, University of California, Berkeley. Represent the curve in polar coordinates by $r = r(\theta)$, a periodic function with $r(\alpha+\pi) = r(\alpha)$. Assume r is differentiable. After multiplying r by a suitable constant, the condition on the centroid is

$$\left[\int_{\alpha}^{\alpha+\pi} r^3 \cos\theta d\theta\right]^2 + \left[\int_{\alpha}^{\alpha+\pi} r^3 \sin\theta d\theta\right]^2 = \left[\int_{\alpha}^{\alpha+\pi} r^2 d\theta\right]^2.$$

Differentiation with respect to α yields

$$\cos\alpha\int_{\alpha}^{\alpha+\pi} r^3\cos\theta d\theta + \sin\alpha\int_{\alpha}^{\alpha+\pi} r^3\sin\theta d\theta = 0,$$

and a second differentiation,

$$-2r^3 - \sin\alpha\int_{\alpha}^{\alpha+\pi} r^3\cos\theta d\theta + \cos\alpha\int_{\alpha}^{\alpha+\pi} r^3\sin\theta d\theta = 0.$$

Another differentiation gives $r'=0$, whence r is a constant and the curve is a circle.

Also solved by the proposer.

Editorial Note. The proposer remarks that if the locus of the centroid were an ellipse instead of a circle, the same argument proves that the original curve must be a homothetic ellipse. He also conjectures that an analogous hypothesis regarding a surface in three-space will lead to a sphere.

A Locus Defined in Complex Coordinates

4870 [1959, 816]. *Proposed by D. S. Mitrinovitch, Belgrade, Yugoslavia*

If the variable z traces the circle $|z|=1$, find and describe the path of the point $w=(z^2-az)/(az-1)$, where the parameter a is a complex constant.

Solution by D. C. B. Marsh, Colorado School of Mines. Multiplication by $(az-1)\bar{z}$ enables one to write the transformation as $wa-w\bar{z}=z-a$, for points of the unit circle, $|z|=1$. Setting $z=x+yi$, $w=u+vi$, $a=A+Bi$, and separating reals and pure imaginaries, one finds

$$(u+1)x+vy=(u+1)A-vB, \qquad vx-(u-1)y=vA+(u+1)B,$$

assuming that a is truly complex (*i.e.*, $B\neq 0$). This linear system yields

$$x=A+\frac{2Bv}{u^3+v^2-1}, \qquad y=-B-\frac{2B(u+1)}{u^2+v^2-1}.$$

so that $x^2+y^2=1$ becomes

$$(u^2+v^2-1)^2=\{A(u^2+v^2-1)+2Bv\}^2+B^2\{(u^2+v^2-1)+2(u+1)\}^2.$$

Establishing a T-axis orthogonal to the w-plane, this quartic may be considered as the orthgonal projection on the w-plane of the curve of intersection of the paraboloid $t+1=u^2+v^2$ and the oblique cone $t^2=(At+2Bv)^2+B^2(t+2u+2)^2$.

The quartic may also be constructed pointwise by assigning values to the parameter m, constructing the circles $u^2+v^2=1+2mB$ and $(u+1+mB)^2+(v+mA)^2=m^2$, and having their points of intersection as points on the quartic.

Also solved by A. J. Kokar.

A Fixed Point Theorem

4873 [1959, 817]. *Proposed by W. R. Utz, University of Missouri*

Let S be a connected topological space in which there is a relation $<$ which satisfies: (1) S has a least and a greatest element relative to $<$, (2) $<$ satisfies the trichotomy law, (3) $<$ is continuous. In other words, if $x<y$ in S, then there exist neighborhoods N_x and N_y of x and y, respectively, such that $z\in N_x$ implies $z<w$ for each $w\in N_y$.

Show that any continuous transformation of S into itself has a fixed point, and give an example of S which is not an arc.

Solution by Robert Speira, University of California, Berkeley. Let f be the continuous transformation. Assume that f has no fixed point. Let 0 and 1 be the least and greatest elements, respectively. Thus $f(0)\neq 0$ and $f(1)\neq 1$. Hence $0<f(0)$, and $f(1)<1$. Thus the sets $\{z: z<f(z)\}$ and $\{z: f(z)<z\}$ are both non-empty, and exhaust S, and are disjoint (using trichotomy and the assumption on f).

We now show both sets are open. Let $x\in\{z: f(z)<z\}$. Let $y=f(x)$. Then there exist open N_x, N_y satisfying the hypothesis of the problem. Now $f^{-1}[N_y]$ is an open set containing x. Thus $f^{-1}[N_y]\cap N_x$ is open and contains x; moreover it is included in $\{z: f(z)<z\}$. For if t is in the intersection, then $t\in N_x$ and $f(t)\in N_y$, hence $t<f(t)$. Thus $\{z: f(z)<z\}$ is open. A similar argument shows $\{z: z<f(z)\}$ is open. But then, S is partitioned into two open sets, contrary to the hypothesis of connectedness. Hence f must have a fixed point.

An example of such an S, not an arc, is the plane square, $0\leqq x\leqq 1$, $0\leqq y\leqq 1$. For the topology on S we take as base all vertical open segments. This topology contains the usual topology, as an open neighborhood can be written as the union of the open vertical segments contained within it. Define $(x_1, y_1)<(x_2, y_2)$ if and only if $(x_1<x_2)\vee(x_1=x_2\wedge y_1<y_2)$. This $<$ clearly satisfies the requirements, where the greatest and least elements are (1, 1) and (0, 0) respectively. This space is not homeomorphic to an arc, as any such mapping would be continuous in the usual sense since this topology contains the usual topology, and there is no one-to-one continuous function mapping the unit interval onto the unit square (Hobson, *Functions of a Real Variable*, Vol. I, §325).

Also solved by Ben Fitzpatrick, Jr., Harley Flanders, Gerald Leibowitz and Louis Hodes, P. S. Landweber, Elliott Mendelson, Jerrold Siegel, James V. Whittaker, and the proposer.

Fourier Coefficients

4874 [1959, 817]. *Proposed by D. J. Newman, Brown University*

Let c_n be a given sequence of constants such that $c_n=O(1/n)$, $n=1, 2, \cdots$. Show that there is a bounded function $f(x)$ with c_n as its positive Fourier coefficients, *i.e.*, such that

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = c_n.$$

Solution by the proposer. It is a theorem of F. Riesz that if (and only if)

$$\left|\sum_{n=1}^{N} c_n\lambda_n\right| \Big/ \int \left|\sum_{n=1}^{N} \phi_n\lambda_n\right| < M,$$

with M independent of N and the λ's, then the moment problem

$$\int \phi_n \cdot f = c_n, \qquad n = 1, 2, \cdots$$

has a solution with f a bounded function.

It is also a theorem of G. H. Hardy that

$$\sum_{n=1}^{N} |\lambda_n| /n \leqq M \int_0^{2\pi} \left|\sum_{n=1}^{N} \lambda_n e^{-inx}\right| dx,$$

where M is an absolute constant.

Since $2\pi c_n = O(1/n)$, an application of these two theorems gives the result.

The Coefficients of $x \cosh x/\sin x$

4875 [1959, 920]. *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India*

Prove the following conjecture: if

$$\frac{x \cosh x}{\sin x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{D_{2n}x^{2n}}{(2n)!},$$

then

$$D_{2n} = G_{2n} + \sum_{\substack{(p-1)\mid 2n \\ p\equiv 1(4)}} \frac{1}{p} + \sum_{\substack{p\leqq 2n+1 \\ p\equiv 3(4)}} \frac{\epsilon_p}{p},$$

where the G_{2n} are integers and the ϵ_p are integers depending on p and n. Indeed, an explicit formula for ϵ_p can be obtained.

Solution by Leonard Carlitz, Duke University. Put

$$\frac{x \cosh x}{\sin x} = -\sum_{n=0}^{\infty} (-1)^n \frac{D_{2n}x^{2n}}{(2n)!}, \quad \frac{x}{\sin x} = \sum_{n=0}^{\infty} (-1)^n \frac{R_{2n}x^{2n}}{(2n)!}.$$

Then it is known (see Nörlund, *Differenzenrechnung*, pp. 27, 28) that

$$R_{2n} = 2(1 - 2^{2n-1})B_{2n}, \tag{1}$$

where B_{2n} denotes the Bernoulli number in the even suffix notation. It follows that

$$-D_{2n} = \sum_{r=0}^{n} (-1)^{n-r} \binom{2n}{2r} R_{2r}. \tag{2}$$

Now from (1) and the Staudt-Clausen theorem we have, for $n>0$,

$$pR_{2n} \equiv \begin{cases} -1 \pmod{p} & (p-1 \mid 2n) \\ 0 & (p-1 \nmid 2n); \end{cases}$$

thus (2) yields

$$pD_{2n} \equiv \sum_{\substack{r>0 \\ p-1 \mid 2r}} (-1)^{n-r} \binom{2n}{2r} \pmod{p}. \tag{3}$$

If $p \equiv 1 \pmod 4$, (3) becomes

$$pD_{2n} \equiv (-1)^n \sum_{\substack{r>0 \\ p-1 \mid 2r}} \binom{2n}{2r} \pmod{p}. \tag{4}$$

But by a formula of Hermite and Bachmann (for a short proof see *Mathematics Magazine*, Sept.-Oct. 1959, p. 7)

$$\sum_{0<r(p-1)<n} \binom{n}{r(p-1)} \equiv 0 \pmod{p}.$$

Thus (4) becomes

$$pD_{2n} \equiv \begin{cases} 1 \pmod{p} & (p-1 \mid 2n) \\ 0 & (p-1 \nmid 2n). \end{cases} \tag{5}$$

On the other hand, when $p \equiv 3 \pmod 4$, we get

$$pD_{2n} \sum_{r>0} (-1)^{n-r} \binom{2n}{r(p-1)} \pmod{p}.$$

Hence, if we put

$$\epsilon_p = \sum_{r>0} (-1)^{n-r} \binom{2n}{r(p-1)},$$

the conjectured result follows.

The formula for ϵ_p does not seem capable of simplification.

For similar results, see *Mathematics Magazine*, Mar.–April 1956, pp. 193–197; May–June 1957, problem 311; Mar.–April 1958, pp. 185–190.

Discontinuous Function with Partial Derivatives Everywhere

4876 [1959, 921]. *Proposed by Naoki Kimura, University of Washington*

If a real valued function $f(x, y)$ of two real variables possesses all of its partial derivatives $\partial^{m+n} f(x, y)/\partial x^m \partial y^n$ at every point, is it necessarily continuous?

Solution by John Burr, University of New England, Australia. The following

example shows that the function need not be continuous. The function $f(x, y) = \exp(-x^2/y^2 - y^2/x^2)$ $(xy \neq 0)$, $f(0, y) = f(x, 0) = 0$, is discontinuous at $(0, 0)$, since when $t \to 0$, $f(t, t) \to e^{-2} \neq f(0, 0)$. Suppose that it has been proved that a particular partial derivative $\phi(x, y)$ has the properties (i) $\phi(0, y) = \phi(x, 0) = 0$, (ii) if $xy \neq 0$ then $\phi(x, y) = R(x, y)f(x, y)$ where $R(x, y)$ is rational with denominator of the form $x^p y^q$. Then, by (i), $\phi_y(0, y) = \phi_x(x, 0) = 0$; by (ii) $\phi_x(0, y) = 0$ when $y \neq 0$, since when $x \to 0$, $f(x, y) \to 0$ more rapidly than any power of x; similarly $\phi_y(x, 0) = 0$. Hence $\phi_x(x, y)$ and $\phi_y(x, y)$ both have the property (i), and it is clear from (ii) that they also have the property (ii). Since $f(x, y)$ has these properties, it follows by induction that all the partial derivatives have them, and that these derivatives exist at every point.

It may be noted that this function satisfies conditions more stringent than those prescribed in the problem; namely the additional conditions that the mixed partial derivatives all exist, and are independent of the order in which the differentiations are performed.

Also solved by George Piranian, and the proposer.

Editorial Note. Piranian asks: What are the point sets E in the xy-plane for which there exists a function $(f(x, y; E)$ that is discontinuous everywhere on E and possesses all partial derivatives everywhere in the plane? For example, can E be everywhere dense? Can it be the entire plane?

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Analytic Geometry and an Introduction to the Calculus. By A. Clyde Schock and B. S. Warshaw. Prentice-Hall, Englewood Cliffs, N. J., 1960. 165 pp. $3.96.

This book is an attempt to present the material of the title in a form which the authors consider suitable for a high school class. The result is a weak selection of material supported by definitions which frequently are grossly incorrect and by incomplete demonstrations. A course based on this book can only result in a considerably weakened high school mathematics program and an impossible problem for the student who continues on to college. May this reviewer strongly recommend "What to do about a new kind of freshman," published in the August 1959 MONTHLY, on page 584, which details the problem which results.

DONALD A. NORTON
University of California, Davis

Introduction to Advanced Dynamics. By S. W. McCuskey. Addison-Wesley, Reading, Mass., 1959. viii+263 pp. $8.75.

The material covered in this book is split about equally between what is clearly elementary and what is usually regarded as advanced. The author states in the preface that the contents have been used as a one-semester course for seniors in physics, and feels that it paves the way for the study of modern physics or serves to introduce the nascent engineer or mathematician to domains of classical mechanics not treated in a first course in this subject.

Chapters 1, 3, 4, and part of 2 are elementary, and cover approximately the material on particle and rigid body mechanics contained in standard treatises on theoretical physics such as Page or Lindsay.

Chapters 2 (the latter portion), 5, 6, and 7 cover advanced material. In Chapter 2 we find Hamilton's principle, generalized coordinates, Lagrange's equations, non-holonomic and nonconservative systems, and a few pages on relativistic dynamics. To the reviewer this chapter seems misplaced, and should follow Chapter 4. Chapter 5 contains essentially the standard treatment of normal coordinates. A discussion of the stability of the linear system is found here, a rather novel feature in a book of this type. Chapters 6 and 7 get to the heart of the matter for the physicist, taking up Hamilton's equation, phase space, the Liouville theorem, canonical transformations, the Hamilton-Jacobi equation, and action and angle variables.

Criticisms are the following: The discussion of virtual work could be clarified. The notions of degrees of freedom and constraints of a system should be explained more fully before this principle is discussed. The definition of a position of stable equilibrium on page 46 is not clearly stated—at least for the mathematically inclined. On page 109, the author appears to use the terms moment and couple without distinction. From the discussion of the inertia tensor, it may be inferred (correctly) that this tensor depends upon the choice of origin of the coordinate system with respect to which the tensor is calculated. Since the parallel axis theorem is not mentioned, the important point cannot be made that it is really unnecessary to consider inertia tensors other than those associated with the center of mass. As the author admits in the preface, Chapter 5 is rendered somewhat cumbersome by the failure to use the matrix theory and notation.

The typography of the book is up to the usual high standard of its publishers; in particular, the figures are superb. Numerous exercises are given at the end of each chapter. Some of these are the well-worn academic standards, but a considérable number will be of real interest to the physicist and engineer.

This volume should serve very well as the basis for a terminal course in mechanics for the undergraduate engineering or mathematics student, and for a course prefatory to graduate work in modern physics.

WILLIAM H. PELL
National Bureau of Standards

How to Take a Chance. By Darrell Huff and Irving Geis. Norton, New York, 1959. 173 pp. $2.95.

This book is in the same light style as its predecessor *How to Lie with Statistics.* The illustrations are again by Irving Geiss and add a clever touch for the reader's enjoyment.

This new book presents a popularized approach to some applications of probability theory. In it the author dwells on points of a deeper nature than he did in his previous book. In fact, he touches on Type I errors, variances, confidence limits, sample size, validity of inferences, randomization and quite a number of others. They are, of course, not discussed in this terminology; indeed, a layman would never realize that he was being exposed to some of the basic concepts of statistical theory. Few of us could write in this style.

Specific applications of probability theory in this book are bridge, poker, molecular motion, game theory, E.S.P.—to mention just a few. The book also has quite a number of references to people in the various fields and consequently serves to acquaint the reader with some of the outstanding names.

Like all books on probability theory, it contains a section on problems and puzzles.

Do not expect a treatise on probability theory, but do read this book; you will enjoy it.

W. H. Williams
McMaster University

Abelian Groups. By L. Fuchs. Publishing House of the Hungarian Academy of Sciences, Budapest, 1958. 367 pp. $9.00.

The combination in one book of treatise and textbook is not a new phenomena in mathematical publishing. However, rarely has it been done with such felicity and success as in Fuch's book. Starting with the fundamental ideas, the book carries the beginning student through all the principal areas such as cyclic groups, divisible groups, pure subgroups, torsion and torsion-free groups, etc. At the same time—and particularly in the latter half of the book—the specialist will find an almost exhaustive treatment of current research in Abelian groups, *e.g.* the structure of additive groups of rings and multiplicative groups of fields.

The treatment throughout is detailed and extremely clear. The subject matter is so comprehensive that it might be advisable for teachers to point out to a beginning student those parts that might best be omitted on a first reading lest he be lost in too much detail. One of the happiest features of the book is the abundance of exercises and problems at the end of each chapter. The exercises are never trivial and range from the fairly direct to the quite complicated. The problems (86 in all) are all problems which were unsolved as of the writing of this book. This book should do a great deal to stimulate interest in these problems. A thorough bibliography is given. Finally, the book seems to be free from any major misprints.

A listing of the subject matter covered is as follows: Basic concepts; Direct sums of cyclic groups; Divisible groups; Direct summands and pure subgroups; Basic subgroups; The structure of p-groups; Torsion-free groups; Mixed groups; Homomorphism groups and Endomorphism rings; Group extensions; Tensor products; The additive group of rings; The multiplicative group of fields; The lattice of subgroups; Decompositions into direct sums of subsets; Miscellaneous questions.

E. H. BATHO
University of New Hampshire

Homotopy Theory. By S. T. Hu. Academic Press, New York. 1959, xiii+347 pp., $11.00.

With the exception of Hilton's *An Introduction to Homotopy Theory*, a part of Steenrod's *Topology of Fibre Bundles*, some seminar and course notes, and a few papers that contain expository material, there has been almost nothing available to provide an adequate and up-to-date background in one of the most vigorous fields of algebraic topology. Hilton's and Steenrod's books are somewhat condensed and have many gaps, whereas the notes and papers are usually quite specialized in their treatment of homotopy theory. Because of this, Professor Hu's book fills a definite need and is more than welcome.

Presupposing the spirit and much of the material of Eilenberg-Steenrod's *Foundations of Algebraic Topology*, the book gives a readable and modern introduction to homotopy theory. The first two chapters introduce the main problems of homotopy theory and illustrate their treatment. Fibre spaces and mapping spaces are discussed in Chapter III and the homotopy groups are defined geometrically and also axiomatized in Chapter IV. The elementary exact sequences and techniques of computing homotopy groups is discussed in Chapter V. Obstruction theory is considered in the next chapter and the cohomotopy groups in Chapter VII. The next two chapters are devoted to spectral sequences and their applications in homotopy theory and Chapter V discusses classes of abelian groups and the generalized Hurewicz theorem. The last chapter is devoted to the computation of some of the homotopy groups of spheres.

A noteworthy feature of the book is the numerous exercises. A few are exercises in the usual sense but most serve as a supplement to the text and a guide to the journal literature. Thus, for example, Steenrod squares, Eilenberg-MacLane complexes, Kan complexes, the generalizations of obstruction theory (developed in the text for finite cell complexes only) and other topics are outlined—with references—in the exercises.

Although some topics are omitted entirely, the book provides a good foundation for their study in the literature. On the whole, the book not only fills a need, it does so very well.

JOHN B. GIEVER
New Mexico State University

Statistical Estimates and Transformed Beta-Variables. By Gunnar Blom. Wiley, New York, 1959. 176 pp. $5.00.

This book presents certain results in the theory of transformed beta-variables, which are a generalization of the notion of order statistics. General properties of transformed beta-variables are discussed, including approximations to the first two moments and the asymptotic distribution of a linear combination. The construction of linear estimates of location and scale parameters in any continuous distribution is studied and a method is presented for finding "nearly best" linear estimates of these parameters.

The first two chapters present a theory of estimation generalized to cover asymptotic minimum variances of estimates for all continuous distributions. This general theory includes the Cramer-Rao inequality as a special case.

The next six chapters discuss moments and probability distributions for transformed beta-variables. Such variables include order statistics and the transformation of the binomial and related distributions to others which may be easier to handle, e.g. the normal distribution, as special cases. Means and variances are approximated for "nonsingular" and "singular" transformed beta-variables, and finally for linear combinations of these variables.

Chapters 9 through 14 discuss applications of the theory of transformed beta-variables to the theory of linear estimation. Here, the notion of "nearly-best" linear estimates is used. Comparisons are made with other methods of estimation. A discussion is given of the connection of these ideas with the problems of plotting points on probability papers.

Dr. Blom's book brings together known results in these fields, adds many results of his own and points out relationships among these ideas not generally realized. It is a useful addition to the reference literature of both order statistics and the theory of estimation.

PAUL D. MINTON
Southern Methodist University

The World of Ki. By John D. Goodell, Riverside Research Press, Greenwich, Conn., 1957. 215 pp. $1.00.

This is a curious and interesting combination of two books. One deals with the game of Go, originally Chinese (wei-chi) and later adopted by the Japanese. The other is a loosely connected sequence of essays on games and game-playing, treating such topics as thought, the origin of board games, games played by machines, special properties of games, game theory, the logical invention of games, complete information games, elements of luck, and possible applications of games to learning and teaching.

Go is a board game with very simple rules which can be learned in ten minutes, but is complex and subtle enough to present a considerable challenge to the highest intelligence. In this respect it compares very favorably with chess, and many devotees of both games would regard it as far superior. For the beginner,

this book has certain virtues, such as a clear exposition of the rules, given in several forms, and a discussion of many of the troublesome questions often raised in the early stages of learning. On the other hand, a disproportionate amount of attention and space is given to the relatively rare situations which may arise. The author might well have used some of this space to give the scores of several typical games together with annotations. While it is true that no one will learn to play well just by reading this—or any—book, the beginner would have been helped more by concrete examples than by wide generalizations. Also, assuming that the reader has become sufficiently interested in Go to wish to pursue it further, a reference to the American Go Association (96 Cedar Avenue, Hackensack, N. J.) would be appropriate. Equipment and books can be obtained through the Association. There is an excellent book for beginners by Takagawa (in English).

One may disagree with the author about some of his comments in the general sections, but they are always thought-provoking. Scattered throughout are descriptions of various esoteric games. One in particular caught this reviewer's fancy, the one described on pages 169 and 170 (called tawlbrdd in Wales, tablut in Lapland). It appears to be extremely interesting and worth trying. Anyone for tawlbrdd?

N. J. FINE
The Institute for Advanced Study

Lectures on Fourier Integrals. By Salomon Bochner. (Translated by Morris Tenenbaum and Harry Pollard.) Ann. of Math. Studies No. 42, Princeton University Press, Princeton, 1959. 333 pp. $5.00.

This edition of Bochner's well-known book is a faithful translation into English of the original (1932) German edition. A translation of the author's supplement on Monotone Functions, Stieltjes Integrals and Harmonic Analysis, originally published in German [Mathematische Annalen, vol. 108, 1933, pp. 378–410] appears as an appendix in this edition.

The material covered in the body of the book is indicated by the chapter headings: Basic Properties of Trigonometric Integrals, Representation and Sum Formulas, The Fourier Integral Theorem, Stieltjes Integrals, Operations with Functions of Class $\mathfrak{F}_0$, Generalized Trigonometric Integrals, Analytic and Harmonic Functions, Quadratic Integrability, and Functions of Several Variables.

The typing and its reproduction are both done well.

RUEL V. CHURCHILL
University of Michigan

Testing Statistical Hypotheses. By E. L. Lehmann. Wiley, New York, 1959. xiii+369 pp. $11.00.

Lehmann presents the testing of hypotheses in the framework of Neyman-Pearson theory. The closely related problem of confidence intervals is also in-

cluded. The author deals only briefly with the modern theory of statistical decisions and points out from time to time the resultant restrictions of his approach.

The book begins with a discussion of the general decision framework and basic probability theory. The next chapter deals with uniformly most powerful tests, the Neyman-Pearson fundamental lemma, and includes a section on sequential probability ratio tests. Next, come chapters on unbiasedness, invariance, linear hypotheses and finally, the minimax principle.

Measure theory is a requirement of the book and, strictly speaking, *no* knowledge of statistics is needed. However, in general, it would not be advisable to attempt to read this book without some background of statistical theory. Indeed, Lehmann recognizes that most readers will have such a background and as a result he has not included numerical illustrations.

This book would be excellent for an advanced text. It is clearly and concisely presented and has ample good problems for the student's consideration. The problems are divided into sections which correspond to the sections of the chapter.

The author has taken the very worthwhile effort to annotate most of the references cited. Consequently, this bibliography can be used as an excellent, accessible guide to further reading on the various topics.

This book is a significant contribution to statistical literature and will no doubt be one of the standard works in the area.

W. H. WILLIAMS
McMaster University

BRIEF MENTION

A Source Book in Mathematics. (Two volumes.) By David Eugene Smith. Dover, New York, 1959. xiii+701 pp. $1.85 each or $3.50 set.

This reprint of Smith's well-known 1929 book needs no review other than an indication of its availability.

A Course in Mathematical Analysis. (Three volumes.) By Edouard Goursat, translated by E. R. Hedrick. Dover, New York, 1960. 548+300+259 pp. $5.00 set.

The reprinting of the Hedrick and Dunkel translations of Goursat's work from the early 1900's should make these volumes easily available to many. Included in the series are: Volume I, Applications to Geometry, Expansion in Series, Definite Integrals, Derivatives and Differentials. Volume II, Part I, Functions of a Complex Variable. Volume II, Part II, Differential Equations.

Functions of a Complex Variable. The Theory of Functions of Real Variables, Volumes I and II. By James Pierpont. Dover, New York, 1960. 583+560+645 pp. $2.45 each.

Three additional early twentieth-century books which have been out of print for some time.

Mathematics: A House Built on Sand? By Ivan Niven. University of Oregon Books, Eugene, Oregon, 1959. 38 pp. $1.00.

This transcription of a lecture delivered at the University of Oregon could well be in every library, public and private. It provides an excellent popular (*i.e.*, nonmathematical) discussion of a series of important events in mathematical thought.

Ramanujan, 12 Lectures on Subjects Suggested By His Life and Work. By G. H. Hardy. Chelsea, New York, 1959. 236 pp. $3.95.

This Chelsea reprint of Hardy's 1940 work on the famous Indian number theorist, Ramanujan, will be welcomed by number theorists everywhere.

Business Mathematics, Exercises, Problems and Tests. By R. Robert Rosenberg. Gregg Publishing Division, McGraw-Hill, New York, 1959. 218 pp. $2.20.

It seems inconceivable that, when the *Harvard Business Review*, *Fortune*, and *Higher Education for Business* are advocating that business students be given some real mathematics, even calculus and beyond, a book such as this could appear. It is strictly a workbook in simple arithmetic. The last few pages contain a two-part final examination, the first part of which is mostly arithmetical. Section 2 of the final examination contains similar arithmetic, but is well loaded with business terms. Typical Problem: "A man sold for $9.00 a cigarette lighter that cost him $7.50. The markup was what part of the retail." (sic)

Hydrodynamics. By D. H. Wilson. St. Martin's Press, New York, 1959. viii+149 pp. $5.50.

A concise introduction to "classical" hydrodynamics based upon vector theory, which is assumed.

Probability and Statistics. The Harald Cramér Volume. Edited by Ulf Grenander. Wiley, New York, 1959. 434 pp. $12.50.

Those who prefer a more mathematical version of statistics may value this book of studies presented to Harald Cramér in honor of his 65th birthday. The authors of its 19 articles include some of the best-known mathematicians and mathematical statisticians in the country. Most of these articles will undoubtedly be reviewed in *Mathematical Reviews*, hence no extensive review is anticipated here.

Introduction to Theoretical Meteorology. By Seymour L. Hess. Holt, Rinehart and Winston, New York, 1959. xiv+362 pp. $8.50.

An excellent text in modern dynamic meteorology. The preface claims that no mathematical demands other than a thorough knowledge of differential and integral calculus are made; however, the author is apparently well-grounded in mathematics and uses partial differential equations freely. How many of our readers can derive formulas to determine how fast a billiard ball 2 inches in diameter must be propelled directly at another billiard ball 102 inches away in order that it shall just barely miss striking the second ball, due to Coriolis force, if the table is located at 43° North latitude? (See problem 6, page 174).

Introduction to Mathematical Thinking. By Friedrich Waismann. Harper, New York, 1959. $x+260$ pp. $1.40.

This is a new printing of Theodore J. Benac's translation of Waismann's 1951 work. The purpose of the book is "to give an insight into the nature of mathematical concept formation, that is, to point out the activities of the mathematician that might be of interest to a philosophically minded observer."

1960 Catalog—Educational Motion Pictures. Audio-Visual Center. Indiana University, Bloomington, Indiana, 1960. 664 pp.

This appears to be a reasonably careful compilation of 16 mm educational motion pictures. By checking through various portions of the subject matter index as suggested under the general title "Mathematics" a fairly extensive list of mathematical films can be compiled; a useful reference in these days of urgent requests from high school teachers on how to spend Title III money of the Defense Education Act. This may be an appropriate time to again mention that supplementary library books in mathematics for the enrichment of high school curriculum are also approved for purchase under this act.

Qualifications and Teaching Loads of Mathematics and Science Teachers, in Maryland, New Jersey, and Virginia. By Kenneth E. Brown and Ellsworth S. Obourn. U. S. Department of Health, Education, and Welfare Circular No. 575. U. S. Government Printing Office, Washington, D. C., 1959. viii+101 pp. 70¢

A careful compilation of statistics from the three states named, to determine the background preparation, teaching load and teaching combinations of mathematics and science teachers. This study could well serve as a pattern for similar studies in other states.

Basics of Digital Computers. (Three volumes.) By John S. Murphy. John F. Rider, New York, 1958. 116+133+136 pp. (soft covers) $2.80 each; three volumes in a single cloth binding, $9.50.

An easy-to-read "picture book" which provides in three volumes an excellent introduction to some of the basic ideas of computer hardware, both past and present.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Butler University: Assistant Professor Juna L. Beal retired June 6, 1960; Mr. Robert Lamberson and Mrs. Joy F. Martin have been appointed Instructors.

College of the Holy Cross: The Reverend R. J. Swords has been appointed President of Holy Cross; Professor V. O. McBrien has been appointed Chairman of the Department of Mathematics; Professor J. A. Nestor will serve as Acting Chairman during the academic year 1960–61; Dr. D. L. McQuillan, Johns Hopkins University, has been appointed Assistant Professor; Assistant Professor D. G. Dewey, Aquinas College, has been appointed Assistant Professor.

Cornell University: Dr. Zbigniew Ciesielski, Assistant in Mathematics, University of Pozan, has been appointed Instructor; Dr. R. C. Curjel, Research Assistant, Swiss Federal Institute, has been appointed Instructor; Assistant Professor R. V. Chacon, University of Wisconsin, has been appointed Visiting Assistant Professor ; Dr. Leonard Gross, Yale University, has been appointed Assistant Professor; Assistant Professor S. V. Parter, University of Indiana, has been appointed Assistant Professor of Mathematics

1960 Preliminary Actuarial Examinations are as follows:

FIRST PRIZE OF $200

Gitlin, Todd A.	Harvard University

ADDITIONAL PRIZES OF $100 EACH

Emerson, William R.	California Institute of Technology
Goodman, Richard H.	Harvard University
Landman, Maurice A.	Harvard University
Lorden, Gary A.	California Institute of Technology
McDonnell, Robert N.	University of Chicago
Newmeyer, John A.	California Institute of Technology
Sampson, Schuyler S.	Bowdoin College
Shulsky, Abram N.	Cornell University

The Society of Actuaries has authorized a similar set of nine prizes for 1961.

Beginning in 1961, the Preliminary Actuarial Examinations will consist of two examinations: The General Mathematics Examination (based on the first two years of college mathematics), and The Probability and Statistics Examination.

The 1961 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service under the direction of a committee of actuaries and mathematicians, and will be administered by the Society of Actuaries at centers throughout the United States and Canada on November 16, 1960 and May 10, 1961. The closing date for applications for the May examination is April 1, 1961.

Further information concerning these Examinations can be obtained from the Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 112 persons have been elected to membership by the Board of Governors on applications duly certified.

GERALD E. ABRAHMS, B.A. (Colorado) Seismic Computer, The Atlantic Refining Company

ALBERT S. ANDERSON III, Student, Emory University

DALE E. ANDERSON, Student, University of Nebraska

WILLIAM N. ANDERSON, JR., Student, Carnegie Institute of Technology

JACK L. BALDWIN, Student, Southwest Missouri State College

MARILYN MAE BALDWIN, Student, duPont Manual High School, Louisville, Kentucky

MICHEL L. BALINSKI, Ph.D. (Princeton) Instr., Princeton University; Consultant, Mathematica

ROBERT P. BALLES, Student, Fordham University

JAMES M. BARKO, Student, St. Vincent College

HAROLD M. BELL, B.S. (Kentucky) Grad. Student, Eastern Kentucky State College

RUDOLPH J. W. BERAN, Student, St. Michael's College School

JOSEPH B. BOHAC, B.S. (St. Louis) Engr., Horner and Shifrin Consulting Engineers

DON LOREN BOHMONT, B.S. (Nebraska) Teaching Asst., Colorado State University

HOWARD I. BOWN, B.S. (Beloit Coll.) Grad. Student, Beloit College

DENNISON R. BROWN, B. S. (Duke) Asso. Math., Louisiana State University

EUGENE B. BRUNNER, Student, Indiana State College

ROBERT S. BURDICK (M.I.T.) Teacher, Iolani School, Honolulu, Hawaii

EDWARD A. BURFINE, Student, California State Polytechnic College

PHILIP R. CARLSON, B.S. (Minnesota) Grad. Student, University of Minnesota

RUU-KWANG CHANG, M.S. (Colorado) Teaching Asst., Colorado State University

WARREN E. CHASE, B.S. (Franklin and Marshall Coll.) Grad. Student, Franklin and Marshall College

FRANKLIN D. CHEEK, Student, University of Wisconsin

RUDOLPH J. CHERKAUER, Ed.D. (Buffalo) Prof., State University of New York, College of Education, at Buffalo

HERMAN CHERNOFF, Ph.D. (Brown) Prof., Stanford University

ROBERT E. CLAPP, Student, Reed College

MRS. EMAGENE W. CLOTHIER, B.S. (Marquette) Teacher, Divine Savior High School, Milwaukee, Wisconsin

LEONARD COHEN, Ph.D. (Columbia) Instr., College of the City of New York

GORDON L. COLE, M.A. (Baltimore) Asst. Prof., Indiana Technical College

CRAIG COMSTOCK, B.E.A. (Cornell) Instr., U. S. Naval Postgraduate School, Monterey, California
JOHN S. CONNER, B.S. (U.S. Military Acad.) Grad. Student, Monterey Peninsula College
DONALD W. COUNSELLOR, Student, Rutgers College of South Jersey
WILLIAM C. CUTLER, B.S. (Sanford) Menlo Park, California
CARL P. DEAN, B.S. (Oklahoma) Instr., University of Oklahoma
ROBERT E. DODGE, JR., M.A. (Missouri) Instr., University of Missouri
JOANNA M. ENZMANN, B.S. (M.I.T.) Engr., Sylvania Electric Products
VOIGT M. EPTING, M.S. (S. Carolina) Superintendent, Pacolet Schools, South Carolina
LLOYD R. ERICKSON, JR., Student, Rice University
BRUCE FELDMEYER, Student, Sir Francis Drake High School, San Anselmo, California
MARIE FEORANZ, M.A. (N.Y.U.) Teacher, Lodi High School, New Jersey
JOSEPH C. FERRAR, Student, Michigan State University
SHERIDAN K. FERRIERA, A.A. (Sacramento Jr. Coll.) Math. Analyst, Aerojet-General Corp.
OSBURN R. FLENER, M.S. (Illinois) Teacher, Western Kentucky State College
HAROLD W. FLESNER, B.S. (S. Dakota) Grad. Student, South Dakota State College
MRS. MARIAN E. FLESNER, M.S. (S. Dakota) Instr., South Dakota State College
THOMAS W. FORGET, B.S. (St. Johns) Research Engr., Lockheed
JOHN H. FOX, JR., B.A. (U.C.L.A.) Math. Consultant, The Martin Company
JOSEPH A. FRISBIE, A.S. (Danbury) Lab. Tech., Barden Corporation
SEYMOUR GOLDBERG, Ph.D. (California) Asst. Prof., New Mexico State University
RICHARD L. GOODRICH, A.B. (Harpur) Programmer, I.B.M., Owego
BERNICE M. GORDON, M.A. (Minnesota) Classroom Consultant, Edina-Morningside Public Schools, Minnesota
ROBERT B. GRAFTON, Sc.B. (Brown) Lt. J.G., U. S. Navy
PRANAS A. GRUSAS, M.A. (Syracuse) Staff Mathematician, I.B.M., Bethesda
JAMES E. HARDIN, M.Ed. (Trinity) Teacher, MacArthur High School, San Antonio, Texas
JAMES F. HAWLEY, B.A. (Kansas) Grad. Asst., Kansas State College
ROBERT W. HEATH, Ph.D. (N. Carolina) Asst. Prof., University of Georgia
JOYCE HOCHSTEIN, Student, Hunter College
RONALD L. HOFFMANN, Student, duPont Manual High School, Louisville, Kentucky
FREDRICK A. HOPPERSTEAD, Student, North Texas State College
MRS. LUCILLE A. HORTON, M.A. (N.Y.S. Teachers Coll.) Head of Dept., Tri-Valley Central School, Grahamsville, New York
GERALD A. HUTCHISON, B.S. (Creighton) Teaching Asst., University of California
JENS A. JENSEN, M.S. (Wyoming) Instr., Iowa State Teachers College
DANIEL R. JESSUP, Student, Kent State University
RICHARD R. KARCH, Student, Case Institute of Technology
TERENCE J. KELLY, Student, University of Maryland
JEROME R. KIRK, Student, Reed College
PAUL W. KLIPFEL, B.S. (Ohio) Teacher, A. P. Giannini Junior High School, San Francisco California
DAVID H. KNOWLES, M.A. (San Jose) Dept. Chairman, Samuel Ayer High School, Milpitas, California
NICOLAS KONOPLIV, B.M.E. (Minnesota) Teaching Asst., University of Minnesota
FRED KRAKOWSKI, D. Sc. (Swiss Fed.) Asst. Prof., University of California, Davis
RONALD K. LACEY, Student, California State Polytechnical College
JOHN D. LEADLEY, M.S. (Washington) Asst. Prof., Reed College
PAUL AARON LEE, M.B. (Mukden) Medical Resident, St. Joseph's Hospital, Bronx
CONSTANCE J. LEUER, Student, Beloit College
PAUL J. LOATMAN, B.S. (Siena) Chief, Computer Unit, Watervliet Arsenal
FREDERICK W. LONG, Student, Boston Technical High School, Massachusetts
ROBERT A. LOOP, B.S. (Marquette) Teacher, Wrightstown High School, Wisconsin
RICHARD S. LYNN, M.A. (Long Beach State Coll.) Programmer, North American Aviation, Inc.
JANE MATTHEWS, M.A. (Columbia) Teaching Fellow, Hunter College
WARREN B. MCCLANAHAN, Student, Alabama College
IRVING MODER, B.A. (George Washington) Sr. Economist, Council for Economic and Industry Research, Inc.
DORLIN J. MOELLER, B.A. (Wartburg) Teacher, Allison-Bristow Common School District, Allison, Iowa
CLAY NICHOLS, Student, University of Tulsa
KAZUO OGAWA, B.S. (California Poly.) Student Teaching Asst., California State Polytechnical College
HEBERTO PACHON, B.S. in E.E. (Illinois) Engr. Asst., The Cleveland Electric Illuminating Company
HAROLD D. PENDERGRAFT, Student, University of Oklahoma
CARL A. PERSINGER, Student, Virginia Polytechnical Institute
MRS. CHARLOTTE E. PICKETT, B.S. (Auburn) Asso. Engr., Boeing Airplane Company
DEMETRIOS POROS, diploma (Athens) Grad. Student, Boston University
WALTER K. RAPP, B.S. (Missouri) Grad. Student, University of Missouri
WINSTON RILEY III, B.A. (Swarthmore) Mathematician, Council for Economic and Industry Research, Inc.
CHARLES A. RITCHER, B.A. (Bellarmine Coll.) Teacher, Catholic Country Day School, Louisville, Kentucky
CECIL T. RODNEY, D.E. (Buffalo) Prof., State University College of Education at Buffalo
WILLIAM H. ROW, JR., Student, Knox College
N. JAMES SCHOONMAKER, Ph.D. (Pittsburgh) Prof. and Chairman of Dept., University of Vermont
SARAH SCHWARTZ, B.A. (Hunter) Teacher, Rabbi Jacob Joseph School, New York City, New York
RICHARD M. SHANK, A.B. (U.S.C.) Analyst, Hughes Aircraft
JAMES R. SILVA, Ed. M. (Harvard Head of Dept., Pelham High School, New York
ANNE T. SILVERSTEIN, M.A. (Columbia) Brooklyn, New York
WILLIAM S. SIMS, B.Sc. (Wales) Instr., Ryerson Institute of Technology
SISTER MARY JOHN M.S. (Marquette) Teacher, Father Ryan High School, Nashville, Tennessee
GASTON SMITH, M.A. (Alabama) Asst. Prof., Mississippi Southern College
CHARLES H. TAFT, M.S. (Michigan) Branch Manager, I.B.M., Corpus Christi
DENNIS D. THUTE, Student, University of Nebraska
ALAN F. TURNER, B.A. (Yale) Editor, Holt, Rinehart and Winston
GEORGE A. VANBUSKIRK, Student, University of California, Riverside
ROBERT C. VOWELS, M.A. (Howard) Instr., Howard University
OLEN J. WARD, JR., Student, Jacksonville, University
SETH L. WARNER, Ph.D. (Harvard) Asst. Prof., Duke University
LONNIE WHITEHEAD, JR., B.A. (Emory) Statistician, Emory University
FREDERICK W. WILKE, M.A. (Washington) Instr., University of Missouri
LAWRENCE A. WOLD, B.A. (Connecticut) Teaching Asst., University of Washington
WINSTON L. WONG, B.A. (San Jose State) Grad. Student, San Jose State College

THE NOVEMBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held on November 14, 1959 at Concordia College in Moorehead, Minnesota. Professor Jerry Heuer of Concordia College, Section Chairman, presided at the

morning session. Professor Sigurd Mundhjeld of Concordia College presided at the afternoon session. There were 50 people registered for the meeting, including 37 members of the Association.

During the business meeting the secretary read a report from the Minnesota National Laboratory for the Improvement of Secondary School Mathematics. The report discussed the past and future plans of the Laboratory which is under the direction of Professor P. C. Rosenbloom.

The following papers were presented:

1. *Estimation of correlation coefficients from scattergrams*, by Professor C. R. Perisho, Mankato State College.

If a scattergram is plotted from a bivariate distribution, the area covered by the points is, in general, elliptical. The correlation coefficient can be estimated from the shape of any of the elliptical contour lines. A formula is developed which gives the correlation coefficient if the angle of inclination of the major axis and the ratio of the major and minor axes of the ellipse are known.

2. *The generalized tower of Hanoi problem*, by Mr. Don Olivier, Carleton College, introduced by Professor F. L. Wolf, Carleton College.

The classical problem (a number of disks of different sizes are stacked on one of three needles; they must be transferred to another by a series of moves where only one disk may be moved at a time and a larger disk may never rest on a smaller disk) is generalized to use m disks and n needles. On the basis of a preliminary simplifying assumption (which is plausible but has yet to be proved) a formula for the minimum number of moves required to effect the transfer is derived and proved inductively.

3. *Euler's conjecture*, by Dr. E. T. Parker, Remington-Rand Univac (by invitation of the Executive Committee).

Euler conjectured in 1782 that there exists no pair of orthogonal Latin squares of any even order not a multiple of four. Earlier this year Professors R. C. Bose and S. S. Shirkhande and the speaker constructed pairs of squares of all orders $4t+2 \geqq 10$. (Papers are in Proc. Nat. Acad. Sci., May and June, 1959. Longer papers are pending.) Some history of the problem is presented, and the recent work is summarized. Particular reference is made to the connection with finite projective planes.

4. *Other results on orthogonal Latin squares*, by Professors N. S. Mendelsohn, S. Nathan, A. L. Dulmage, and D. M. Johnson, University of Manitoba.

Using as a basic square a direct product of cyclic squares, a method of constructing pairwise orthogonal Latin squares by means of permutations of rows is obtained. The results lead to larger numbers of squares than that predicted by the now defunct MacNeish conjecture. The squares are quite different from those constructed by Parker, Shrikhande and Bose.

F. L. WOLF, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky on April 30, 1960. Professor W. C. Royster, Chairman of the Section, presided at both the morning and afternoon sessions. Seventy persons attended the meeting, including fifty-seven members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor A. G. Anderson, Western Kentucky State College; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor A. W. Goodman, University of Kentucky.

By invitation, Professor W. J. Thron of the University of Colorado delivered an hour address "*Convergence Criteria for Continued Fractions*" at the afternoon session.

The following papers were presented:

1. *On the applications of B_0-spaces to summability theory*, by Mr. H. R. Coomes, University of Kentucky.

A discussion of the work of Zeller together with some new criteria in order that $B(AX) = BA(X)$ for matrices A and B and sequences X.

2. *A note on the convergence of* $1+\sum_{n=0}^{\infty}\binom{m}{n}x^n$ *for* $x=\pm 1$, by Professor Aughtum Howard, Eastern Kentucky State College.

The author gives an elementary proof of the convergence of the binomial series $1+\sum_{n=0}^{\infty}(-1)^n\binom{m}{n}$, $0<m<1$, and of $1+\sum_{n=0}^{\infty}\binom{m}{n}$, $-1<m<0$.

3. *On the fundamental theorem of linear equivalences in kinematic relativity*, by Professor W. S. Krogdahl, University of Kentucky.

4. *On half-groups*, by Professor Frank Levin, University of Kentucky.

5. *Concerning the Euler differential equation*, by Miss Helen S. Weihe, University of Kentucky.

This talk was concerned with solving the Euler equation by means of the substitution $y=(ax+b)^m$.

6. *On the Visiting Lecturer Program*, by Professor T. J. Pignani, University of Kentucky.

7. *Revisions in the mathematics curriculum at Western Kentucky State College*, by Professor A. G. Anderson, Western Kentucky State College.

Changes in the mathematics offerings at Western Kentucky State College, to become effective next fall, were outlined. Emphasis is on providing separate programs at the first year level for students majoring and minoring in mathematics and/or science and the group categorized under the general heading "non-science" students. At the junior-senior level, a similar separation is made to provide distinct programs meeting the needs of the group preparing for positions in secondary education and those having as objectives positions in business or industry or further training in mathematics at the graduate level.

8. *Generalization of an ordered group*, by Mr. W. K. Cope, University of Kentucky.

V. F. COWLING, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-seventh annual spring meeting of the Indiana Section of the Mathematical Association of America was held Saturday, May 7, 1960 at Earlham College, Richmond, Indiana. Approximately 50 members of the Association attended. Professor Kermit Carlson of Valparaiso University, Chairman of the Section, presided at both the morning and afternoon sessions.

The following officers were elected: Chairman, Professor M. E. Shanks, Purdue University; Vice-Chairman, Professor J. E. Yarnelle, Hanover College; Secretary-Treasurer, Professor P. T. Mielke, Wabash College.

Professor J. C. Polley reported on the activities of the State School and College Committee for Mathematics. This is a joint Committee of the Indiana Section of the Association, the Indiana Academy of Science, and the Indiana Council of Teachers of Mathematics. The purpose of the committee is to foster cooperation between school and college mathematics teachers and the State Department of Public Instruction. It was announced that a grant had been received from the National Science Foundation to support committee activities for the coming year.

A report on the National Test sponsored by the Association and the Actuarial Society, prepared by Professor Jack Forbes, was presented. It was decided that next year the Indiana School and College Committee would sponsor this test within the state.

Professor R. H. Bing of the University of Wisconsin presented the guest lecture on *Recent Developments in Euclidean Topology.*

The following short papers were presented:

1. *The Earlham statistics program*, by Professors Roland Smith and Howard Alexander, Earlham College.

2. *Trends in teacher education*, by Professor Charles Brumfiel, Ball State Teacher's College.

3. *The additive group of real numbers*, by Professor Johannes DeGroot, Purdue University.

4. *Approximate solutions of the Dirichlet problem*, by Professor Donald Greenspan, Purdue University.

5. *On repeated integrals*, by Professor J. H. B. Kemperman, Purdue University.

6. *The recursive relation in Pascal's triangle*, by Professor Dale Mesner, Purdue University.

CHARLES BRUMFIEL, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual spring meeting of the Minnesota Section of the Mathematical Association of America was held on May 7, 1960 at St. Olaf College in Northfield, Minnesota. Professor C. S. Carlson of St. Olaf College presided at the morning session. The Section Chairman, Professor Jerry Heuer of Concordia College, presided at the afternoon session. There were 68 registered for the meeting, of whom 55 were members of the Association.

At the business meeting Professor Seymour Schuster of Carleton College reported on the 1960 Minnesota High School Mathematics Contest which was sponsored, in part, by the Minnesota Section. 261 high schools and 9,655 students participated in the contest this year. The thanks of the section were expressed to Professor Schuster and his committee for doing an excellent job in handling the contest.

It was suggested that it would be helpful to the colleges and high schools in Minnesota if the Association discovered and published the natures of the available mathematics programs at the colleges and universities in the state. It was moved and passed that a committee be appointed to study this matter.

The motion was made and passed to accept with thanks a donation made to the section for the purpose of purchasing name tags to be used at section meetings. The donation was made by Dr. E. J. Parker of Remington-Rand Univac.

The following officers were elected to serve during the school year of 1960–61: Chairman, Professor Fulton Koehler of the University of Minnesota; Secretary-Treasurer, Professor Murray Braden of Macalaster College; Members of the Executive Committee, Professor O. E. Stanaitis of St. Olaf College and Professor Warren Thomsen of Mankato State College.

The following papers were presented:

1. *Density theorems for sets of integers, and a generalization*, by Professor William Moser, University of Manitoba. (by invitation of the Executive Committee).

Let $f(x)$ be defined, integrable and $0 \leqq f(x) \leqq 1$ on $[0, 1]$; $g(x) \equiv 1 - f(x)$ on $[0, 1]$; $f(x) \equiv g(x) \equiv 0$ outside $[0, 1]$. Furthermore let $\int_0^1 f(x)dx \geqq \alpha t$, $0 \leqq t \leqq 1$, α a fixed real positive number, $0 < \alpha < 1$. Let $D(t) = \int_0^1 f(x)g(x+t)dx$ and $D = \inf_{f(x)} \sup_t D(t)$. Bounds for D are found. The methods used and the results obtained are generalizations of similar results regarding the density of the sum of two sequences of integers (See Kasch, Math. 3 vol. 62, 1955 pp. 368–387). Related problems are discussed and a brief outline of the history of these problems is given.

2. *An obvious—but untrue—algebraic result related to geometry*, by Professor Seymour Schuster, Miss Alison Krotter, and Mr. Don Olivier, Carleton College.

A paper, *The lines and planes connecting the points of a finite set*, by T. Motzkin (Trans. Amer. Math. Soc., 1951), contained the following statement: "Let $\{a\}=\{b\}=\{c\}$ be finite subsets of field F without 0 such that every product ab is in $\{c\}$, every bc in $\{a\}$, every ca in $\{b\}$; a simple computation shows that one of them, say $\{a\}$, consists of the m mth roots of 1 for some m, and either $\{b\}=\{c\}=\{a\}$ or $\{b\}=\{c\}=-\{a\}$." The authors, believing that no really "simple computation" exists, exhibit two proofs of the first part of Motzkin's assertion, and show that the second part is true only if m is odd. If m is even, then either $\{b\}=\{c\}=\{a\}$ or $\{b\}=\{c\}=ae^{\pi i/n}$)

3. *On the definition of surface area*, by Professor James Serrin, University of Minnesota.

It is a common opinion that the well-known formula for surface area cannot be proved rigorously except by extremely delicate arguments. In consequence, calculus textbooks tend to give only heuristic justifications of this formula. Beginning with Lebesgue's definition of area by means of approximating polyhedral surfaces, however, it is actually not difficult to prove that the area of a piecewise continuously differentiable surface $z=f(x, y)$ is given by the usual formula. Indeed, this fact follows at once from a simple inequality which (roughly speaking) asserts that averaging is an area-shrinking operation.

F. L. WOLF, *Secretary*

THE MAY MEETING OF THE OHIO SECTION

The forty-fourth annual meeting of the Ohio Section of the Mathematical Association of America was held at Kent State University, Kent, Ohio, on Saturday, May 7, 1960. Professor W. R. Van Voorhis, Chairman of the Section, presided at the morning and afternoon sessions. There were 84 persons registered in attendance, including 71 members of the Association.

Officers elected for the coming year are: Chairman, Professor Wade Ellis, Oberlin College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor Wendell Johnson, Hiram College; Program Committee, Professor C. W. Topp, Fenn College, Chairman, Professor W. E. Restemeyer, University of Cincinnati, and Professor Clarence Heinke, Capital University.

The following papers were presented:

1. *The impact of modern mathematics upon college teachers*, by Professor W. R. Van Voorhis, Fenn College (Chairman's Address).

The growing emphasis upon mathematics in our secondary and college programs calls for critical evaluation of course content, and changes in instructional methods and requirements. Soon course offerings may depart from the conventional listings of specific topics and be presented under broad headings such as mathematical structures, models, processes, transformations, etc. Instructional methods for adequately handling larger numbers of students with more extensive and variable backgrounds will be found by the teachers of mathematics rather than by those trained primarily in methodology. There will be greater emphasis upon independent study under the supervision of experienced instructors with the aid of assistants.

2. *A procedure for optimizing a function involving uncertain parameters*, by S. K. Gupta, Case Institute of Technology.

Exact values of parameters appearing in mathematical decision models are seldom, if ever, known, and must be estimated whenever a given decision function is to be used. Traditional statistical techniques of estimation generally fail to take into account consequences of the effect of overestimation and underestimation of the parameters. Consequently, it is desirable to establish procedures for using estimates in such a way that the "cost of error" resulting from their use is reduced. Pragmatically, then, it is important to find a procedure which will minimize the expected cost of error and/or its variance. The nature of a general solution to this problem is discussed, and a specific method of *adjusting* parameter estimates is developed. In particular, techniques are given for models involving one parameter, and suggestions are given for extending these techniques to the cases of two or more parameters.

3. *Reflections in plane geometry*, by Professor P. B. Yale, Oberlin College.

Elementary congruence and incidence relations between points and lines in the Euclidean plane are characterized by group theoretic relations between the corresponding point and line reflections. Proofs of some theorems of elementary geometry are given using these characterizations.

4. *Remarks on some problems in the American Mathematical Monthly*, by Professor C. F. Pinzka, University of Cincinnati.

Several unsolved problems of an elementary nature were discussed. The birthdays problems, requiring the minimum number of people n such that the probability of a birthday coincidence exceeds $\frac{1}{2}$, is known to have the solution $n=23$. If the problem is extended to the cases of sets of 3, 4, 5 people with the same birthday, then a Poissonian approximation yields the values $n=88$, 187, 314, respectively, the first of which is easily verified by the formula

$$p(n) = 1 - (365!n!/365^n) \sum_{k=0}^{[\frac{1}{2}n]} 1/(2^k k!(n-2k)!(365-n+k)!).$$

5. *Quantum theory of gravitation*, by Professor S. N. Gupta, Wayne State University.

The conflict between Einstein's theory of gravitation and quantum mechanics can be resolved by following a new approach, which is based on the fundamental assumption that the physical space is flat and not curved. It can be shown that Einstein's gravitational field questions can be transformed from curved space to flat space by means of a Lorentz-covariant expansion. It is then possible to quantize the gravitational field, and the resulting theory predicts the existence of a new type of elementary particle called the graviton. The quantum theory of gravitation also throws new light on several physical phenomena.

6. *Mathematics—Is it a prerequisite to a college degree?* by Professor E. A. Tabler, The University of Akron.

Many surveys disclose a lack of precollege accomplishment in the quantitative language of mathematics. One such national survey concludes, "In our elementary and secondary schools, the coverage in mathematics is inadequate and the too little that is required is not well taught." Believing that college graduates need proficiency in mathematical concepts to be encountered, that mathematics is not just a prerequisite to certain other subjects, but a language, prerequisite for being efficient citizens in a modern world, The University of Akron includes Numbers Communication in its long-established General Education Program. Evaluation procedures indicate a needed service is being provided.

7. *Curvature without the second derivative*, by Professor L. C. Graue, Bowling Green State University.

Methods for finding curvatures of plane curves without involving the second derivative are presented. These are applied to extending the notion of curvature to some examples where the second derivative does not exist.

8. *Twelfth-grade mathematics program at Hiram College*, by Professor M. A. Moore, Hiram College.

Hiram College has for two years sponsored a twelfth-grade mathematics program for gifted students of nearby small high schools. The program was supported by the Wean Foundation and by N. S. F. The course blended algebra, analytic geometry, and trigonometry, with about equal emphases on the three subjects. The class met daily on the Hiram campus, the students furnishing their own transportation. Problems that arose involved road conditions, variation among participating schools in holidays and field trips, etc. Both classes maintained a good level of interest and attainment; those who continued into college unanimously report that the course was valuable precollege training.

FOSTER BROOKS, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The sixteenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at the University of Rochester, Rochester, New York, on May 7, 1960. Professor Dis Maly, the Chairman of the Section, presided at the morning session and Professor B. H. Gere, Vice-Chairman of the Section, presided at the afternoon session. There were 122 persons in attendance,including 81 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor B. H. Gere, Hamilton College; Vice-Chairman, Professor D. E. Kibbey, Syracuse University; Secretary-Treasurer, Professor N. G. Gunderson, University of Rochester. The Committee on the Strengthening of Mathematics in the Section was discharged with thanks. Professor Nura Turner, of the State University College of Education at Albany, as Chairman of the Contest Committee, reported another successful contest. She also reported the plans for bringing to Albany on May 14, 1960 for a day of lectures and discussions on Mathematics as a Career the students in the upstate section of New York who placed in the top one percent in the 1960 contest.

The program was as follows:

1. *A problem in probability*, by Professor A. G. Davis, Clarkson College of Technology.

The speaker evaluated the probability that a triangle can be formed with length of sides equal to numbers x_i obtained by drawing from a uniform distribution 0 to 1.

2. *Approximately continuous functions*, by Professor A. G. Fadell, University of Buffalo.

A Lebesgue-measurable subset S of Euclidean n-space E_n is called a *density set* if every point of S is a point of density of S. Then the class of density sets D_n topologizes E_n. Hence, approximately continuous functions become continuous relative to the density set topology. Theorems on continuous functions for general topological spaces therefore apply to approximately continuous functions. A few such theorems are cited.

3. *Lagrange's theorem and stationary functions*, by Mr. R. J. Pegis, Bausch and Lomb Optical Company.

Methods are given for evaluating a function when it is stationary with respect to one or more of its variables. It is shown through an extension of Lagrange's Theorem that we do not need to know the values of the variables which make the function stationary. Applications are indicated, especially in optics.

4. *Methods to treat mathematical problems with symmetry of rotation*, by Dr. M. J. Herzberger, Eastman Kodak Company.

Given two vectors $\vec{b}_1$ and $\vec{b}_2$ and their symmetric functions e_1, e_2, e_3; given a function E of the e_i developed into a power series of the e_i; a method is described for computing the coefficients of E for the symmetric functions of B_1 and B_2 where $\vec{B}_1=\alpha\vec{b}_1+\beta\vec{b}_2$, $\vec{B}_2=\gamma\vec{b}_1+\delta\vec{b}_2$, and α, β, γ, δ are general functions of the e_i. The method utilizes certain polynomials first developed by T. Smith. The solution has many applications in the field of geoemtrical optics as well as in all problems connected with the extremals of a variation.

5. *A useful course in geometry for future high school teachers*, by Professor Israel Halperin, Queen's University.

The current agitation to modernize the high school mathematical curriculum by introducing modern concepts such as those of set theory, may, in practice, lose more than it gains. The advantages of the old-fashioned material should be well understood; namely, good students, at least, are brought face to face with difficult problems which they have the ability and the mathematical equipment to solve. Foremost in geometry should be the challenging problem, correctly posed. The speaker's course for future teachers, based on these ideas, follows closely the axiomatic development given in Hilbert's "Grundlagen der Geometrie."

6. *Mathematics in industrial research*, by Dr. E. W. Marchand, Eastman Kodak Company.

The speaker emphasized the wide gulf which separates the fields of pure and applied mathematics, the great diversity of talents an industrial mathematician should have, and the inadequacy of a classical mathematical education in preparation for industrial mathematics.

7. *The case for a general education course in mathematics*, by Professor Harriet F. Montague, University of Buffalo.

This paper describes the results of a study made to determine relations between grades received in a course entitled Perspectives in Mathematics taught to some 500 students from 1951 to 1959 and such items as previous training in mathematics, rank in high school graduating class, and beginning of course attitudes toward mathematics, such as fear, indifference, enjoyment.

8. *Teaching experiences with a high school supplementary mathematics program*, by Mr. G. R. Rising, Greece Olympia High School.

A program of mathematics enrichment for selected students from four Monroe County secondary schools is described with special reference to problems in administration of the program, selection of topics to be taught and evaluation of the results.

9. *A modification of Birkhoff's axioms for Euclidean geometry*, by Professor Roland Brossard, Universite de Montreal.

As a result of a weakening of the axiom of angular measure an angle can be defined without any orientation. However, it now has two measures, α and $2\pi-\alpha$, the smallest called the interior measure and the other, the exterior measure. The fact that the measures of a straight angle are both π is taken as an axiom. To permit the construction of space geometry the notion of plane bundle of half-lines is introduced, and the axioms on angles are formulated in terms of plane bundles of half-lines.

10. *Some elemtary aspects of integral geometry*, by Professor W. R. Baum, Syracuse University.

The concept of figure-measure, in particular line-measure, is discussed from geometric and analytic points of view and illustrated by simple examples like Crofton's classical result on the line-measure of a closed convex domain in the plane. After describing the fundamental formula of Santaló-Blaschke and extensions due to Hadwiger, recent applications of integral geometry to multi-dimensional information theory and toward a quantitative treatment of the problem of shape of sets (in particular in "pattern recognition") are sketched.

11. *Finite geometries and the four color problem*, by Professor W. T. Tutte, University of Toronto.

The problem of coloring a map in four colors can be reduced in various ways to that of solving a system of linear inequalities over a finite field. This approach leads to the problem of classifying those sets of points in $PG(n, 2)$ which have a member on each $(n-2)$-dimensional hyperplane. This classification problem is solved in 2 and 3 dimensions; results obtained in 4 and 5 dimensions are stated; and a conjecture, which implies the Four Color Theorem, is made for space of n dimensions.

N. G. GUNDERSON, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The thirteenth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Montana State University, Missoula, Montana, on June 17, 1960 in conjunction with the 570th meeting of the American Mathematical Society and jointly with the Pacific Northwest Section of the Society for Industrial and Applied Mathematics. Professor A. E. Livingston, Chairman of the Section, presided over the meetings. There were 35 members of the Association registered.

The business meeting was held at 8:00 P.M., June 17. The following officers were elected: Chairman, Professor T. G. Ostrom, Washington State University; Vice-Chairman, Professor R. E. Gaskell, Oregon State College; Secretary-Treasurer, Professor K. S. Ghent.

nents and structures. Three possible levels of concentration requiring interdepartmental cooperation are: (1) construction (mathematics, electrical engineering, physics, chemistry); (2) application (mathematics, economics, business administration, sciences); (3) implication (mathematics, philosophy, psychology). The third area has hitherto received insufficient attention.

Contributed ten minute paper:

6. *An extension of Graeffe's method*, by Professor J. P. Ballantine, University of Washington.

$P(x, h)$ is called a factornomial of degree n if it is the product of n linear factors in x and h. If $P_0(x)$ and $P_1(x)$ in the Maclaurin expansion of a factornomial, $P(x, h)=P_0(x)+hP_1(x)+\cdots$, are known, then $P(x, h)$ is unique. $G(P(x, h))$ is defined as the unique factornomial determined by $P_0(s)P_0(-s)+\frac{1}{2}h[P_0(s)P_1(-s)+P_0(-s)P_1(s)]$, where $s^2=x$. An x-root of $P(x, h)$ is any number x_i such that $P(x_i, 0)=0$. An h-root is any number such that $P(0, h_i)=0$. Under G, x-roots are squared and h-roots remain unchanged. To find the roots of $P(x)$, find the h-roots of the factornomial $P(x+h)$, which are the same as the h-roots of $G^kP(x+h)$. The latter is factored by inspection, as in Graeffe's method, whence the h-roots are easily found.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–31, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA

IOWA, Simpson College, Indianola, April 21, 1961.

KANSAS

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, December 3, 1960.

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA, University of North Dakota, Grand Forks, November 5, 1960.

MISSOURI, University of Missouri, Columbia Spring, 1961,

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 5, 1960.

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, University of Colorado, Boulder, May, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

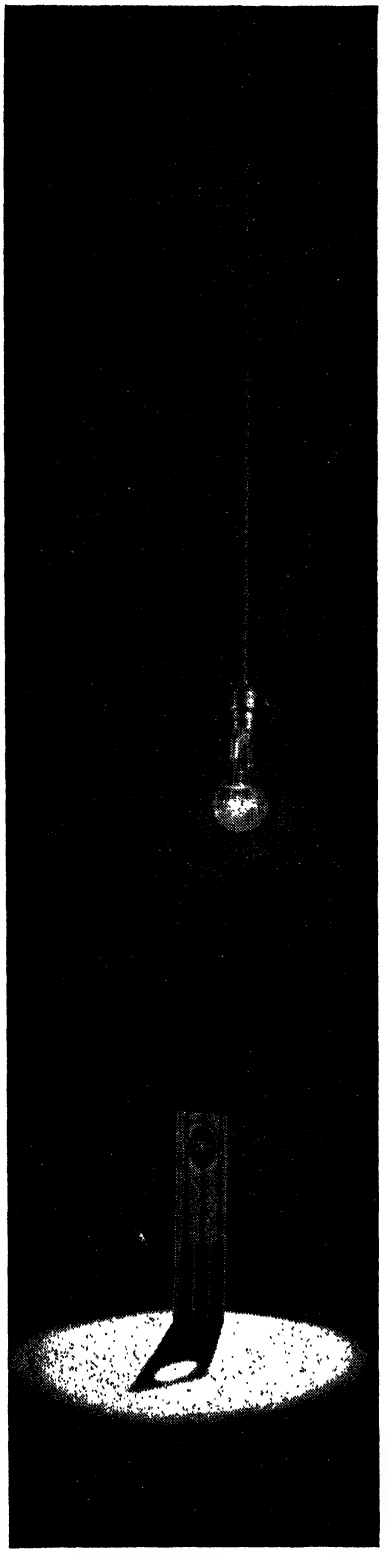
SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, April, 1961.

TEXAS, Stephan F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.



One of a series

New Light on a Compound Semiconductor

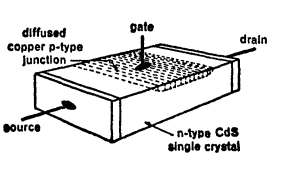
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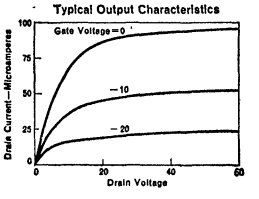
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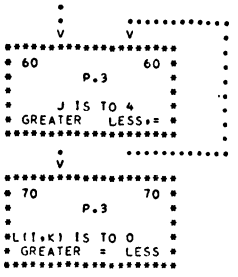
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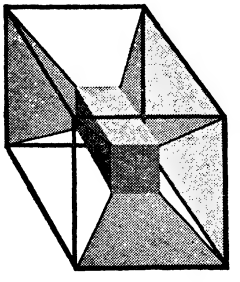
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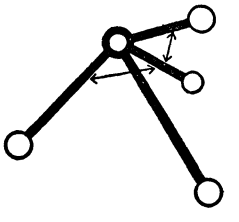
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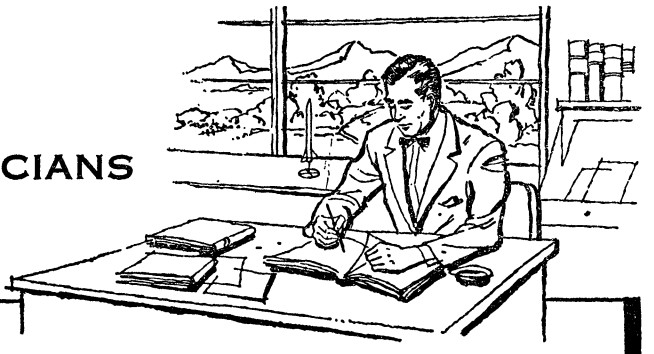
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ON THE COMPLETENESS OF STURM-LIOUVILLE EXPANSIONS

GARRETT BIRKHOFF, Harvard University, AND GIAN-CARLO ROTA,* Massachusetts Institute of Technology

1. Historical remarks. A very old difficulty in the theory of ordinary differential equations is the proof of the fundamental assertion about the eigenfunctions of a Sturm-Liouville system: that such a sequence of functions is complete. In the special case of $y''+\lambda y=0$, this difficulty can be resolved by elementary methods taught in advanced calculus (See, for example, [4], Ch. IX or [10], Ch. XIII). However, it should be remembered that, even in this case, the difficulty perplexed such mathematical giants as Euler and Lagrange, being finally resolved rigorously by Dirichlet and Riemann† only after a century of controversy.

The general case is correspondingly more difficult, though the difficulty is commonly glossed over by physicists. In his celebrated *Theory of Sound* (Art. 187), Rayleigh discussed the number of zeros of the remainder of an "arbitrary function" after the first n terms of the Sturm-Liouville series have been subtracted from it. He then asserted that "the application of this theorem to demonstrate the possibility of expanding an arbitrary function in an infinite series of normal functions would proceed exactly as in Art. 142," that is, as in the case of Fourier series. Sommerfeld (*Partial Differential Equations*, p. 179) refers to this result as the *Ohm-Rayleigh Principle.*

Apparently Bôcher queried Lord Rayleigh about his statement. His question received the following reply, in 1899; a photostat of it is reproduced in Figure 1 because of its historical interest.

> Dear Prof. Bôcher
>
> I doubt if I had any special method in my mind when I wrote §187. At any rate if I had then, I have not now.
>
> I think I took this case as covered by the general theory as in Ch. IV. I have often wished that mathematicians would devote themselves to improving the rigour of this and similar demonstrations, rather than to inventing special methods for the particular cases.
>
> Prof. Klein, who was here a few months ago, spoke of some recent work of Poincaré in this direction. Probably you know of this. I think Klein specially mentioned something published at Palermo.

Actually, Rayleigh could have justified his statement quite easily, had he combined already available results with some simple observations about Hilbert space. It is the purpose of this note to supply the needed argument; it follows lines originally suggested by George D. Birkhoff [1].

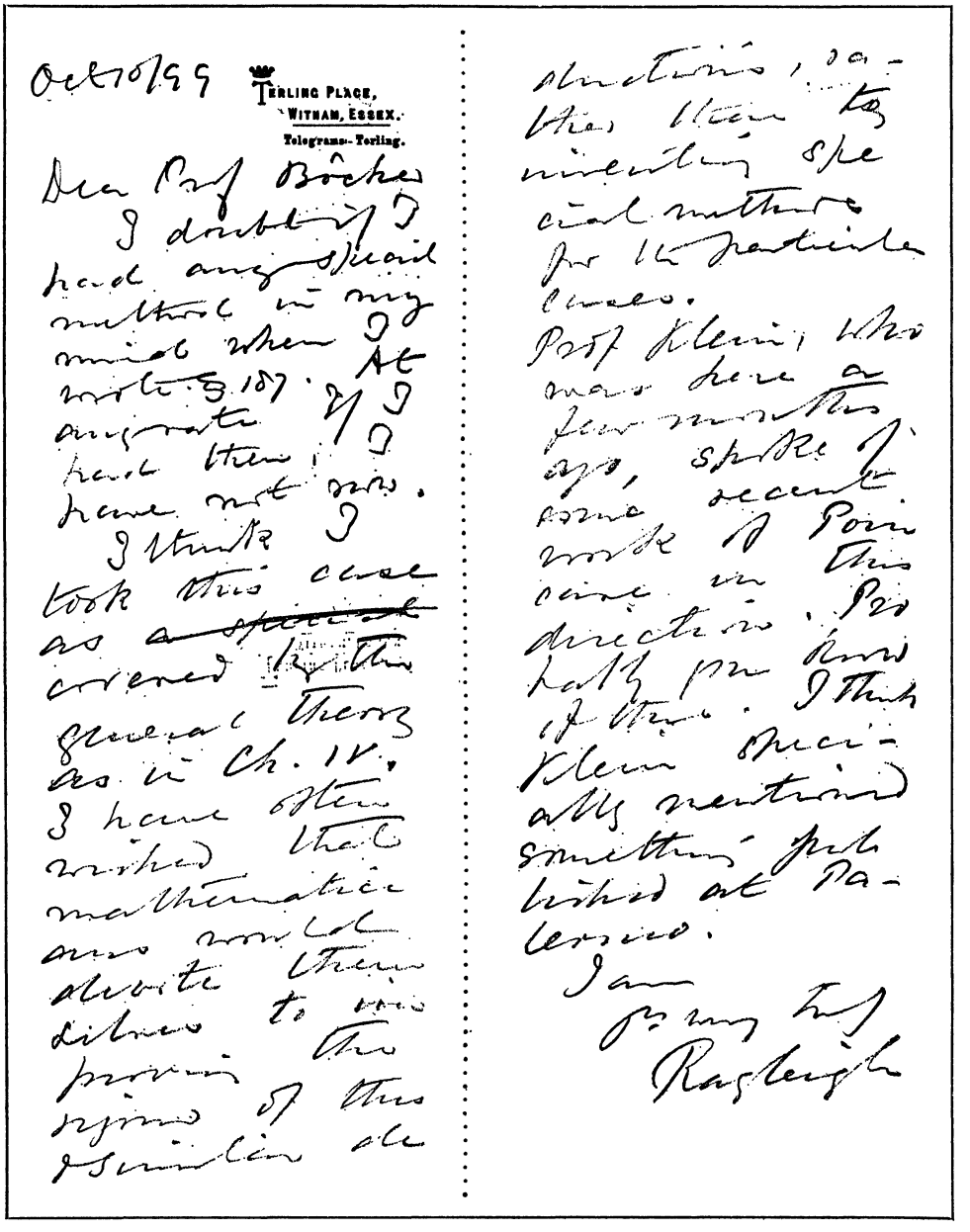
2. Role of Hilbert space. Let $\mathfrak{C}$ be the Euclidean vector space ([1], p. 189) of all real functions $y=f(x)$ continuous on $a\leqq x\leqq b$. Given $f\in\mathfrak{C}$ and a sequence

* Professor Rota's work was partially done under Contract 7667, Office of Naval Research.

† See [6]. Fourier's arguments were hardly rigorous; he referred loosely to "all possible functions" (Analytical Theory of Heat, end of Art. 224).

$\{\phi_k\}$ of normalized orthogonal functions, one can define the formal expansion

$$f \sim c_1\phi_1 + c_2\phi_2 + \cdots, \qquad c_k = (f, \phi_k) = \int_a^b f\phi_k dx. \tag{1}$$



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Dear Prof Bôcher

I doubt if I had any special matter in my mind when I wrote S 107. At any rate I had then I have not now. I think I took this case as a special case covered by the general theorem as in Ch. IV. I have often wished that mathematicians would devote themselves to the proving of theorems of this description etc. etc. This leaves to my investigations, as the most important specials matters published at Paris.

I am

Yours truly

Rayleigh

FIG. 1

The coefficients c_k always satisfy the Bessel inequality

$$\sum_1^\infty c_k^2 \leqq \int_a^b f^2(x)dx < +\infty. \tag{2}$$

The sequence $\{\phi_k\}$ is called *complete* when, for every $f \in \mathfrak{C}$, the Bessel inequality (2) is an equality (the Parseval equality):

$$\sum_1^\infty c_k^2 = \sum_1^\infty (f, \phi_k)^2 = \int_a^b f^2(x)dx. \tag{2*}$$

It is proved in advanced calculus that the cosine functions

$$\phi_k = \sqrt{(2/\pi)} \cos kx \qquad (k = 0, 1, \cdots)$$

are complete on $0 \leqq x \leqq \pi$, as are the sine functions

$$\tilde{\phi}_k = \sqrt{(2/\pi)} \sin kx \qquad (k = 1, 2, \cdots);$$

these are correlated with the even and odd components $\frac{1}{2}[f(x)+f(-x)]$ and $\frac{1}{2}[f(x)-f(-x)]$, respectively, of a general continuous function on $-\pi \leqq x \leqq \pi$.

The expansion (1) establishes a linear isomorphism $\theta: f \to (c_1, c_2, \cdots)$ between $\mathfrak{C}$ and a subspace $\theta(\mathfrak{C}) = \mathfrak{G}$ of the space $\mathfrak{H}$ of all infinite sequences $(c_1, c_2, \cdots)$ satisfying $\sum_1^\infty c_k^2 < +\infty$. Moreover, if (real) inner products are defined in $\mathfrak{H}$ by

$$(\gamma, \tilde{\gamma}) = \sum_1^\infty c_k \tilde{c}_k, \qquad \gamma = (c_1, c_2, \cdots), \qquad \tilde{\gamma} = (\tilde{c}_1, \tilde{c}_2, \cdots), \tag{3}$$

then the mapping θ preserves inner products: if $\theta(f) = \gamma$ and $\theta(\tilde{f}) = \tilde{\gamma}$, then $\int_a^b f(x)\tilde{f}(x)dx = (\gamma, \tilde{\gamma})$. The definition (3) makes $\mathfrak{H}$ a Euclidean vector space, and in fact a *Hilbert space* l_2; relative to the usual distance formula

$$\|\gamma - \tilde{\gamma}\| = (\gamma - \tilde{\gamma}, \gamma - \tilde{\gamma})^{1/2} \tag{3'}$$

defined in $\mathfrak{H}$ by (3), θ is distance-preserving (an *isometry*).

Now consider a general regular Sturm-Liouville system. It suffices to consider the differential equation

$$y'' + (\lambda - q(x))y = 0, \qquad 0 \leqq x \leqq \pi, \tag{4}$$

since other cases can be reduced to this by Liouville's transformation ([9], p. 238; [5], pp. 291–295.) We assume *separated* endpoint conditions, of the form

$$\alpha f(0) + \alpha_1 f'(0) = \beta f(\pi) + \beta_1 f'(\pi) = 0. \tag{4'}$$

As essentially shown by Liouville, one can then derive the following asymptotic estimate ([9], p. 244; [3], Part B) for the eigenfunctions of (4)–(4'):

$$\psi_k(x) = \sqrt{(2/\pi)} \cos kx + O(1/k), \tag{5}$$

provided $\alpha_1 \neq 0$ and $\beta_1 \neq 0$. (If $\alpha_1 = 0$ or $\beta_1 = 0$, use sine in place of cosine.) As a

corollary, we have the relation $\|\psi_k-\phi_k\|^2=O(1/k^2)$, where ϕ_k is an orthonormal basis* of cosine or sine functions. Hence, in particular,

$$\sum_{k=1}^{\infty}\|\psi_k-\phi_k\|^2<+\infty. \tag{5'}$$

3. Main theorem. The completeness of the ψ_k now follows from a general theorem about bases in Hilbert space. This can be made to follow from more general theorems on biorthogonal systems ([8], p. 206), but we have not seen a simple direct proof in the literature.

THEOREM 1. *Let ϕ_k $(k=0, 1, \cdots)$ be an orthonormal basis in Hilbert space $\mathfrak{H}$ and let ψ_k $(k=0, 1, \cdots)$ be an orthonormal sequence in $\mathfrak{H}$ satisfying*

$$\sum_{k=1}^{\infty}\|\psi_k-\phi_k\|^2<+\infty. \tag{6}$$

Then the sequence $\{\psi_k\}$ is also an orthonormal basis.

LEMMA. *Let $\phi_0, \phi_1, \cdots$ be an orthonormal basis in the Hilbert space $\mathfrak{H}$. Let $\psi_0, \psi_1, \cdots$ be an orthonormal sequence in $\mathfrak{H}$ satisfying the conditions*

$$\sum_{k=1}^{\infty}\|\psi_k-\phi_k\|^2<1. \tag{7}$$

Then the sequence $\{\psi_k\}$ is also an orthonormal basis in $\mathfrak{H}$.

Proof. If the sequence $\{\psi_k\}$ were not a basis, then we could find a nonzero function g orthogonal to every ψ_k. For this function, $(g, \phi_k)=(g, \psi_k)+(g, \phi_k-\psi_k)$ $=(g, \phi_k-\psi_k)$. Squaring and using the Schwarz inequality,

$$(g, \phi_k)^2=(g, \phi_k-\psi_k)^2\leqq\|g\|^2\|\phi_k-\psi_k\|^2. \tag{7'}$$

Summing with respect to k, we get

$$\sum_{k=0}^{\infty}(g, \phi_k)^2\leqq\|g\|^2\sum_{k=0}^{\infty}\|\phi_k-\psi_k\|^2<\|g\|^2,$$

the $\{\phi_k\}$ being a basis, in evident violation of Parseval's equality.

COROLLARY 1. *Replace condition (7) above by the weaker condition*

$$\sum_{k=N+1}^{\infty}\|\psi_k-\phi_k\|^2<1$$

for some integer N. Then every element of $\mathfrak{H}$ orthogonal to $\phi_0, \phi_1, \cdots, \phi_N$ and to $\psi_{N+1}, \psi_{N+2}, \cdots$ must vanish.

* By a *basis* for the Hilbert space $\mathfrak{H}$, we mean a subset, whose finite linear combinations are dense in $\mathfrak{H}$.

Proof. Any such element g would satisfy the inequality (7′) for all $k>N$. Hence, summing over all k, since $(g, \phi_k)=0$ for $k=1, \cdots, N$, we have

$$\sum_{k=1}^{\infty}(g, \phi_k)^2 = \sum_{k=N+1}^{\infty}(g, \phi_k)^2 \leqq \|g\|^2 \sum_{k=N+1}^{\infty}\|\phi_k - \psi_k\|^2 < 1,$$

again contradicting Parseval's equality (2*).

COROLLARY 2. *Under the assumptions of Corollary* 1, *every element of* $\mathfrak{H}$ *orthogonal to* $\psi_{N+1}, \psi_{N+2}, \cdots$ *and to the elements*

$$\eta_n = \phi_n - \sum_{k=N+1}^{\infty}(\phi_n, \psi_k)\psi_k, \qquad n = 0, 1, \cdots, N, \tag{8}$$

must vanish.

Proof. For any such element g we have

$$(g, \phi_n) = (g, \eta_n) + \sum_{k=N+1}^{\infty}(\phi_n, \psi_k)(g, \psi_k) = 0; \quad n = 0, 1, \cdots, N.$$

Thus g also satisfies the conditions of Corollary 1, and therefore vanishes.

The proof of Theorem 1 can now be completed as follows.

Choose an integer N so that

$$\sum_{k=N+1}^{\infty}\|\psi_k - \phi_k\|^2 < 1.$$

By Corollary 2, any element g of $\mathfrak{H}$ which is orthogonal to the elements ψ_{N+1}, $\psi_{N+2}, \cdots$ and to the elements η_n ($n=0, 1, \cdots, N$) defined by (8) must vanish. Denote by S the set of all elements of $\mathfrak{H}$ orthogonal to $\psi_{N+1}, \psi_{N+2}, \cdots$. Evidently S is a vector space containing $\eta_0, \eta_1, \cdots, \eta_n$. By virtue of the above remark, the vector space S contains only the linear combinations of these elements. In other words, S is a finite-dimensional vector space whose dimension is at most N.

But the elements $\psi_1, \cdots, \psi_N$ also belong to the vector space S, and they are linearly independent (they are an orthonormal sequence!). Therefore the elements $\psi_1, \cdots, \psi_N$ are a *basis* for the vector space S ([2], pp. 168–9). Hence the elements $\eta_0, \eta_1, \cdots, \eta_N$ are linear combinations of $\psi_1, \cdots, \psi_N$. It follows that any element of $\mathfrak{H}$ which is orthogonal to all the ψ_k must vanish, because such an element is also orthogonal to $\eta_0, \eta_1, \cdots, \eta_N$ and to $\psi_{N+1}, \psi_{N+2}, \cdots$. We conclude that the sequence $\{\psi_k\}$ is complete.

4. Further discussion. As Rayleigh, like most of his contemporaries, probably interpreted "completeness" as referring to pointwise or uniform convergence, it is relevant to discuss the relation between the "mean-square convergence" defined by the Parseval equality, and pointwise or uniform con-

vergence. Some facts can be proved easily, without an appeal to the Lebesgue integral.

Thus, from the Weierstrass approximation theorem (surely known to Rayleigh!), it follows immediately that the set $\mathfrak{P}$ of *polynomial* functions is dense in $\mathfrak{H}$. Therefore, the functions of class $\mathfrak{C}'$, $\mathfrak{C}''$, etc. (the number of primes referring to continuous differentiability) are also dense in $\mathfrak{H}$. By a slight modification near the endpoints, one can however approximate any such function by one satisfying the separated endpoint conditions (4'), arbitrarily closely in $\mathfrak{H}$. We now observe

THEOREM 2. *If $f \in \mathfrak{C}''$ satisfies the endpointcon ditions* (4'), *then the kth term of the expansion of f in the eigenfunctions ψ_k of the Sturm-Liouville system* (4)–(4') *is $O(1/k^2)$.*

Proof. Equation (4) implies

$$c_k = \int_0^\pi f(x)\psi_k(x)dx = (1/\lambda_k)\int_0^\pi [q(x)\psi_k(x) - \psi_k''(x)]f(x)dx. \tag{9}$$

On the other hand, by the Lagrange identity and (4')

$$\int_0^\pi (\psi_k f'' - f\psi_k'')dx = [\psi_k f' - f\psi_k']_{x=0}^{x=\pi} = 0,$$

since f and the ψ_k both satisfy (4'). Hence (9) implies

$$c_k = (1/\lambda_k)\int_0^\pi [q(x)\psi_k(x) - \psi_k(x)f''(x)]dx = O(1/k^2), \tag{9*}$$

where the last equality follows since $\lambda_k = O(k^2)$, while the ψ_k, q, and f'' are uniformly bounded, all by (5). Uniform convergence follows from (9*) by the Weierstrass M-test.

So far, Lebesgue integration has not been invoked. This is needed to identify $\mathfrak{H}$ as constructed above with the class $L_2(0, \pi)$ of functions whose squares are Lebesgue integrals. Specifically, given any formal sum $\sum_1^\infty c_k\phi_k(x)$ with $\sum_1^\infty c_k^2 < +\infty$, one can extract from the partial sums $\sigma_n(x) = \sum_1^n c_k\phi_k(x)$ a subsequence $\sigma_{n(k)}(x)$ converging almost everywhere to a (Lebesgue integrable) limit $\sigma_\infty(x)$; this is the essence of the proof of the Riesz-Fischer Theorem ([7], p. 185); any two such limits are equal a.e., and determine the same σ_∞ in $\mathfrak{H}$. Using these facts, one can prove

THEOREM 4. *If the nth partial sums $\sigma_n(x) = \sum_1^n c_k\phi_k(x)$ in* (1), *converge pointwise to $f(x)$ then they converge in the mean square.*

For, by hypothesis, the $\sigma_{n(k)}(x)$ mentioned above converge pointwise to $f(x)$; hence $f(x)$ coincides with $\sigma_\infty(x)$ almost everywhere, implying the Parseval equality (2*). We omit the details.

COROLLARY. *If the nth partial sums* $\sigma_n(x)$ *in* (1) *converge pointwise to* $f(x)$, *for all* f *in any dense subset* $\mathfrak{D}$ *of* $\mathfrak{H}$, *then* $\{\phi_k\}$ *is complete.*

It is not clear that the above results can be proved without using the Lebesgue integral. They provide a final link between classical pointwise convergence and the modern theory of mean square convergence.

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OPERATORS AND THE r-WAY CROSSED CLASSIFICATION

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1. Introduction. The development of the analysis of variance for the r-way crossed classification usually involves some tedious algebra. It is our purpose to clear much of this algebra away by means of an operational approach to the problem. The resulting presentation has been used successfully with undergraduates.

2. The 2-way crossed classification. It will be easier to understand what follows if we first review some of the results for the case where $r=2$. Here we make observations, X_{ijk}, and are concerned about the effect upon these observations of two factors, temperature and pressure, say. We make a selection of p temperatures and q pressures and make n observations at each of the pq possible combinations of temperature and pressure. We assume that the expected value of X_{ijk} is given by the equation

$$(1)\quad E(X_{ijk}) = \mu + \alpha_i + \beta_i + (\alpha\beta)_{ij} \quad (i = 1, \cdots, p; j = 1, \cdots, q; k = 1, \cdots, n),$$

where μ is a constant, α_i and β_j are the *main effects* of the ith temperature and jth pressure while $(\alpha\beta)_{ij}$ is called the *interaction* between the ith temperature and the jth pressure. As we shall see in the general case, we may assume without loss of generality that

$$\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = \sum_{i=1}^{p} (\alpha\beta)_{ij} = \sum_{j=1}^{q} (\alpha\beta)_{ij} = 0. \tag{2}$$

We are interested in testing three hypotheses; the first is that the temperatures all have the same effect and hence, by (2), all α_i are equal to zero. The remaining hypotheses are $\beta_j = 0$ and $(\alpha\beta)_{ij} = 0$ for all i and j. To test the hypotheses we form an analysis of variance table containing the quadratic forms

$$SSA = nq \sum_{i=1}^{p} (\overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot\cdot\cdot})^2, \qquad SSB = np \sum_{j=1}^{q} (\overline{X}_{\cdot j\cdot} - \overline{X}_{\cdot\cdot\cdot})^2,$$

$$SS(AB) = n \sum_{i,j} (\overline{X}_{ij\cdot} - \overline{X}_{i\cdot\cdot} - \overline{X}_{\cdot j\cdot} + \overline{X}_{\cdot\cdot\cdot})^2, \qquad SSE = \sum_{i,j,k} (X_{ijk} - \overline{X}_{ij\cdot})^2,$$

$$SST = \sum_{i,j,k} (X_{ijk} - \overline{X}_{\cdot\cdot\cdot})^2,$$

where subscripts replaced by dots have been averaged over. The above quadratic forms satisfy the identity $SST = SSA + SSB + SS(AB) + SSE$ which may be rewritten as

$$\sum_{i,j,k} X_{ijk}^2 = npq\overline{X}_{\cdot\cdot\cdot}^2 + SSA + SSB + SS(AB) + SSE. \tag{3}$$

The ranks, or degrees of freedom, of the quadratic forms are $p-1$, $q-1$, $(p-1)(q-1)$, $pq(n-1)$ and $npq-1$, respectively. The mean sums of squares, MSA, MSB, $MS(AB)$ and MSE are obtained by dividing the corresponding sums of squares by their degrees of freedom and it is the ratio of the first three mean sums of squares to the fourth which are used to test the three hypotheses. More detail and proofs will be given in the succeeding sections of this paper when we consider the general case, our approach being one which was used by Wald [2].

3. The r-way crossed classification. The reader may find it helpful to set $r = 2$ in the work that follows and compare the results with those given in Section 2. An observation is denoted by $X_a = X_{a_1 \cdots a_{r+1}}$, $a_i = 1, \cdots, t_i$, where the last subscript is used to indicate replications. Main effects and interactions are represented by $\mu(I, a_I) = \mu(i_1, \cdots, i_k; a_{i_1}, \cdots, a_{i_k})$, where $I = (i_1, \cdots, i_k)$ is a subset of $R = (1, \cdots, r)$ and $\mu(I, a_I) = \mu$ if I is the null set.

We introduce two operators, $D_i = D_{a_i}$ which drops i and a_i from $\mu(R, a_R)$ and $M_i = M_{a_i}$ which averages a function over a_i, if a_i appears, and otherwise leaves the function unchanged. These operators obey all the ordinary laws of algebra, excluding division. Corresponding to equations (1) and (2), we assume that

$$E(X_a) = (1 + D)_R \mu(R, a_R), \tag{4}$$

$$M_{i_j} \mu(I, a_I) = 0, \qquad (j = 1, \cdots, k), \tag{5}$$

where $(1+D)_R = (1+D_1) \cdots (1+D_r)$.

It is easy to establish

LEMMA 1. *Independent of condition* (5),

(6) $M_iD_i=D_i$, $(1-M_i)D_i=0$, $M_i(1+D_i)=M_i+D_i$, $(1-M_i)(1+D_i)=1-M_i$.

Subject to condition (5),

(7) $(M_i+D_i)\mu(R, a_R)=D_i\mu(R, a_R)$, $(1-M_i)\mu(R, a_R)=\mu(R, a_R)$,

provided that these expressions are not multiplied by D_i.

We define the sums of squares

(8) $$SS(I)=tM_IA^2(I, a_I),$$

where $t=t_1\cdots t_{r+1}$, I is a subset, not necessarily proper, of the set $R+1=(1, \cdots, r+1)$, $M_I=M_{i_1}\cdots M_{i_k}$,

(9) $$A(I, a_I)=M_{R+1-I}(1-M)_IX_a,$$

$M_{R+1-I}=M_1\cdots M_{r+1}/M_{i_1}\cdots M_{i_k}$ and $(1-M)_I=(1-M_{i_1})\cdots(1-M_{i_k})$. We note that, if I is the null set, $A=A(0, a_0)=M_{R+1}X_a=\overline{X}$, where $\overline{X}$ is the mean of all observations and $SS(0)=t\overline{X}^2$. We also define an *error sum of squares*, SSE, as the sum of all sums of squares such that $r+1$ is a member of the set I so that

(10) $$SSE=(1+D)_RSS(R+1).$$

We now establish the fundamental result

LEMMA 2. $A(I, a_I)=M_b(t\delta_{ab}-1)_IX_b$, *where* $M_b=M_{b_1}\cdots M_{b_{r+1}}$.

Proof. Note that $(1-M_1)X_a=M_{b_1}(t_1\delta_{a_1b_1}-1)X_{b_1a_2\cdots a_{r+1}}$. Then the desired result follows from definition (9).

We are now in a position to establish a result corresponding to (3). It is

THEOREM 1. $tMX_a^2=(1+D)SS(R+1)$, *where* $t=t_1\cdots t_{r+1}$, $M=M_{R+1}=M_1\cdots M_{r+1}$, *and* $(1+D)=(1+D)_{R+1}=(1+D_1)\cdots(1+D_{r+1})$.

Proof. Making use of equations (8), (9) and Lemma 2, we have

$$\begin{aligned}(1+D)SS(R+1)&=t(1+D)_{a_{R+1}}M_aA^2(R+1, a_{R+1})\\&=t(1+D)_{a_{R+1}}M_aM_bM_c(t\delta_{ab}-1)_{R+1}(t\delta_{ac}-1)_{R+1}x_bx_c\\&=t(1+D)_{a_{R+1}}M_bM_c(t\delta_{bc}-1)_{R+1}x_bx_c\\&=tM_bM_c(1+t\delta_{bc}-1)_{R+1}x_bx_c=tM_bx_b^2.\end{aligned}$$

COROLLARY 1. *The analysis of variance table adds up since*

$$SST=SS(1)+\cdots+SS(r)+SS(1, 2)+\cdots+SS(r, r-1)+\cdots+SS(1, \cdots, r)+SSE,$$

where $SST=tM(X_a-\overline{X})^2$.

COROLLARY 2. $SS(I) = tM_bM_c(t\delta_{bc}-1)_IX_bX_c$.

We also note that

$$\sum_{a_{ij}=1}^{t_{ij}} A(I, a_I) = M_b \sum_{a_{ij}=1}^{t_{ij}} (t\delta_{ab} - 1)_I X_b = 0 \tag{11}$$

since

$$\sum_{a_{ij}=1}^{t_{ij}} (t_{ij}\delta_{a_{ij}b_{ij}} - 1) = 0.$$

Mathematical induction and (11) may be used to verify that any $A(I, a_I)$, which has at least one of the $a_{i_j}=1$, can be expressed as a linear combination of terms such as $A(I, b_I)$ where no b_{i_j} is equal to unity. This result leads to a set of linearly independent linear homogeneous equations in the $A(I, a_I)$ and we may conclude that the rank of $SS(I)$, a quadratic form in the X_a, does not exceed $(t-1)_I$. Referring to Theorem 1, we see that

$$t = \text{rank } (tMX^2) \leqq (1 + D)(t - 1)_{R+1} = [1 + (t - 1)]_{R+1} = t \tag{12}$$

and this is only possible if the rank of $SS(I)$ is $(t-1)_I$.

To simplify (10), our definition of SSE, we apply the second corollary of Theorem 1 to obtain

$$\begin{aligned} SSE &= tM_bM_c[(1 + D)_R(t\delta_{bc} - 1)_{R+1}]X_bX_c \\ &= tM_bM_c(t\delta_{b_{r+1}c_{r+1}} - 1)t_R\delta_{bc_R}X_bX_c = tM(X_b - \overline{X}_{b_R})^2 \end{aligned}$$

where $\overline{X}_{b_R}$ is the mean obtained by averaging X_b over b_{r+1}. We observe that the degrees of freedom of SSE cannot exceed $(1+D)_R(t-1)_{R+1}=t_R(t_{r+1}-1)$, and an argument similar to that associated with (12) shows that $t_R(t_{r+1}-1)$ is indeed the rank of SSE. Equation (12) also shows that the degrees of freedom in our analysis of variance table add up to those of the total sum of squares, SST.

4. The r-way crossed classification without replication. All definitions and results of the preceding section hold with the understanding that $t_{r+1}=1$. We denote our observations by X_{a_R} and define

$$\mu_{a_R} = E(X_{a_R}) = (1 + D)_R\mu(R, a_R)$$

and

$$\begin{aligned} \bar{\mu}_{a_I} &= M_{R-I}\mu_{a_R} = M_{R-I}(1 + D)_R\mu(R, a_R) = (1 + D)_I(M + D)_{R-I}\mu(R, a_R) \\ &= (1 + D)_ID_{R-I}\mu(R, a_R) = (1 + D)_I\mu(I, a_I), \end{aligned}$$

making use of Lemma 1. Multiplying both sides of this result

$$\bar{\mu}_{a_I} = (1 + D)_I\mu(I, a_I) \tag{13}$$

by $(1-M)_I$ and applying Lemma 1 and equation (5) we find that

$$\mu(I, a_I) = (1 - M)_I \bar{\mu}_{a_I} = M_{R-I}(1 - M)_I \mu_{a_R}. \tag{14}$$

It is easy to verify that (14) does give us the unique solution of (13) and, if we make (14) our definition of $\mu(I, a_I)$, that our earlier definitions (4) and (5) are satisfied. Wald defined the main effects by $\mu(i_1; a_{i_1}) = \bar{\mu}_{a_{i_I}} - \bar{\mu}$, and the interactions by

$$\mu(I, a_I) = \bar{\mu}_{a_I} - [(1 + D)_I - 1]\mu(I, a_I) \tag{15}$$

where he subtracted interactions of lower order corresponding to subsets of a_I from the mean. This is a recursive way of obtaining (14) since (15) is merely (13) rewritten. Since the formulas for $A(I, a_I)$ and $\mu(I, a_I)$ are of the same form, $A(I, a_I)$ is an unbiased estimator of $\mu(I, a_I)$.

Up to this point, our work has been algebra rather than statistics; we now turn to the statistical aspects of the subject. We shall see that, when there is no replication, we must assume that all the $(r-1)$st order interactions, $\mu(R, a_R)$, are equal to zero. We introduce the hypotheses

$$H(I): \mu(I, a_I) = 0 \qquad (a_{i_j} = 1, \cdots, t_{i_j}; j = 1, \cdots, k).$$

We define $\epsilon_{a_R} = X_{a_R} - \mu_{a_R}$, and assume that the X_{a_R} are $NID(\mu_{a_R}; \sigma^2)$, that is, they are normally and independently distributed with means μ_{a_R} and a common variance σ^2. Also $Z_{a_R} = (X_{a_R} - \mu_{a_R})/\sigma$ and hence are $NID(0; 1)$. We define

$$A^*(I, a_I) = M_{R-I}(1 - M)_I Z_{a_R}, \qquad SS^*(I) = t_R M_I A^{*2}(I, a_I).$$

An application of Theorem 1 shows that $t_R M_R Z^2_{a_R} = (1+D)_R SS(R)$ and a consequence of Cochran's theorem ([3], p. 107) is

THEOREM 2. *The quadratic forms $SS^*(I)$, $0 \leqq k \leqq r$, are independently distributed as χ^2 with $(t-1)_I$ degrees of freedom (d.f.).*

It is easy to verify, with the help of (14), that

$$A(I, a_I) = \mu(I, a_I) + \sigma A^*(I, a_I) = \mu(I, a_I) + M_{R-I}(1 - M)_I \epsilon_{a_R}, \tag{16}$$

and this result leads to

THEOREM 3. *If $H(I)$ holds, $SS(I)/\sigma^2 = SS^*(I)$ and is distributed as χ^2 with $(t-1)_I$ d.f. and is independent of $SS(R)$ provided $H(R)$ holds.*

Proof. This theorem is an immediate consequence of (16) and Theorem 2.

It follows, if we assume that the highest order interactions, $\mu(R, a_R)$, are zero, that $F(I) = MS(I)/MSE$ has the F-distribution with $(t-1)_I$ and $(t-1)_R$ d.f., where $MS(I) = SS(I)/(t-1)_I$, $MSE = SS(R)/(t-1)_R$. This result makes it possible to test the hypothesis $H(I)$ at the $100(1-\alpha)\%$ level of significance by rejecting $H(I)$ if $F(I) > F_\alpha$, where $\Pr(F > F_\alpha) = \alpha$. This test seems reasonable if we recall ([1]) that

$$E(MSE) = \sigma^2, \qquad E[MS(I)] = \sigma^2 + t_R \sigma^2(I)/t_I,$$

where $\sigma_2(I)$ is not a variance but is defined by $\sigma^2(I) = t_I M_I M_I \mu^2(I, a_I)/(t-1)_I$.

Suppose we wish to test the hypothesis that both $H(I)$ and $H(J)$ are true while making our usual assumption that $H(R)$ holds. Then $SS(I)$ and $SS(J)$ are distributed as $\chi^2\sigma^2$, independent of each other and of $SSE = SS(R)$, with $(t-1)_I$ and $(t-1)_J$ d.f., respectively. Hence their sum is distributed as $\chi^2\sigma^2$, independent of SSE, with $(t-1)_I + (t-1)_J$ d.f. Thus $F(I, J) = MS(I, J)/MSE$ has the F-distribution with $(t-1)_I + (t-1)_J$ and $(t-1)_R$ d.f., where $SS(I, J) = SS(I) + SS(J)$ is the *pooled* sum of squares, corresponding to the two hypotheses being tested, with degrees of freedom obtained by pooling the corresponding degrees of freedom, and $MS(I, J) = SS(I, J)/[(t-1)_I + (t-1)_J]$.

There may be cases where not only the $(r-1)$st order interactions, $\mu(R, a_R)$, are assumed to be zero but also certain others, say the $\mu(J, a_J)$. The preceding argument leads to replacing $SSE = SS(R)$ by $SSE_1 = SS(J) + SS(R)$ with a corresponding pooling of the degrees of freedom.

5. The r-way crossed classification with replication. We return to the notation of Section 3 and find that, with minor modifications, the results of Section 4 still hold. We note that $\mu_a = E(X_a) = \mu_{a_R}$ and that, in Theorem 2, we should extend the range of k to $r+1$. Also (14) becomes

$$\mu(I, a_I) = M_{R+1-I}(1 - M)_I \mu_a, \tag{17}$$

a result based on the assumption that

$$\mu_a = \mu_{a_R} = (1 + D)\mu(R + 1, a_{R+1}). \tag{18}$$

But then $\mu(I, r+1; a_I, a_{r+1}) = M_{R-I}(1 - M_{r+1})(1 - M)_I \mu_a = 0$, since $(1 - M_{r+1})\mu_a = (1 - M_{r+1})\mu_{a_R} = 0$, and it follows that (4) and (18) are consistent. We conclude that any interaction, whose last index is $r+1$, is *a priori* zero. We have seen in the preceding section that, in this case, the error sum of squares, SSE, is the sum of all sums of squares with last index $r+1$. This is simply the definition of SSE given in (10). Theorem 3 leads us to the conclusion that $SS(I)$ and SSE are independently distributed as $\chi^2\sigma^2$ with $(t-1)_I$ and $t_R(t_{r+1} - 1)$ d.f. Thus we may still use the test of $H(I)$ given in Section 4.

6. Computation of the sums of squares. We first establish

THEOREM 4. $t_I \sum_{a_I} T_{a_I}^2/t = (1+D)_I SS(I)$, where $SS(0) = t\overline{X}^2$ and $T_a = \sum_b (\delta_{ab})_I X_b$.

Proof. Consider the set of t_I variables $\overline{X}_{a_I}$ and apply Theorem 1 and its second corollary to obtain

$$\sum_{a_I} \overline{X}_{a_I}^2 = (1 + D)_I SS(I) = t_I M_{b_I} M_{c_I}[(1 + D)_I (t\delta_{bc} - 1)_I]\overline{X}_{b_I}\overline{X}_{c_I}$$

$$= t_I M_b M_c[(1 + D)_I (t\delta_{bc} - 1)_I] X_b X_c = (t_I/t)(1 + D)_I SS(I).$$

Then, since $\overline{X}_{a_I} = t_I T_{a_I}/t$, the desired result follows.

In order to obtain the entries in the analysis of variance table, we first compute the *correction term* $SS(0) = T^2/t$, where T is the sum of all the observations. Then, by Theorem 4,

$$SS(i) = (t_i/t) \sum_{a_i=1}^{t_i} T_{a_i}^2 - T^2/t \qquad (i = 1, \cdots, r).$$

Next,

$$(t_i t_j/t) \sum_{a_i=1}^{t_i} \sum_{a_j=1}^{t_j} T_{a_i a_j}^2 = SS(i, j) + SS(i) + SS(j) + SS(0)$$

and we can solve for $SS(i, j)$. Continuing this procedure, we can compute all of the sums of squares save SSE and SST. It is easy to verify that $SST = \sum_a X_a^2 - T^2/t$ and then SSE may be obtained by subtraction since Corollary 1 of Theorem 1 holds.

The reader who is interested in further applications of this operational method will find them in another paper [1]. Models other than the Type I model of this paper are considered, expected values of mean squares are obtained, and the variances of variance components are determined.

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SOME INEQUALITIES FOR POLYNOMIALS*

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1. Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be an arbitrary polynomial of degree n in $z = x + iy$ and let $|p(z)| \leqq M$ on a set $\mathcal{E}$. The modulus of $p'(z)$ on $\mathcal{E}$ has an upper bound depending on M, on n, and on the set $\mathcal{E}$. In this connection Sewell [3] has proved the following theorem.

THEOREM A. *Let $p(z)$, a polynomial of degree n in z, be in modulus less than or equal to a constant M on a set $\mathcal{E}$ which has no isolated points and whose complement has finite connectivity. Then $|p'(z)| \leqq MK_1(\mathcal{E})n^2$ on $\mathcal{E}$, where the constant $K_1(\mathcal{E})$ depends only on $\mathcal{E}$.*

This theorem generalizes the following theorem of A. Markoff (see [2]).

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THEOREM B. *Let* $|p(x)| \leqq M$ *in the interval* $-1 \leqq x \leqq 1$. *Then* $|p'(x)| \leqq Mn^2$ *for* $-1 \leqq x \leqq 1$.

The following theorem due to Bernstein (see [2]) gives another estimate for $|p'(x)|$.

THEOREM C. *Let* $|p(x)| \leqq M$ *in the interval* $-1 \leqq x \leqq 1$. *Then* $|p'(x)| \leqq Mn/(1-x^2)^{1/2}$ *for* $-1 < x < 1$.

For every subinterval of $(-1, 1)$ this result is very much better than the theorem of Markoff if n is sufficiently large.

It was later on proved by Sewell [4] that if the set $\mathfrak{S}$ is bounded by an analytic Jordan curve $\mathfrak{C}$ then $|p'(z)| \leqq MK_2(\mathfrak{C})n$, where $K_2(\mathfrak{C})$ is a constant depending only on $\mathfrak{C}$. If $\mathfrak{C}$ is the unit circle, we know by a theorem of Bernstein that $K_2(\mathfrak{C}) = 1$. It may be interesting to consider some well-known sets $\mathfrak{S}$ and to solve the problem in each case.

We prove first of all the following theorem

THEOREM 1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is an arbitrary polynomial of degree* n *in* z *and* $\mathfrak{C}$ *is the boundary of an ellipse in the* z*-plane then for every* $k \geqq 1$

$$\left(\int_{\mathfrak{C}} |p'(z)|^k |dz|\right)^{1/k} \leqq \frac{n}{b}\left(\int_{\mathfrak{C}} |p(z)|^k |dz|\right)^{1/k},$$

where b *denotes the semiminor axis of the ellipse.*

Proof. Let us suppose that the ellipse has its foci in the points c and $-c$ on the real axis. The Laurent polynomial

$$p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right] \tag{1}$$

takes on $|\zeta| = 1$ the same set of values as $p(z)$ does on the boundary $\mathfrak{C}$ of the ellipse. Hence for $k \geqq 1$

$$\begin{aligned}
&\int_{\mathfrak{C}} |p'(z)|^k |dz| \\
&= \int_{|\zeta|=1} \left|\frac{d}{d\zeta} p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right]\frac{d\zeta}{dz}\right|^k \left|\frac{dz}{d\zeta}\right| |d\zeta| \\
&= \int_{|\zeta|=1} \left|\frac{d}{d\zeta} p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right]\right|^k \left|\frac{d\zeta}{dz}\right|^{k-1} |d\zeta| \\
&= \int_{|\zeta|=1} \left|\frac{d}{d\zeta} p\left[\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c}\zeta + \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta}\right\}\right]\right|^k \\
&\qquad \cdot \left|\frac{c}{2}\left\{\frac{b+(b^2+c^2)^{1/2}}{c} - \frac{c}{b+(b^2+c^2)^{1/2}}\frac{1}{\zeta^2}\right\}\right|^{1-k} |d\zeta|
\end{aligned}$$

$$\leqq \int_{|\zeta|=1} \left| \frac{d}{d\zeta} p\left[\frac{c}{2} \left\{ \frac{b+(b^2+c^2)^{1/2}}{c} \zeta + \frac{c}{b+(b^2+c^2)^{1/2}} \frac{1}{\zeta} \right\} \right] \right|^k \cdot \left[\frac{2\{b+(b^2+c^2)^{1/2}\}}{\{b+(b^2+c^2)^{1/2}\}^2 - c^2} \right]^{k-1} |\, d\zeta \,|$$

$$= \frac{1}{b^{k-1}} \int_{|\zeta|=1} \left| \frac{d}{d\zeta} p\left[\frac{c}{2} \left\{ \frac{b+(b^2+c^2)^{1/2}}{c} \zeta + \frac{c}{b+(b^2+c^2)^{1/2}} \frac{1}{\zeta} \right\} \right] \right|^k |\, d\zeta \,|.$$

A theorem of Zygmund [6] states that if $s_n(z) = \sum_{-n}^{n} b_\nu z^\nu$, then for every $k \geqq 1$

$$\left(\int_{|z|=1} |\, s_n'(z) \,|^k |\, dz \,| \right)^{1/k} \leqq n \left(\int_{|z|=1} |\, s_n(z) \,|^k |\, dz \,| \right)^{1/k}.$$

Applying this result to the Laurent polynomial (1), we get from above for $k \geqq 1$

$$\int_{\mathcal{C}} |\, p'(z) \,|^k |\, dz \,| \leqq \frac{n^k}{b^{k-1}} \int_{|\zeta|=1} \left| p\left[\frac{c}{2} \left\{ \frac{b+(b^2+c^2)^{1/2}}{c} \zeta + \frac{c}{b+(b^2+c^2)^{1/2}} \frac{1}{\zeta} \right\} \right] \right|^k |\, d\zeta \,|$$

$$= \frac{n^k}{b^{k-1}} \int_{\mathcal{C}} |\, p(z) \,|^k \left| \frac{d\zeta}{dz} \right| |\, dz \,| \leqq \frac{n^k}{b^k} \int_{\mathcal{C}} |\, p(z) \,|^k |\, dz \,|,$$

and the theorem follows by taking the kth root of both the sides.

The case $k = \infty$ is $\max_{z \in \mathcal{C}} |\, p'(z) \,| \leqq (n/b) \max_{z \in \mathcal{C}} |\, p(z) \,|$, also proved by Sewell [5].

By applying Zygmund's theorem to $p(c_1\zeta + c_0 + c_{-1}\zeta^{-1})$, where c_1, c_0 and c_{-1} are complex numbers, analogous results for various other curves may be deduced. We also prove

THEOREM 2. *If $\mathcal{E}$ is the arc of the parabola $x^2 = ay (a > 0)$ with endpoints $(-1, 1/a)$, $(1, 1/a)$, and if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n in z and $|\, p(z) \,| \leqq M$ on $\mathcal{E}$, then*

$$|\, p'(z) \,| \leqq \max \left[\frac{4Mn^2 a}{\{a^2 + 4[1 - (4n^2)^{-1}]\}^{1/2}}, 2Mn \right]. \tag{2}$$

Proof. The polynomial $p(x + i(x^2/a))$ is of degree $2n$ and takes in the interval $-1 \leqq x \leqq 1$ the same set of values as $p(z)$ does on the arc of the parabola. Hence from Theorem B it follows that for $z \in \mathcal{E}$, $|\, 1 + 2i(x/a) \,|\, |\, p'(z) \,| \leqq 4Mn^2$ or

$$|\, p'(z) \,| \leqq \frac{4Mn^2 a}{(a^2 + 4x^2)^{1/2}}. \tag{3}$$

If we apply Theorem C to the polynomial $p(x+i(x^2/a))$ we get

$$|p'(z)| \leqq \frac{2Mna}{(a^2+4x^2)^{1/2}(1-x^2)^{1/2}}. \tag{4}$$

Case i. If $a \geqq 2$, the term on the right side of (4) increases with $|x|$ and therefore it follows from (4) that for $x^2 \leqq 1-(4n^2)^{-1}$

$$|p'(z)| \leqq \frac{4Mn^2a}{\{a^2+4[1-(4n^2)^{-1}]\}^{1/2}}. \tag{5}$$

On the other hand the term on the right side of (3) decreases as $|x|$ increases and so from (3) we have (5) for $x^2>1-(4n^2)^{-1}$.

Case ii. $a<2$. As in Case i, (5) holds for $|x|>[1-(4n^2)^{-1}]^{1/2}$. As $|x|$ increases from 0 to $[\frac{1}{8}(4-a^2)]^{1/2}$, the term on the right side of (4) decreases and attains a minimum at $|x|=[\frac{1}{8}(4-a^2)]^{1/2}$. For $|x|>[\frac{1}{8}(4-a^2)]^{1/2}$, it is monotone increasing. Therefore, for $|x| \leqq [1-(4n^2)^{-1}]^{1/2}$

$$|p'(z)| \leqq \max[p'\{[1-(4n^2)^{-1}]^{1/2}\}, p'(0)],$$

which is equal to the right side of (2) and the theorem is proved.

2. Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n such that $|p(z)| \leqq 1$ for $|z| \leqq 1$. If $R>1$ then by the maximum principle

$$\max_{|z|=1/R} |z^n p(z^{-1})| \leqq \max_{|z|=1} |z^n p(z^{-1})| = \max_{|z|=1} |p(z^{-1})| = \max_{|z|=1} |p(z)| \leqq 1$$

so that $\max_{|z|=1/R} |p(z^{-1})| \leqq R^n$. But $\max_{|z|=1/R} |p(z^{-1})| \equiv \max_{|z|=R} |p(z)|$ and therefore $\max_{|z|=R} |p(z)| \leqq R^n$.

The example $p(z)=z^n$ shows that the estimate is best possible. Thus, the upper bound of $|p(z)|$ over a certain circle being given, we can determine its upper bound over a larger concentric circle.

An upper bound for $|p(z)|$ over the linear interval $(-1, 1)$ having been given, Bernstein determined the upper bound for $|p(z)|$ over an ellipse having foci at the points 1 and -1. He proved ([1], vol. 1, p. 137, problem III 271):

THEOREM D. *Let $p(z)$ be a polynomial of degree n or less such that in the interval $(-1, 1)$ of the real axis $|p(z)| \leqq 1$. If R is the sum of the semiaxes of an ellipse passing through the point z and having foci at the points -1 and 1, then $|p(z)| \leqq R^n$.*

Here we prove

THEOREM 3. *Let $p(z)$ be a polynomial of degree n. If $\mathcal{C}$ denotes the boundary of an ellipse whose foci lie at the points -1 and 1 and the sum of whose semiaxes is R, then for a positive integer m,*

$$\int_{\mathcal{C}} |p(z)|^m |dz| \leqq 2R^{mn+1} \int_{-1}^{1} |p(x)|^m dx. \tag{6}$$

The proof makes use of the following lemma.

LEMMA. *Let* $f(z)=\sum_{\nu=-M}^{N} a_\nu z^\nu$, *where* M *and* N *are positive integers. If* $\mu(r,f)$ *denotes the mean value of* $|f(z)|$ *over the circle* $|z|=r$, *then*

$$\mu(r_1,f)/r_1^N \geqq \mu(r_2,f)/r_2^N, \qquad 0<r_1<r_2.$$

Proof. $F(z)=z^{-N}f(z)$ is regular in the punctured plane $|z|>0$, the point $z=\infty$ included. $\mathfrak{F}(z)=F(z^{-1})$ is regular in the finite circle $|z|<\infty$ and $\mu(r, F(z))$ is the mean value of $|\mathfrak{F}(z)|$ on the circumference $|z|=r^{-1}$. Consequently

$$\mu(r_1^{-1}, \mathfrak{F}(z)) \geqq \mu(r_2^{-1}, \mathfrak{F}(z)), \qquad r_2^{-1}<r_1^{-1}<\infty,$$

or $\mu(r_1, F(z)) \geqq \mu(r_2, F(z))$, $0<r_1<r_2$, from which the lemma follows.

Proof. If m is a positive intger, then

$$\left\{p\left(\frac{\zeta+\zeta^{-1}}{2}\right)\right\}^m (1-\zeta^{-2})$$

is a function of the form $f(\zeta)=\sum_{\nu=-M}^{N} a_\nu \zeta^\nu$ with $M=mn+2$, $N=mn$. Therefore, it follows from the lemma that

$$\begin{aligned}\int_{\mathcal{C}} |p(z)|^m |dz| &= \int_{|\zeta|=R} \left|p\left(\frac{\zeta+\zeta^{-1}}{2}\right)\right|^m \frac{1}{2} |1-\zeta^{-2}|\,|d\zeta| \\ &\leqq R^{mn+1}\int_0^{2\pi} |p(\cos\theta)|^m |\sin\theta|\,d\theta \\ &= 2R^{mn+1}\int_0^{\pi} |p(\cos\theta)|^m \sin\theta\, d\theta = 2R^{mn+1}\int_{-1}^{1} |p(x)|^m dx.\end{aligned}$$

By taking the mth root of both the sides of (6) and making $m\to\infty$ we get Theorem D.

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MEAN VALUE THEOREMS FOR FUNCTIONS WITH FINITE DERIVATES

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The law of the mean and many of its extensions require the existence of the derivative. However, since the derivative may fail to exist, it seems desirable to have expressions which may serve us when there is no derivative. The purpose of this paper is to consider extensions of the law of the mean to functions which have finite right and left "Dini derivates." Theorems of Karamata [1] and Vučković [4] as corrected by Utz [3] are contained in our results.

In some cases we need not require that all derivates be finite. Since these cases are easily resolved, they will be left for the consideration of the reader.

Let f be a function defined on a finite closed interval $a \leqq x \leqq b$. Throughout this paper we will use D^+f, D_+f, D^-f, and D_-f to denote the upper right, lower right, upper left, and lower left derivates ([2], p. 188), respectively, of f. In addition, if p and q are real numbers, we will use $pD(+)f+qD(-)f$ to denote any one of the expressions

$$p_1D^+f + q_1D^-f, \qquad p_2D^+f + q_2D_-f, \qquad p_3D_+f + q_3D^-f, \qquad p_4D_+f + q_4D_-f.$$

Finally, we will use $pD(\dagger)f+qD(\ddagger)f$ to denote all four of the above expressions.

Our first theorem is an extension of Rolle's theorem.

THEOREM 1. *Let f be a function which is continuous at each point of the finite closed interval $a \leqq x \leqq b$, let $f(a)=f(b)=0$ and let D^+f, D_+f, D^-f, and D_-f be finite at each point of the open interval $a<x<b$. Then there exist numbers ξ, p, and q with $a<\xi<b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which $pD(+)f(\xi)+qD(-)f(\xi)=0$.*

Proof. To avoid unnecessary repetition we shall prove the theorem for only one possible choice of derivates. The relations for other pairs of derivates can be similarly proved.

If $f(x) \equiv 0$ on $a \leqq x \leqq b$, then we may take $\xi=\frac{1}{2}(a+b)$ and $p=q=\frac{1}{2}$. If $f(x) \not\equiv 0$ on $a \leqq x \leqq b$, then, since f is continuous, it assumes a positive maximum or a negative minimum or both on $a<x<b$. Let ξ, $a<\xi<b$, be such that f assumes a maximum or a minimum at ξ. If $D^+f(\xi)=0$, let $p=1$ and $q=0$. If $D^+f(\xi) \neq 0$, then the product $D^+f(\xi)D^-f(\xi) \leqq 0$ and we may choose

$$p = \frac{D^-f(\xi)}{D^-f(\xi) - D^+f(\xi)} \geqq 0, \qquad q = \frac{D^+f(\xi)}{D^+f(\xi) - D^-f(\xi)} \geqq 0,$$

which completes the proof.

Theorem 2 is an extension of the law of the mean. Its proof, as well as the proofs of our remaining theorems, is based on Theorem 1.

* The author is indebted to the referee for his helpful suggestions.

THEOREM 2. *Let f be a function which is continuous at each point of the finite closed interval $a \leqq x \leqq b$ and let the derivates D^+f, D_+f, D^-f, and D_-f be finite at each point of the open interval $a < x < b$. Then there exist numbers ξ, p, and q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which*

$$f(b) - f(a) = [b - a][pD(+)f(\xi) + qD(-)f(\xi)].$$

Proof. We shall prove the theorem for only one possible choice of derivates. The other cases can be proved in a similar manner. Let

$$F(x) = [b - a][f(x) - f(a)] - [x - a][f(b) - f(a)].$$

The function F is continuous on $a \leqq x \leqq b$, F has finite one-sided derivates on $a < x < b$, and $F(a) = F(b) = 0$. Hence F satisfies the conditions of Theorem 1 and there exist numbers ξ, p, and q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ such that $pD^+F(\xi) + qD^-F(\xi) = 0$. But

$$D^+F(x) = [b - a]D^+f(x) - [f(b) - f(a)],$$
$$D^-F(x) = [b - a]D^-f(x) - [f(b) - f(a)],$$

and the required result now follows.

Our next theorem is an extension of Taylor's formula with remainder.

THEOREM 3. *Let the function f, defined on the closed interval $a \leqq x \leqq b$, be such that its first $n-1$ derivatives exist and are continuous on $a \leqq x \leqq b$ (only right and left derivatives are required at a and b, respectively). In addition, let f be such that the derivates $D^+f^{(n-1)}$, $D_+f^{(n-1)}$, $D^-f^{(n-1)}$, and $D_-f^{(n-1)}$ of its $(n-1)$st derivative $f^{(n-1)}$ are finite at each point of the open interval $a < x < b$. Then for any number v such that $n - v > 0$ there exist numbers ξ, p, and q with $a < \xi < b$, $p \geqq 0$, $q \geqq 0$, $p+q=1$ for which*

$$f(b) = f(a) + (b - a)f'(a) + \cdots + \frac{(b - a)^{n-1}f^{(n-1)}(a)}{(n - 1)!}$$
$$+ \frac{(b - a)^{n-v}(b - \xi)^v}{(n - v)(n - 1)!}[pD(+)f^{(n-1)}(\xi) + qD(-)f^{(n-1)}(\xi)].$$

Proof. We shall prove the theorem for only one possible choice of derivates. The relations for other pairs of derivates can be similarly proved. Let

$$F(x) = -f(b) + f(x) + (b - x)f'(x) + \cdots + \frac{(b - x)^{n-1}}{(n - 1)!}f^{(n-1)}(x) + \frac{(b - x)^{n-v}}{(b - a)^{n-v}}K,$$

where

$$K = f(b) - f(a) - (b - a)f'(a) - \cdots - \frac{(b - a)^{n-1}}{(n - 1)!}f^{(n-1)}(a).$$

Since F satisfies the conditions of Theorem 1, there exist numbers ξ, p, and q with $a<\xi<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$ such that $pD^+F(\xi)+qD^-F(\xi)=0$. But

$$D^+F(x) = \frac{(b-x)^{n-1}}{(n-1)!}D^+f^{(n-1)}(x) - \frac{(n-v)(b-x)^{n-v-1}}{(b-a)^{n-v}}K,$$

$$D^-F(x) = \frac{(b-x)^{n-1}}{(n-1)!}D^-f^{(n-1)}(x) - \frac{(n-v)(b-x)^{n-v-1}}{(b-a)^{n-v}}K,$$

and the required result then follows.

Let f and g be functions which are continuous at each point of the closed interval $a\leqq x\leqq b$ and which have finite derivates D^+f, D_+f, D^-f, D_-f, and D^+g, D_+g, D^-g, D_-g, respectively, at each point of the open interval $a<x<b$. We shall say that $D(+)f+D(-)f$ is partially bounded above by $D(+)g+D(-)g$ provided there exist numbers ξ, p, and q with $a<\xi<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$ such that

$$[g(b)-g(a)][pD(+)f(\xi)+qD(-)f(\xi)] \leqq [f(b)-f(a)][pD(+)g(\xi)+qD(-)g(\xi)].$$

Moreover, we shall say that the four expressions $D(\overset{+}{+})f+D(\overset{-}{-})f$ are partially bounded above by $D(+)g+D(-)g$ provided each of the expressions is partially bounded above by $D(+)g+D(-)g$. The notion of partially bounded below is defined by replacing *above* by *below* and $\leqq$ by $\geqq$ in the preceding definition.

Our final results, which are extensions of Cauchy's generalized law of the mean, will be stated as three related parts of Theorem 4. We shall prove only one relation from Part (a). The other relations can be similarly proved.

THEOREM 4. *Suppose that the functions f and g are continuous at each point of the closed interval $a\leqq x\leqq b$ and that the one-sided upper and lower derivates of f and g are finite at each point of the open interval $a<x<b$.*

(a) *If $f(b)-f(a)\geqq 0$, $g(b)-g(a)\geqq 0$, then $D(\overset{+}{+})f+D(\overset{-}{-})f$ is partially bounded above by D^+g+D^-g and partially bounded below by D_+g+D_-g. Moreover, D^+f+D_-f and D_+f+D^-f are partially bounded above and below by D^+g+D_-g and D_+g+D^-g, respectively.*

(b) *If $f(b)-f(a)\leqq 0$, $g(b)-g(a)\leqq 0$, then $D(\overset{+}{+})f+D(\overset{-}{-})f$ is partially bounded above by D_+g+D_-g and partially bounded below by D^+g+D^-g. Moreover, D^+f+D_-f and D_+f+D^-f are partially bounded above and below by D^+g+D_-g and D_+g+D^-g, respectively.*

(c) *If $f(b)-f(a)\leqq 0$, $g(b)-g(a)\geqq 0$, then $D(\overset{+}{+})f+D(\overset{-}{-})f$ and $D(\overset{+}{+})g+D(\overset{-}{-})g$ are partially bounded above by D_+g+D_-g and D^+f+D^-f, respectively, and partially bounded below by D^+g+D^-g and D_+f+D_-f, respectively. Moreover, D^+f+D_-f and D_+f+D^-f are partially bounded above and below by D_+g+D^-g and D^+g+D_-g, respectively.*

Proof (a). Let $F(x)=[g(b)-g(a)][f(x)-f(a)]-[f(b)-f(a)][g(x)-g(a)]$. Since F satisfies the conditions of Theorem 1, there exist numbers ξ, p, q, with

$a<\xi<b$, $p\geqq 0$, $q\geqq 0$, $p+q=1$, such that $pD_+F(\xi)+qD_-F(\xi)=0$. But ([2], p. 190)

$$D_+F(x) \leqq [g(b) - g(a)]D^+f(x) - [f(b) - f(a)]D^+g(x),$$
$$D_-F(x) \leqq [g(b) - g(a)]D^-f(x) - [f(b) - f(a)]D^-g(x).$$

It follows that

$$[g(b) - g(a)][pD^+f(\xi) + qD^-f(\xi)] \geqq [f(b) - f(a)][pD^+g(\xi) + qD^-g(\xi)],$$

which was to be shown.

Added in proof. The reader will find of interest a paper by S. R. Mukhoti, *On a generalization of some theorems of the differential calculus*, Bull. Calcutta Math. Soc., vol. 50, 1958, pp. 87–93.

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DIRECT PROOF OF LANCZOS' DECOMPOSITION THEOREM

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In his recent paper [1], C. Lanczos has proved the following general

THEOREM. *For every real $m\times n$ matrix A of rank $p>0$ it is possible to find an $n\times p$ column-orthogonal matrix Y and an $m\times p$ column-orthogonal matrix X such that*

$$A = X\Lambda Y', \tag{1}$$

*where Λ is a $p\times p$ diagonal matrix of rank p.**

While the inherent interest of Lanczos' construction of the column-orthogonal matrices X, Y cannot be denied, it seems to be desirable to have a more direçt proof of the theorem. Some new applications are discussed in the later sections of the present paper.

1. We interpret the matrix A as representative of a linear mapping of the real n-dimensional vector space R_n into the space R_m, more accurately: onto a

* The prime indicates transposition. For all notations see the author's book [2].—Theorem and proof remain valid in the case of complex matrices if one replaces "orthogonal" by "unitary-(or hermitian-) orthogonal", X' by $\bar{X}'=X^*$, etc.

p-dimensional linear subspace L_p of R_m. We show first that in R_n there is an orthonormal system of vectors $y^{(1)}, \cdots, y^{(p)}$ such that their images in R_m, *i.e.* the vectors

$$Ay^{(1)} = z^{(1)}, \cdots, Ay^{(p)} = z^{(p)}$$

form an orthogonal basis of L_p. Indeed, let us choose $y^{(1)}, \cdots, y^{(p)}$ as the first p column vectors of the orthogonal $n \times n$ matrix

$$\mathfrak{y} = (Y, Y_0), \quad Y = y^{(1)}, \cdots, y^{(p)}, \quad Y_0 = (y^{(p+1)}, \cdots, y^{(n)}),$$

which reduces the $n \times n$ Gram matrix $A'A$ (positive definite if $p = n$, semidefinite if $p < n$) by orthogonal congruence transformation into its normal form

$$\mathfrak{y}'A'A\mathfrak{y} = (A\mathfrak{y})'A\mathfrak{y} = \begin{pmatrix} \mu_1 & & & & & \\ & \ddots & & & 0 & \\ & & \mu_p & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} \qquad (\mu_i > 0)$$

where $\mu_1, \cdots, \mu_p$ are the nonzero eigenvalues of $A'A$.† Hence the vectors $z^{(1)}, \cdots, z^{(p)}$ form an orthogonal system:

$$z^{(i)\prime}z^{(j)} = \begin{cases} 0 & \text{if } j \neq i \\ \mu_i & \text{if } j = i \end{cases} \qquad (i, j = 1, \cdots, p).$$

Further let $\lambda_i = \sqrt{\mu_i}$ and $z^{(i)} = \lambda_i x^{(i)}$. Then $x^{(1)}, \cdots, x^{(p)}$ form an orthonormal vector system in R_m and

$$AY = (\lambda_1 x^{(1)}, \cdots, \lambda_p x^{(p)}) = X\Lambda, \tag{2}$$

where

$$X = (x^{(1)}, \cdots, x^{(p)}), \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{pmatrix}.$$

Since in general Y is an $n \times p$ nonsquare column-orthogonal matrix, (2) is not immediately equivalent to (1). But we observe that $y^{(p+1)}, \cdots, y^{(n)}$ are the eigenvectors of $A'A$ for the eigenvalue 0, *i.e.*, a complete system of linearly independent solutions of the linear system $A'Ay = 0$. As there are also $n - p$ (and not more) linearly independent solutions of the system $Ay = 0$, and these obviously satisfy the former system, it follows that

$$Ay^{(p+1)} = 0, \cdots, Ay^{(n)} = 0 \quad \text{or} \quad AY_0 = 0. \tag{3}$$

† The matrices A and $A'A$ always have the same rank p; *cf.* [2], p. 142, Theorem 19.5.

Therefore $A\mathfrak{y}=A(Y,\ Y_0)=(AY,\ 0)=(X\Lambda,\ 0)$ and

$$A = (X\Lambda, 0)\mathfrak{y}' = (X\Lambda, 0)\begin{pmatrix} Y' \\ Y_0' \end{pmatrix} = X\Lambda Y',$$

i.e., the relation (1).

Since $X'X$ is equal to the $p\times p$ unit matrix, one has immediately from (2)

$$X'AY = \Lambda. \tag{4}$$

This may be considered as the generalized "principal-axes transformation" of an arbitrary real matrix A. As pointed out by Lanczos it neglects the eigenvalue zero.

Applying transposition to (1) and to (4) we obtain the relations

$$A' = Y\Lambda X', \tag{1}'$$

$$Y'A'X = \Lambda, \tag{4}'$$

showing that X and Y are interchanged if A is replaced by A'. Thus X is the $m\times p$ matrix whose columns represent a system of orthonormal eigenvectors of AA' which belong to the nonzero eigenvalues of AA'.

Also X may be completed to an $m\times m$ orthogonal matrix $\mathfrak{x}=(X,\ X_0)$, where $A'X_0=0$ (*cf.* [3]) in the same way as Y to an $n\times n$ orthogonal matrix. We note that because $\mathfrak{x}$ and $\mathfrak{y}$ are orthogonal matrices

$$X'X_0 = 0, \qquad X_0'X = 0; \qquad Y'Y_0 = 0, \qquad Y_0'Y = 0. \tag{5}$$

2. Assuming that for a given matrix A the matrices X, Y, have been determined one can obtain an explicit solution of the linear system $Ay=b$. This has been dealt with by Lanczos [1]. It is briefly resumed here in order to make the connection with a treatment given elsewhere (*cf.* [3], [4]).

With regard to (1) the given system may be written in the form

$$X\Lambda Y'y = b. \tag{6}$$

If a solution y exists then by (5) $X_0'b=0$ and this condition is also known to be sufficient. Thus b is orthogonal to the column vectors of X_0 and is therefore a linear combination of the column vectors of X, *i.e.*, $b=Xb^{(0)}$ so that $b^{(0)}=X'b$ and the system (6) is reduced to $Y'y=\Lambda^{-1}b^{(0)}$.

Using unknown column vectors

$$u = \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_p \end{pmatrix}, \qquad u^{(0)} = \begin{pmatrix} u_{p+1} \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix},$$

we may put

$$v = Yu + Y_0u^{(0)} = u_1y^{(1)} + \cdots + u_py^{(p)} + u_{p+1}y^{(p+1)} + \cdots + u_ny^{(n)}. \tag{7}$$

With regard to (5) it follows that $Y'y = u + Y'Y_0u^{(0)} = u = \Lambda^{-1}b^{(0)}$. Thus $u^{(0)}$ remains unrestricted. By (7) one has then the general solution of (6) in the form

$$y = Y\Lambda^{-1}X'b + Y_0u^{(0)}. \tag{8}$$

The vector $Y\Lambda^{-1}X'b$ represents the "orthogonal solution" (*cf.* [3], [4]) which is orthogonal to every solution $Y_0u^{(0)} = v$ of the homogeneous system $Ay = 0$. This is at the same time the solution vector y of minimum length: $y'y = \min$.

3. Now let us assume that the linear system $Ay = b$ has no solution, *i.e.*, $X_0'b \neq 0$. Instead of asking for a solution one may then ask for a vector y such that the deviation vector $Ay - b$ has minimum length, *i.e.*,

$$\delta^2 = (Ay - b)'(Ay - b) = \min.$$

For A we use again the "orthogonal representation" (1) and for abbreviation we put $Y'A'b = d$, $Y'y = z$. Making use of a device of [5] we can write the sum of the squares of the deviation components in the form

$$\begin{aligned}\delta^2 &= z'\Lambda^2z - z'd - d'z + b'b \\ &= z'\Lambda^2z - z'\Lambda^2\Lambda^{-2}d - d'\Lambda^{-2}\Lambda^2z + b'b \\ &= (z - \Lambda^{-2}d)'\Lambda^2(z - \Lambda^{-2}d) - d'\Lambda^{-2}d + b'b.\end{aligned}$$

Thus δ^2 will assume its minimum value

$$\delta^2_{\min} = b'b - d'\Lambda^{-2}d = b'b - b'X\Lambda Y'Y\Lambda^{-2}Y'Y\Lambda X'b = b'(E - XX')b$$

if and only if $z = \Lambda^{-2}d = \Lambda^{-2}Y'A'b$, which by (1)' is equal to $\Lambda^{-1}X'b$.

It remains to solve the system $Y'y = z$. Making use of (7) as in Section 2 we obtain

$$y = Yz + Y_0u^{(0)} = Y\Lambda^{-1}X'b + Y_0u^{(0)},$$

where $u^{(0)}$ is arbitrary. This is an explicit solution of the central problem in the "method of least squares," valid in all cases. It has the same form as the solution (8) in the case of solvability of the linear system $Ay = b$.

We observe that in the case of nonsolubility the vector b cannot be written in the form $Xb^{(0)}$; thus $b = Xb^{(0)} + X_0b^{(1)}$, where $b^{(1)} \neq 0$ because $X_0'b \neq 0$. Hence one obtains the minimum deviation vector

$$\begin{aligned}Ay - b &= X\Lambda Y'(Y\Lambda^{-1}X'Xb^{(0)} + Y\Lambda^{-1}X'X_0b^{(1)}) - Xb^{(0)} - X_0b^{(1)} \\ &= Xb^{(0)} + 0 - Xb^{(0)} - X_0b^{(1)} = -X_0b^{(1)},\end{aligned}$$

and, with regard to the column-orthogonality of X_0: $\delta^2_{\min} = b^{(1)\prime}b^{(1)}$.

4. We conclude with a few examples.

(a) Extending Lanczos' example (*cf.* [1], p. 672) let A be a matrix of rank $p = 1$. Then there are two nonzero column vectors a, b such that $A = ab'$ and its orthogonal representation according to the theorem will be

$$A = \frac{a}{\|a\|} \Lambda \frac{b'}{\|b\|}, \qquad \Lambda = \|a\|\|b\|$$

if $\|a\| = \sqrt{(a'a)}$. Indeed the Gram matrix $A'A = a'a \cdot bb'$ has only one eigenvalue different from zero, its trace $a'a \cdot b'b = \|a\|^2\|b\|^2$.

The linear system $ab'y = c$ has a solution y if and only if $c = \gamma a$, where γ is a number. By (8) one has

$$y = \frac{b}{\|b\|} \frac{1}{\|a\|\|b\|} \frac{a'}{\|a\|} \gamma a + v = \gamma \frac{b}{\|b\|^2} + v,$$

where v is any vector perpendicular to b.

In the case of nonsolvability any vector y yielding minimum deviation is represented by

$$y = \frac{a'c}{\|a\|^2\|b\|^2} b + v$$

and the minimum deviation vector,

$$Ay - c = \frac{a'c}{a'a} a - c,$$

does not depend on b. The square of its length equals

$$\delta^2_{\min} = \frac{a'a \cdot c'c - (a'c)^2}{a'a}.$$

(b) Let A be a skew-symmetric matrix of rank 2. With regard to [2], p. 153, there are two linearly independent vectors a, b in R_n such that

$$A = ba' - ab'.$$

The Gram matrix

$$A'A = b'b \cdot aa' + a'a \cdot bb' - a'b(ba' + ab')$$

has a and b as eigenvectors, both with the eigenvalue $\mu = a'a \cdot b'b - (a'b)^2 = \lambda^2 > 0$ (by Cauchy-Schwarz's inequality). Thus

$$y^{(1)} = \frac{a}{\|a\|}, \qquad y^{(2)} = \frac{a'a \cdot b - a'b \cdot a}{\|a\|\lambda}$$

form a system of orthonormal eigenvectors of $A'A$. Now, since $Ay^{(1)} = \lambda y^{(2)}$, $Ay^{(2)} = -\lambda y^{(1)}$, one has $x^{(1)} = y^{(2)}$, $x^{(2)} = -y^{(1)}$ and it is readily verified that if $X = (y^{(2)}, -y^{(1)})$, $Y = (y^{(1)}, y^{(2)})$ then

$$X'AY = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \Lambda = \lambda E$$

and also

$$(9) \qquad A = X\Lambda Y' = (\lambda y^{(2)}, -\lambda y^{(1)}) \begin{pmatrix} y^{(1)\prime} \\ y^{(2)\prime} \end{pmatrix} = \lambda(y^{(2)}y^{(1)\prime} - y^{(1)}y^{(2)\prime}).$$

We consider again the linear system $Ay=c$. It has a solution y if and only if $c=Xc^{(0)}=c_1^0x^{(1)}+c_2^0x^{(2)}=c_1^0y^{(2)}-c_2^0y^{(1)}$. By (8)

$$\begin{aligned} y &= Y\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} X'Xc^{(0)} + v \\ &= \lambda^{-1}(c_1^0 y^{(1)} + c_2^0 y^{(2)}) + v \\ &= \lambda^{-1}\left(c_1^0 \frac{a}{\|a\|} + c_2^0 \frac{a'a\cdot b - a'b\cdot a}{\|a\|\lambda}\right) + v. \end{aligned}$$

In the case of nonsolvability, that is if the three vectors a, b, c are linearly independent, the y yielding minimum deviation is given by

$$\begin{aligned} y &= Y\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} X'c + v \\ &= \lambda^{-1}(y^{(1)}y^{(2)\prime} - y^{(2)}y^{(1)\prime})c + v = -\lambda^{-2}Ac + v. \end{aligned}$$

The minimum deviation itself is $Ay-c=-\lambda^{-2}(A^2+\lambda^2E)c$ and

$$\delta_{\min}^2 = \lambda^{-4}c'(A^2 + \lambda^2E)^2c.$$

Now by (9)

$$A^2 = -\lambda^2(y^{(1)}y^{(1)\prime} + y^{(2)}y^{(2)\prime}), \qquad A^4 = \lambda^4(y^{(1)}y^{(1)\prime}+y^{(2)}y^{(2)\prime}) = -\lambda^2A^2.$$

Therefore $(A^2+\lambda^2E)^2=\lambda^2A^2+\lambda^4E=\lambda^2(A^2+\lambda^2E)$ and

$$\begin{aligned} \delta_{\min}^2 &= \lambda^{-2}c'(A^2 + \lambda^2E)c \\ &= \lambda^{-2}(\lambda^2c'c + 2b'c\cdot c'a\cdot a'b - a'a(b'c)^2 - b'b(a'c)^2) \\ &= \lambda^{-2}\begin{vmatrix} a'a & a'b & a'c \\ b'a & b'b & b'c \\ c'a & c'b & c'c \end{vmatrix}. \end{aligned}$$

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THE DIAGONALIZATION OF CERTAIN NORMAL MATRICES

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1. Introduction. A numerical method for the determination of the characteristic values of arbitrary matrices was recently developed by the author, and is described in references [1] and [2]. The basis of this method is a theorem of I. Schur, which states that any arbitrary matrix A of complex elements may be transformed by means of unitary transformations into a triangular matrix having the characteristic values of A along its diagonal. The method proceeds as follows: An arbitrary off-diagonal nonzero element $a_{ij} \equiv b$ is chosen to be the "pivot" of a unitary transform U which differs from the unit matrix only in the positions (i, i), (i, j), (j, i), and (j, j). These positions in U are taken, respectively, by the elements $t = r\exp(i\theta)$, $-(1-r^2)^{1/2}$, $(1-r^2)^{1/2}$, and $\bar{t}$. The quantities r, $0 < r < 1$, and θ are to be determined in such a manner as to minimize the difference

$$M_1 - M = \sum_{\substack{i,j=1 \\ i<j}}^{n} (|a_{ij}^{(1)}|^2 - |a_{ij}|^2),$$

where $A_1 \equiv (a_{ij}^{(1)}) = U^{-1}AU$. It turns out that

$$M_1 - M < H(R, \theta) = \sum_{s=1}^{4} C_{s-1}(\theta) R^s, \tag{1}$$

where $R = r^{-1}(1-r^2)^{1/2}$ and the $C_s(\theta)$ are well-defined expressions in θ involving certain elements of A. The determination of R and θ on the basis of $M_1 - M \leqq \min H(R, \theta)$ leads to the norm-reducing cubic polynomial

$$F(R, \theta) \equiv 4C_3R^3 + 3C_2R^2 + 2C_1R + C_0 = 0. \tag{2}$$

For matrices that are symmetric this method reduces to the well-known and widely used procedure due to Jacobi. Now further investigation of this method has led to additional interesting results, some of which are described below. It was found, namely, that certain classes of normal matrices can be reduced by this method to diagonal form by means of a finite number of transformations only.

2. Arbitrary normal matrices of order 2. Let us consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3}$$

where the elements are arbitrary complex numbers. The assumption that A is normal, *i.e.*, $AA^{CT} = A^{CT}A$ (with A^{CT} denoting the conjugate transpose of A), leads to the two conditions

$$|b| = |c|, \tag{4}$$

$$b(\bar{a} - \bar{d}) = \bar{c}(a - d). \tag{5}$$

Putting $a=|a|\exp(i\alpha)$, $b=|b|\exp(i\beta)$, $c=|c|\exp(i\gamma)$, $d=|d|\exp(i\delta)$, $a-d=|a-d|\exp(i\epsilon)$, it is seen that (5) leads to

$$2\epsilon = \beta + \gamma. \tag{6}$$

Let us now construct the norm-reducing polynomial (2). From [1],

$$C_3 = |c|^2,$$

$$C_2 = -2|c|\,[|d|\cos(\theta+\gamma-\delta) - |a|\cos(\theta+\gamma-\alpha)],$$

$$C_1 = |d-a|^2 - 2|bc|\cos(2\theta-\beta+\gamma) + \sum_{i+1\leq k\leq j-1}[|a_{ki}|^2 + |a_{jk}|^2],$$

$$C_0 = 2|b|\,[|d|\cos(\theta+\delta-\beta) - |a|\cos(\theta+\alpha-\beta)] + 2\sum_{i+1\leq k\leq j-1}[|a_{ik}|\,|a_{jk}|\cos(\theta+\alpha_{jk}-\alpha_{ik}) - |a_{ki}|\,|a_{kj}|\cos(\theta+\alpha_{ki}+\alpha_{kj})].$$

Choosing $\theta=\frac{1}{2}(\beta-\gamma)$ and utilizing (4), (5), and (6), we obtain for the matrix A

$$C_3 = |b|^2, \qquad C_2 = -2|b|\,[|d|\cos(\epsilon-\delta) - |a|\cos(\epsilon-\alpha)],$$

$$C_1 = |d-a|^2 - 2|b|^2, \qquad C_0 = 2|b|\,[|d|\cos(\epsilon-\delta) - |a|\cos(\epsilon-\alpha)].$$

However, expansion shows that

$$|d|\cos(\epsilon-\delta) - |a|\cos(\epsilon-\alpha) = |d-a|. \tag{7}$$

Consequently,

$$F(R,\theta) = 4|b|^2R^3 - 6|b|\,|d-a|\,R^2 + 2(|d-a|^2 - 2|b|^2)R + 2|b|\,|d-a|$$
$$= 2[|b|\,R^2 - |d-a|\,R - |b|\,][2|b|\,R - |d-a|\,].$$

We choose then for R one of the roots of

$$|b|\,R^2 - |d-a|\,R - |b| = 0 \tag{8}$$

and for the transformation variable z, the number $R\exp(i\theta)$, where $\theta=\frac{1}{2}(\beta-\gamma)$. In the transformed matrix

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix},$$

by [1], $b_1=-\bar{t}^2[cz^2-(d-a)z-b]$, $c_1=-t^2[b\bar{z}^2-(d-a)\bar{z}-c]$, where $t=r\exp(i\theta)$, $r^2=(1+R^2)^{-1}$. But

$$cz^2 - (d-a)z - b = \exp(i\beta)[|b|\,R^2 - |d-a|\,R - |b|\,],$$

$$b\bar{z}^2 - (d-a)\bar{z} - c = \exp(i\gamma)[|b|\,R^2 - |d-a|\,R - |b|\,],$$

so that $b_1=c_1=0$. Thus A_1 is diagonalized, with a_1 and d_1 representing the characteristic values of A,

3. Another class of normal matrices. Let us next consider the matrix

$$B = \begin{bmatrix} u & v & w \\ w & u & v \\ v & w & u \end{bmatrix}, \tag{9}$$

where $u = u_1 + iu_2$, v, w are arbitrary complex numbers. Clearly, B is normal: $B^{CT}B = BB^{CT}$. The matrices B are circulants; such matrices are known to have characteristic values, all of which are expressible in closed form in terms of the primitive roots of unity. What is considered of interest here is the fact that the method of [2] diagonalizes these matrices in exactly three well-defined unitary transformations. Causey [3] stated that it takes four sweeps, *i.e.* twelve transformations to triangularize a matrix of the type (9) by Greenstadt's method.

Let us now turn to the diagonalization of B by the method of [2].

Step 1. We choose first $\theta_1 = 0$, $i = 1$, $j = 3$, and put

$$U_1 = \begin{bmatrix} r_1 & 0 & -s_1 \\ 0 & 1 & 0 \\ s_1 & 0 & r_1 \end{bmatrix}. \tag{10}$$

Consequently,

$$B_1 \equiv U_1^{-1}BU_1 = \begin{bmatrix} u + r_1s_1(v + w) & r_1v + s_1w & r_1^2w - s_1^2v \\ r_1w + s_1v & u & r_1v - s_1w \\ r_1^2v - s_1^2w & r_1w - s_1v & u - r_1s_1(v + w) \end{bmatrix}. \tag{11}$$

The value of r_1 will be appropriately determined later on.

Step 2. Let us choose next in (11), $\theta_2 = 0$, $i = 1$, $j = 2$, or

$$U_2 = \begin{bmatrix} r_2 & -s_2 & 0 \\ s_2 & r_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

The value of r_2 is to be fixed in such a manner as to reduce the norm $\sum_{i<j} |a_{ij}^{(2)}|^2$ of the super-diagonal elements $a_{ij}^{(2)}$ in $B_2 = U_2^{-1}B_1U_2$, as outlined in [2]. A short calculation shows that, for this step, the coefficients of the norm-reducing cubic are

$$\begin{aligned} C_3 &= r_1^2|w|^2 + s_1^2|v|^2 + 2r_1s_1(v_1w_1 + v_2w_2), \\ C_2 &= 2r_1s_1[r_1|w|^2 + s_1|v|^2 + (r_1 + s_1)(v_1w_1 + v_2w_2)], \\ C_1 &= r_1^2s_1^2|v + w|^2 - 2[r_1s_1(|v|^2 + |w|^2) + (v_1w_1 + v_2w_2)], \\ C_0 &= -2r_1s_1[s_1|w|^2 + r_1|v|^2 + (r_1 + s_1)(v_1w_1 + v_2w_2)]. \end{aligned}$$

These expressions become greatly simplified for the choice

$$r_1 = s_1 \tag{13}$$

or $r_1 = 2^{-\frac{1}{2}}$. In that case we have

$$C_3 = \tfrac{1}{2}\,|\,v + w\,|^2, \qquad C_2 = (\tfrac{1}{2})^{1/2}\,|\,v + w\,|^2,$$
$$C_1 = -\tfrac{3}{4}\,|\,v + w\,|^2, \qquad C_0 = -(\tfrac{1}{2})^{1/2}\,|\,v + w\,|^2.$$

The cubic polynomial (2) may now be factored into

$$|\,v + w\,|^2[2^{1/2}R^2 + r - 2^{1/2}][2^{1/2}R + 2^{-1}], \tag{14}$$

leading to the roots $R_2 = 2^{-\frac{1}{2}}, -2^{\frac{1}{2}}, -(2\cdot 2^{\frac{1}{2}})^{-1}$. Of these roots R_2, that particular value is to be chosen which makes

$$H(R, \theta) \equiv C_3R^4 + C_2R^3 + C_1R^2 + C_0R \tag{15}$$

a minimum. Thus we take $R_2 = r_1 = 2^{-1/2}$. Then $r_2 = (\frac{2}{3})^{1/2}$ and, consequently,

$$B_2 \equiv U_2^{-1}B_1U_2 = \begin{bmatrix} u + v + w & 0 & 0 \\ 0 & u - {}^{1/2}(v + w) & \tfrac{1}{2}3^{1/2}(v - w) \\ 0 & -\tfrac{1}{2}3^{1/2}(v - w) & u - \tfrac{1}{2}(v + w) \end{bmatrix}. \tag{16}$$

The super-diagonal norm has decreased from $\frac{1}{2}|v+w|^2+\frac{3}{4}|v-w|^2$ in B_1 to $\frac{3}{4}|v-w|^2$ in B_2, *i.e.*, by the norm $|b|^2$ of the pivot b in B_1.

Step 3. We now choose $i = 2, j = 3$, and put $\theta = \frac{1}{2}(\beta - \gamma)$. Since $\beta = \gamma + \pi$, $\theta = \frac{1}{2}\pi$, (2) becomes $R^2 = 1$, so that $R_3 = 1$, whence $r_3 = 2^{-1/2}$, $s_3 = i2^{-1/2}$, and

$$U_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i2^{-1/2} & -2^{-1/2} \\ 0 & 2^{-1/2} & -i2^{-1/2} \end{bmatrix}. \tag{17}$$

It is seen that

$$B_3 \equiv U_3^{-1}B_2U_3 = \begin{bmatrix} u+v+w & 0 & 0 \\ 0 & u-\frac{1}{2}v+w+i3^{1/2}(v-w) & 0 \\ 0 & 0 & u-\frac{1}{2}v+w-i3^{1/2}(v-w) \end{bmatrix}. \tag{18}$$

The diagonalization of B has thus been accomplished by means of the unitary transformation $B_3 = U^{-1}BU$, where

$$U = U_1U_2U_3 = \begin{bmatrix} 3^{-1/2} & -\frac{1}{2} - i\frac{1}{2}3^{-1/2} & \frac{1}{2}3^{-1/2} + i\frac{1}{2} \\ 3^{-1/2} & i3^{-1/2} & -3^{-1/2} \\ 3^{-1/2} & \frac{1}{2} - i\frac{1}{2}3^{-1/2} & \frac{1}{2}3^{-1/2} - i\frac{1}{2} \end{bmatrix}. \tag{19}$$

The diagonal elements of B_3 are, naturally, the characteristic values of B.

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ON THE METHOD OF CONTRACTANTS

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The history of the recently reported "method of contractants" ([2], [3]) is much older than is apparently appreciated. A paper by Charles L. Dodgson (Lewis Carroll) [1] appears to contain all the essential information of the papers by Macmillan [2] and Lotkin [3]. In fact, it goes rather further into the matter of transposing to remove internal zeros in such a manner as to retain, as useful, most of the quantities previously calculated.

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MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

HOW TO PLAY BASEBALL

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1. Introduction. This is supposed to be a mathematics paper, our preposterous title notwithstanding. Now there is very little which mathematics can do about certain aspects of the great American game; we do not expect to make any vital contributions to the axioms "Swing hard," "Pitch good," "Run fast," "Sleep lots," "Don't drink," "Play heads up," etc.

But there is one facet of baseball which does lend itself to the intrusion of

$$V_{32} = \frac{V_{33}^2}{2V_{33} - V_{42}} = \frac{1}{2\cdot 1 - 0} = \frac{1}{2},$$

and so $p_{22} = (1-\frac{1}{2})/(2-\frac{1}{2}) = \frac{1}{3}$. Thus, when the count is 2:2, pitch a strike with probability $\frac{1}{3}$.

Batter's strategy. By setting up the same conditions for the batter, we find that the equations become identical, and we conclude that $q_{m,n} = p_{m,n}$. Therefore, a knowledge of the $p_{m,n}$ is all that is required.

5. Computations. We now give the approximate values obtained for some of the $p_{m,n}$.

	0	1	2	3	m
0	.300	.347	.410	.5	
1	.262	.312	.385	.5	
2	.2	.25	.333	.5	
n					

The exact values are rather complicated fractions, *e.g.*, $p_{20} = \frac{89}{217}$.

One further computation is of interest. We find that $V_{0,0} = .4995$, and so a one-bagger is just about as likely as an out. From this we can find the expected number of runs per inning. The result is .75 runs.

ENUMERATION OF THE POSITIVE RATIONALS

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Let $p_1, p_2, \cdots$ be the primes in increasing order. We define a 1-1 mapping of the positive integers onto the positive rationals by

$$G\colon\ n = \prod_1^\infty p_i^{a_i} \xrightarrow{G} r = \prod_1^\infty p_i^{b_i},$$

where $b_i = a_{2i-1}$ if $a_{2i-1} \neq 0$ and $b_i = -a_{2k}$ if $a_{2i-1} = 0$ and is the kth such a_{2j-1}.

But this seems a bit cumbersome. A more elegant and general set of mappings may be described as follows: Let f be a 1-1 mapping of the positive integers and 0 onto the positive and negative integers such that $f(0) = 0$. Then

$$F\colon\ n = \prod_1^\infty p_i^{a_i} \xrightarrow{F} r = \prod_1^\infty p_i^{f(a_i)}$$

is a 1-1 mapping of the positive integers onto the positive rationals. The obvious choice for f is $f(n) = (-1)^{n+1}[\frac{1}{2}(n+1)]$ where $[x]$ is the greatest integer function.

Thus the enumeration of the positive rationals is reduced to the enumeration of the integers.

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A PROPERTY OF THE SALTUS OPERATOR

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Let f be a bounded real valued function defined on the interval $[0, 1]$. The saltus operator S is defined by the formula

$$(1) \qquad Sf(x) = \limsup_{\epsilon \to 0} \{ |f(y) - f(z)| : |x - y| < \epsilon, |x - z| < \epsilon \}.$$

Thus the saltus operator maps the set of all real bounded functions into itself. In this note we shall prove a conjecture of S. K. Stein, namely, that S satisfies the relationship $S^2 = S^3$; *i.e., for any real valued bounded function f defined on* $[0, 1]$ *we have* $S[Sf] = S[S[Sf]]$.

While this relationship is easily proved, and is possibly well known, it does not seem to be stated explicitly in the literature. The result will be demonstrated with the help of three well-known facts which will be stated as lemmas.

LEMMA 1. *If f is an arbitrary bounded function, defined on* $[0, 1]$, *then Sf is an upper semicontinuous function* ([1], p. 95).

LEMMA 2. *The points of discontinuity of a semicontinuous real function defined on an interval form a set of the first category* ([1], p. 110). *Hence every subinterval of* $[0, 1]$ *must contain a point of continuity.*

LEMMA 3. *Let f be a bounded real function defined on* $[0, 1]$. *Then f is continuous at* $x \in [0, 1]$ *if and only if* $Sf(x) = 0$ ([1], p. 96).

Using the fact that $S^2 f = g$ is a nonnegative function which is equal to zero on a set which is dense in $[0, 1]$, we have for any $x \in [0, 1]$

$$\begin{aligned} Sg(x) &= \limsup_{\epsilon \to 0} \{ |g(y) - g(z)| : |x - y| < \epsilon, |x - z| < \epsilon \} \\ &= \limsup_{\epsilon \to 0} \{ g(y) | : |x - y| < \epsilon \}. \end{aligned}$$

Using the upper semicontinuity of g we have

$$\limsup_{\epsilon \to 0} \{ g(y) | : |x - y| < \epsilon \} = g(x).$$

We have now shown $Sg = g$, which is the desired result.

Reference

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ON THE METHOD OF VARIATION OF PARAMETERS—II

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In [1], the method of variation of parameters was modified and applied to linear differential equations of second order. Here, this method is modified and applied to linear difference equations of second order:

$$y(n+2)+Q(n)y(n+1)+P(n)y(n)=R(n), \tag{1}$$

where $P(n)\neq 0$ and $R(n)\not\equiv 0$ for $n\geqq 0$. Let the complementary solution, $y_c(n)$ be given by $y_c(n)=au(n)+bv(n)$, where $u(n)$ and $v(n)$ are two linearly independent solutions of (1) with $R(n)\equiv 0$. Let the particular integral, $y_p(n)$, be given by $y_p(n)=A(n)u(n)+B(n)v(n)$, where $A(n)$ and $B(n)$ are functions of n to be determined. Then the method of variation of parameters yields the system of two equations [2],

$$u(n+1)\Delta A(n)+v(n+1)\Delta B(n)=0, \tag{2}$$

$$u(n+2)\Delta A(n)+v(n+2)\Delta B(n)=R(n). \tag{3}$$

Since the Wronskian, $W(n)=u(n)v(n+1)-u(n+1)v(n)\neq 0$ for $n\geqq 0$, we find that

$$y_p(n)=-u(n)\sum_{k=0}^{n-1}\frac{v(k+1)R(k)}{W(k+1)}+v(n)\sum_{k=0}^{n-1}\frac{u(k+1)R(k)}{W(k+1)}. \tag{4}$$

It is recalled that (2) is arbitrary, that (3) is determined from (1), and that the general solution of (1) is $y(n)=y_c(n)+y_p(n)$. We note that (2) and (3) remain the same if $y_p(n)=[A(n)+\alpha]u(n)+[B(n)+\beta]v(n)$, where α and β are arbitrary constants. In practice, one chooses $\alpha=\beta=0$.

Let $f(n)\equiv f^*(n, u(n), v(n))$ be defined for $n\geqq 0$. If we introduce $f(n)$ into the righthand side of (2), then (3) assumes a new form. Assuming a particular integral, $Y_p(n)=[A(n)+\alpha]u(n)+[B(n)+\beta]v(n)$, where α and β are constants whose values will be specified later, the new system of equations is

$$u(n+1)\Delta A(n)+v(n+1)\Delta B(n)=f(n), \tag{5}$$

$$u(n+2)\Delta A(n)+v(n+2)\Delta B(n)=R(n)-f(n+1)-Q(n)f(n). \tag{6}$$

We will now show that (5) and (6), accompanied with a proper specification of α and β, yields the particular integral $Y_p(n)\equiv y_p(n)$, as given by (4).

Since $u(n)$ and $v(n)$ satisfy (1) with $R(n)\equiv 0$, we find that

$$\begin{aligned} W(n+1)\Delta A(n) &= -v(n+1)R(n)+v(n+1)f(n+1)-P(n)v(n)f(n),\\ W(n+1)\Delta B(n) &= u(n+1)R(n)-u(n+1)f(n+1)+P(n)u(n)f(n),\\ Y_p(n) &= y_p(n)+g(n), \end{aligned}$$

where

$$\begin{aligned} g(n)=u(n)&\sum_{k=0}^{n-1}\frac{v(k+1)f(k+1)-P(k)v(k)f(k)}{W(k+1)}\\ &-v(n)\sum_{k=0}^{n-1}\frac{u(k+1)f(k+1)-P(k)u(k)f(k)}{W(k+1)}+\alpha u(n)+\beta v(n). \end{aligned} \tag{7}$$

Given $f(n)$, there exist suitable choices of α and β such that $g(n)\equiv 0$. If $f(n)\equiv 0$,

then $g(n)\equiv 0$ if $\alpha=\beta=0$. We show now that there exists no $f(k)\not\equiv 0$ such that $v(k+1)f(k+1)-P(k)v(k)f(k)\equiv 0$ and $u(k+1)f(k+1)-P(k)u(k)f(k)\equiv 0$ simultaneously. Suppose now that such a $f(k)\not\equiv 0$ exists. Since $W(k+1)\equiv P(k)W(k)$, we have $v(k)f(k)\equiv C_1W(k)$ and $u(k)f(k)\equiv C_2W(k)$, where $C_1\neq 0$, $C_2\neq 0$ are fixed constants. From this we have that $C_1u(k)\equiv C_2v(k)$, which contradicts the assumption that $u(k)$ and $v(k)$ are linearly independent solutions of (1).

Recalling the formula for summation by parts,

$$\sum_{k=0}^{n-1}\sigma(k)\Delta H(k)=[\sigma(k)H(k)]_0^n-\sum_{k=0}^{n-1}H(k+1)\Delta\sigma(k),$$

we have

$$(8)\qquad \sum_{k=0}^{n-1}\frac{\Delta[v(k)f(k)]}{W(k+1)}=\left[\frac{v(k)f(k)}{W(k+1)}\right]_0^n-\sum_{k=0}^{n-1}\frac{[1-P(k+1)]v(k+1)f(k+1)}{W(k+2)}.$$

Since $v(k+1)f(k+1)-P(k)v(k)f(k)=\Delta[v(k)f(k)]+(1-P(k))v(k)f(k)$, we have, using (8),

$$(9)\qquad \begin{aligned}&\sum_{k=0}^{n-1}\frac{\Delta[v(k)f(k)]+(1-P(k))v(k)f(k)}{W(k+1)}\\&\quad=\left[\frac{v(k)f(k)}{W(k+1)}\right]_0^n-\sum_{k=0}^{n-1}\Delta\left[\frac{(1-P(k))v(k)f(k)}{W(k+1)}\right]\\&\quad=\left[\frac{v(k)f(k)-(1-P(k))v(k)f(k)}{W(k+1)}\right]_0^n=\left[\frac{v(k)f(k)}{W(k)}\right]_0^n.\end{aligned}$$

Since (9) holds if $v(k)$ is replaced by $u(k)$, we have

$$(10)\qquad g(n)=\alpha u(n)+\beta v(n)+u(n)\cdot\left[\frac{v(k)f(k)}{W(k)}\right]_0^n-v(n)\cdot\left[\frac{u(k)f(k)}{W(k)}\right]_0^n.$$

Thus, $g(n)=\{\alpha-[v(0)f(0)/W(0)]\}u(n)+\{\beta+[u(0)f(0)/W(0)]\}v(n)$. Hence, if $\alpha=v(0)f(0)/W(0)$, $\beta=-u(0)f(0)/W(0)$, then $g(n)\equiv 0$. If $f(0)=0$, then $\alpha=\beta=0$. This can always be done by replacing $f(n)$ by $[f(n)-f(0)]$ in (5) and (6). We note that $u(0)$ and $v(0)$ cannot both be zero for then $W(0)=0$, $W(k)\equiv 0$. In only two cases, when $f(n)\equiv 0$ or $f(n)\not\equiv 0$, $f(0)=0$, can one choose $\alpha=\beta=0$.

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A NEW PLANET

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1. In our times the conception of a new planet is no longer a fantasy *à la* Jules Verne. Such a body is, of course, subjected to the fundamental theorems of mechanics and moves according to Kepler's laws. The planets in existence may be divided into two classes, the *inner* and the *outer* ones, and whatever the differences in their appearance from a geocentric point of view may be, they have in common that their orbits on the celestial sphere are rather complicated; all of them have a period of *direct* motion and a stage in which they are *retrograde*. We consider now a body which is a *new* planet in the sense that it does not belong to either of the two classes because its distance to the sun is the same as the earth's. We simplify the matter by assuming that the orbits in space of all planets are circles (but we allow them to be in different planes) and by neglecting their mutual influences. We ask for the apparent motion of the new planet when it is observed from a terrestial standpoint.

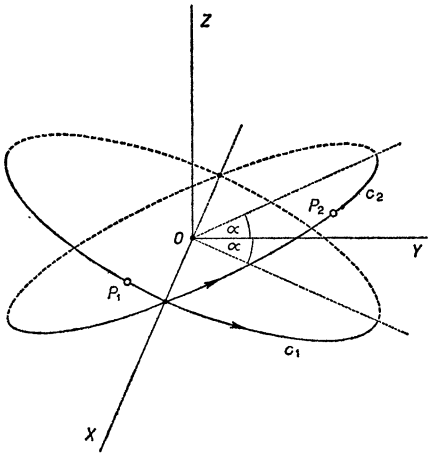


FIG. 1

2. We consider two planets, P_1 (the earth) and P_2 (the new planet) which move in concentric circular orbits about O (the sun) with the same radius R, lying in different planes V_1 and V_2, the angle between which is 2α $(0 \leqq \alpha < \frac{1}{2}\pi)$ (Fig. 1). According to Kepler's laws both planets move with uniform velocity and, moreover, this is the same for both; the latter circumstance is the reason why the relative motion of P_1 and P_2 is rather simple. The two circles c_1 and c_2 have two points of intersection S_1 and S_2. We assume that the two circular motions have a phase-difference so that collision is excluded. Let this difference be 2ψ $(0 < \psi < \pi)$; that means that if P_1 is in S_1, the arc S_1P_2 is 2ψ.

We introduce a cartesian frame as follows. OS_1 is taken as the x-axis, and the bisecting planes of V_1 and V_2 are $y=0$ and $z=0$, such that, seen from the

point (0, 0, 1), the planets are both circulating in the left-hand direction. If ω is the common angular velocity and $t=0$ the moment when both arcs P_1S_1 and S_1P_2 are equal to ψ, the coordinates of P_1 and P_2 are

$$(1)\quad \begin{aligned} x_1 &= R\cos(\omega t-\psi), \quad y_1 = R\sin(\omega t-\psi)\cos\alpha, \quad z_1 = -R\sin(\omega t-\psi)\sin\alpha,\\ x_2 &= R\cos(\omega t+\psi), \quad y_2 = R\sin(\omega t+\psi)\cos\alpha, \quad z_2 = R\sin(\omega t+\psi)\sin\alpha. \end{aligned}$$

The components of the vector P_1P_2 are

$$(2)\quad \begin{aligned} p_1 &= x_2 - x_1 = -2R\sin\omega t\sin\psi,\\ p_2 &= y_2 - y_1 = 2R\cos\alpha\cos\omega t\sin\psi,\\ p_3 &= z_2 - z_1 = 2R\sin\alpha\sin\omega t\cos\psi. \end{aligned}$$

From this it follows immediately that $\sin\alpha\cos\psi\cdot p_1+\sin\psi\cdot p_3=0$, which means that the direction of P_1P_2 is always parallel to the plane through the y-axis with the equation

$$(3)\qquad \sin\alpha\cos\psi\cdot x+\sin\psi\cdot z=0.$$

Hence, seen from the earth, *the new planet moves along a great circle on the celestial sphere.*

The plane V_1 has the equation $\sin\alpha\cdot y+\cos\alpha\cdot z=0$. Therefore, if δ is the acute angle between V and V_1, we have

$$(4)\qquad \cos^2\delta = (\cos^2\alpha\sin^2\psi)/(\sin^2\alpha\cos^2\psi+\sin^2\psi),$$

$$(5)\qquad \operatorname{tg}\delta = (\operatorname{tg}\alpha)/(\sin\psi).$$

Hence *the angle δ between the new planet's apparent orbit and the ecliptic is given by* (5) and it is easily seen that the angle between the x-axis and the line of intersection of V and V_1 is $\frac{1}{2}\pi-\psi$.

3. The fact that all vectors P_1P_2 are parallel to a fixed plane gives rise to the question: What is the locus of the line P_1P_2 in space? The points P_1 and P_2 move along circles which are intersecting but lie in different planes; the correspondence between the points of c_1 and c_2 is a metrically specialized projectivity. Now it is well known from line geometry that the line joining corresponding points of two projective conics generates a *quartic ruled surface.* In our case this F_4 is of a special type because of the way in which the two conics intersect. It is the type numbered VIII in the classification of Salmon.* F_4 has a double conic and a double line, which have a point in common; all generators meet both and the double line is not itself a generator. Applying this to our problem we see that in V_1 there are two generators P_1P_2, namely, when P_2 is either in S_1 or in S_2; obviously these two are parallel. Furthermore there are two generators in V_2, namely, when P_1 is in S_1 or S_2; they too are parallel. When two generators meet their point of intersection is on the double line of F_4. The conclusion is:

* G. Salmon, A Treatise on the Analytic Geometry of Three Dimensions, Dublin, 1882, p. 520.

the double line is in the plane at infinity and therefore all generators are parallel to a fixed plane. The generators in V_1 have the direction $x:y:z=\sin\psi:\cos\psi\cos\alpha:-\cos\psi\sin\alpha$, those in V_2: $x:y:z=-\sin\psi:\cos\psi\cos\alpha:\cos\psi\sin\alpha$ and the plane parallel to both is indeed $\sin\alpha\cos\psi\cdot x+\sin\psi\cdot z=0$. The equation of F_4 can be found from (1). If we put $L=\sin\alpha\cos\psi\cdot x+\sin\psi\cdot z$ and $y=y_1\cos\alpha$ it reads

$$(x^2+y_1^2)L^2-2\sin\alpha\cos\psi\cdot xLw^2+\sin^2\alpha\cdot w^2(\cos^2\psi\cdot w^2-y_1^2)=0.$$

The double line is given by $L=w=0$ and the double conic is the hyperbola $y=xL-\sin\alpha\cos\psi\cdot w^2=0$; their common point is $L=y=w=0$.

4. We return now to the problem proper and ask how the new planet moves along its great circle c lying in V. In order to study the motion in the apparent orbit we can take an arbitrary position as the initial one and we choose for this the y-axis which is in V. We determine the position of the planet by the angle ϕ which P_1P_2 makes with this axis. Then from (2)

$$\cos\phi=\frac{\cos\alpha\sin\psi\cos\omega t}{\sqrt{\{\sin^2\omega t(\sin^2\alpha\cos^2\psi+\sin^2\psi)+\cos^2\omega t\cos^2\alpha\sin^2\psi\}}} \tag{6}$$

or in view of (4)

$$\operatorname{tg}\phi=(\operatorname{tg}\omega t)/(\cos\delta). \tag{7}$$

which gives *the position of the planet as a simple function of the time.* It has the same period as the apparent motion of the sun (which was obvious beforehand); it is seen from (7) that ϕ goes from 0 to $\frac{1}{2}\pi$ if ωt does. The motion however is not uniform; for the angular velocity we have

$$\dot{\phi}=\frac{\omega\cos\delta}{\cos^2\delta+\sin^2\delta\sin^2\omega t}, \tag{8}$$

$\dot{\phi}$ always has the same sign; the new planet always goes in the same direction. *The angular velocity varies between the minimum* $\omega\cos\delta$ *and the maximum* $\omega/\cos\delta$. It is equal to ω when $\operatorname{tg}^2\omega t=\cos\delta$, that is when $\operatorname{tg}\phi=\cos^{-1/2}\delta$. When the planet passes the ecliptic we have $\omega t=-\psi$ or $\omega t=\pi-\psi$ and therefore $\operatorname{tg}\phi=-(\operatorname{tg}\psi)/(\cos\delta)$ and $\dot{\phi}=(\omega\cos^2\alpha)/(\cos\delta)$.

5. We consider some special cases. If $\alpha=0$ then V_1, V_2 and V coincide. The apparent motion of the new planet is along the ecliptic and with uniform angular velocity ω. The distance between P_2 and the sun, as seen from the earth, is $\frac{1}{2}\pi-\psi$ which is an arc between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. The planet moves along the sun's orbit at a constant distance to it which corresponds to a period of at most three months on either side. Such a planet is either a morning star ($\psi<\frac{1}{2}\pi$) or an evening star ($\psi>\frac{1}{2}\pi$); for $\psi=\frac{1}{2}\pi$ it is permanently in conjunction with the sun. Another special case is given by $\psi=\frac{1}{2}\pi$, $\alpha\neq0$. Then if P_1 is in S_1, P_2 passes S_2 and *vice versa*. We have now $\delta=\alpha$ and the plane V coincides with $z=0$. On

the moments that the planet passes the ecliptic it is in conjunction with the sun and its velocity has its minimum value $\omega \cos \delta$. From (1) and (2) it follows that the planet always coincides with the sun's orthogonal projection on the plane V. In this case the double hyperbola of the quartic ruled surface degenerates into two lines. Finally we remark that a new planet's orbit may coincide with the celestial equator. Then we must have $\delta = \epsilon (= 23.5°)$ and the line of intersection of V and V_1 must pass through the equinoxes.

THE TRACES OF CERTAIN MATRICES AND CHROMATIC GRAPHS

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Let M be a symmetric matrix of order n whose diagonal elements are zero, and whose elements are either zero or unity. Let N be the matrix associated with M such that $M+N$ is a matrix whose diagonal elements are zero, but whose off-diagonal elements are unity. Denote $\operatorname{tr}(M^3)+\operatorname{tr}(N^3)$ by t.

Then we have for t the sharp lower bound

$$t \geqq \begin{cases} 2u(u-1)(u-2), & \text{if } n = 2u, \\ 4u(u-1)(4u+1), & \text{if } n = 4u+1, \\ 4u(u+1)(4u-1), & \text{if } n = 4u+3. \end{cases} \tag{1}$$

As we shall see, this result is an easy consequence of the theorem proved in [1].

Since M is an interpretation of a gathering of n people, with unity in position (ij) denoting that the ith and jth persons are acquainted and zero that they are strangers, then the elements in the diagonal of M^3 determine the number of "full triangles" (*i.e.*, mutual acquaintances involving three persons) there are in the gathering. Such triangles will be counted twice (once in each direction) from each vertex. Consequently each "full triangle" will be counted six times, and $\operatorname{tr}(M^3) = 6F$, where F is the number of full triangles. Similarly, the trace of N^3 will determine the number E of "empty triangles" (*i.e.*, mutual strangers involving three persons), and $\operatorname{tr}(N^3) = 6E$. The result (1) then quickly follows from [1].

The question raised in [1] concerning the number of full and empty polygons may possibly be answered by a close investigation of traces of other powers of M and N.

If n points are joined pairwise by lines, some of which are colored red and some colored blue, the resulting configuration is called a chromatic graph [2]. Then with an appropriate interpretation (see [1]), $\frac{1}{6}t$ is the number of solidly colored triangles in the graph. It follows that $t \leqq n(n-1)(n-2)$ and this upper bound is sharp.

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SOME PROPERTIES OF THE FIBONACCI NUMBERS

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The Fibonacci series is defined by $u_0=0$, $u_1=1$, and $u_n=u_{n-1}+u_{n-2}$. Its first few terms are: 0, 1, 1, 2, 3, 5, 8, 13 $\cdots$. The aim of this paper is to prove the

THEOREM. *If* $r>1$ *and* $n>1$, $u_{nr}>u_n^r$.

First we prove

LEMMA 1. $u_n=u_{k+1}u_{n-k}+u_ku_{n-k-1}$ $(0<k+1\leqq n)$.

We shall prove this by induction. The lemma is obvious for $k=0$ and $k=1$. If it is true for $k=i$, we have

$$u_n = u_{i+1}u_{n-i} + u_iu_{n-i-1} = u_{i+1}(u_{n-i-1} + u_{n-i-2}) + u_iu_{n-i-1}$$
$$= u_{n-i-1}(u_{i+1} + u_i) + u_{n-i-2}u_{i+1} = u_{i+2}u_{n-i-1} + u_{i+1}u_{n-i-2},$$

which completes the induction proof.

LEMMA 2. $u_{m+n}>u_mu_n$ $(m, n>0)$.

For if we substitute in the formula of Lemma 1, $m+n$ for n and m for k we have $u_{m+n}=u_{m+1}u_n+u_mu_{n-1}>u_{m+1}u_n\geqq u_mu_n$. We now have $u_{rn}>u_{(r-1)n}u_n$ $>u_{(r-2)n}u_n^2>\cdots>u_n^r$ by repeated application of Lemma 2.

SELF-ADJOINT DIFFERENTIAL EXPRESSIONS

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The differential expression $L(y)=\sum_{i=0}^{n}A_i(x)y^{(i)}(x)$ is self-adjoint if its adjoint $M(y)=\sum_{i=0}^{n}(-1)^i(A_i(x)y(x))^{(i)}$ is identical with $L(y)$. Some properties of self-adjoint expressions are easy to develop, *e.g.*,

(a) There is no odd ordered self-adjoint expression; *i.e.*, if n is odd, then $A_n=0$.

(b) The sum (or difference) of two self-adjoint expressions is self-adjoint.

(c) If $L(y)=Ay^{(2n)}+\cdots$, and $\mathfrak{L}(y)=Ay^{(2n)}+\cdots$, are both self-adjoint, the order of $L(y)-\mathfrak{L}(y)$ is at most $2n-2$.

(d) For a given $A(x)$ (having $2n$ derivatives) there is at most one self-adjoint expression of the form

$$L(y) = Ay^{(2n)} + \sum_{i=1}^{n} A_{2i-1}y^{(2i-1)}. \tag{1}$$

Following up this last observation, one finds that ay, $by''+b'y'$, $cy^{\text{iv}}+2c'y'''-c'''y'$ are self-adjoint and that $cy^{\text{iv}}+2c'y'''+by''+(b'-c''')y'+ay$ is the most general self-adjoint differential expression of the fourth order. It becomes reasonable to expect that for given $A(x)$ there is exactly one self-adjoint expression of

type (1). This expectation is correct but the existence does not seem to be obvious and the exact form of the expression does not seem to appear in the literature. To obtain the desired expression, we start with

THEOREM I. *Let* $\{x_{2r-1}\}$ *and* $\{y_{2r-1}\}$ *be the solution of the algebraic equations*

$$x_{2r-1} + \sum_{i=1}^{r} \binom{2r-1}{2i-1} x_{2i-1} = 1, \qquad \sum_{i=1}^{r} \binom{2r}{2i-1} y_{2i-1} = 1 \qquad (r = 1, 2, \cdots).$$

Then $x_{2r-1} = y_{2r-1} = r^{-1}(2^{2r}-1)B_{2r}$; $B_2, B_4, \cdots$, *are the Bernoulli numbers.** If we let $\tanh \frac{1}{2}x = \sum_{i=1}^{\infty} (c_{2i-1}x^{2i-1})/(2i-1)!$ we get

$$\sum_{r=1}^{\infty} \frac{x^{2r}}{(2r)!} = \cosh x - 1 = \tanh \tfrac{1}{2}x \sinh x$$

$$= \sum_{i=1}^{\infty} \frac{c_{2i-1}x^{2i-1}}{(2i-1)!} \sum_{s=0}^{\infty} \frac{x^{2s+1}}{(2s+1)!} = \sum_{i=1}^{\infty}\sum_{s=0}^{\infty} \frac{c_{2i-1}x^{2(i+s)}}{(2i-1)!(2s+1)!}$$

$$= \sum_{i=1}^{\infty}\sum_{r=1}^{\infty} \frac{c_{2i-1}x^{2r}}{(2i-1)!(2r-2i+1)!} = \sum_{r=1}^{\infty} x^{2r} \sum_{i=1}^{r} \frac{c_{2i-1}}{(2i-1)!(2r-2i+1)!}.$$

Equating coefficients, we get

$$\sum_{i=1}^{r} \frac{c_{2i-1}}{(2i-1)!(2r-2i+1)!} = \frac{1}{(2r)!}; \qquad \sum_{i=1}^{r} \binom{2r}{2i-1} c_{2i-1} = 1; \qquad y_{2r-1} = c_{2r-1}.$$

A similar procedure gives us

$$\sum_{r=1}^{\infty} \frac{x^{2r-1}}{(2r-1)!} = \sinh x = \tanh \tfrac{1}{2}x + \tanh \tfrac{1}{2}x \cosh x$$

$$= \sum_{i=1}^{\infty} \frac{c_{2i-1}x^{2i-1}}{(2i-1)!} + \sum_{i=1}^{\infty} \frac{c_{2i-1}x^{2i-1}}{(2i-1)!} \sum_{s=0}^{\infty} \frac{x^{2s}}{(2s)!}$$

$$= \sum_{r=1}^{\infty} \frac{c_{2r-1}x^{2r-1}}{(2r-1)!} + \sum_{r=1}^{\infty} x^{2r-1} \sum_{r=1}^{r} \frac{c_{2i-1}}{(2i-1)!(2r-2i)!};$$

$$1 = c_{2r-1} + \sum_{i=1}^{r} \binom{2r-1}{2i-1} c_{2i-1}; \qquad x_{2r-1} = c_{2r-1}.$$

Finally, the relation

$$\tanh \frac{1}{2}x = \sum_{i=1}^{\infty} \frac{(2^{2i}-1)B_{2i}x^{2i-1}}{(2i-1)!i}$$

produces the desired result.

* Professor E. J. McShane called my attention to the theorem: *Any result about the binomial coefficients is in the literature.* I cannot produce a suitable reference so I give a proof of this result.

THEOREM II. *The differential expression*

$$L(y) = Ay^{(2n)} + \sum_{i=0}^{n-1} \binom{2n}{2i+1} c_{2n-2i-1} A^{(2n-2i-1)} y^{(2i+1)}, \quad c_{2r-1} = r^{-1}(2^{2r}-1)B_{2r},$$

is self-adjoint.

To prove this theorem, we simplify the adjoint

$$M(y) = \sum_{s=0}^{2n} \binom{2n}{s} A^{(2n-s)} y^{(s)} - \sum_{i=0}^{n-1} \sum_{s=0}^{2i+1} \binom{2n}{2i+1} \binom{2i+1}{s} c_{2n-2i-1} A^{(2n-s)} y^{(s)}$$

by using

$$\sum_{s=0}^{2n} a_s = a_{2n} + \sum_{k=0}^{n-1} (a_{2k} + a_{2k+1}); \quad \binom{2n}{2i+1}\binom{2i+1}{s} = \binom{2n}{s}\binom{2n-s}{2n-2i-1};$$

$$\sum_{i=0}^{n-1} \sum_{s=0}^{2i+1} b_{is} = \sum_{k=0}^{n-1} \sum_{i=k}^{n-1} (b_{i,2k} + b_{i,2k+1}).$$

Thus

$$\begin{aligned} M(y) &= Ay^{(2n)} + \sum_{k=0}^{n-1} \binom{2n}{2k} A^{(2n-2k)} y^{(2k)} \left[1 - \sum_{i=k}^{n-1} \binom{2n-2k}{2n-2i-1} c_{2n-2i-1} \right] \\ &\quad + \sum_{k=0}^{n-1} \binom{2n}{2k+1} A^{(2n-2k-1)} y^{(2k+1)} \left\{ 1 - \sum_{i=k}^{n-1} \binom{2n-2k-1}{2n-2i-1} c_{2n-2i-1} \right\} \\ &= Ay^{(2n)} + \sum_{k=0}^{n-1} \binom{2n}{2k} A^{(2n-2k)} y^{(2k)} \left[1 - \sum_{n=1}^{n-k} \binom{2n-2k}{2r-1} c_{2r-1} \right] \\ &\quad + \sum_{k=0}^{n-1} \binom{2n}{2k+1} A^{(2n-2k-1)} y^{(2k+1)} \left\{ 1 - \sum_{r=1}^{n-k} \binom{2n-2k-1}{2r-1} c_{2r-1} \right]. \end{aligned}$$

Theorem I (with the r replaced by $n-k$) is applicable; the expression in brackets becomes zero, the expression in braces becomes $c_{2n-2k-1}$, and we have $M(y) = L(y)$.

Theorem II along with the remark (d) allows us to state that the most general self-adjoint differential expression of order $2n$ is

$$\sum_{s=0}^{n} A_s y^{(2s)} + \sum_{s=1}^{n} \sum_{k=0}^{s-1} \binom{2s}{2k+1} \frac{2^{2s-2k}-1}{s-k} B_{2s-2k} A_s^{(2s-2k-1)} y^{(2k+1)}.$$

Using $B_1 = -\frac{1}{2}$, $0 = B_3 = B_5 = \cdots$, this can also be written as

$$\sum_{s=0}^{n} \sum_{k=0}^{2s} (-1)^{k+1} \binom{2s}{k} \frac{2^{2s-k+1}-1}{2s-k+1} 2B_{2s-k+1} A_s^{(2s-k)} y^{(k)}.$$

THE NUMBER OF UNITARY DIVISORS OF AN INTEGER

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In this note n will represent a positive integral variable, and x a real variable, $x \geqq 2$. Let $\tau(n)$ denote the number of positive divisors of n. It is a classical elementary result of Dirichlet, dating from 1849, that

$$T(x) \equiv \sum_{n \leqq x} \tau(n) = x(\log x + 2\gamma - 1) + O(\sqrt{x}), \tag{1}$$

where γ is Euler's constant.

Suppose now that $t(n)$ represents the number of *unitary* divisors of n; that is, the number of (positive) divisors d of n such that d is relatively prime to the complementary divisor n/d. In 1874 Mertens proved that

$$T^*(x) \equiv \sum_{n \leqq x} t(n) = \frac{x}{\zeta(2)}\left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + O(\sqrt{x} \log x), \tag{2}$$

where $\zeta(s)$ is the Riemann ζ-function and $\zeta'(s)$ its derivative.

Proofs of (1) have appeared in several texts in English, for example, LeVeque ([4], Th. 6.31). Although proofs of (2) have been reproduced in German texts (compare Landau [3], p. 666), none seems to be readily accessible in English. It is the purpose of the present note to correct this situation.

Mertens's proof of (2) is based on the Dirichlet estimate (1) for $T(x)$. In the more direct proof below we shall require only the much weaker estimate

$$T(x) = O(x \log x). \tag{3}$$

In place of the divisor function, our proof will be based mainly upon properties of totient functions.

Let $\phi(x, n)$ denote the number of positive integers $\leqq x$ which are prime to n, and place $\phi(n) = \phi(n, n)$. We note the following elementary results:

$$\phi(x, n) = \phi(n)(x/n) + O(\tau(n)), \tag{4}$$

uniformly in $x \geqq 1$;

$$\sum_{n \leqq x} \frac{\phi(n)}{n} = \frac{x}{\zeta(2)} + O(\log x); \tag{5}$$

$$\sum_{n \leqq x} \frac{\phi(n)}{n^2} = \frac{1}{\zeta(2)}\left(\log x - \frac{\zeta'(2)}{\zeta(2)} + \gamma\right) + O\left(\frac{\log x}{x}\right). \tag{6}$$

For a proof of (6), see [2], Lemma 2.4. The result (5) is well known; a generalization may be found in Wintner ([5], $(29)_4$, $f(n) = \phi(n)/n$). The relation (4) actually holds with $\tau(n)$ replaced by $t(n)$ and follows readily from the familiar evaluation of $\phi(x, n)$ in terms of the Möbius function ([1], Lemma 3.4; also *cf.* [6]).

Proof of (2). With (d, δ) denoting the greatest common divisor of d and δ, we have

$$T^*(x) = \sum_{n\leqq x} \sum_{d\delta=n}{}' 1 = \sum_{d\delta\leqq x}{}' 1,$$

where the prime indicates that in the summation $(d, \delta)=1$. If $d\delta\leqq x$, then clearly d and δ cannot simultaneously assume values $>\sqrt{x}$. Hence

$$T^*(x) = 2 \sum_{d\delta\leqq x, d\leqq\sqrt{x}}{}' 1 - \sum_{d\leqq\sqrt{x}, \delta\leqq\sqrt{x}}{}' 1 = 2 \sum_{d\leqq\sqrt{x}} \phi(x/d, d) - \sum_{d\leqq\sqrt{x}} \phi(\sqrt{x}, d),$$

the prime again indicating that $(d, \delta)=1$. Applying (4), $T^*(x)$ becomes

$$T^*(x) = 2x \sum_{n\leqq\sqrt{x}} \frac{\phi(n)}{n^2} - \sqrt{x} \sum_{n\leqq\sqrt{x}} \frac{\phi(n)}{n} + O(T\sqrt{x}). \tag{7}$$

Application to (7) of (3), (5), and (6) yields $(x\geqq 4)$

$$T^*(x) = 2x\left(\frac{1}{\zeta(2)}\left(\frac{1}{2}\log x - \frac{\zeta'(2)}{\zeta(2)} + \gamma\right) + O\left(\frac{\log x}{\sqrt{x}}\right)\right) - \sqrt{x}\left(\frac{\sqrt{x}}{\zeta(2)} + O(\log x)\right) + O(\sqrt{x}\log x),$$

from which (2) results, on simplification. The extension of the result to the full range $x\geqq 2$ is immediate.

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ASYMPTOTICS I: A NOTE ON LAPLACE'S METHOD*

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Laplace's method has long been an important tool for examining the behavior of

$$I(x) = \int_a^b e^{-xh(t)} g(t)\,dt$$

as $x\to\infty$. This note is for the purpose of weakening the hypotheses under which an asymptotic expansion of $I(x)$ can be obtained. We treat the case where $[a, b]$ contains only one critical point (*i.e.*, minimum of h). For this case we have the best possible result in a certain sense: all we ask is that the integral exist for

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some x_0 and that the asymptotic behavior of h and g be known at the critical point, but we make no smoothness assumptions.

We make no attempt to survey the literature of this problem, but refer to Erdélyi (*Asymptotic Expansions*, New York, 1956) who treats the problem in Section 2.4. We are indebted to the referee for the remark that the possibility of removing the smoothness conditions was mentioned by Bekessy (Magyar Tud. Akad., vol. 2, 1957), and that a remark by Erdélyi at the end of Section 2.4 hints of this possibility.

For simplicity, and without loss of generality, we make the following normalizations: (1) the lower limit of integration is the critical point, (2) the lower limit of integration is 0, and (3) $f(0)=0$.

THEOREM. *Let h and g be measurable functions on $[0, b]$ which satisfy the following conditions*:

(1) *There is an x_0 for which $e^{-x_0 h}g \in L[0, b]$.*

(2) *For each a, $0<a<b$, there is an $A>0$ so that $h(t) \geqq A$ if $t \geqq a$.*

(3) $h(t)=t^{\nu}\sum_0^N a_k t^k+o(t^{N+\nu})$ *as* $t\to 0$, *where* $\nu>0$, $a_0>0$.

(4) $g(t)=t^{\lambda-1}\sum_0^N b_k t^k+o(t^{N+\lambda-1})$ *as* $t\to 0$, *where* $\lambda>0$ *and not all b's are zero.*

Then $I(x)=\int_a^b e^{-xh}g\,dt$ *exists for* $x \geqq x_0$ *and*

$$I(x) = \sum_0^N c_k x^{-(k+\lambda)/\nu} + o(x^{-(N+\lambda)/\nu}) \text{ as } x \to \infty,$$

where the c's are determined by the a's and b's.

Proof. $I(x)$ exists for $x \geqq x_0$: $e^{-x_0 h}g$ forms a dominating function since $h \geqq 0$. For any a, $0<a<b$ we have

$$I(x) = \int_0^a e^{-xh}g\,dt + \int_a^b e^{-xh}g\,dt = I_1(x) + I_2(x).$$

Now

$$|I_2(x)| = \left|\int_a^b e^{-xh}g\,dt\right| \leqq e^{-(x-x_0)A}\int_a^b e^{-x_0 h}|g|\,dt = \text{const. } e^{-Ax}.$$

For any given ϵ choose a so small that

$$\left|h(t) - t^{\nu}\sum_0^N a_k t^k\right| \leqq \epsilon t^{N+\nu} \text{ in } [0, a],$$

$$h_{\pm}(t) \equiv t^{\nu}\sum_0^N a_k t^k \pm \epsilon t^{N+\nu}$$

are both strictly increasing in $[0, a]$, and $g(t)$ is of one sign throughout $[0, a]$. This can be done since g will have the sign of the first nonvanishing b in its asymptotic development. Without loss of generality we may assume that $g(t)$ is nonnegative.

We next define I_+ and I_- by

$$I_+(x) = \int_0^a e^{-xh+g}\,dt, \qquad I_-(x) = \int_0^a e^{-xh-g}\,dt.$$

Now h_+ and h_- both satisfy the conditions of Erdélyi's theorem. Thus

$$I_\pm(x) = \sum_0^{N-1} c_k x^{-(k+\lambda)/\nu} + c_N(\epsilon)x^{-(N+\lambda)/\nu} + o(x^{-(N+\lambda)/\nu}),$$

where it is easy to see from Erdélyi's proof that

(1) c_k is independent of ϵ for $k=0, 1, \cdots, N-1$,

(2) $c_N(\epsilon)$ is linear in ϵ: $c_N(\epsilon) = c_N \mp \epsilon c_N'$,

(3) the o-term depends on ϵ.

Thus, since $I_+ \leqq I_1 \leqq I_-$ we easily get

$$\left| \left(I - \sum_0^N c_k x^{-(k+\lambda)/\nu} \right) x^{(N+\lambda)/\nu} \right| \leqq \epsilon c_N' + o(1) + O(e^{-Ax} \cdot x^{(N+\lambda)/\nu}).$$

Letting $x \to \infty$ and then $\epsilon \to 0$ we see that $I = \sum_0^N c_k x^{-(k+\lambda)/\nu} + o(x^{-(N+\lambda)/\nu})$.

The precise evaluation of the c's in terms of the a's and b's is described by Erdélyi. In particular we might mention that if b_n is the first nonvanishing b then c_n is the first nonvanishing c and then $c_n = a_0^{(n+\lambda)/\nu}\Gamma((n+\lambda)/\nu)b_n/\nu$. If all the b's vanish then these same calculations show that $I(x) = o(x^{-(N+\lambda)/\nu})$.

ON A PROBLEM OF L. FEJES-TÓTH*

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L. Fejes-Tóth has shown† that a point in 3-space may be "hidden" by six congruent spheres; more precisely, six congruent, nonintersecting closed spheres in E^3 may be placed around a point P (not belonging to any of the spheres) in such a way that each ray issuing from P intersects at least one of the spheres§. He also raised the question as to the least number of spheres for which such an arrangement is possible.

We shall show that five congruent, disjoint spheres may not "hide" a point, and prove the slightly stronger

THEOREM. *Let a point P and five congruent, open, disjoint spheres S_i, $1 \leqq i \leqq 5$, be given in E^3. If P belongs to none of the spheres S_i then there exists a ray R, issuing from P, such that $R \cap (\bigcup_{i=1}^5 \overline{S}_i)$ is either the point P, or is empty.*

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† Unpublished to the best of our knowledge. We are indebted to Professor V. L. Klee for this information.

§ It is not difficult to see that such configurations may be obtained by arbitrarily small, suitable transformations of the six spheres of radius 1 whose centers are (in cylindrical coordinates (r, ϕ, z)): $(0, 0, 1)$, $(0, 0, -1)$, $(\sqrt{3}, \phi_0, 0)$, $(\sqrt{3}, -\phi_0, 0)$, $(\sqrt{3}, 3\phi_0, 0)$, $(\sqrt{3}, -3\phi_0, 0)$ where $\phi_0 = \arccos \sqrt{\frac{2}{3}}$. The "hidden" point is $(0, 0, 0)$.

The proof of the theorem will follow from some simple lemmata.

Throughout this note we denote by S_i congruent, open, disjoint spheres, and by P a (fixed) point belonging to none of the spheres S_i. With any family of spheres S_i we associate the family of (open) spherical caps C_i on the surface of a fixed sphere S with center P, where C_i is obtained by projecting S_i onto Bd S by rays issuing at P. We denote by α_i the (angular) diameter of C_i, and by A_i the center of S_i.

LEMMA 1. *For any two spheres S_1 and S_2 we have* $\operatorname{diam}(C_1 \cap C_2) < \frac{1}{2}\pi$.

Proof. Since $\operatorname{diam}(C_1 \cap C_2)$ does not decrease if the centers A_i of the spheres S_i are moved toward P we may, without loss of generality, assume that P belongs to Bd S_1 and that the closed spheres $\overline{S}_1$ and $\overline{S}_2$ touch in a point. Let T be the plane passing through P and tangent to $\overline{S}_1$. If A_2 belongs to T or to that open half-space determined by T which contains A_1, an easy computation yields $\operatorname{diam}(C_1 \cap C_2) = \alpha_2 \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{1}{2}\pi$. On the other hand, if T separates A_1 from A_2 it follows by elementary trigonometry that the angle ϕ, spanned at P by $T \cap \overline{S}_2$, is given by

$$\phi = 2 \arcsin \frac{[(1 - \cos\vartheta)\cos\vartheta]^{1/2}}{\sin\vartheta},$$

where $\vartheta = \measuredangle PA_1A_2$. Since in this case $\operatorname{diam}(C_1 \cap C_2) \leqq \phi$, and $\phi < \frac{1}{2}\pi$ for $\vartheta > 0$, we have to settle only the case $\vartheta = 0$. But then $C_1 \cap C_2 = \varnothing$, *i.e.*, $\operatorname{diam}(C_1 \cap C_2) = 0$. This completes the proof of the lemma.

LEMMA 2. *If* $\overline{C}_1 \cap \overline{C}_2 \cap \overline{C}_3 \neq \varnothing$ *then* $\min(\alpha_1, \alpha_2, \alpha_3) \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{5}{12}\pi$.

Proof. From the assumption of the lemma it follows that there exists a ray R, issuing from P, such that $R \cap \overline{S}_i \neq \varnothing$ for $i = 1, 2, 3$. The centers A_1, A_2, A_3, of the spheres are contained in one of the closed half-spaces determined by the plane T passing through P and orthogonal to R. Let A_3 be not nearer to P than A_1 and A_2 (*i.e.*, $\alpha_3 \leqq \alpha_1, \alpha_2$). Since any motions of S_1 and S_2 (subject to the limitations $S_i \cap S_j = \varnothing$ for $i \neq j$, $\overline{S}_i \cap R \neq \varnothing$ for $i = 1, 2, 3$, and $\alpha_1, \alpha_2 \geqq \alpha_3$) do not change $\alpha_3 = \min \alpha_i$, it is easily seen that S_1 and S_2 may be moved (keeping S_3 fixed) in such a way that $A_1, A_2 \in T$ (and therefore R is their common tangent). But in this case a simple computation shows that $\alpha_3 \leqq \arcsin \frac{2}{3}\sqrt{2}$, which ends the proof of Lemma 2.

LEMMA 3. *For any three spheres S_i, $i = 1, 2, 3$, we have* $\min(\alpha_1, \alpha_2, \alpha_3) \leqq \frac{2}{3}\pi$.

Proof. Let T be a plane containing the centers A_i of the spheres S_i. (If the A_i are not collinear, T is unique.) If P is orthogonally projected onto T, the angles α_i increase. But if $P \in T$, then α_i obviously equals the angle spanned at P by the circle $T \cap \overline{S}_i$, and the assertion of the lemma is clearly true. Indeed, we have $\min \alpha_i < \frac{2}{3}\pi$ unless $\overline{S}_i \cap \overline{S}_j \neq \varnothing$ and P is the centroid of the (equilateral) triangle A_1, A_2, A_3, in which case $\alpha_i = \frac{2}{3}\pi$ for $i = 1, 2, 3$.

Proof of the theorem. Let us assume that there exists a configuration of five spheres S_i contradicting the theorem. If $P \in \overline{S}_1 \cap \overline{S}_2$ then $\overline{C}_3 \cap \overline{C}_4 \cap \overline{C}_5$ must cover a great circle on Bd S. But this is impossible since, by Lemma 2, $\alpha_i \leqq \arcsin \frac{2}{3}\sqrt{2} < \frac{1}{2}\pi$ for $i=3, 4, 5$. Therefore P belongs at most to one of the spheres $\overline{S}_i$. By the assumption made on S_i we have $\bigcup_{i=1}^{5} \overline{C}_i \supset \text{Bd } S$. We shall prove this to be impossible. Let $\alpha_1 \geqq \alpha_2 \geqq \alpha_i$ for $i=3, 4, 5$. Since in this case $\operatorname{diam}(\overline{C}_1 \cap \overline{C}_2) = \operatorname{diam}(C_1 \cap C_2) < \frac{1}{2}\pi$ by Lemma 1, the complement of $C_1 \cup C_2$, and thus also $\overline{C}_3 \cup \overline{C}_4 \cup \overline{C}_5$, contains an arc $> \frac{3}{2}\pi$ of a great circle on Bd S. Now, at least two of the three numbers $\alpha_3, \alpha_4, \alpha_5$ are smaller than $\arcsin \frac{2}{3}\sqrt{2}$ (by Lemma 2) since obviously there exist points of Bd S covered by three of the sets $\overline{C}_i$ and not all such points are contained in one of the sets $\overline{C}_i$. Assume $\alpha_4, \alpha_5 \leqq \arcsin \frac{2}{3}\sqrt{2}$; then $\alpha_3 > \frac{3}{2}\pi - 2 \arcsin \frac{2}{3}\sqrt{2} > \frac{2}{3}\pi$. But since $\alpha_1 \geqq \alpha_3$, $\alpha_2 \geqq \alpha_3$, we have reached a contradiction to Lemma 3. This completes the proof of the theorem.

Remark. It is easily seen that 4 spheres of different sizes may "hide" a point. On the other hand it seems probable that 4 spheres of two (possibly also three) different sizes may not "hide" a point.

THE FALSITY OF A CERTAIN ASYMPTOTIC RELATION

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On page 56 of the booklet* by Ernst Trost there is the statement that $p_1 \cdot p_2 \cdot \cdots \cdot p_{\pi(N)} \sim e^N$, which for $N = p_n$ implies

$$M_n = \prod_{p \leqq p_n} p \sim e^{p_n}. \tag{1}$$

This statement was suspect because its acceptance together with some newly derived results of F. B. Correia implied the truth of Cramér's conjecture which is $d_n = p_{n+1} - p_n = O(\log^2 p_n)$. The following theorem is therefore offered to show the falsity of Trost's statement.

THEOREM. *The relation* (1) *is false.*

Proof. If we suppose the contrary, then $M_n = e^{p_n}(1+o(1))$ and $\log M_n = p_n + \log(1+o(1)) = p_n + o(1)$. Thus

$$\log p_{n+1} = \log M_{n+1} - \log M_n = p_{n+1} - p_n + o(1) = d_n + o(1),$$

$$\frac{d_n}{\log p_n} = \frac{\log p_{n+1}}{\log p_n} + o\left(\frac{1}{\log p_n}\right) = \frac{\log p_{n+1}}{\log p_n} + o(1).$$

Since $\log p_{n+1} \sim \log p_n$ we have $(\log p_{n+1})/(\log p_n) = 1 + o(1)$ and $d_n/\log p_n = 1 + o(1) + o(1) = 1 + o(1)$, so that $\lim_{n\to\infty} d_n/\log p_n = 1$. This is a contradiction since it is known† that $\limsup_{n\to\infty} d_n/\log p_n = \infty$.

* Primzahlen, Bd. II, Basel-Stuttgart, 1953.

† Karl Prachar, Primzahlverteilung, Berlin-Gottingen-Heidelberg, 1957, p. 157.

CLASSROOM NOTES

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THE CENTROID IN ABSOLUTE GEOMETRY

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This paper deals with properties of the centroid of a tetrahedron that do not require any assumption about parallelism. We need only Hilbert's axioms of incidence, order, and congruence ([3], pp. 3–14).

Let us state specifically two theorems that are simple consequences of the axioms. (I) Every line segment has a midpoint ([3], p. 25). (II) The line joining the midpoints of two sides of a triangle is orthogonal to the perpendicular bisector of the third side ([1], Sec. 77; [3], p. 44). Furthermore, without quoting them verbally, we will make use of almost all the usual theorems on perpendicular lines and planes whose proof does not involve parallels ([2], pp. 182–183; [4], p. 43). We are now ready to prove the

THEOREM. *The lines joining the midpoints of opposite edges of a tetrahedron intersect in the centroid.*

For the proof let A, B, C, D be the vertices of a tetrahedron. Midpoints (I) are simply denoted by (AB), etc. Then the line through (AC), (BC) is perpendicular to the perpendicular bisector of edge AB in the plane ABC (II). A similar statement applies to the perpendicular bisector of AB in plane ABD. Clearly, line AB is perpendicular to the plane of the perpendicular bisectors mentioned and so are planes ABC and ABD. Hence line (AC), (BC) is orthogonal to the plane of the perpendicular bisectors and similarly line (AD), (BD). It follows that the two lines are coplanar. Hence lines (AC), (BD) and (AD), (BC) intersect in a point G. For reasons of analogy line (AB), (CD) passes through this point also. Now the planes $AD(BC)$; $BD(AC)$; $CD(AB)$ pass through point G and line DG. The intersections of these three planes and plane ABC are the medians $A(BC)$; $B(AC)$; $C(AB)$. Thus we see that the medians are concurrent at the point of intersection of line DG and plane ABC. This proves that G is the centroid of the tetrahedron and incidentally that the medians of a triangle intersect.

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ARITHMETIC, GEOMETRIC INEQUALITY

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The fact that, for $x_1, \cdots, x_n$ positive,

$$\sqrt[n]{(x_1 \cdots x_n)} \leqq \frac{x_1 + \cdots + x_n}{n}, \tag{1}$$

has many proofs abounding in the literature. We give a new variant which seems much more direct and which is just the thing for a class equipped with mathematical induction and little else.

(1) is trivial for $n=1$, now suppose it to hold for $n=k$ and assume $x_1x_2 \cdots x_{k+1}=1$, $x_i>0$. It is our burden to prove $x_1+\cdots+x_{k+1}\geqq k+1$.

Since $x_1x_2 \cdots x_{k+1}=1$ it follows that one of the x_i's is $\geqq 1$. We may take this to be x_{k+1} and obtain $x_1x_2 \cdots x_k=1/x_{k+1}$. By (1), which is assumed to hold for $n=k$, we obtain $x_1+x_2+\cdots+x_k\geqq k/\sqrt[k]{x_{k+1}}$ and so $x_1+x_2+\cdots+x_{k+1} \geqq k/\sqrt[k]{x_{k+1}}+x_{k+1}$. The proof will be complete, then, once we establish

LEMMA. *If* $x\geqq 1$ *then* $k/\sqrt[k]{x}+x\geqq k+1$.

(The fact is that, for $x\geqq 1$, $k/\sqrt[k]{x}+x$ is an increasing function, as is easily shown by differentiation, but if differentiation is outlawed then the following proof may be used.)

Proof. First recall that, for $a\geqq 0$, $(1+a)^{k+1}\geqq 1+(k+1)a$. This is easily established by induction or, for that matter, by the binomial theorem. Hence, with $a=\sqrt[k]{x}-1$, we obtain $x\cdot\sqrt[k]{x}\geqq(k+1)\sqrt[k]{x}-k$, or, transposing and dividing, $k/\sqrt[k]{x}+x\geqq k+1$.

That the equality holds in (1) only when $x_1=x_2=\cdots=x_n$, can also be obtained by a trivial modification of the above proof.

ON SIMPLE CONTINUED FRACTIONS

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In most texts on higher algebra the expression $p_nq_{n-r}-p_{n-r}q_n$ is evaluated in terms of a's for $r=1, 2, 3, 4$, where p_n/q_n is the nth approximant (*i.e.*, convergent) to the continued fraction:

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n+}\cdots. \tag{1}$$

The aim of this paper is to evaluate that expression for any positive integer $r<n$ in the case of two particular simple continued fractions.

The numerator p_n and denominator q_n of the nth approximant p_n/q_n to the simple continued fraction (1) are defined by the equations:

$$p_n = a_np_{n-1} + p_{n-2}, \qquad q_n = a_nq_{n-1} + q_{n-2}, \qquad (n > 2).$$

From this it follows that $p_{n-2}=p_n-a_np_{n-1}$, $q_{n-2}=q_n-a_nq_{n-1}$. Now letting $K_{n,s}=p_nq_s-p_sq_n$, we have

$$K_{n,s} = K_{n,s+2} - a_{s+2}K_{n,s+1}. \tag{2}$$

Putting $s=n-1, n-2, \cdots, n-r$ in succession in (2) and then adding these r relations we get

$$K_{n,n-r} = (K_{n,n-1} - K_{n,n-r+1}) - \sum_{i=1}^{r-1} a_{n+1-i}K_{n,n-i} \qquad (1 < r < n),$$

since $K_{n,n+1}=K_{n,n-1}$ and $K_{n,n}=0$.

Now consider the following simple continued fraction with one recurring element:

$$a + \frac{1}{a+}\,\frac{1}{a+}\cdots. \tag{3}$$

Knowing that $\binom{r}{m}=r!/m!(r-m)!$, and $\binom{r}{m}=0$, $m>r$, $m<0$; and $\binom{r}{m}=1$ for $m=0$, $r=m=0$; we claim that $K_{n,n-r}=K_{n,n-r+2}-a_{n-r+2}K_{n,n-r+1}$ satisfies the relation

$$K_{n,n-r} = (-)^{r-1} \sum_{p=0}^{[\frac{1}{2}(r-1)]} \binom{r-1-p}{p} a^{r-1-2p}K_{n,n-1} \qquad (1 < r < n),$$

which can be restated as follows:

$$\begin{aligned} K_{n,n-2r} &= -\sum_{p=0}^{r-1}\binom{2r-1-p}{p} a^{2r-1-2p}K_{n,n-1}, \\ K_{n,n-(2r+1)} &= \sum_{p=0}^{r}\binom{2r-p}{p} a^{2r-2p}K_{n,n-1}. \end{aligned} \tag{4}$$

We prove (4) by induction on r. It is easily seen that (4) is true for $r=1$. Suppose it is true for $r=R$. Then

$$\begin{aligned} K_{n,n-2(R+1)} &= K_{n,n-2R} - aK_{n,n-(2R+1)} \\ &= -\left[\sum_{p=0}^{R-1}\binom{2R-1-p}{p} a^{2R-1-2p} + \sum_{p=0}^{R}\binom{2R-p}{p} a^{2R+1-2p}\right]K_{n,n-1}. \end{aligned}$$

But

$$\sum_{p=0}^{R-1}\binom{2R-1-p}{p} a^{2R-1-2p} = \sum_{q=1}^{R}\binom{2R-q}{q-1} a^{2R+1-2q} = \sum_{q=0}^{R}\binom{2R-q}{q-1} a^{2R+1-2q}.$$

Therefore

$$K_{n,n-2(R+1)} = -\left[\sum_{p=0}^{R}\left\{\binom{2R-p}{p-1} + \binom{2R-p}{p}\right\} a^{2R+1-2p}\right]K_{n,n-1}$$

$$= -\sum_{p=0}^{R}\binom{2R+1-p}{p}a^{2R+1-2p}K_{n,n-1}.$$

Next

$$K_{n,n-\{2(R+1)+1\}} = K_{n,n-(2R+1)} - aK_{n,n-2(R+1)}$$

$$= \left[\sum_{p=0}^{R}\binom{2R-p}{p}a^{2R-2p} + \sum_{p=0}^{R}\binom{2R+1-p}{p}a^{2R+2-2p}\right]K_{n,n-1}$$

$$= \left[\sum_{q=1}^{R+1}\binom{2R+1-q}{q-1}a^{2R+2-2q} + \sum_{p=0}^{R}\binom{2R+1-p}{p}a^{2R+2-2p}\right]K_{n,n-1}$$

$$= \left[\sum_{q=0}^{R+1}\binom{2R+1-q}{q-1}a^{2R+2-2q} + \sum_{p=0}^{R+1}\binom{2R+1-p}{p}a^{2R+2-2p}\right]K_{n,n-1}$$

$$= \sum_{p=0}^{R+1}\binom{2R+2-p}{p}a^{2R+2-2p}K_{n,n-1}.$$

Thus (4) is true for $r=R+1$ whenever it is true for $r=R$. This completes the proof.

We thus get the

THEOREM. *If p_n/q_n and p_{n-r}/q_{n-r} $(r=2, 3, \cdots, n-1)$ are the nth and $(n-r)$th approximants to the continued fraction* (3), *then*

$$p_n/q_n - p_{n-r}/q_{n-r} = \frac{(-)^{n+r-1}}{q_n \cdot q_{n-r}} \cdot \left\{\sum_{p=0}^{[\frac{1}{2}(r-1)]}\binom{r-1-p}{p}a^{r-1-2p}\right\},$$

since $K_{n,n-1} = (-)^n$.

For the simple continued fraction with two recurring elements:

$$a + \frac{1}{b+}\ \frac{1}{a+}\ \frac{1}{b+}\cdots$$

$K_{n,n-(2r+1)}$ and $K_{n,n-2r}$ satisfy the relations

$$K_{n,n-(2r+1)} = \sum_{p=0}^{r}\binom{2r-p}{p}(ab)^{r-p}K_{n,n-1} \quad (n \text{ odd or even}),$$

$$K_{n,n-2r} = -\sum_{p=0}^{r-1}\binom{2r-1-p}{p}a^{r-p}b^{r-1-p}K_{n,n-1} \quad (n \text{ odd}),$$

$$K_{n,n-2r} = -\sum_{p=0}^{r-1}\binom{2r-1-p}{p}b^{r-p}a^{r-1-p}K_{n,n-1} \quad (n \text{ even}).$$

The proof by induction on r is omitted.

A METHOD OF CALCULATING THE CHARACTERISTIC EQUATION OF A MATRIX

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The purpose of this note is to make readily available to students and teachers of matrix algebra an interesting, and relatively simple method of finding the characteristic equation and then the inverse of a matrix. This method is briefly mentioned in references [1] and [5], but the proof is omitted and does not seem to be readily available in the literature. The total number of arithmetic operations required by this method is distinctly greater than for many routines designed purely for matrix inversion. However, the method is straightforward, finite and easily checked. I have found that programming this procedure for both small and large digital computers is relatively easy. The greatest advantage of this method is the bulk of useful information obtained.

DEFINITION. *The characteristic polynomial* $f(\epsilon)$ *of the square matrix* A *is defined by* $f(\epsilon)=\det[\epsilon I-A]$. *The characteristic equation of* A *is* $f(\epsilon)=0$, *and the roots of this equation are the characteristic roots, or eigenvalues, of* A.

It is a simple matter to show that $f(\epsilon)$ is monic, of degree equal to the order of the matrix, and that the constant term is $(-1)^n \det[A]$, n being the order of A [6]. The remaining coefficients in the characteristic equation are not so readily obtained, and it is this topic that I shall pursue further. Once the characteristic equation has been obtained, it may be used to obtain the inverse of the matrix. The theoretical justification here is the classic Cayley-Hamilton theorem: *A matrix satisfies, in the matrix sense, its own characteristic equation* [6].

DEFINITION. *The trace of a matrix* A, $\operatorname{tr}(A)$, *is the sum of the diagonal elements of* A.

The result in question, Theorem 5 below, follows from the following familiar theorems. Consider

$$f(\epsilon) = \det[\epsilon I - A] = \prod(\epsilon - \epsilon_i) = \epsilon^n + a_1\epsilon^{n-1} + \cdots + a_{n-1}\epsilon + a_n.$$

THEOREM 1. [6]. $\sum\epsilon_i = -a_1 = \operatorname{tr}(A)$.

THEOREM 2. [3]. $a_j = (-1)^j \sum\epsilon_1 \cdots \epsilon_j$.

THEOREM 3 [2]. (Newton's formula for symmetric polynomials.) *If* S_k *denotes the symmetric polynomial* $\sum(\epsilon_j)^k$, *and if* p_j *denotes the symmetric polynomial* $\sum\epsilon_1 \cdots \epsilon_j$, *then, for* $r \leqq n-1$

$$S_r - p_1S_{r-1} + p_2S_{r-2} - p_3S_{r-3} + \cdots + (-1)^r(r)p_r = 0.$$

THEOREM 4 [6]. *If* ϵ_i *is a characteristic root of* A, *then* ϵ_i^k *is a characteristic root of* A^k *and* $\operatorname{tr}(A^k) = \sum(\epsilon_i)^k = S_k$.

THEOREM 5. *The coefficients of* $f(\epsilon)$ *may be evaluated by the recursion formula*

$$a_k = [-1/k][a_{k-1}S_1 + a_{k-2}S_2 + \cdots + S_k]; \qquad S_k = \operatorname{tr}(A^k).$$

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Proof. By Theorem 2, $a_j=(-1)^j\sum\epsilon_1\cdots\epsilon_j=(-1)^jp_j$, and by Theorem 4, $S_j=\operatorname{tr}(A^j)=\sum(\epsilon_i)^j$. Using Newton's formula we have,

$$S_j - p_1S_{j-1} + p_2S_{j-2} + \cdots + (-1)^{j-1}p_{j-1}S_1 + (-1)^jjp_j = 0.$$

Since $a_j=(-1)^jp_j$, this is equivalent to $S_j+a_1S_{j-1}+a_2S_{j-2}+\cdots+a_{j-1}S_1+ja_j=0$, from which it follows that $a_j=[-1/j][a_{j-1}S_1+a_{j-2}S_2+\cdots+S_j]$.

Example.

$$A=\begin{bmatrix}1&0&2&-1&4\\5&3&-1&0&1\\8&5&-3&1&4\\6&2&0&0&1\\0&1&4&2&0\end{bmatrix},\qquad A^2=\begin{bmatrix}11&12&12&9&11\\12&5&14&-4&19\\15&6&36&-3&26\\16&7&14&-4&26\\49&27&-13&4&19\end{bmatrix},$$

$$A^3=\begin{bmatrix}221&125&18&23&113\\125&96&53&40&105\\315&218&20&73&207\\139&109&87&50&123\\104&43&186&-24&175\end{bmatrix},\qquad A^4=\begin{bmatrix}1128&624&715&23&1104\\1269&738&415&138&848\\2003&1107&1180&119&1631\\1680&985&400&194&1063\\1663&1186&307&432&1179\end{bmatrix},$$

$$A^5=\begin{bmatrix}10106&6597&3903&1795&8019\\9107&5413&3947&842&7612\\17692&11090&5883&2439&13958\\10969&6406&5427&846&9499\\12641&7136&5935&1002&9498\end{bmatrix}.$$

We then find $S_1=1$, $S_2=67$, $S_3=562$, $S_4=4419$, and $S_5=31746$. The recursion formula of Theorem 5 then yields $a_1=-1$, $a_2=-33$, $a_3=-154$, $a_4=-373$, and $a_5=382=(-1)^5\det[A]$. The characteristic polynomial is

$$\epsilon^5 - \epsilon^4 - 33\epsilon^3 - 154\epsilon^2 - 373\epsilon + 382.$$

From the Cayley-Hamilton theorem we have

$$A^{-1} = -1/382[-373I - 154A - 33A^2 - A^3 + A^4],$$

$$A^{-1}=\frac{-1}{382}\begin{bmatrix}17&103&-7&-143&12\\-22&-358&54&230&-38\\-39&-79&61&-9&-50\\89&337&-149&-97&-72\\-58&98&-66&16&4\end{bmatrix}.$$

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PROPOSITION 2. *Let A and B be totally ordered finite sets having p and q elements, respectively. Denote by F_{pq}^{+} (resp., I_{pq}^{+}, S_{pq}^{+}, B_{pq}^{+}) the number of the increasing functions (resp., injections, surjections, bijections) defined in A with values in B. The following formulae hold (with $\binom{-1}{0}=1$):*

$$F_{pq}^{+}=\binom{p+q-1}{p},\quad I_{pq}^{+}=\binom{q}{p},\quad S_{pq}^{+}=\sum_{i=0}^{q}(-1)^{i}\binom{q}{i}\binom{p+q-i-1}{p},$$

$$B_{pq}^{+}=\delta_{pq}.$$

The proofs of the second and fourth formulae can be found in [1], the first formula is an immediate consequence of the second one, and the proof of the third formula is similar to that one given in Proposition 1.

COROLLARY. *Let p and q be natural numbers. Then*

a) $$\sum_{i=0}^{p}(-1)^{i}\binom{p}{i}\binom{2p-i-1}{p}=1.$$

b) $$p<q \text{ implies } \sum_{i=0}^{q}(-1)^{i}\binom{q}{i}\binom{p+q-i-1}{p}=0.$$

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EXISTENCE OF A MAXIMUM OF A CONTINUOUS FUNCTION

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The existence theorem for a maximum of a function continuous on a closed interval is often stated in terms of the boundedness of the function and the attainment of its least upper bound. Perhaps because of this many of the standard proofs of the theorem depend essentially on the existence and properties of the least upper bound of a bounded function. This note contains a version of the familiar bisection proof in which the existence of a maximum is proved directly and the idea of a least upper bound is not mentioned. The proof is based on the fact that a bounded increasing sequence has a finite limit.

There seem to the author to be two advantages in using such a proof in an elementary course in analysis. Firstly, the required result is obtained in one stage and not two. Secondly, such a proof makes possible the postponement of the definition of a least upper bound until after the basic theorems on continuity and the derivative (and even the Riemann integral if it is defined as a limit) have been proved. Of course, the standard "compactness" proof based on the Heine-Borel theorem has both these properties and is much more general, but it cannot be called elementary.

THEOREM. *If the real-valued function f is continuous on the closed interval $[a, b]$, there is a number $\xi \in [a, b]$ such that $f(x) \leqq f(\xi)$ for every $x \in [a, b]$.*

Proof. Let $a_0 = a$ and $b_0 = b$. We define a_{n+1} and b_{n+1} inductively in terms of a_n and b_n. Let $c_n = \frac{1}{2}(a_n + b_n)$. If there is a number $u \in [a_n, c_n]$ such that $f(v) \leqq f(u)$ for every $v \in [c_n, b_n]$, let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. If not (that is, if for each $u \in [a_n, c_n]$ there is a number $v \in [c_n, b_n]$ such that $f(v) > f(u)$), let $a_{n+1} = c_n$ and $b_{n+1} = b_n$. In either case

(i) $a_n \leqq a_{n+1} \leqq b_{n+1} \leqq b_n$;
(ii) $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$;
(iii) if $x \in [a_n, b_n]$, there is a number $y \in [a_{n+1}, b_{n+1}]$ such that $f(x) \leqq f(y)$.

This inductive process defines a sequence of intervals $[a_n, b_n]$ $(n = 0, 1, 2, \cdots)$ with the properties:

(i)′ $a_m \leqq a_n \leqq b_m \leqq b_0$ if $n \geqq m \geqq 0$;
(ii)′ $b_n - a_n \to 0$ as $n \to \infty$;
(iii)′ if $x \in [a, b]$ and N is a nonnegative integer, there is a number $y \in [a_N, b_N]$ such that $f(x) \leqq f(y)$.

By (i)′, the sequence a_n tends to a finite limit, ξ say, as $n \to \infty$, and $\xi \in [a_m, b_m]$ for $m = 0, 1, 2, \cdots$. In particular, $\xi \in [a, b]$. We shall prove that $f(x) \leqq f(\xi)$ for every $x \in [a, b]$.

Let $x^* \in [a, b]$. Choose $t > f(\xi)$. Since f is continuous on $[a, b]$ and $\xi \in [a, b]$, there is a positive number δ such that $f(y) < t$ whenever $y \in [a, b]$ and $|y - \xi| < \delta$. By (ii)′ we can choose N so that $b_N - a_N < \delta$, and then, by (iii)′, we can choose $y^* \in [a_N, b_N]$ so that $f(x^*) \leqq f(y^*)$. Since $\xi \in [a_N, b_N]$, we have $|y^* - \xi| \leqq b_N - a_N < \delta$. Also $y^* \in [a, b]$. Hence, $f(y^*) < t$ and, consequently, $f(x^*) < t$. It follows that $f(x^*) \leqq f(\xi)$, for if this were not so we could take $t = f(x^*)$ and obtain a contradiction.

We remark, in conclusion, that the above proof applies without essential change to an upper-semicontinuous mapping f of $[a, b]$ into any linearly ordered set T. (One extra step is required to deal with the case when there is no $t \in T$ such that $t > f(\xi)$; in this case we cannot have $f(x^*) > f(\xi)$ since $f(x^*) \in T$, and so we must have $f(x^*) \leqq f(\xi)$.) Also, it is not difficult to replace $[a, b]$ by any closed bounded subset of the real line (or of real Euclidean n-space). But for such results it is more natural to use the Heine-Borel theorem.

A POWER SERIES DEVELOPMENT OF THE CONVOLUTION THEOREM

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Let the function $F(t)$ have the Laplace transform

$$L[F(t)] = f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

and recall the well-known formula

$$(1)\qquad L\left[\int_0^t\int_0^t\cdots\int_0^t F(t)dt^{n+1}\right] = f(s)/s^{n+1},\ n \text{ a nonnegative integer.}$$

This formula is really just a special case of the convolution theorem. For, if the identity*

$$\int_0^t\cdots\int_0^t F(t)dt^{n+1} = \int_0^t \frac{F(z)(t-z)^n}{n!}\,dz,$$

is used to replace the multiple integral in (1) by a single integral, there follows

$$(2)\qquad L\left[\int_0^t F(z)(t-z)^n dz\right] = f(s)(n!/s^{n+1}) = L[F(t)]\cdot L[t^n],$$

which we recognize as the convolution theorem as it pertains to $F(t)$ and nonnegative powers of t.

In order to extend the theorem to $F(t)$ and the polynomial

$$P(t) = \sum_{k=0}^{n} a_k t^k, \qquad a_1, a_2, \cdots, a_n \text{ constants},$$

we observe, in view of (2) and the linearity of the operator L, that

$$L\left[\int_0^t F(z)P(t-z)dz\right] = \sum_{k=0}^{n} a_k L\left[\int_0^t F(z)(t-z)^k dz\right]$$

$$= f(s)\sum_{k=0}^{n} a_k k!/s^{k+1} = L[F(t)]L[P(t)].$$

Now let $G(t)$ be any function which admits of a Maclaurin series representation $G(t) = \sum_{k=0}^{\infty} G^{(k)}(0)t^k/k!$, $t>0$. Then, proceeding formally, we obtain

$$L\left[\int_0^t F(z)G(t-z)dz\right] = L\left[\int_0^t F(z)\sum_{k=0}^{\infty} G^{(k)}(0)(t-z)^k/k!\,dz\right]$$

$$= \sum_{k=0}^{\infty} G^{(k)}(0)/k!\int_0^t F(z)(t-z)^k dz$$

$$= f(s)\sum_{k=0}^{\infty} G^{(k)}(0)/s^{k+1} = L[F(t)]L[G(t)].$$

Having arrived at the convolution theorem by means of these elementary considerations, it is natural to suspect that the theorem is true under less stringent conditions on $G(t)$. Thus, the stage is set for a more general, and more rigorous treatment of the theorem using classical methods.

* F. B. Hildebrand, Methods of Applied Mathematics, Englewood Cliffs, N. J., 1952, p. 384.

(4.5) THEOREM. *The hyperbola with foci f_1 and f_2 and semitransverse axis of length a is the union of two ellipses, each with semimajor axis of length $\frac{1}{4}c-a$. The foci of one ellipse are the points f_1 and $\pi(f_2)$. The foci of the other are f_2 and $\pi(f_1)$.*

Consequently:

(4.6) THEOREM. *The hyperbola with foci f_1 and f_2 and semitransverse axis of length a is identical to the hyperbola with foci $\pi(f_1)$ and $\pi(f_2)$ and semitransverse axis of length a.*

We also observe that one elliptic component of a hyperbola is the image under the projection π of the other.

From (1.3), (2.4), and (4.5) we have: *the hyperbola is not empty if and only if*

$$-d(f_1, f_2) \leqq 2a \leqq d(f_1, f_2), \tag{4.7}$$

There are several special cases:

(4.8) If $f_1=f_2$, then $2a=0$ and the hyperbola is the whole sphere.
(4.9) If $f_1\neq f_2$ and $2a=0$, then the hyperbola is a single great circle.
(4.10) If $f_2=\pi(f_1)$ and $2a\neq 0$ then the hyperbola is a pair of circles the planes of which are perpendicular to the diameter joining f_1 and f_2.
(4.11) If $2a=d(f_1, f_2)$ or $-d(f_1, f_2)$, then the hyperbola is the union of the two arcs $[f_1, \pi(f_2)]$ and $[f_2, \pi(f_1)]$.

ON THE DEFINITION OF THE RIEMANN INTEGRAL*

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The now classic approach to the theory of the Riemann integral of a bounded function f on a closed interval $[a, b]$, $a<b$, is based on the concept of a partition. A *partition* P of $[a, b]$ is any finite ordered set of numbers $\{x_0, x_1, \cdots, x_p\}$ satisfying

$$a = x_0 < x_1 < \cdots < x_p = b.$$

The intervals $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{p-1}, x_p]$ are the *subdivisions* of P; the *norm* of P is the length of its longest subdivision.

Using P, we now form a *Riemann sum*

$$\sum_{k=1}^{p} f(\xi_k)(x_k - x_{k-1}), \qquad x_k \text{ in } P, \tag{1}$$

where ξ_k is *any* number in the kth subdivision of P. We shall say that P *underlies* the sum (1). (Note that in general may different sums may have the same

* The writing of this paper was facilitated by the Carnegie Foundation, under a grant to the Illinois Institute of Technology.

underlying partition.) What happens now has been aptly characterized by E. J. McShane ([2], p. 6):

" . . . We take some kind of limit of this sum. The question is, what kind of limit. Sometimes one hears 'the limit as the number of subdivisions tends to infinity,' with some qualifying remark about the lengths of the subdivisions. But this sum is not a single-valued function of the number of subdivisions. Some kind of extension of the concept of limit is called for. One alternative is to look sternly at the student and say 'That's perfectly clear, isn't it?' . . . A better way is to widen the theory to cover multiple-valued functions, treating the value of a sum as a multiple-valued function of the length of the largest subinterval A third way . . . is to use the Moore-Smith limit"

The purpose of this note is to discuss a fourth way, in which the integral is defined, rigorously and simply, as an ordinary limit of an ordinary sequence of numbers. Now "sequence" definitions of integrals are not new. Both N. Wiener ([4], p. 366) and K. Menger ([3], p. 141) have defined

$$\int_a^b f = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + (k-\alpha)\frac{b-a}{n}\right)\frac{b-a}{n}, \tag{2}$$

with $\alpha=0$, $\alpha=\frac{1}{2}$, respectively, whenever the limit exists. This definition yields the Riemann integral of all Riemann-integrable functions, and has the advantages of simplicity, adaptability to manual or machine computations, and ease of proof of certain properties of the integral, *e.g.*, linearity. Against these advantages, however, are certain drawbacks. For example, the function g defined by setting

$$g(x) = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}, \end{cases}$$

is integrable under (2), (though not Riemann-integrable). However, we have $\int_0^1 g = 1$, but $\int_0^c g = 0$ for *any* irrational c; and c may be arbitrarily close to 1. Thus under (2), the integral is not a continuous function of the interval of integration.

This fact is related to the fact that under (2) the basic property of *interval-additivity*, expressed by

$$\int_a^b f + \int_b^c f = \int_a^c f, \tag{3}$$

will not be generally valid. Taking g to be the function defined above and c to be any irrational number, we have, for example: $\int_0^1 g + \int_1^c g = 1 + 0 \neq 0 = \int_0^c g$. Furthermore, as pointed out by Menger ([3], p. 142), even for functions for which (3) is valid, this validity is not easy to derive from (2).

These drawbacks can be overcome by the following procedure: For any integer n, let $P_n(a, b)$ consist of the numbers a, b, and all numbers between a and b of the form $m/2^n$, m an integer. If there are $p+1$ numbers (including a and b) in $P_n(a, b)$, let these be denoted in their order of magnitude by x_0,

$x_1, \cdots, x_p$. Thus $P_n(a, b)$ is a partition of $[a, b]$, but of a special type. We now define Darboux-type upper and lower sums

$$ {}_n\overline{S}_a^b f = \sum_{k=1}^{p} M_k(x_k - x_{k-1}), \qquad {}_n\underline{S}_a^b f = \sum_{k=1}^{p} m_k(x_k - x_{k-1}), $$

where M_k and m_k denote, respectively, the least upper bound and the greatest lower bound of f on the kth subdivision of $P_n(a, b)$. We can show without difficulty that the ${}_n\overline{S}$'s form a bounded nonincreasing sequence and the ${}_n\underline{S}$'s a bounded nondecreasing sequence. Since *any* bounded monotonic real sequence has a real limit, it follows that

$$ \lim_{n\to\infty} {}_n\overline{S}_a^b f \quad \text{and} \quad \lim_{n\to\infty} {}_n\underline{S}_a^b f $$

both exist. We write

$$ \lim_{n\to\infty} {}_n\overline{S}_a^b f = \overline{\int_a^b} f, \qquad \lim_{n\to\infty} {}_n\underline{S}_a^b f = \underline{\int_a^b} f, \tag{4} $$

and call these limits, respectively, the *upper* and the *lower integral* of f on $[a, b]$. If the upper and lower integrals are equal, we call their common value the *integral* of f on $[a, b]$ and denote it by the usual symbol. Accordingly, whenever the integral exists, we have

$$ \int_a^b f = \lim_{n\to\infty} {}_n\overline{S}_a^b f = \lim_{n\to\infty} {}_n\underline{S}_a^b f. \tag{5} $$

The upper and lower integrals of (4) coincide with the Darboux-Riemann upper and lower integrals; while the integral of (5) coincides with the ordinary Riemann integral. To see this, we need only observe, firstly, that the partitions P_n underlying the sums ${}_n\overline{S}$ and ${}_n\underline{S}$ have norms which are not greater than 2^{-n} (and consequently approach zero as n tends to infinity). We then appeal to the following theorems:* Any sequence of upper (lower) sums converges to the upper (lower) Darboux-Riemann integral, provided only that the underlying partitions have norms which approach zero; if the upper and lower Darboux-Riemann integrals are equal, their common value is the Riemann integral, and conversely, if the Riemann integral exists, the upper and lower Darboux-Riemann integrals are equal.

However, it should be noted that *the entire theory of integration can be based on* (4) *and* (5) *without any direct appeal to the customary theory.* For example, the fundamental property (3) can be derived from (4) and (5) on the basis of the following simple observation: By virtue of the definitions of P_n, ${}_n\overline{S}$ and ${}_n\underline{S}$, we have, for any n, and for $a \leqq b \leqq c$,

* Proofs may be found in, *e.g.*, Graves [1], pp. 86–87.

(6) $$ {}_n\overline{S}_a^b f + {}_n\overline{S}_b^c f = {}_n\overline{S}_a^c f + \alpha_n, \qquad {}_n\underline{S}_a^b f + {}_n\underline{S}_b^c f = {}_n\underline{S}_a^c f + \beta_n, $$

where α_n and β_n satisfy $|\alpha_n| \leqq 2^{-n} \sup |f|$, $|\beta_n| \leqq 2^{-n} \sup |f|$. Letting n tend to infinity in (6), we therefore obtain

$$ \overline{\int}_a^b f + \overline{\int}_b^c f = \overline{\int}_a^c f, \qquad \underline{\int}_a^b f + \underline{\int}_b^c f = \underline{\int}_a^c f, $$

from which (3) follows as a corollary.

It is likewise not hard to prove the *existence* of the integral for continuous functions (if the concept of uniform continuity is available); it is easier still, indeed almost trivial, to demonstrate this existence for *monotonic* functions. Finally, it may be noted that, without unduly great effort, it is possible to show that for functions integrable under (4) and (5), the partitions P_n can be replaced by arbitrary partition schemes, and the sums ${}_n\overline{S}$, ${}_n\underline{S}$ by general Darboux-Riemann sums, or by ordinary Riemann sums. In this way, if desired, the classic definition of the Riemann integral can be recovered. In particular, the partitions P_n can be modified by replacing the factor 2^{-n}, which occurs in their definition, by the factors 3^{-n}, or 10^{-n}, or indeed by k^{-n}, where k is any integer $\geqq 1$.

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A SYSTEM ISOMORPHIC TO THE REALS

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In certain constructive approaches to the real number system, the nonnegative (or, strictly, unsigned) reals are encountered prior to the introduction of the (signed) reals—positive, zero, and negative. Various technical devices are available to effect the extension from the nonnegative reals to the reals. The reals can, for example, be introduced as equivalence classes of couples (ordered pairs) of nonnegative reals where two such couples, (u, v) and (r, s), are defined to be equivalent if and only if $u+s=v+r$. The traditional algebraic approach postulates the existence of additive inverses and introduces the negative reals as "tagged" positive reals, the minus sign playing a dual role as an operator (for subtraction) and as a tag for the negative reals.

In this paper we proceed heuristically and informally from the unsigned reals in decimal form $N \cdot a_1 a_2 a_3 \cdots$ (N integral and the a_i's digits from 0 to 9) to a

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novel system exhibiting the formal properties of the signed reals. Our method of development initially will be one of blind faith and unabashed juggling of symbols (in the tradition of the eighteenth century analysts), following where our intuitions lead. Only later will a definition introduce meaning and show that we have not been led to absurdity.

Consider now the equation $x+3=0$. In a system of tagged reals $x=-3$ is a solution. If we remove the restriction, in our decimal expansions, that N be an integer—*i.e.*, that our decimals terminate on the left—there is another (formal) solution, namely, $x=\cdots 9997.$, as suggested by the formal calculation:

$$\begin{array}{r} \cdots 9997. \\ +\ 3 \\ \hline \cdots 0000. \end{array}$$

providing we take $\cdots 0000.$ in our new system as an expansion of zero. If further we take $\cdots 0001.=1$, $\cdots 0002.=2$, etc., we have by similar experimentation the following equivalences among the elements of our new system and the system of tagged reals

$$\begin{aligned} \cdots 9999. &= -1 \\ \cdots 9998. &= -2 \\ \cdots 9997. &= -3 \\ &\ \ \vdots \quad\ \vdots \\ \cdots 9990. &= -10 \\ &\ \ \vdots \quad\ \vdots \\ \cdots 9900. &= -100 \\ &\ \ \vdots \quad\ \vdots \end{aligned}$$

As another example, a formal solution to the equation $x+5=10x$, namely $x=\cdots 4445.$, suggests the equivalence $\cdots 4445.=\frac{5}{9}$, since by formal calculation we have

$$\begin{array}{r} \cdots 4445. \\ +\ 5 \\ \hline \cdots 44450. \end{array} \qquad \text{and} \qquad \begin{array}{r} \cdots 4445. \\ \times\ 10 \\ \hline \cdots 44450. \end{array}$$

The same equivalence is suggested by the calculations:

$$\begin{array}{r} \cdots 4445. \\ \times\ 9. \\ \hline \cdots 0005. \end{array} \qquad\qquad 9/\overline{\cdots 4445.}\ \cdots 0005.$$

and $\cdots 4445.=1+(\cdots 444.)=1+\frac{4}{9}(\cdots 999.)=1+\frac{4}{9}(-1)=\frac{5}{9}$.

If we consider symbols nonterminating in either direction—right or left—from the decimal point we have, for example, $\cdots 999.999\cdots = (\cdots 999.) + (.999\cdots) = -1+1=0$ and thus $\cdots 999.999\cdots$ reveals itself as another representation of zero. Representations in our system are thus seen to be not unique. Indeed we have by similar considerations

$$0 = \cdots 111\cdot 111\cdots = \cdots 222\cdot 222\cdots = \cdots 121212\cdot 121212\cdots$$
$$= \cdots xxx\cdot xxx\cdots \text{ (where x is any digit or block of digits).}$$

Indeed any real number has an infinitude of representations as the following suggest:

$$\tfrac{1}{2} = \cdots 000.5000\cdots = \cdots 000.4999\cdots = \cdots 111.6111\cdots$$
$$= \cdots 222.7222\cdots = \cdots 5556.0555\cdots = \cdots 6667.1666\cdots$$
$$= \cdots 101010.601010\cdots = \cdots 121212.621212\cdots$$
$$\tfrac{1}{7} = \cdots 142857142857143.000\cdots = \cdots 000.142857142857\cdots$$
$$= \cdots 1428571428571.571428571428\cdots$$
$$\sqrt{2} = \cdots 0001.4142\cdots = \cdots 1112.5253\cdots = \cdots 2223.6364\cdots$$
$$-\sqrt{2} = \cdots 9998.5857\cdots = \cdots 8887\cdot 4746\cdots$$

(where the representations of $\sqrt{2}$ and $-\sqrt{2}$ are meant to be periodic to the left but not to the right).

Examples could be multiplied ad nauseam. We have displayed a number sufficient to motivate the following:

DEFINITION. *A double-decimal is a symbol composed of a decimal point preceded and followed by an infinite array of digits, the array to the left of the decimal point being periodic from some position to the left on.*

A double-decimal is thus seen to be of the form

$$\cdots \overline{c_nc_{n-1}\cdots c_2c_1}\,\overline{c_nc_{n-1}\cdots c_2c_1}b_mb_{m-1}\cdots b_1b_0.a_1a_2a_3\cdots$$

where the a's, b's, and c's are digits, 0 to 9, and $\overline{c_nc_{n-1}\cdots c_2c_1}$ is the part of the number repeating to the left. Thus $\cdots 000.101001000100001\cdots$ (where the number of 0's between two 1's increases by one at each stage) is a double-decimal whereas $\cdots 100001000100101.000\cdots$ is not.

We define the *value* of the double-decimal given above to be

$$\frac{1'}{1-10^n}(10^{m+1}c_1 + 10^{m+2}c_2 + \cdots + 10^{m+n}c_n)$$
$$+ b_0 + 10b_1 + \cdots + 10^mb_m + \sum_{i=1}^{\infty}\frac{a_i}{10^i}.$$

The double-decimal is said to *represent* its value. Definitions of equality,

the arithmetic operations, and order can be readily supplied (see [3]). For example, two double-decimals are to be *equal* if their values are equal, *i.e.*, if their difference is some expansion of zero.

We conclude with the following observations:

(1) Of course there is nothing special about the radix ten. Any other base could be used.

(2) Every real number has an infinitude of double-decimal representations. Every double-decimal represents a unique real number.

(3) In view of (2), the reals can be defined as equivalence classes of double-decimals, where equivalence should be so defined as to constitute equality of value. The arithmetic operations are readily defined along well-known lines [3]. In the resulting system the reals (or their double-decimal representations) are not tagged and the minus sign reverts to its aboriginal role as operator. Of course the double-decimals can be tagged if it is desirable to do so.

(4) Considerations of technical efficacy would mitigate against the use of double-decimals in computations. For example, order loses the easy perceptibility of ordinary decimal expansions. Thus $\cdots 999.000 \cdots < \cdots 8889.000 \cdots$. Indeed the former number is negative (-1) whereas the latter is positive $(+\frac{1}{9})$.

(5) The process of "rounding off" can be carried out on the right as usual but not on the left. $\cdots 333.333000 \cdots$ is a fair approximation to $\cdots 333.333 \cdots$ but $\cdots 000333.333 \cdots$ is poor. And the more 3's we take on the left (short of all of them) the poorer it becomes!

(6) Notations of course have such meaning as we choose to give them (within limits imposed by requirements of consistency and adequacy). We have assigned no meaning to decimal arrays which are not periodic to the left.

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REMAINDER FORMULAE IN TAYLOR'S THEOREM

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It is not uncommon in an elementary calculus course to develop Taylor's formula with a remainder described on the one hand in integral form and on the other by use of an intermediate value of a derivative of the function being represented. Thus if we write

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n,$$

we have

$$R_n = \int_a^x \frac{(s-t)^n}{n!} f^{(n+1)}(t)dt \tag{1}$$

or

$$R_n = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(X), \tag{2}$$

where $(X-a)(X-x)<0$. The first form is often arrived at by a succession of integrations by parts; the second by an application of Rolle's theorem to a suitable function.

The comparison of these two representations of the remainder provides an opportunity for introducing by example the generalized mean-value theorem for integrals (to use the terminology of Courant [1]), but we wish to show that it can also be used in an elementary course to motivate a consideration of distinctions between types of discontinuities.

For convenience (1) will be called the integral form of R_n, (2) Lagrange's form. Further we shall assume $a<x$; the necessary changes in the discussion will be obvious in the alternative case.

In an elementary treatment, it is usual to assume in arriving at the integral form that $f^{(n+1)}$ is continuous in the closed interval from a to x. At the same time, the generalized mean-value theorem for integrals, *viz.*,

$$\int_a^x p(t)\phi(t)dt = \phi(X)\int_a^x p(t)dt, \qquad (a < X < x), \tag{3}$$

is also usually proven under the assumptions: (a) $p(t)\geqq 0$, (b) p and ϕ continuous in the closed interval from a to x. Then one establishes directly Lagrange's remainder from the integral form. But in applying Rolle's theorem, Lagrange's form of the remainder is obtained under a weaker assumption on $f^{(n+1)}$—namely its existence in the open interval from a to x. This suggests that Taylor's formula with Lagrange's remainder applies to a wider class of functions.

However, Taylor's formula with the integral remainder could be established by integration by parts assuming, say, only Riemann integrability for $f^{(n+1)}$. This puts that form of Taylor's theorem in a little more favorable light; but the question now arises as to whether, if we wish, we can establish Lagrange's remainder by an integral mean-value theorem. To do this we extend the generalized mean-value theorem for integrals.

We shall say ϕ assumes intermediate values over an interval from a to b if for each pair x_0, x_1 of numbers selected from this interval and for any number y between $\phi(x_0)$ and $\phi(x_1)$, there is an x between x_0 and x_1 such that $\phi(x)=y$. This was, implicitly, the definition of continuity in less sophisticated times. It is easy to prove in the usual way that if in the closed interval from a to x: (i) $p(t)$

$\geqq 0$, (ii) ϕ assumes intermediate values, and (iii) the indicated integrals in (3) exist, then (3) is true.

Using this form of the mean-value theorem, Lagrange's remainder can be established as a consequence of the integral remainder under the assumptions that $f^{(n+1)}$ satisfies the conditions (ii) and (iii) just stated. But the interesting point which can be made is that (ii) is *automatically* satisfied when ϕ is a derivative, *e.g.*, $f^{(n+1)}$. That is, a derivative, even though discontinuous, must assume intermediate values, or put another way, a derivative cannot have jump discontinuities, (*cf.* [2]). To summarize, Taylor's theorem with integral remainder is valid if $f^{(n+1)}$ is, say, Riemann integrable; with Lagrange's remainder it is valid under the weaker condition that $f^{(n+1)}$ exist in the open interval from a to x.

The pedagogical point of the discussion is this. It is often observed that the integral remainder is better suited to some purposes and this furnishes an obvious motivation to the above extension of its range of validity. At the same time, it is an opportunity to show reasons for introducing finer distinctions in discussions of discontinuities. And finally there is perhaps some attraction in the device of displaying an apparent discrepancy of results followed by at least a partial resolution of the discrepancy.

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FEUERBACH'S THEOREM: A NEW PROOF

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In 1822 Feuerbach published at Nürnberg a little book of 62 pages entitled "Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks und weiterer durch sie bestimmten Linien und Figuren." The British Museum has a copy of this rare book (which has not been translated into English). In it the author sets forth many properties of a triangle and its inscribed and escribed circles. Using the values he has found (Fig. 1) for OI, IH and NH in the formula $OI^2+IH^2=2NI^2+2NH^2$, it follows that $NI=\frac{1}{2}R-r$, which shows that the Nine-point Circle touches the inscribed circle—his famous theorem—extended easily to the escribed circles.

Since 1822 many new proofs have been provided. In 1842 Stubbs and Ingram, of Dublin, gave to the world the fruitful method of Inversion, which yielded simpler proofs than Feuerbach's. Properties of tangents and of conics leading to interesting generalizations have been applied suggestively (See, *e.g.*, Baker, *Introduction to Plane Geometry*, Appendix to Ch. XII). The simplest proof is one given in McClelland's *Geometry of the Circle* page 225, and in Lachlan's *Modern Pure Geometry* page 74.

As there are so many proofs—now I can reckon nine!—I was naturally surprised to be able to evolve a new proof which I give herewith.

R=rad. circum. centre O. r=rad. in-circ. centre I. H=orthocentre. K, G mid. pts. HE, HD, and lie on $N.P.C.$ $KG=R$=diam. $N.P.C.$ perp. to FP. GFK =right angle. N=centre $N.P.C.$ QX=diam. in-circle perp. to FP. XL is perp. to GK. $LM=XQ=2r$. FG, Simson line of D, is perp. to CD at S, and bisects DH at G. $DI.IC=DB.IC=2Rr$. Angles marked α are equal.

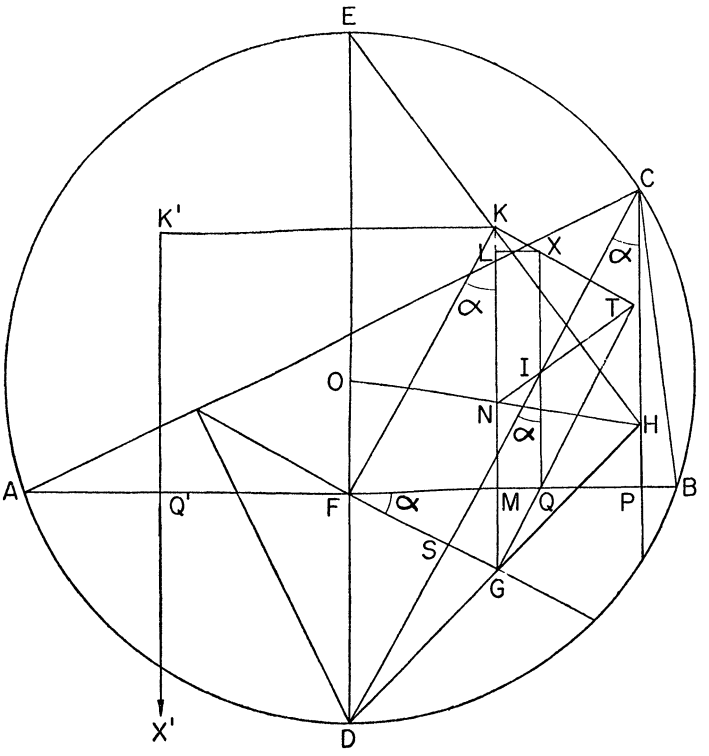


FIG. 1

To prove that GQ and KX, meeting at T, are at right angles. For then circles on diameters KG and QX, ($N.P.C.$ and in-circle) touch at T. $FQ=DI \sin\alpha$; $QP=IC \sin\alpha$; therefore $FQ.QP=DI.IC \sin^2\alpha=2Rr \sin^2\alpha$. But $R \sin^2\alpha = GK \sin^2\alpha=FG \sin\alpha=GM$. Hence $FQ.QP=2r.GM=LM.GM$. $FQ.QP+MQ^2 = MP^2 = MF^2 = KM.GM = (KL + LM)GM = LM.GM + KL.GM = FQ.QP + KL.GM$. Therefore $MQ^2=KL.GM=LX^2$.

Hence $(LX)/(KL)=(GM)/(MQ)$. Therefore, KX and GQ meet at right angles, and therefore the circles touch at T.

Mutatis mutandis it can be shown that GQ' and KX' which join ends of diameters of $N.P.C.$ and escribed circle cross at right angles. In this case $2R.r_3 \sin^2\alpha = Q'P.FQ$. $FQ = FQ'$. $Q'X' = 2r_3$. Hence easily $(X'K')/(K'K) = (Q'M)/(GM)$ and the result follows.

MATHEMATICAL EDUCATION NOTES

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TEACHING OF MATHEMATICS IN SWITZERLAND

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In the last years, my teaching activity was almost entirely devoted to additional training of teachers of mathematics and science. Participants in my classes often asked me questions about European education, and so I gave a few talks, outside class time, about this subject with some aspects of which I am well acquainted from personal experience. I wanted to use a few months' trip to Switzerland to gather additional information, especially about recent developments. I am greatly indebted to the Executive Committee of the Mathematical Association of America and to the U.S. Commission on Mathematical Instruction which designated me their official representative to visit Swiss secondary schools to gain an up-to-date picture of current trends in mathematical education in Switzerland.

The present report summarizes my impressions. It is so written that most of it can be submitted to any group of high school teachers or interested parents and can be read by people who have no time or patience to absorb many details or many unfamiliar terms.

Summary. In such complex human matters as the one before us, simplifications and generalizations are never very safe. Yet one cannot summarize without simplifications and generalizations. I am keenly conscious of all this when, in comparing the Swiss schools *with the average American high school*, I find the following four points the most significant:

1. *Division into specialized branches.* After primary school, that is, on the average from the seventh grade on, the school is divided into several specialized branches, and the curriculum within each branch leaves no choice, or little choice, to the student.

2. *Teacher training* emphasizes mastery of the subject-matter much more, practice teaching more, and general didactical lectures much less.

3. *Performance.* In those branches of the Swiss schools that prepare for university studies the programs in mathematics (and also in science and languages) go considerably further.

4. *Change.* No major change has taken place in the recent past, and no major change is likely to take place in the near future, although there is some gradual change, some discussion, and some experimentation.

The foregoing refers to mathematics as taught in an average Swiss school compared with an average U.S. high school for boys and girls of the same age. Yet, in fact, points 1, 2 and 3 (not point 4) are approximately correct to a much

further extent: the teaching of mathematics, science and languages in Central European schools shows, on the whole, the same contrast with the American high schools. By the way, all this is no news; it can be seen by any not quite casual and somewhat open-minded observer, it has been stated time and again —what is needed here is not so much fact-finding but rather fact-facing.

The following four sections illustrate the four points by adding appropriate details. An appendix says a few words about the personal experience of the author on which his opinions are based.

1. Division of the school into specialized branches. All the children (the future medical doctor, the future salesman, the future housewife, and so on) go to the same primary school, but eventually they must study different things and so their studies must be separated from a certain point onward: the question is when and how. The basic difference between the European and U.S. school systems concerns this *When* and *How*.

The annexed diagram "The Zurich School System" represents the system of public schools in the Canton of Zurich. (There is some simplification; more complex and less typical details are neglected.) I believe that it is a fairly representative example; any school system in Central Europe would show the same kind of contrast with American high schools.

The diagram shows clearly enough, I hope, how the school splits successively into several branches; it shows the number of years the students have to spend in each branch, and the students' age; it shows moreover (sketchily) the subjects covered by the mathematical instruction in each branch. It emphasizes the grades 7 through 13; it gives less space to the first six grades, says nothing about the university level. I wish to explain the abbreviations and add a few remarks. (I add, in parentheses, the original German terms and the corresponding terms used in the French-speaking Swiss cantons.)

PRIMARY: Primary school (Primarschule, école primaire) is for all children; it lasts six years. The separation into several branches begins after it, with the seventh grade.

PREP. A, B, C: Preparatory schools; they prepare the pupils especially for university studies.

PREP. A: Preparatory school of type A (Literargymnasium, gymnase littéraire) with Latin, Greek, and one living foreign language (French). Mathematics is taught in all grades through $6\frac{1}{2}$ years.

PREP. B: Preparatory school of type B (Realgymnasium; gymnase) with Latin (no Greek) and two living foreign languages (usually French and English). The program in mathematics is the same as for type A.

PREP. C: Preparatory school of type C (Oberrealschule, gymnase scientifique). No Latin, two living foreign languages, more mathematics and science than in types A and B. It prepares principally for the study of Science and Engineering on university level.

SECONDARY: Secondary school (Sekundarschule, école secondaire). It prepares principally for professions which need no university studies. It is entered by the greatest fraction of students (about 60%) and has more resemblance to the American junior high school than other branches of the Swiss school system. Mathematics is taught in each grade.

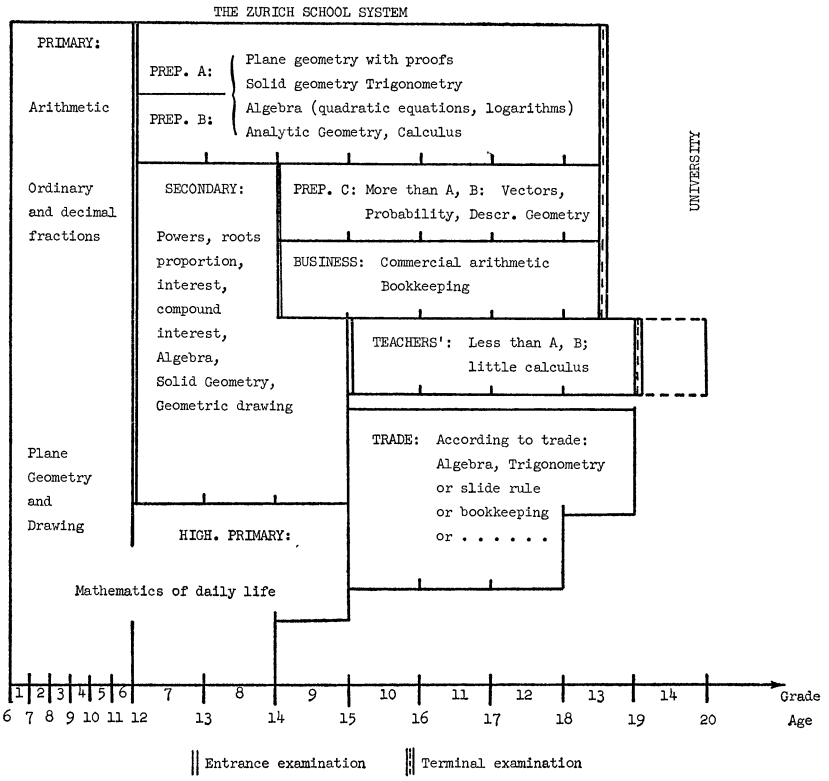
HIGH. PRIMARY: Higher primary school (Oberschule, école primaire supérieure). Still less theoretical than the secondary school.

TRADE: Trade school (Gewerbeschule, école professionelle). It has several branches, accord-

ing to the different professions, of varying duration and intensity. It is an obligatory part-time school (8 to 10 hours weekly) for "apprentices" who are trained in shops, workshops, offices or factories (they work also part time and receive some salary which increases as their training advances). Mathematical instruction varies greatly: quite strong for some trades, very little for some others.

BUSINESS: Business school (Handelsschule, école de Commerce).

TEACHERS': Teachers' preparatory school (Unterseminar, école normale). (See Sect. 2.)



Entrance examinations are marked in the diagram by double vertical line. For example, a pupil who has completed the primary school and wishes to enter the preparatory school of type A has to pass such an examination; he can enter the higher primary school without such an examination.

Terminal examinations which entitle the student to enter the university, or some branches of it (Maturitätsprüfung, maturité) are marked in the diagram by a triple vertical line (the middle line is dotted).

There is *full-time obligatory schooling* for 9 years to the age of 15 (in some cases only 8 years). Moreover, there is additional *part-time* obligatory schooling to the age 18; apprentices fulfill this obligation by attending trade school.

2. Teacher training.

Primary school. There is a special preparatory school (with two living foreign languages, no Latin) for prospective teachers of the primary and secondary schools (see TEACHERS' in diagram). In view of the future profession of the pupils, the program of this preparatory school contains subjects which are not obligatory in other preparatory schools (music, art drawing, manual skills, technique of speech) and teaches other subjects to a lesser extent. Mathematics is taught in all grades (5 hours per week in grade 10, four hours in the following grades): Algebra to quadratic equations and logarithms (also slide rule), plane geometry, solid geometry, some descriptive geometry (acceptable representation of solids on the blackboard), analytic geometry, differential calculus up to easy problems in maxima and minima.

Only the last year of this preparatory school brings some distinctively professional instruction (4 hours weekly in one semester, 3 in the other) to the prospective teachers and this instruction is mostly practical: visits to schools and practice teaching with following discussion; general didactical ideas presented in class are closely tied up with the practical exercises.

The TEACHERS' preparatory school is followed by the *one year teachers' training school on university level* (Oberseminar) which, again, is predominantly practical: lectures on general didactics, history of pedagogy, psychology and child psychology take weekly 5 hours out of 34 in one semester and $8\frac{1}{2}$ hours out of $32\frac{1}{2}$ in the other. The special didactics of various subjects are strongly practical; exercises are emphasized.

The main emphasis is on *practice teaching*: each candidate teaches 3 hours per week through the whole year under the guidance of an experienced teacher. Moreover, between the two semesters, he teaches 3 weeks full time in a city school and another 3 weeks in a country school. Having passed the training school, the candidate must teach successfully at least 40 weeks in the following two years to receive his final certificate.

Secondary schools. The prospective teacher in a secondary school must have the primary teacher's final certificate and two years practice in a primary school, after which he receives *two years training at the university.* The program of this training contains didactical lectures, but the emphasis is on the mastery of the subject.* There are six branches of science teaching, divided into two groups: (1) Mathematics, Physics, Chemistry;(2) Botany, Zoology, Geography. Any secondary teacher of science must take two subjects from group (1) and two subjects from group (2), do practice teaching in both groups and, in his final examination, teach a class for one hour in group (1) before the examiners, and another class in group (2). One of the branches chosen by any science teacher must be mathematics and the minimum requirement is a course in calculus.

Preparatory schools. A major fraction of the mathematics teachers for pre-

* For instance, language teachers must have lived in a place where the language is spoken at least 6 months, and speak it fluently.

paratory schools in all Swiss cantons receive their training at the Swiss Federal Institute of Technology (E.T.H., Eidgenössische Technische Hochschule, Ecole Polytechnique Fédérale) in Zurich (a high class Technical University).

The program for the final certificate (mathematician's diploma) demands solid knowledge both in classical and in modern branches of mathematics; there are also courses and examinations in mechanics and in experimental and theoretical physics. An extensive paper on a special mathematical subject, for which three months' work is allowed, is also part of the final examination. In short, a mathematician's diploma at the E.T.H. is roughly equivalent to a strong M.S. degree.

About 250 semester-hours' work is needed for the mathematican's diploma;† only 7 additional semester-hours are needed for the additional pedagogical examination which is also required for teaching in preparatory schools. These are distributed as follows:

General didactics, 3 semester hours: most of the time is spent on discussing the established school system, its purpose, its practical aspects.

Topics from school-mathematics, 2 semester-hours: discussion of the most important topics which should be viewed differently at the university and in the preparatory school.

Practice teaching, 2 semester-hours: either all candidates listen to a class conducted by an experienced teacher, or one candidate teaches and the others observe. Each hour of practice teaching is followed by a discussion (particularly useful if the instructor is experienced).

In the final examination, the candidate must teach twice, one hour each time, in presence of the examiners.

Before the present teacher shortage (which began in Switzerland relatively late, two or three years ago) the (unwritten) requirements for a preparatory school teacher were even higher: a Ph.D. degree in the better-paying city schools.

The teacher in a preparatory school has some advantages over the teacher in a secondary school: less teaching load, more salary, a little more security, but the differences are not so great: the salaries are in the proportion 5 to 4, or thereabout. Yet the preparatory school teacher can do research and may advance to a university position not infrequently. (A personal note: among my former students who wrote their Ph.D. dissertation at the E.T.H. under my guidance, three are now professors of the E.T.H.; all three taught several years in preparatory schools).

3. Performance. It is easy to compare programs, but it is not easy to judge the performance. For instance, there is a subject which is treated in Switzerland even in secondary schools, but which is conspicuously (and for the outsider, sur-

† A student takes around thirty hours per week, but several hours are devoted to practice sessions or to laboratory work. For instance, freshman calculus is taught 9 hours weekly: 6 hours lectures, 3 hours practice sessions. As a result there is relatively less homework than in an American university.

prisingly) absent from the program of the average American high school: divisibility and the decomposition into prime factors. Yet to compare in performance the primary schools on both sides of the Atlantic, or the Swiss secondary schools with the junior high schools, is not so easy. At any rate, I am not in a position to undertake such a comparison.

There is, however, a point where the comparison is possible: the young people who passed the terminal examination (Reifeprüfung, Maturitätsprüfung, maturité) of a Swiss (or German, or Austrian, . . .) preparatory school, are about two years ahead of those boys and girls who, having finished high school, enter the university: they are about the same age and attended school the same number of years, but the boy with the "Maturitätsprüfung" can be admitted to an American university as a junior* whereas the boy from the high school will be a freshman. As far as I can judge on the basis of my extensive experience on both sides of the Atlantic, this difference of two years persists: the boy who gets a B.A. or a B.S. after four years of study at one of the American universities I know intimately, is about as far advanced as the boy who, after two years of study, passes his intermediate examination at the E.T.H.

To see the full picture, we must realize that the minority of young people who pass the terminal examination (Reifeprürung, maturité, . . .) of the Swiss, or European, preparatory schools (Gymnasium, lycée, liceo, . . .) results from a *selection*. To begin with, only a minority of children is able to pass the entrance examination into the preparatory schools. Of those who, having surmounted this obstacle, enter the preparatory schools, a considerable percentage will be dropped along the road because of insufficient achievements. (About 50% in Zurich: of 100 children entering the preparatory school about 50 succeed in arriving at, and passing, the terminal examination.) Thus the preparatory school acts as a selecting agent, as a kind of sieve: it picks out the students who can be expected to deserve university education.

[It is obvious that any such procedure of selection, however beneficial it may be on the average, inavoidably involves hardships and undesirable results in some cases. It must be mentioned that such young people as do not succeed in the preparatory school have a second chance: they can attempt the "terminal examination" given by the state in regular intervals outside, and independently of, the preparatory schools (Eidgenössische Maturitätsprüfung, maturité fédérale). There are several private schools (some of them are rather good) preparing people for these state examinations.]

Some readers of this report may raise the question: How is the performance of the teachers in preparatory schools affected by the circumstance that only a very small percentage of their training has been devoted to didactical subjects? (*Cf.* Sec. 2.) This question was often in my mind when I spoke with teachers but I met just one teacher who was really interested in it and had a decided opinion. Here it is, in a nutshell: "As a student at the university, I was impressed

* This is, to some extent, actual practice as far as I can judge from the cases I know personally.

by some of my teachers. Then, when I started teaching in the school, I had difficulties. The trouble was, as I see it now, that I imitated my university professors: I did not realize that teaching in a school must differ in many ways from teaching at the university. The generalities I heard in our short course of general didactics gave me no help. I imagine that a good didactical course by a competent professor could have helped me to understand the situation sooner." Let us hope that my interlocutor was right.

4. Change. In this section, I speak only about the preparatory schools.

The program of the "state terminal examinations" (Eidgenössische Maturitätsprüfung; see Section 3) is fixed by a federal ordinance (of 1925). This program provides a minimum standard for the schools: the materials on this federal program are included in the curriculum of the schools, by law or by consent.

A committee of the Union of Swiss teachers of Mathematics and Physics recommended a carefully phrased new minimum standard for preparatory schools (in 1958; with the avowed purpose that the federal ordinance of 1925 should be correspondingly modified). The recommended changes are by no means revolutionary; here are a few.

For all three types, A, B and C: Less time should be spent on computations with logarithms, and the slide rule should be extensively used.

For the types A and B: Compound interest and especially annuities should be dropped, and the elements of the calculus introduced.

For the type C: Spherical Trigonometry should be dropped, and several other things introduced, especially vectors; geometric transformations should be treated, and so treated that they may serve as a preparation for the concepts of group theory.

These recommendations have been adopted by the schools to a good extent (most of them by the better schools, I am inclined to believe). At any rate, these recommendations have been widely discussed. There is a lively discussion and a strong interest in some schools, but the changes envisioned are by no means subversive. For instance, in the program of a school, mathematics and science gained a few semester-hours at the expense of Latin. Some teachers would like to teach a little more probability and add some statistics; others would insist more on geometric transformations and the group concept; and so on. In several schools, the programs are sufficiently elastic so that the teachers have a possibility to put their ideas into practice.

Some teachers in the French speaking Swiss cantons, especially in Geneva and Neuchâtel, would go much farther and, under French influence, they would like to "bourbakize" the curriculum; in fact, they have started to do so to some extent or will start next year. There were a few clashes; the "progressives" charge the others with "stabilism." I give the word to one of the "stabilists" who explained to me why he has objected so strongly to a talk by a "progressist": "He need not tell us what is a set, a ring, a field or a mapping, we have heard

all that and more at the university. He should have told us how and when, with which suggestive examples, to which use, he did, or intends to, introduce such concepts, and how much resistance he has encountered from the students."

I refrain, of course, from taking sides. I am inclined to believe that some "Bourbakism" will penetrate into the Swiss schools eventually, by degrees. On the other hand, it seems to me that the Swiss preparatory schools function, on the whole, satisfactorily, and so some "stabilism" is justified: experience and precaution are very much needed when new concepts or methods are introduced at the expense of tried and tested ones.

Appendix. Opinions such as have been voiced on the preceding pages cannot be "proved": they necessarily depend on the experience and the background of the author. Therefore, the reader is entitled to know through what experience the author has acquired his opinions.

I attended a school (in Budapest, Hungary) similar to the Swiss preparatory school of type A. I did practice teaching for a year in a school (also in Budapest) similar to the Swiss preparatory school of type C. In the Zurich preparatory schools of types A and B, I was part-time teacher for about three years around 1920. Later, for more than ten years, I was in charge of the mathematical part of the admission examinations to the E.T.H. (see Sec. 2) and there I examined candidates coming from schools all over Europe.

Since the fall of 1958 I spent about 8 months in Switzerland. I visited several schools, both in the German-speaking and the French-speaking parts, gave talks to pupils and teachers, and discussed their problems with the teachers. (Many of them attended my classes at the E.T.H. or I was well acquainted with them otherwise.) I spoke also with some key people in charge of the mathematical training of future primary and secondary teachers.

A MATHEMATICS INSERVICE EDUCATION PROJECT FOR ELEMENTARY SCHOOL TEACHERS

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College mathematics departments will be assuming increasing responsibility for the success of elementary school mathematics programs as the rising demands for improvement spotlight the need, not only to improve the inservice teacher's methods, but also to augment his mathematical knowledge. This report of a project undertaken in Austin is being presented in—and should be justified by—the hope that such a report will (1) illustrate the possible cooperative role of the mathematics department of any college in improving elementary school mathematics, and (2) stimulate other mathematics departments to initiate like inservice courses in their communities to meet the rising tide of need and desire for such programs. The project was carried out jointly by the following University of Texas personnel: M. Vere DeVault, Associate Professor of Elementary Education; Roger Osborn, Assistant Professor of Mathematics;

W. Robert Houston and Claude C. Boyd, Research Associates. Its purpose was to increase the teacher's knowledge of mathematics in order that he might more successfully teach mathematics.

Procedure. In order most nearly to achieve the project's aim, it was thought mandatory to provide instruction both in mathematical content and in method, as the mathematics teacher's excellence was seen to be dependent upon a close union of mathematical knowledge and educational know-how.

The wide variance in the level of mathematical preparation of the teachers (from no college credit to fifteen semester hours), and in the length of time since they had received mathematical training, necessitated the use of elementary content for the programs. This content, however, was treated on a reasonably rigorous basis. The topics chosen for the five hour-and-a-half sessions of the course, and covered by means of closed-circuit TV, panel discussions, smaller group discussions, and lectures on mathematics and methods, were as follows:

(1) Numbers and numerals.
(2) Base and place (including bases other than 10).
(3) The structure of the number system and its relation to the four fundamental operations.
(4) Some laws of arithmetic (including the commutative, associative, and distributive laws and their uses in the elementary classroom).
(5) The use of models in teaching mathematics.

Effort was made to base the programs on experiences common to all teachers, as well as to give each session, insofar as was possible, a modern flavor. Also, attention was given to accommodating the participating teachers in that (1) no charge was made for the course, (2) the programs were presented on a night chosen by a majority of the teachers, (3) the sessions were begun and closed punctually, (4) no reports were required, and (5) previously prepared materials were given to each class member to minimize the amount of note-taking required.

Conclusions. Evaluation of each program and of the project as a whole was based on written comments from participating teachers representing the various Austin schools. Although not all were favorable, the majority of these informal appraisals indicated that the teachers judged the course to be of significant value and utility to themselves as teachers of mathematics. General conclusions reached as a result of the project were the following:

(1) Teachers recognize their need for improvement. (This fact was demonstrated by the enrollment alone, which had to be *limited* to 250.)
(2) Teachers will grasp at an opportunity for self-improvement.
(3) The level of effectiveness of the classroom teacher can be raised by even so limited a project as this.

LOGIC: GAME OR TOOL?

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A good many texts intended as an introduction to mathematics at a first-year university level now include what the authors are pleased to call "symbolic logic." It is this logic, its presentation and its function, that I wish to discuss here.

The usual treatment of symbolic logic in such texts is first to produce the truth tables for conjunction, disjunction and negation. The content of these tables is justified by a short exposition designed to show that this, in tabular, symbolic form, is the way we use "and," "or," "not" in the English language. The only reason authors get away with this (to me) obvious falsehood is that students at first-year level do not think critically about anything, least of all their own language. In the exposition it is usual to ignore the fact that the "inclusive" use of "or" in the language is relatively rare and that "and" usually carries a far heavier load of meaning than the truth table allows it; *e.g.*, consider the use of "and" in "He put out the cat and went to bed"—where it implies temporal succession. In my experience, students do not notice these discrepancies, but is logic introduced as an example of how large a camel they will swallow?

In some texts, the truth tables are followed by a list of axioms (usually incomplete). No mention is made of the relationship between the axioms and the truth tables except, sometimes, to set as problems the "proving" of the axioms! That is, not only are the fundamentals of the subject left in total obscurity but the function of an axiom set as basic to a system is implicitly denied. Finally, problems are set which assume that "and," "or," "not" as used in the English language are always used as logicians use their symbols and that material implication is translatable: "p implies q" or "if p then q."

These are certainly not all the muddles perpetrated by such treatments of symbolic logic, but they are a fair selection. They will suffice for my present purposes.

To me, one thing seems entirely certain in the teaching of elementary mathematics: that we should on no account give muddled accounts of anything to students, that mathematics, apart from its obvious utilitarian function, should parade as a triumph of clear thinking. I do not mean we should prove every theorem we ever use (impossible at a first-year level), nor that we should not foster good, practical mathematical intuition; I do mean that we should not present as obvious that which is false or that which is so inexplicit as to be rightly neither true nor false.

It should not be supposed from the foregoing that I am against teaching symbolic logic at an elementary level; I am, in fact, strongly for it. But I do not believe that the justification for doing so is that it introduces students to "modern mathematics." I feel that the justification is as follows: in the past, elementary mathematics courses were often solely manipulative in content. I

do not depreciate the value of this. However, I feel that two weeks to a month of a full course could profitably be devoted to (a) axiomatic methods and valid proof, (b) the relationship between mathematical system based on axioms and the world (of the physicist, of the chemist, of everyday life). It happens that symbolic logic is ideal for illustrative purposes in doing this and has the added advantage of providing an excellent entrance to set theory, which may be used in turn for counting and probability. The symbolism with which the student becomes familiar may also prove of value later.

For such a treatment of logic, I would suggest the following outline: First the elementary truth tables would be developed but it would be *emphasized* that the "and" and "or" of logic are heavily truncated when compared to ordinary language. Once the student has some facility with the truth tables, it would be pointed out that working with truth tables we are in a no-man's land; we have seen that the justification cannot be ordinary language and what else can it be? This offers an opportunity for a discussion of the function of an axiom set, followed by the presentation of an axiom set for symbolic logic with all the trappings, including meta- and object-languages and a light discussion of the difference between theorems within the system defined by the axioms and outside that system (meta-theorems). It would be insisted that although the psychological ground of the axiom set is the truth tables and though every axiom could be "proved" by the truth table method, the symbols used in the object language are uninterpreted. As the aim here would not be formal simplicity, a far longer and more complicated axiom set than necessary (on the model of that for the real numbers) would be best. On the basis of these axioms, various theorems could be proved with absolute validity. We then have a complete, elementary system of symbolic logic, securely founded on axioms and with the psychological origin of the axioms laid bare. Now, by interpreting this system in terms of propositions and connectives, the relationship between mathematics and the world can be discussed.

From this point the work can be extended. The idea of isomorphism (alternative interpretations of the object language) can be introduced with reference to switching circuits, Venn diagrams and set theory. If more complicated work of this type is desired, the set builder notation can be introduced and used to define quantified propositions. What happens at this stage is of far less interest to me here. My point is that with minor changes in what is taught, but major changes in how it is taught, some of the most important ideas of mathematics can be introduced without making the treatment unduly difficult. Then symbolic logic would become an important part of the student's training at an elementary level instead of being a rather poor game.

INSERVICE RE-EDUCATION OF MATHEMATICS TEACHERS

KENNETH E. BROWN, Office of Education

"Inservice re-education of mathematics teachers is a must if the improved mathematics programs are to be successful" was the conclusion of a recent conference on inservice education of mathematics teachers held at the U. S. Office of Education.

It was pointed out that the University of Illinois Committee on School Mathematics had been training teachers in the content and methods of teaching their program since its beginning in 1952. The School Mathematics Study Group, the University of Illinois Mathematics Project, the Ball State Teachers Project, the Commission on Mathematics of the College Entrance Examination Board, and others, recognize the need for re-education of teachers before the new and improved mathematics materials can be used effectively in the classrooms.

It is estimated that perhaps 90% of all the teachers need some re-education in mathematics. The conferees supported one speaker when he said "schools contemplating the introduction of new mathematics material should make adequate plans for a change by establishing an intensive inservice program. The inservice program should give the teachers the needed background for the new courses, and it should supply the mathematical and psychological reasons for the change. Without a carefully planned inservice education program, the introduction of new material into the classroom might end in disaster and subsequently prevent desirable changes."

The role of the administrator in inservice education programs was discussed. "Leadership is the key to the solution of our inservice problems" declared Dr. Henry Van Engen, specialist in teaching mathematics at the University of Wisconsin in an address to the conferees. "Superintendents and principals set the intellectual atmosphere in which a staff must work, and this atmosphere will condition the attitudes, the working habits, and the instructional goals of the staff. Although the supervisor should know the school and children, he should also know mathematics. This is particularly true of elementary supervisors. A sound mathematical program cannot be developed by one who is insecure about the mathematical principles to be taught."

A keynote speaker, Dr. W. L. Duren, Jr., Dean of the College of Arts and Science at the University of Virginia, stated that because of the failure to keep the high school mathematics content up-to-date "we find that we have been losing ground for thirty years. Despite their willingness to work, our school mathematics teachers do not have enough mathematical power to adapt to the demands of a new curriculum. If a student is not already pretty good in mathematics before he leaves high school, there is little that college mathematics can do to retrieve him as a scientist. College mathematics can add to the competence of those who have already achieved some mastery of school mathematics, but it cannot recover many who failed to get a good high school mathematical edu-

cation. Thus, the secondary school teacher is perhaps the key person in determining our national scientific and technological strength."

One discussion group concluded that "While there can be no one best plan to suit the needs of every local situation, any good inservice program may be described as one that:

1. Encompasses the needs of *all persons* engaged in the teaching of mathematics in the schools—teachers, supervisors and administrators.
2. Is organized on an all-school, or system-wide basis.
3. Is coordinated within school levels (elementary, junior high and high school) and from one grade to the next, through grade 12.
4. Is developmental in that it is continuous and adjustable to the needs of the classroom teacher and changes in curriculum development and implementation.
5. Recognizes the individual differences among teachers—differences that reflect preservice training, length and variety of teacher experience and grasp of subject matter.
6. Stimulates wholesome teacher growth and understanding of various viewpoints and competencies.
7. Makes provision for teachers to play some part in setting up the program of inservice training (surveys, suggestions and expression of preferences).
8. Is developed in an emotional climate which gives confidence to the participants and indicates administrative support.
9. Takes in account the needs and abilities of all the children—the slow learner, the retarded, the under-achiever, the average and the gifted.
10. Has the leadership of a competent and fair-minded administrator.
11. Is closely integrated with the supervisory and guidance program of the school.
12. Develops a variety of approaches to teacher growth (workshops, seminars, institutes, study groups, committee work).
13. Is judged for its effectiveness at regular intervals through various techniques of appraisal.
14. Draws upon the resources of the community, the colleges and universities and the professional staff itself.
15. Is aware of the limitations on teacher time and daily load and makes reasonable adjustments for teachers involved in the inservice program.
16. Gives recognition and encouragement to active participants in the inservice program in terms of promotional opportunities, financial remuneration, salary increases and professional status.

Other suggestions from the conferees included:

1. Released time from teaching for inservice courses such as the "Syracuse Plan" (New York).
2. Provision for inservice education film or TV kinescopes such as the Mathematics Kinescopes developed by New York City Board of Education.
3. Provision of additional consultants to guide and assist in inservice education programs.
4. Use of Skinner-type teaching machines for teaching improved mathematics, to be used by both student and teacher.
5. Use of visiting lecturers in mathematics such as the visiting lecturer program to secondary school teachers sponsored by the Mathematical Association of America.
6. Increase in inservice education for mathematics teachers through state organizations such as that now sponsored by the Illinois Council of Mathematics Teachers and California Council of Mathematics Teachers.
7. Also, television was suggested as a means of inservice education. The caution was added "it must be used wisely and with the proper teachers in charge."

Another discussion group suggested that the following questions be con-

sidered before an inservice education program be established:

1. Is leadership available either from a principal or superintendent?
2. Are consultants available; consultants for both mathematics content and organization and methods? This should include high school teachers as well as a college mathematician.
3. Is the climate in the school conducive to successful inservice education?
4. Does the teacher have a class using a modern program in mathematics? If not, is there one available for observation?
5. Does the teacher have time to study, organize, and experiment by:
 1. Teaching fewer classes?
 2. Being excused from some outside activities?
 3. Other means?
6. Is some financial support provided for:
 1. Textbooks and supplies?
 2. Consultant service?
 3. Extra teacher time or over-time pay?
 4. Making it possible for teachers to attend institutes, conferences and workshops?

Descriptions of many inservice education programs were given by the conferees. These materials will appear in the report of the conference. The conference, sponsored by the U. S. Office of Education and the National Council of Teachers of Mathematics, was attended by approximately fifty specialists who met in Washington, D. C., for two and one-half days.

NCTM Regional Conferences for School Administrators

Phillip S. Jones, president of the National Council of Teachers of Mathematics, has announced a Council-sponsored plan for sending consultants into the field to help school systems strengthen their mathematics programs through the use of the improved materials of instruction now available. The project, supported by a $48,000 grant from the National Science Foundation, is aimed at providing selected supervisors and school administrators with the information and orientation they need so that they may provide leadership in establishing improved mathematics programs in their school systems.

With this end in view, a series of eight regional invitational conferences has been scheduled during the fall of 1960. Each conference will be under the jurisdiction of a regional director and all conferences will be conducted by the same team of three consultants. This team consists of G. Baley Price, executive secretary of the Conference Board of the Mathematical Sciences; Kenneth E. Brown, mathematics specialist, U. S. Office of Education; and W. Eugene Ferguson, head of the Department of Mathematics, Newton High School, Newtonville, Massachusetts.

Each two-day conference has a four-part program: (1) "Progress in Mathematics and its Implication for the Secondary School," G. B. Price; (2) "The Drive to Improve School Mathematics—Comparisons and Common Elements of Special Programs," K. E. Brown; (3) "Our Experiences with the New Programs in Mathematics"—Panel of teachers to be selected in each region; (4) "Implementing the New Mathematics Program in Your School," W. E. Ferguson. In addition, time will be allowed for question sessions and for the inspection of the sample texts which will be on display. A preview of this program was presented at the Council's summer meeting at Salt Lake City, August 22, 1960.

The project is being directed by Frank B. Allen, member of the Council's Board of Directors and chairman of the Mathematics Department of Lyons Township High School and Junior College, La Grange, Illinois, with the advice of a steering committee including Max Beberman, E. G. Begle, Kenneth E. Brown, Edwin C. Douglas, Phillip

S. Jones, John R. Mayor, G. Baley Price, and Mina Rees. Ex officio members are Philip Peak and Henry Van Engen. Consultants to the committee are Harold P. Fawcett, Ohio State University, and Mary Hovet, supervisor of mathematics, Howard County Schools, Ellicott City, Md.

All Regional Directors have been appointed and the following schedule of Regional Orientation Conferences in Mathematics has been approved by the Steering Committee:

Date	*Location*	*Director*	*Region*
Oct. 3–4	Philadelphia, Pa.	M. Albert Linton, Jr., William Penn Charter School, Philadelphia, Pa.	Connecticut, Delaware, D. C., Maine, Maryland, Massachusetts, New Hampshire, New Jersey, New York, Pennsylvania, Rhode Island, Vermont.
Oct. 10–11	Iowa City, Iowa	H. Vernon Price, State University of Iowa	Illinois, Iowa, Minnesota, Nebraska, North Dakota, South Dakota, Wisconsin
Oct. 27–28	Atlanta, Georgia	H. Mack Huie, Atlanta Board of Education, Atlanta, Georgia	Alabama, Georgia, Louisiana, Mississippi, North Carolina, South Carolina, Virginia
Nov. 3–4	Portland, Oregon	William W. Matson, Mathematics Supervisor, Portland Public Schools, Portland, Oregon	Alaska, Idaho, Montana, Oregon, Washington, Wyoming
Nov. 18–19	Los Angeles, California	Clifford Bell, U.C.L.A.	Arizona, California, Hawaii, Nevada, Utah
Dec. 2–3	Topeka, Kansas	Marjorie L. French, Topeka High School, Topeka, Kansas	Arkansas, Colorado, Kansas, Missouri, New Mexico, Oklahoma, Texas
Dec. 9–10	Miami, Florida	Agnes Y. Rickey, Dade County Public Schools, Miami, Florida	Florida, Puerto Rico
Dec. 15–16	Cincinnati, Ohio	Mildred Keiffer, Supervisor of Mathematics, Cincinnati, Public Schools, Cincinnati, Ohio	Indiana, Kentucky, Michigan Ohio, Tennessee, West Virginia

The results of these conferences will be summarized in a brochure which is to be distributed by the National Council of Teachers of Mathematics.

Frank B. Allen

An Experiment in Individualized Mathematics in Grade 9

Queens College, the Division of Teacher Education, and Syosset High School in New York will start a research project in the teaching of mathematics in the 9th grade with the aid of a number of high schools in the New York metropolitan area. The experiments will be supported by a grant from the U. S. Office of Education. The object of the experiment is to develop methods and materials for individualized learning with a regular class so that pupils may advance at their own rate. The experiment will not be concerned with curriculum, but primarily with the use of perceptual materials. The development of the experiment is the result of work done by Dr. Gwladys C. Crosby of Queens College and Mr. Herbert Fremont, chairman of the mathematics department of Syosset High School, during the past two years.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1436. *Proposed by Mok-Kong Shen, Karlsruhe, W. Germany*

Through the vertices of a given $\triangle ABC$ draw straight lines l, m, n, respectively, such that n and l intersect in D, l and m in E, m and n in F inside $\triangle ABC$ and

(1) $\triangle ABE = \triangle BCF = \triangle CAD = \triangle DEF = (1/4)\triangle ABC$.

(2) $\triangle DEF$ is similar to $\triangle ABC$ and has an area equal to a given fraction of $\triangle ABC$.

E 1437. *Proposed by Nickolas Konopliv, University of Minnesota*

Test each of the following infinite series for convergence:

(1) $x+\sin x+\sin(\sin x)+\sin[\sin(\sin x)]+\cdots$,
(2) $x-\sin x+\sin(\sin x)-\sin[\sin(\sin x)]+\cdots$,
where $0<x<\pi$.

E 1438. *Proposed by S. H. Gould, Mathematical Reviews, Providence, R. I.*

A car travelling at a speed of v miles per hour is required by law to remain at a distance (in Rhode Island one car-length for every 10 miles per hour) of cv miles behind the car ahead, where c is a constant depending upon the state. Find, as a function of time, the maximum allowable acceleration for a car starting from rest immediately behind an unobstructed car which accelerates at a constant rate of a miles per hour per hour.

E 1439. *Proposed by J. M. Elkin, Long Island University*

Construct a bounded strictly monotonic function $f(x)$ such that $f'(x)$ exists for all real x and $\lim_{x\to\pm\infty} f'(x)\neq 0$.

E 1440. *Proposed by W. B. Carver, Cornell University*

Find a closed-form expression for $\sum_{s=1}^{n} s^3 r^{s-1}$, $r\neq 1$.

SOLUTIONS

Dissecting an Obtuse Triangle into Acute Triangles

E 1406 [1960, 290]. *Proposed by Michael Goldberg, Washington, D. C.*

Cut an obtuse triangle into the least number of acute triangles.

Solution by Wallace Manheimer, Franklin K. Lane High School, Brooklyn, N. Y. Let S be a set of acute triangles satisfying the problem. One line segment of S must divide the obtuse angle A, say, of the given triangle ABC. It may not terminate in the opposite side since otherwise another nonacute triangle would be formed and S would not be minimal. Therefore a vertex of S must lie within the triangle. At least five line segments of S must meet at this vertex, and a subset of S must form a centrally divided pentagon. It follows that S must contain at least seven triangles.

To construct a configuration with seven triangles, draw tangents DE and FG (D on AB, G on AC, E and F on BC) to the inscribed circle so that BDE and FGC will be acute isosceles triangles with vertex angles at B and C respectively. The bisectors of the angles of the pentagon $ADEFG$ will complete the required figure, since a triangle is acute if it contains two acute angles each greater than 45°.

Also solved by J. W. Baldwin, D. R. Bedgood, W. F. Cheney, Rufus Crane, Thomas Curry, R. E. Denman and V. E. Hoggatt (jointly), Jane Evans, P. J. Federico, C. E. Franti, L. D. Goldstone, D. E. Knuth, Harry Langman, D. C. B. Marsh, D. A. Moran, C. S. Ogilvy, S. W. Saunders, William Schoening, Walter Zayachkowski, and the proposer.

Federico gave an analysis of the problem (and also of that of dissecting a square into a minimum of eight acute triangles) based upon the Euler formula $V-E+F=1$ for a plane network. Denman and Hoggatt established the following allied results: (1) A nonacute triangle can be dissected into seven acute triangles; if $B>90°$, and $B-A<90°$ and $B-C<90°$, then the seven triangles can be isosceles. (2) An obtuse triangle can be dissected into eight acute isosceles triangles. (3) A right triangle can be dissected into nine acute isosceles triangles. (4) An acute nonisosceles triangle can be dissected into four congruent acute triangles or nine acute isosceles triangles. (5) An acute isosceles triangle may be dissected into four congruent triangles similar to the original triangle. (6) Any triangle can be dissected into two right triangles or three obtuse triangles.

The given problem was proposed by Martin Gardner in "Mathematical Games," *Scientific American*, vol. 202, Feb. 1960, p. 150, and a dissection into seven parts appeared in the Mar. 1960 issue, pp. 177–178.

An Application of Fermat's Little Theorem

E 1407 [1960, 290]. *Proposed by Leo Moser, University of Alberta*

Prove that, for $p>3$ and prime, ab^p-ba^p is divisible by $6p$.

I. *Solution by E. T. Parker and David Zeitlin, Remington Rand Univac, St. Paul, Minn.* Since $b(b^2-1)$ is a factor of b^p-b, we have $b^p-b\equiv 0 \pmod 6$. From Fermat's little theorem, $b^p-b\equiv 0 \pmod p$; and since $(6, p)=1$, it follows that $b^p-b\equiv 0 \pmod{6p}$. The addition of $ab^p-ab\equiv 0 \pmod{6p}$ and $-ba^p+ab \equiv 0 \pmod{6p}$ yields $ab^p-ba^p\equiv 0 \pmod{6p}$, the desired result.

II. *Solution by the proposer.* We will show more generally that $ab^p - ba^p \equiv 0 \pmod{\prod_{i=1}^{k} q_i}$, where the q_i are primes and $q_i - 1 \mid p-1$. The required result will then follow from the fact that $2-1 \mid p-1$, $3-1 \mid p-1$, and $p-1 \mid p-1$. To prove the more general result we need only show that q a prime and $q-1 \mid p-1$ implies $ab^p - ba^p \equiv 0 \pmod{q}$. If $ab \equiv 0 \pmod{q}$ the result is trivial, while if $ab \not\equiv 0 \pmod{q}$ then $a^{p-1} = a^{(q-1)t} \equiv 1 \pmod{q}$ and $b^{p-1} \equiv 1 \pmod{q}$ so that $(a^{p-1} - b^{p-1})ab \equiv 0 \pmod{q}$ in any case. Thus, for example, $ab^{61} - ba^{61} \equiv 0 \pmod{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 31 \cdot 61}$.

Also solved by A. N. Aheart, Ray Authement, D. W. Bailey, J. W. Baldwin, Alan Beal, William Becker, Brother Joseph Heisler, Brother Louis Francis, T. C. Brown, James Burling, R. G. Buschman, Richard Cottle, D. F. Criley, Mary Ann Davis, Bob DeVore, J. W. Ellis, C. E. Franti, Anton Glaser, Michael Goldberg, L. D. Goldstone, R. Gramann, S. H. Greene, Bernard Greenspan, E. H. Grossman, Emil Grosswald, John Jordan, M. S. Klamkin, D. E. Knuth, Sidney Kravitz, J. P. Labute, Harry Langman, Joe Lipman, Gerald Leibowitz, Wallace Manheimer, D. C. B. Marsh, R. W. Means, M. V. Mielke, J. W. Moon, J. B. Muskat, Sidney Penner, D. J. Persico, J. L. Pietenpol, C. F. Pinzka, J. H. Rascoff, R. M. Rippey, Ray Rogers, R. T. Sandberg, William Schoening, J. A. Schumaker, D. L. Silverman, Vencil Skarda, E. L. Spitznagel, Jr., Robert Strichartz, E. A. Sturley, Carl Superko, R. P. Tapscott, L. Thomas, Dmitri Thoro, W. C. Waterhouse, Alan Wayne, Charles Wexler, Dale Woods, and W. Zayachkowski. Late solution by A. S. Gregory.

Mielke pointed out that the special case $p=5$ of this problem appears in *The Mathematics Teacher*, vol. LI, No. 8, Dec. 1958, p. 587, in the article "The Seventh Mathematical Olympiad for Secondary School Students in Poland." A labored solution for this case had been given by Olga Mitrinović in "Sur une congruence," *Bulletin de la Société des Mathématiciens et des Physiciens de la R. P. Macédoine*, t. vii, 1956.

The Highest Power of 2 in the Numerator of $\sum_{i=1}^{k} 1/(2i-1)$

E 1408 [1960, 290]. *Proposed by J. L. Selfridge, IBM, Yorktown Heights, N. Y.*

Find the highest power of 2 which divides the numerator of

$$1 + 1/3 + 1/5 + \cdots + 1/(2k-1).$$

Solution by D. L. Silverman, Defense Dept., Ft. Meade, Md. We have

$$1 + 1/3 + \cdots + 1/(2k-1) = \sum_{i=0}^{M-1} \{1/(i \cdot 2^{r+1} + 1) + 1/(i \cdot 2^{r+1} + 3) + \cdots + 1/((i+1) \cdot 2^{r+1} - 1)\},$$

where $k = 2^r M$, M odd. If in each of the M bracketed terms, the 2^r constituent terms are collected using as common denominator P_i, the product of the denominators within the ith term, then the resulting numerators will each consist of 2^r terms, each of the form P_i over a distinct odd residue of 2^{r+1}. These ratios must themselves be the odd residues of 2^{r+1} in some order. Since the sum of the odd residues of 2^N is 2^{2N-2}, it follows that each of the M numerators is of the form $2^{2r}M_i$, M_i odd. The numerator of $1+1/3+\cdots+1/(2k-1)$ is thus of the form $2^{2r}Q$ where Q is the sum of an odd number of odd terms, hence odd. Hence

if 2^r is the highest power of 2 dividing k, 2^{2r} is the highest power of 2 dividing the numerator of $1+1/3+\cdots+1/(2k-1)$.

Also solved by J. P. Ballantine, Leonard Carlitz, N. J. Fine, D. E. Knuth, D. C. B. Marsh, R. T. Sandberg, and the proposer.

Ballantine proved, more generally, that if $S(n, k)=\sum_{i=1}^{k}(2i-1)^n$ and if, for $N\neq 0$, $T(N)$ is the exponent of the highest power of 2 that divides N, and if $T(N/D)$ is defined as $T(N)-T(D)$, then $T(S(n, k))=T(k)$, if $T(n)>0$; $T(S(n, k))=2T(k)$, if $T(n)=0$. The answer to the given problem is found by taking $n=-1$.

The proposer pointed out that the given problem is left unsolved in H. S. Shapiro and D. L. Slotnick, "On the mathematical theory of error-correcting codes," *IBM J. Res. Develop.*, v. 3 (1959), p. 31 (Remark 1). There the authors say, "an estimate [of this] power of $2\cdots$ seems in general to be a difficult number theoretic problem."

Bidding Sequences in Bridge

E 1409 [1960, 290]. *Proposed by Paul Lukacs, Tel-Aviv, Israel*

In the game of bridge North is dealer. (a) What is the number of the different bidding sequences, assuming that West and East are silent? (b) What is the number of the different bidding sequences if all four players must participate in the bidding?

Solution by Walter Zayachkowski, University of Alberta. (a) North has 36 possible opening bids. The number of different bidding sequences is $2^{36}-1$, the number of subsets of $\{1, 2, \cdots, 36\}$ excluding the null subset.

(b) Labelling 7 No Trump as bid 1, 7 Spades as bid 2, $\cdots$, 1 Diamond as bid 34, 1 Club as bid 35, all possible opening bids, if North opens with bid k we have $7\cdot 22^{k-1}$ different bidding sequences. The total number of different bidding sequences for the 35 opening bids is $(22^{35}-1)/3$. Should North pass originally, there are an additional 22^{35} bidding sequences, giving a total of $(4\cdot 22^{35}-1)/3$ different bidding sequences.

Also solved by L. D. Goldstone, W. C. Guenter, A. R. Hyde, D. E. Knuth, Wallace Manheimer, D. C. B. Marsh, J. L. Pietenpol, D. L. Silverman, and the proposer.

Not all the above solutions agreed. For an allied problem see E 801 [1948, 578].

Isomorphic Groups

E 1410 [1960, 290]. *Proposed by Albert Wilansky, Lehigh University*

Is there a field whose additive and multiplicative groups are isomorphic?

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* No. Because there is implied the existence of a field in which the usual postulates hold and also in which addition is distributive over multiplication. Let 0 be the additive identity, e the multiplicative identity, and x any element of the field. Then $e+0\cdot x=(e+0)(e+x)$ gives $x=0$, which contradicts the fact that a field must contain at least two distinct elements.

II. *Solution by N. J. Fine, Institute for Advanced Study.* No. The equation

$x^2=1$ has either one or two solutions, depending on whether $-1=1$ or $-1\neq 1$, that is, on whether $2=0$ or $2\neq 0$. In the first case the corresponding equation $2x=0$ has at least the two solutions $x=0$, 1, and in the second case it has only one solution, $x=0$.

Also solved by Neil Bernstein, D. M. Bloom, J. L. Brown, Jr., T. C. Brown, J. N. Deeter, R. H. DeVore, J. W. Ellis, R. L. Faber and M. I. Freedman (jointly), E. R. Gentile, L. D. Goldstone, J. D. Haggard, Shigeru Ishii and Joel Levy (jointly), John Jordan, M. S. Klamkin, D. E. Knuth, A. G. Konheim, J. P. Labute, Loren Larson, Joe Lipman, Gerald Losey, J. A. Lutts, D. C. B. Marsh, H. F. Mattson, D. R. Morrison, F. D. Parker, D. J. Persico, J. L. Pietenpol, R. E. Priest, Azriel Rosenfeld, C. P. Sequin, D. L. Silverman, C. F. Stephens, Wu Ta-Sun, Seth Warner, and Alan Wayne.

This problem is the same as Problem 4644 [1956, 587]. In the last line of the solution given there, the words "one to one" should be replaced by the word "isomorphic."

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4929. *Proposed by James W. Brown, University of Michigan*

Given that $y(t)$ satisfies the integral equation

$$y(t) = t^k + \int_0^t y(t-\tau)J_0(\tau)d\tau,$$

where $J_0(t)$ is the first order Bessel function and $k>-1$. Show that

$$\int_0^\infty e^{-t}y(t)dt = \frac{(\sqrt{2})\Gamma(k+1)}{\sqrt{2}-1}.$$

4930. *Proposed by D. F. Rearick, University of British Columbia*

For a fixed positive integer n, consider the numbers $a_m=(m, n)$ as m ranges over a complete residue system (mod n). Prove that the number of a_m having an even number (including none) of distinct prime factors exceeds the number having an odd number of distinct prime factors if n is odd, and that the two are equal if n is even.

4931. *Proposed by Harry Goheen, Oregon State College*

In an algebra for which there is a commutative law but no associative law, how many formally distinct sums are there of n elements?

4932. *Proposed by D. S. Mitrinovitch, Belgrade University, Yugoslavia*

Consider the differential equation $z^2(d^4w/dz^4) = a(d^2w/dz^2)$, where a is a real parameter, $z = x + iy$. Determine the singularities of the complex functions $w = f(z, a)$ which satisfy it.

4933. *Proposed by I. N. Herstein, Cornell University*

Suppose a ring R is the set theoretic union of a finite number of commutative fields; prove that R must then be a commutative field.

4934. *Proposed by I. N. Herstein, Cornell University*

Let F be a field of characteristic 0, let G be a group, and $\Gamma(G, F)$ the group algebra of G over F. Prove that $\Gamma(G, F)$ is an algebraic algebra over F if and only if G is locally finite.

SOLUTIONS

Definite Integrals

4877 [1959, 921]. *Proposed by O. P. Aggarwal and Irwin Guttman, University of Alberta*

Show that

$$\int_0^a e^{-t^2/2}dt = (\pi/2 - aI)^{1/2},$$

where

$$I = \int_{a^2}^{2a^2} \frac{e^{-\omega/2}d\omega}{\omega\sqrt{(\omega - a^2)}}.$$

I. *Solution by E. S. Keeping, University of Alberta.* Let $N = \int_0^a e^{-t^2/2}dt$, $a > 0$. Then

$$N^2 = \int_0^a \int_0^a e^{-(t^2+u^2)/2}dtdu.$$

Changing to polar coordinates and noting the symmetry about $\theta = \frac{1}{4}\pi$, we obtain

$$N^2 = 2\int_0^{\frac{1}{4}\pi} \int_0^{a \sec\theta} e^{-r^2/2}rdrd\theta = 2\int_0^{\frac{1}{4}\pi} (1 - e^{-(a^2 \sec^2\theta)/2})d\theta.$$

Putting $\omega = a^2 \sec^2\theta$, $d\omega = 2a^2 \sec^2\theta \tan\theta d\theta = (2\omega/a)(\omega - a^2)^{\frac{1}{2}}d\theta$, we get the desired result,

$$N^2 = \tfrac{1}{2}\pi - a\int_{a^2}^{2a^2} \frac{e^{-\omega/2}d\omega}{\omega\sqrt{(\omega - a^2)}}.$$

II. *Solution by Marlow Sholander, Carnegie Institute of Technology.* Let

$$P = \int_0^a e^{-t^2/2}dt, \qquad f(w) = \frac{e^{-w/2}}{w\sqrt{(w-a^2)}}, \qquad I = \int_{a^2}^{2a^2} f(w)dw.$$

We integrate the last by parts, differentiate as shown, and substitute $w=t^2+a^2$, obtaining:

$$aI = \tfrac{1}{2}\pi e^{-a^2} + \int_{a^2}^{2a^2} e^{-w/2}\cos^{-1}\frac{a}{\sqrt{w}}\,dw,$$

$$\frac{d(aI)}{da} = -\int_{a^2}^{2a^2} wf(w)dw = -2e^{-a^2/2}P = -\frac{dP^2}{da}.$$

Hence $aI+P^2=\frac{1}{2}\pi$, the value at $a=0$.

Also solved by A. R. Bradley, D. A. Breault, Robert Breusch, J. W. Brown, R. G. Buschman, L. Carlitz, P. R. Chernoff, P. G. Engstrom, N. J. Fine, Todd Gitlin, M. L. Glasser, George Glauberman, Emil Grosswald, M. S. Klamkin, A. G. Konheim, T. V. Lakshminarasimhan, S. Leja, Julius Lieblein, D. C. B. Marsh, J. S. Potter, Barbara Sakitt, E. M. Scheuer, Chih-yi Wang, Robert Weinstock, David Zeitlin, and the proposers. Late solution by E. K. McLachlan.

Helly's Theorem

4878 [1959, 921]. *Proposed by D. J. Newman, Brown University*

A collection of closed bounded convex sets is given in the plane with the property that any three of them have a point in common. Prove they all have a point in common.

Solution by D. S. Greenstein, University of Michigan. The stated proposition is case $n=2$ of Helly's theorem, an elegant proof of which is to be found in H. Rademacher and I. J. Schoenberg, *On Helly's theorem on convex sets and the Tchebycheff approximation problem*, Canad. J. Math., 1950. Helly's original proof appeared in Jber. Deutsch Math. Verein., vol. 32, 1923, pp. 175–176.

Also solved by S. H. Eisman, Irma Esrig, Emil Grosswald, D. W. Henderson, Nicholas Kazarinoff, P. S. Landweber, Yoshio Matsuoko, Marlow Sholander, Robert Weinstock, A. S. Zamanakos, and the proposer. Late solution by E. K. McLachlan.

A Definite Integral Relationship

4880 [1959, 921]. *Proposed by A. W. Goodman, University of Kentucky*

Show that, for all α, β, $0<\alpha<\beta<\pi$,

$$\int_0^\alpha \sqrt{\left(\frac{\cos\theta-\cos\beta}{\cos\theta-\cos\alpha}\right)}\,d\theta + \int_\beta^\pi \sqrt{\left(\frac{\cos\beta-\cos\theta}{\cos\alpha-\cos\theta}\right)}\,d\theta = \pi.$$

Solution by Leonard Carlitz, Duke University. If we put $x=\cos\theta$, $a=\cos\alpha$, $b=\cos\beta$, $-1<b<a<1$, the stated result becomes

$$(1)\qquad \int_{-1}^{b}\sqrt{\left(\frac{b-x}{a-x}\right)}\frac{dx}{\sqrt{(1-x^2)}}+\int_{a}^{1}\sqrt{\left(\frac{x-b}{x-a}\right)}\frac{dx}{\sqrt{(1-x^2)}}=\pi.$$

Since, when $b=a$ (1) obviously holds, it will suffice to show that the derivative (with respect to b) of the left member vanishes, *i.e.*:

$$(2)\qquad \int_{-1}^{b}\frac{dx}{\sqrt{\{(a-x)(b-x)(1-x^2)\}}}-\int_{a}^{1}\frac{dx}{\sqrt{\{(x-a)(x-b)(1-x^2)\}}}=0.$$

We apply the transformation $y=(\lambda x+1)/(x+\lambda)$, choosing λ so that $(a+b)\lambda^2+2(ab+1)\lambda+(a+b)=0$, $|\lambda|>1$. Since the discriminant of the quadratic is $(1-a^2)(1-b^2)>0$, λ is real. Put

$$k=\frac{\lambda a+1}{a+\lambda}=-\frac{\lambda b+1}{b+\lambda};$$

then $0<k<1$ and the intervals $[-1, b]$, $[a, 1]$ are carried into $[-1, -k]$, $[k, 1]$, respectively. The first integral in (2) now becomes

$$\left(\frac{\lambda^2-1}{(\lambda+a)(\lambda+b)}\right)^{\frac{1}{2}}\int_{-1}^{-k}\frac{dy}{\sqrt{\{(y^2-k^2)(1-y^2)\}}}.$$

The transform of the second integral in (2) is the same except that the limits of integration are k to 1. Thus they are equal, and their difference vanishes, as required.

Also solved by Marlow Sholander using contour integration.

Entire Function with Special Property

4881 [1960, 86]. *Proposed by Burton Randol, The Rice Institute*

Is there a nontrivial entire function f, such that $\int_L f(z)dz$ vanishes for all straight lines L infinite in both directions?

Solution by J. C. Kneser, Tübingen, Germany. A nonconstant entire function which tends to zero along every half-line has been exhibited by G. Mittag-Leffler, Acta Math. vol. 42, 1920, pp. 285–308. The derivative of this function has the characteristics required by the present problem.

More general examples may be found in K. Grandjot, Math. Ann. vol. 91, 1924, pp. 316–320 and H. Bohr, S.-B. Preuss. Akad. Wiss., Phys.-Math. Kl. 1929, 565–571.

Also solved by W. H. J. Fuchs and the proposer.

Filters with the Countable Intersection Property

4882 [1960, 86]. *Proposed by I. S. Gal, Cornell University*

An (ω, ∞)-filter in a set X is a filter $\mathfrak{F}$ in X which has the countable intersection property: $F_n\in\mathfrak{F}$ implies $\bigcap F_n\in\mathfrak{F}$ $(n=1, 2, \cdots)$. Prove the following proposition:

If X is a metric space and if every (ω, ∞)-filter in X has a nonvoid adherence then the same holds for every open subspace of X.

Solution by Meyer Jerison, Purdue University. The following is the essential content of problem 8H.5 in L. Gillman and M. Jerison, *Rings of Continuous Functions.*

THEOREM. *For any topological space X, every filter on X with the countable intersection property has nonvoid adherence if and only if X is a Lindelöf space, i.e., every open cover of X has a countable subcover.*

The proof is like the proof of the theorem that every filter on a space has nonvoid adherence if and only if the space is compact—"finite" being replaced by "countable."

Now, a metric space is a Lindelöf space if and only if it is separable. In fact, for each positive integer n, a metric Lindelöf space is covered by countably many balls of radius $1/n$, and the centers of all such balls constitute a countable dense set in the space. The converse follows directly from the existence of a countable base of open sets in a separable metric space.

Thus, the class of spaces considered in the problem is the class of separable metric spaces, and every subspace of such a space, whether open or not, belongs to the class.

Also solved by the proposer.

Symmetric Continued Fraction

4883 [1960, 86]. *Proposed by W. A. Schneider, Milwaukee, Wisconsin*

If

$$F = 1 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n+}\,\frac{1}{a_n+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1}$$

and P_n/Q_n is the nth convergent of F, prove

$$\cot^{-1} Q_{2n} = \cot^{-1} P_{2n} + \cot^{-1} P_{2n+1}.$$

Solution by C. F. Pinzka, University of Cincinnati. From the recurrence relations, $Q_{2n+1} = a_1Q_{2n} + Q_{2n-1}$, $Q_{2n} = a_2Q_{2n-1} + Q_{2n-2}$, $\cdots$, $Q_3 = a_2Q_2 + Q_1$, $Q_2 = a_1Q_1$, we have

$$\frac{Q_{2n+1}}{Q_{2n}} = a_1 + \frac{1}{a_2+}\cdots\frac{1}{a_n+}\,\frac{1}{a_n+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1}$$

$$= \frac{1}{F-1} = \frac{Q_{2n+1}}{P_{2n+1} - Q_{2n+1}}.$$

This gives $Q_{2n} = P_{2n+1} - Q_{2n+1}$. Eliminating Q_{2n+1} from this last relation and $P_{2n+1}Q_{2n} - P_{2n}Q_{2n+1} = -1$, we have

$$Q_{2n} = \frac{P_{2n}P_{2n+1} - 1}{P_{2n} + P_{2n+1}},$$

which is equivalent to the proposed result.

Also solved by Evelyn Frank, Robert Heller, Stanton Philipp, J. H. Toope, and Chih-yi Wang. Late Solution by S. Parameswaran.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association

Mathematical Analysis. By Edwin M. Hemmerling. McGraw-Hill, New York, 1959. xi+332 pp. $5.75.

This book presupposes that the student has completed his full three years of normal high school mathematics and that he will follow this book with a study of analytic geometry and calculus. The conceived function was to review high school mathematics. Thus the author does not feel obligated to develop the sections in a logical sequence (trigonometry is used long before the definition is given), or include topics such as word problems, binomial theorem or inequalities.

In those cases where a student lacks the proficiency to master the calculus course, it is wasteful of time, talent and tuition to compound the difficulty further by repetition of poorly organized remedial drill on algebraic techniques that are deficient in clarity, significance or logical justification. This book's method of presentation is limited to a couple of worked examples per section and a paucity of proofs. Although the author states "Particular emphasis is given to the analytic relationship between trigonometric functions," the student is referred elsewhere for proofs of the addition formulas.

In a fruitless attempt to justify the title, the last two chapters are concerned with infinite series and differential calculus. The common mistakes about limits and the chain rule are present. It is no wonder that high schools are advised not to attempt a discussion of calculus with such inaccurate texts being written for college use.

From among other misprints, mistakes of carelessness and misleading statements, the following are taken as samples: no assumption regarding the rationality of the coefficients precedes the conclusion that the roots are rational if b^2-4ac is a perfect square; the number of inversions in 2431657 is miscounted; and $r=.013$ is used in converting .61313 $\cdots$ to rational form.

JAMES H. MCKAY
Michigan State University Oakland

Fundamental Principles of Mathematics. By John T. Moore. Rinehart, New York, 1960. 630 pp. $7.00.

Here is an interesting agglomeration of "standard" and "modern" topics intended for a two semester college freshman course, either a pre-calculus course or a terminal course.

To give an air of flavor there follows a sketchy sampling of the vocabulary —roughly in the order of first occurrence:

> Proper subset, disjoint, power set, cardinal number, cartesian set, intersection, complement, Boolean algebra, reflexive, equivalence relation, independent postulates, natural numbers, numeral, commutative, monotonic law, rational numbers, real numbers, sequence set, nested intervals (we are on page 39 now!), exponent, principal root, scientific notation, significant digit, logarithm, relation, ordered pair, domain, function, cartesian coordinate, inverse relation, power function, hyperbola, asymptotes, slope of a line, basic circular functions, periodic function, Cap-sine (inverse is function), quadratic equation, completing the square, point-slope equation for line, one parameter family, distance between two points, trigonometric identities, degree circular functions, radian (we are $\frac{1}{3}$ through; remaining terms are general and more sparsely chosen), trigonometry of the general triangle; glimpse of differential calculus; polynomial function, maximum-minimum; glimpse of integral calculus; complex numbers, fundamental theorem of algebra; real polynomial equations; inequalities; systems of equations, matrices, determinants; plane analytic geometry; polar coordinates; solid analytic geometry; binomial theorem; probability; statistics.

Obviously not all of the topics alluded to above can be treated thoroughly.

The specialist in foundations or other purist who believes that a freshman should be told "the truth" will often be horrified. Much of the exposition will be understandable only to students who have already learned the material either in previous courses or in class. Nevertheless, if a book along these lines is indicated, this is a generally creditable job. Probably its strongest feature is the large number of excellent exercises.

DAVID A. PAGE
University of Illinois

Real Variables. By John M. H. Olmsted. Appleton-Century-Crofts, New York, 1959. xvi+621 pp. $9.00.

This work is an extension of Olmsted's *Intermediate Analysis*, which constitutes the first nine chapters of this book. It contains topics normally covered in a first course in advanced calculus. The author has starred certain sections and exercises and double starred others. The starring indicates the level of the work, the unstarred portions being the least difficult, the starred ones more so, and so on. For example, the work which deals with uniform continuity on convergence is in starred sections while sections involving topics such as metric topological spaces are doubly starred. Line integrals are introduced solely in doubly starred exercises while work on the Legendre and Tchebycheff polynomials are introduced in doubly starred sections. Thus this text could be used for a first course in analysis at various degrees of difficulty.

There is an abundance of exercises varying in difficulty from the graphs of simple functions to proofs of the Minkowski inequality for finite forms. The proofs and extensions of many theorems are relegated to the exercises. Thus, exercises are available for the average student and enough of greater difficulty to stimulate the above-average student. Throughout the text the author frequently refers to the exercises and it is sometimes disconcerting when dealing with an unstarred topic to find a reference to one that is starred. As a result, a teacher must take care in assigning exercises to see that those are assigned which are essential to the development of the theory he desires to present.

The book is well written with numerous examples and should prove satisfactory as a text in courses called advanced calculus or introduction to the theory of functions of a real variable.

R. G. SANGER
Kansas State University

The Analysis of Variance. By H. Scheffé. Wiley, New York, 1959. xvi+477 pp. $14.00.

This book presents the theory of the analysis of variance and is not a cookbook, though computational methods are outlined and sufficient problems included to give the reader ample opportunity for practice. The author stresses vector space and geometrical ideas; for instance, the sum-of-squares decomposition may be viewed as resolving the observation vector into components lying in specified subspaces. In the opinion of the reviewer, this book is more comprehensive and complete than its predecessors, and is recommended to all serious students of statistics. It is suitable for either a course at the advanced undergraduate or graduate level, or for self-study. The author suggests as necessary only a minimum mathematics background, but the reviewer strongly recommends a course, or its equivalent, in linear algebra for potential readers of this very fine book.

The contents include the theory of estimation and least squares, and the Gauss-Markov theorem in Chapter 1; the ideas of confidence ellipsoids, tests, and the likelihood ratio are in Chapter 2. Subsequent chapters discuss the analysis of designs ranging from the simple one way to incomplete blocks. Chapter 6 is about covariance. The next three chapters present the analysis of random, mixed, and randomization models. In Chapter 10 he discusses the effect of certain departures from the underlying assumptions. Six appendices deal with vector algebra, matrix algebra, ellipsoids, noncentral distributions, the multivariate normal, and Cochran's theorem. Tables of F, its power, and the studentized range are included. There are numerous literature citations, a long list of references, and a profusion of interesting footnotes.

THOMAS E. KURTZ
Dartmouth College

Introduction to Mathematical Physics. By William Band. Van Nostrand, Princeton, N. J., 1959. x+326 pp. $7.25.

This stimulating work is a well-organized collection of brief surveys of topics in mathematical physics, all of which are currently of cardinal interest and importance. It was designed primarily to supply a preparation for senior and graduate students for more penetrating study in the advanced special treatises. In the reviewer's opinion it does fulfill this purpose very well. It is not a difficult book to read and the author has been able to skim off a considerable amount of the cream from an impressive variety of subjects. The style is relaxed and easy-going and relief from the technical details is provided occasionally by historical and philosophical material. Considerable emphasis is placed on the heuristic value of invariance but full use is not made of this powerful concept. The rigor level of the mathematical developments is about the average for books of this type.

Most works on mathematical physics are not uniformly modern so far as the real variable aspects are concerned. The limit of an infinite numerical process, for example, may be identified with a stage of the process rather than taken as a number associated with the process in a certain way. There are one or two passages in the present text which admit of valid interpretations but nevertheless might tend to strengthen the misconceptions of students who have not had rigorous courses in the calculus, for example, (page 85) . . . "the partial derivative $\partial v/\partial t$ refers to a fixed position in space, so that we are comparing the velocities of two different pieces of material at the two close instants of time." Also in one or two places the student might be led to the erroneous conclusion that the Euler equations constitute sufficient conditions for an extreme value, and the statement (page 4) "The relations between length and area necessitate the use of irrational numbers $2^{\frac{1}{2}}$, pi (π), etc." although true needs revision. These are, of course, not serious matters for the student who is well prepared in mathematics.

This book naturally consists of two parts: the mathematical background and the mathematicophysical theories. Because of limitations of space the mathematical developments are necessarily sketchy. As a time-saving device, vector algebra is presented at the outset by means of a framework of definitions that have their origins in the ordinary geometrical vector algebra, specifically, "The vector is here defined as the set of three numbers, considered as a whole and obeying a certain algebra which consists of mutually consistent rules for adding, multiplying, differentiating such entities." . . . "The *scalar product* of two vectors written $\mathbf{A}\cdot\mathbf{B}$, is defined by $\mathbf{A}\cdot\mathbf{B}=A_1B_1+A_2B_2+A_3B_3$" . . . "The angle θ between two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined by $\cos\theta=(\mathbf{A}\cdot\mathbf{B})/AB$" A set of three ordered real numbers (*i.e.*, a function defined over the discrete set 1, 2, 3) is not the same mathematical concept as a vector in the sense of a directed line segment characterized by postulates extracted from euclidean geometry. A vector in the latter sense has associated with it an infinite collec-

tion of sets of components (one or two sets for each coordinate system). Because of the conceptual differences verifications must be made (if understanding is to be above the manipulative level) before the formulas of the strictly numerical theory can be employed in geometric situations. These verifications are left to the student but pertinent detailed figures and problems are supplied.

Among the topics treated in the part of the book concerned with physical theories are: vibrating strings and membranes, elasticity theory, fluid dynamics, Lagrangian and Hamiltonian mechanics, canonical transformations, phase space, statistical mechanics, electromagnetic theory, Lorentz transformations, invariance of electromagnetic laws, gravitation in general relativity, wave-particle duality, Schrödinger's equation, and relativistic quantum mechanics. There is a slip of the pen in the part on relativity. The author finds it convenient because of previous notational commitments to denote contravariant quantities by subscripts and covariant quantities by superscripts—a reversal of the usual notation. However, on page 283, the expression for interval is written $g_{\mu\nu}dx_\mu dx_\nu$ and here the $g_{\mu\nu}$ cannot be contravariant because the form is invariant. However, it should be borne in mind that the presence of anomalies in advanced original works is almost inevitable and the present text does have many marks of expositional originality. In the reviewer's opinion this book is a valuable contribution to the literature.

HOMER V. CRAIG
Boeing Airplane Co. and The University of Texas

Methods Based on the Wiener-Hopf Technique. By B. Noble, Pergamon Press, New York, 1959. x+246 pp. $10.00.

This book is designed for applied mathematicians, and contains an excellent selection of solutions to mixed boundary value problems from such fields as hydrodynamics, acoustics, electromagnetic theory and potential theory. Throughout the text the differential equations themselves are transformed, thus avoiding the use of too many special functions, Green's functions, and integral equations. This simplified approach (attributed to D. S. Jones) uses only the original differential equations, and the Fourier, Laplace or Mellin transformation applied to these equations, with an application of analytic continuation of the transforms.

There is included a section on approximate solutions, and a section pointing up the connection of the Wiener-Hopf method with the general Hilbert-Riemann problem.

On the whole, there is enough attention to rigor to indicate the appropriate steps, but not so much that the text is bogged down with too many fine details. This makes the book very readable. Further, the discussion of a large number of research papers makes the book an excellent reference for workers in this field.

GORDON LATTA
Stanford University

Finite Mathematical Structures. By John G. Kemeny, Hazleton Mirkil, J. Laurie Snell, and Gerald L. Thompson. Prentice-Hall, Englewood Cliffs, N. J., 1959. xi+487 pp. $7.95.

This book has many points of resemblance to the earlier *Introduction to Finite Mathematics*, by three of the present authors. But where that book was intended for biological and social science students, this one is aimed at the physical sciences and engineering. Furthermore, the earlier book was for freshmen, while this one is for sophomores. There are seven chapters: 1. Compound statements (propositional calculus, truth tables, etc.), 2. Sets and functions, 3. Probability theory, 4. Elementary linear algebra, 5. Convex sets (including in particular an introduction to linear programming), 6. Finite Markov chains, 7. Continuous probability theory. Practically everything starts off in the most elementary way possible, and a knowledge of the calculus is assumed at only a few points, these mostly in the last chapter. The book is designed for a year course, or for a one-semester course in either probability or linear algebra. The book would be less satisfactory as an algebra text than as a probability text, since it doesn't get far in linear algebra. Despite the fact that Chapter 4 is the longest, the flavor of the book is not overly algebraic; elementary logic, probability and related topics get a more thorough treatment.

It is of course refreshing to see a book like this, which recognizes the necessities of the modern world: probability, linear algebra, and linear programming, for instance, are at least as valuable to the physical scientist and engineer today as the subjects we usually stuff them with, such as tricky differential equations. And, by and large, the authors have done a good job of introducing these subjects. The reviewer is impressed with the way they have managed to bring in important mathematical concepts and problems; the continuum hypothesis and Goldbach conjecture, for instance. There are a profusion of problems, mostly attractive and informative, and quite a few are novel.

The reviewer's reservations are of several kinds. First, an even casual inspection reveals statements like: (p. 54) "It would be difficult to give a simple description for the subset containing the elements $\{P1, P4, P14, P30, P34\}$." (In context, this looks better than it does here, but perhaps that makes the sentence even more insidious.) Or, discussing how a function can be defined (p. 71), "And a formula is possible only if both D and the range R consist of numbers." (In Chapter 4, for instance, they give what *look* like formulas for functions on vector spaces.) Second, a good argument can be made that mathematics majors, at least, should not be exposed to truth tables, Venn diagrams, propositional calculus, etc., until they know more honest-to-God mathematics. The treatment must of necessity be elementary and simple (as it is here), yet there is something facile and flashy about such matters which can mislead the student. (By the way, the book does not lend itself easily to the omission of the first two chapters, since the terminology introduced there is used heavily in later chapters. This has the interesting side effect that some statements about

very basic mathematics, later on, sound mysterious and strange to adult mathematicians.) The authors may have discovered that students are not misled, in fact, by their treatment; it is a pragmatic fact about which the reviewer has no definite knowledge.

The reviewer's final feeling: it would be pleasant to try this book on sophomores, but playing down part of the first two chapters, and introducing a more abstract and thorough treatment of linear algebra towards the end of Chapter 4. If one were willing to forego some of the later material on probability, the book might even be suitable for a class of better-than-average freshmen, who presumably know no calculus. Almost certainly this text is a glimpse into the undergraduate near future, with less emphasis than today on gobs of calculus, differential equations, and related analysis topics.

D. R. HUGHES
University of Chicago

Statistical Independence in Probability, Analysis and Number Theory. By Mark Kac. Carus Monograph No. 12. Mathematical Association of America, 1959.* Distributed by Wiley, New York. xiv+93 pp. $4.00.

This book is an outgrowth of the 1955 Hedrick Lectures in which, the author states, his aim was to show that "(a) extremely simple observations are often the starting point of rich and fruitful theories and (b) many seemingly unrelated developments are in reality variations on the same single theme." This is well done indeed in this beautifully written little book.

Starting with the idea of independence implied in a formula of Vieta, which in terms of Rademacher functions shows a case where the integral of a product equals the product of integrals, Kac proceeds through a discussion of the vague early notions of probability and independence, Borel's work marking the beginnings of modern probability theory, and the later developments of this work. He puts such ideas as set-theoretical vs. frequency definitions of probability into perspective, discusses several "laws of large numbers" in terms of Rademacher functions and Lebesgue measure and indicates their connection with "the normal law." Comments and examples tying these ideas to other realms of mathematics and physics abound.

A chapter on the density of prime numbers and the normal law in number theory is followed by one relating the ergodic theorem to continued fractions.

This book should be required reading for students in probability courses and is highly recommended to others as instructive in perspective, methods of proof, lessons to be learned from history and in the fact that mathematics is not a separate discipline but part of a stream of ideas embracing its fields of application.

PAUL D. MINTON
Southern Methodist University

* Each member of the Association may buy one copy at a special price from H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

Analytic Functions. By R. Nevanlinna, H. Behnke and H. Grauert, L. V. Ahlfors, D. C. Spencer, L. Bers, K. Kodaira, M. Heins, J. A. Jenkins. Princeton Mathematical Series, no. 24. Princeton University Press, Princeton, 1960. vii+197 pp. $5.00.

This volume contains the principal addresses presented at a conference held in 1957 at the Institute for Advanced Study. R. Nevanlinna's address contains a new proof of Liouville's theorem on conformal mappings in R^n $(n>2)$. Behnke and Grauert give a very readable introduction to and survey of the theory of analytic functions of several complex variables. The addresses of Ahlfors, Spencer, and Kodaira deal with problems in various complex analytic structures. Bers reports on quasiconformal mappings, Heins on conformal mapping of Riemann surfaces, and Jenkins on coefficients of univalent functions. While the addresses are mainly directed to experts, several, in particular Behnke and Grauert's, will be of interest to the general mathematical reader.

EDWIN HEWITT
University of Washington

An Introduction to Mathematical Statistics. By H. D. Brunk. Ginn, Boston, 1960. xi+403. $7.00.

The heart of this book is intended for a semester course in the beginning of mathematical statistics for advanced undergraduates and beginning graduates. There is an abundance of "starred" material permitting expanded use into a one-year course. The book seems very flexible, not only in being useful for either a one-half year or a full year course, but also in terms of its potential audience. It seems equally suited for students in subject matter fields who want a firm mathematical foundation in statistical procedures, as well as for students who are beginning a serious study of mathematical statistics. Probability foundations and random variables are presented in a fashion which is clear and sufficiently careful for this level. The development of statistical theory well reflects the modern developments of the last several decades. The book is well-written and has many worthwhile problems.

MEYER DWASS
Northwestern University

Linear Programming and Economic Analysis. By R. Dorfman, P. A. Samuelson, and R. M. Solow. McGraw-Hill, New York, 1958. ix + 527 pp. $10.00.

The theory of linear programming centers about a single problem: "Maximize the linear form $L(x)=\sum_{i=1}^{n} b_i x_i$ subject to the constraints $\sum_{j=1}^{n} a_{ij}x_j \leqq c_i$, $i=1, \cdots, M$." The question of existence is simple, and the problem is trivial if N and M are both small. The crux of the problem is that of determining feasible computational algorithms when M and N are large, of the order of several hundred or several thousand. Of some interest is also the problem of determining the analytic structure of the solution when the matrix $A=(a_{ij})$ has a particular form.

Although questions of this nature, as part of the general theory of linear inequalities, date back fifty to seventy-five years, they have only been intensively studied in the last ten years or so. This is a consequence of the great need for their solutions in the field of mathematical economics, in connection with input-output analysis and game theory, and in operations research in connection with allocation of resources, transportation theory, and scheduling theory.

The authors present a detailed account of the theory of linear programming and its economic background with discussion of numerous examples. The writing is leisurely and lucid, and a model of what good scientific writing should be.

The only drawback to the book lies in its treatment of dynamic processes. Although the methods of linear programming are not particularly applicable to this area, the authors have not included a discussion of alternative methods.

The book is thoroughly to be recommended to research workers in the fields of mathematical economics, game theory and operations research, to students entering these expanding fields, and to mathematicians who wish to obtain an overall view of this modern sphere of intellectual activity.

RICHARD BELLMAN
The RAND Corporation

College Algebra and Trigonometry. By William L. Hart. Heath, Boston, 1959. ix+466 pp. $5.50.

The text covers the traditional topics of college algebra and trigonometry, with more than the usual amount of analytic geometry. Graded problem lists are more than adequate. Topics are so arranged and compartmented that the text may easily be adapted to courses of various types and lengths.

Basic concepts of *variable*, *constant*, *function*, etc. are dealt with in traditionally ambiguous language. At one point *variable* is properly defined, but "x" is generally referred to as a "literal number," "letter," "number symbol," "general number." *Function* is initially defined in extension as a set of ordered pairs, etc., but is almost immediately identified with the independent variable, with function value, with "general value," and confused with equations or forms such as "x^2+3x," which, incidentally, are called "number expressions." The notion that a function has a domain and a range comes through rather better, yet the function concept does not assume a proper role throughout. It is not exploited, for instance, in treatment of sequences or "progressions."

Except for a few modern touches (*e.g.* section on probability), the text is a good traditional algebra-trigonometry with traditional virtues and vices. The conscientious student will emerge well drilled in techniques and in solving completely formulated problems. He may still not suspect that algebra has structure. He will probably not have improved his ability to construct algebraic proofs or to distinguish proof from heuristic.

ROBERT M. EXNER
Syracuse University

BRIEF MENTION

Regression Analysis. By E. J. Williams. Wiley, New York, 1959. ix+214 pp. $7.50.

This is not a theory book and not a mathematical book, but is based upon the author's years of consulting experience in applied statistics for experimental scientists.

Methods of Correlation and Regression Analysis. (3rd ed.) By Mordecai Ezekiel and Karl A. Fox. Wiley, New York, 1959, xv+548 pp. $10.95.

Another nonmathematical book on regression analysis; this time as an extension of an earlier book on correlation by Ezekiel. Mathematicians will do well to have a nonmathematical treatment of regression analysis on hand for loaning purposes.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo.

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor A. A. Albert, University of Chicago, has been appointed to the Eliakim Hastings Moore Distinguished Service Professorship of the University.

Professor Howard Eves, University of Maine, represented the Association at the inauguration of Dr. Robert E. L. Strider, II, as President of Colby College on October 11, 1960.

Professor E. E. Moise, University of Michigan, has been appointed James Bryant Conant Professor of education and mathematics at Harvard University.

The Reverend R. J. Swords, S. J., was inaugurated as President of the College of the Holy Cross on June 16, 1960.

Arizona State University: Mr. R. W. Sanders, Lockheed Aircraft Corporation, has been appointed Assistant Professor; Associate Professor B. M. Ingersoll, Lamar State College of Technology, has been appointed Associate Professor; Associate Professor Simon Green, Assumption University of Windsor, Ontario, Canada, has been appointed Professor.

Vanderbilt University: Associate Professor G. F. Clanton, on leave from Baylor University as Lecturer at the University of Minnesota, has been appointed Assistant Professor; Assistant Professors B. F. Bryant and J. R. Wesson have been promoted to Associate Professors.

Washington University: Professor K. A. Hirsch, University of London, England, has been appointed Visiting Professor; Professor Helmut Schiek, University of Bonn, Germany, has been appointed Visiting Assistant Professor; Dr. F. J. Schnitzer, Wayne State University, has been appointed Research Associate.

Dr. R. D. Adams, University of Minnesota, has been appointed Assistant Professor at the University of Kansas.

Dr. R. E. Barlow, Stanford University, has accepted a position with the Communications Research Division of the Institute for Defense Analyses at Princeton University.

Dr. Archie Blake, Bendix Aviation Corporation, Ann Arbor, Michigan, has accepted a position as Manager, Analysis Section of the Raytheon Company, Sudbury, Massachusetts.

Dr. Eugene Butkov, McGill University, has been appointed Assistant Professor at St. John's University, Jamaica, New York.

Associate Professor M. L. Coffman, Abilene Christian College, has accepted a position as Senior Physicist with the Hamilton Standard Division of United Aircraft Corporation, Windsor Locks, Connecticut.

Professor A. J. Coleman, University of Toronto, has been appointed Head of the Department of Mathematics at Queen's University, Kingston, Ontario, Canada.

Mr. W. W. Collier, Harvard University, has accepted a position as Programmer for International Business Machines, Poughkeepsie, New York.

Mr. C. J. Cook, Purdue University, has been appointed Assistant Professor and Head of the Mathematics Department at Highpoint College.

Associate Professor William Craig, Pennsylvania State University, has been appointed Visiting Professor at the University of California, Berkeley.

Professor Wayne Dancer, University of Toledo, has been appointed Professor and Chairman of the Department of Mathematics and Natural Sciences at the Alaska Methodist University.

Mr. H. C. Dixon, Jr., Trinity College, has been appointed Head of the Mathematics Department of Berwick Academy, South Berwick, Maine.

Mr. J. W. Dutton, University of Kansas, has been appointed Teacher of Mathematics at the San Juan Unified School District, Carmichael, California.

Dr. C. W. Foard, Eastman Kodak Company, Rochester, New York, has been appointed Professor at the Rochester Institute of Technology.

Miss Phyllis R. Gansz, Baldwin School, Bryn Mawr, Pennsylvania, has been appointed Assistant Director of Admissions at Wilson College.

Professor P. C. Hammer, Director of the Numerical Analysis Laboratory of the University of Wisconsin, has returned from spending the academic year 1959–60 on a research leave in Zurich, Switzerland.

Professor Duncan Harkin, U. S. Naval Academy, has been appointed Professor and Chairman of the Department of Mathematics and Statistics at the American University, Washington, D. C.

Mr. K. P. Howe, University of Illinois, has accepted a position as Actuary Trainee with the New York Life Insurance Company, New York, New York.

Mr. Leonard Klafter, Army Map Service and the Department of the Air Force, has accepted a position as Head of Scientific Programming at the Bureau of Naval Weapons, Washington, D. C.

Mr. K. S. Kretschmer, Imperial Oil Limited, Toronto, Ontario, Canada, has accepted a position as Planning Associate with the Socony Mobil Oil Company, Incorporated, New York, New York.

Mr. C. J. Labovites, Headquarters, United States Air Force, has accepted a position with the Bureau of Naval Weapons, Washington, D. C.

Mr. J. A. Marlin, Assistant Instructor, University of Missouri, has accepted a position as a member of the Technical Staff of the Bell Telephone Laboratory, Winston-Salem, North Carolina.

Mr. J. W. Milsom, Southwestern Louisiana Institute, has accepted a position as Instructor at Texas College of Arts and Industries.

Mr. D. L. Muench, McQuaid Jesuit High School, Rochester, New York, has been appointed Assistant Professor at the United States Naval Academy.

Dr. D. E. Myers, Millikin University, has been appointed Assistant Professor at the University of Arizona.

Mr. P. T. Nugent, Assistant Instructor, Miami University, has been appointed Instructor at Vincennes University.

Mr. H. E. Pickett, Assistant Instructor, University of California, Berkeley, has accepted a position as Research Mathematician in Analytical and Physical Measurements, Richmond Laboratory of the California Research Corporation.

Dr. R. F. Rinehart, Case Institute of Technology, has returned from a two-year leave of absence as Director of Special Research and Operations Research at Duke University.

Mr. G. R. Rising, Greece Olympia High School, Rochester, New York, has been appointed Mathematics Coordinator for the Norwalk Public Schools, Norwalk, Connecticut.

Mr. R. L. Sauder, Virginia Military Institute, has accepted a position as Programmer with the General Electric Computer Department, Phoenix, Arizona.

Associate Professor N. T. Seeley, Jr., Agricultural and Technical College of North Carolina, has been appointed Assistant Professor at Morgan State College.

Associate Professor R. L. Shively has been awarded a leave of absence from Western Reserve University to accept a National Science Foundation Faculty Fellowship for study at the University of California, Berkeley.

Mr. J. R. Singletary, Fishburne Military School, Waynesboro, Virginia, has been appointed Master in Field of Mathematics at Perkiomen School, Pennsburg, Pennsylvania.

Miss Ann A. Steinbright, Eastern Baptist College, has accepted a position as Mathematician-consultant with the E. I. DuPont de Nemours and Company, Wilmington, Delaware.

Mr. R. F. Steinen, United States Rubber and Development Research Center, Wayne Township, New Jersey, has been appointed Head of the Mathematics Department of Hawken School, Lyndhurst, Ohio.

Mr. R. G. Tobey, Harvard University, has been appointed Instructor at the College of Wooster.

Mr. C. B. H. Watson, Canada Life Insurance Company, Toronto, Ontario, Canada, has accepted a position as Actuary with the Wyatt Company, Consulting Actuaries, Washington, D. C.

Associate Professor J. C. Wilson, Central College, Pella, Iowa, has been appointed Assistant Professor at Florida Presbyterian College.

Miss Edla G. Berger, Mathematician, Equitable Life Assurance Society, died June 17, 1960. She was a charter member of the Association.

Dr. R. P. Peterson, Jr., Senior Mathematician, Burroughs Electrodata, died June 19, 1960. He was a member of the Association for fourteen years.

Mr. George Tipton, Sr., Head of the Mathematics Department, Henderson County Junior College, died May 6, 1960. He was a member of the Association for twelve years.

Professor Oswald Veblen, Institute for Advanced Study, died August 11, 1960. He was a Charter Member of the Association.

Professor G. A. William, Oregon State College, died April 22, 1960. He was a member of the Association for thirty-five years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 87 persons have been elected to membership by the Board of Governors on applications duly certified.

NED PAUL ANDERSON, Student, Euclid Senior High School, Ohio

RUDOLPH F. APPEL, Programmer, Dime Savings Bank, Brooklyn, New York

ENRIQUE BAYÓ, M.A. (Columbia) Asso. Prof., University of Puerto Rico

MRS. LYDIA T. BILLS, A.B. (Wheaton Coll.) Teacher, Windsor Mountain School, Lenox, Massachusetts

DENNIS K BOSWELL, B.A. (Vanderbilt) Arcadia, California

JAMES O. BROOKS, M.A. (Michigan) Asst. Prof., Haverford College

SARAH BROOKS, B.A. (Albany State) Teacher, Utica, New York

ROBERT V. BUDNY, Student, Massachusetts Institute of Technology

RONALD C. BZOCH, Ph.D. (Illinois I.T.) Asst. Prof., University of Utah

ALICE M. CALHOUN, B.A. (Illinois) Programmer-Analyst, General Telephone and Electronic Labs., Bayside, New York

DAVID M. CARGILL, M.Ed. (Boston) Head of Dept., Dean Junior College

ARNOLD M. CHANDLER, M.S. (Wisconsin) Supervisor, Department of Public Instruction, Madison, Wisconsin

JOHN D. CLOUD, B.S. (Alabama) Grad. Asst., University of Alabama

ROBERT K. COBB, Student, Southwest Missouri State College

ROBERT E. COOMBS, B.S. (U.S. Naval Academy) Instr., Lakeside School, Seattle, Washington

ROBERT CROTEAU, M.Sc. (Montreal) Asst. Prof., Laval University

GEORGE S. CUNNINGHAM, M.Ed. (Maine) Director of Math. Education, New Hampshire State Department of Education

LEROY M. DAMEWOOD, M.S. (Oregon Coll.) Instr., Bowdoin College

GABRIEL F. DELVILLAR, M.S. (Mexico) Prof., University of the State of Lower California

FRANK R. DEMEYER, Student, Seattle University

DANIEL G. DEWEY, M.A. (Kansas) Asst. Prof., College of the Holy Cross

HENRY H. DIEHL, M.A. (Ohio) Asst. Prof., Wittenberg University

L. M. EVANS, M.S. (Texas State) Teacher, Highland Park High School, Dallas, Texas

ARTHUR G. EYLES, B.Eng. (McGill) Chief Computer Engr., Analdata Engineering, Boston, Massachusetts

SARAH A. FLUCK, Student, Temple University

ARNE HARTWICK GARNESS, M.S. (Wisconsin) Instr., Concordia College

JOHN W. GAROFALO, B.S. (St. Bonaventure) Programmer Math., Thiokol Chemical Corp., Elkton, Maryland

JOSEPH GELLER, B.A. (Montclair Teacher, Central High School Newark, New Jersey

VIOLA GRIFFIN, M.S. (Texas State) Teacher, Queen City Independent Schools, Texas

LOUIS J. GRIMM, M.S. (Georgia I.T.) Chemist, U.S. Public Health Service, Savannah, Georgia

LOUIS N. GROSS, B.S. (Chicago) Mathematician, Mitre Corporation, Lexington, Massachusetts

HARRY P. HALE, M.S. (Pittsburgh) Asso. Prof., Wayne State University

THOMAS E. HANRAHAN, M.S. (DePaul) Research Asst., American Medical Association, Chicago, Illinois

HAROLD HENDLER, A.B. (Brooklyn Coll.) Teacher, Andrew Jackson High School, Canbria Heights, New York

ROBERT F. HENKEL, Technical Analyst, McDonnell Aircraft Corporation, St. Louis, Missouri

NEIL W. HENRY, M.A. (Dartmouth Coll.) Research Asst., Sociology, Columbia University

EARL W. HESSEE, B.N.E. (N. Carolina) Manager, Lockheed Aircraft Corporation, Dawsonville, Georgia

THOMAS P. HETTMANSPERGER, Student, Indiana University

KONRAD J. HEUVERS, Student, Stanford University

ROGER B. HOOPER, M.A. (Wesleyan) Head of Dept., Trinity-Pawling School, Pawling, New York

FRANK L. JENKINS, JR., M.A. (Illinois) Instr., Fenn College

MIKE KAPLAN, Student, Hunter College

MOSES S. KARMAN, M.A. (Brooklyn Coll.) Head of Dept., Ramaz Upper School, New York, New York

RUBIE L. KELLEY, B.S.E. (Arkansas) Teacher, Monahans High School, Texas

BRUCE W. KING, B.S. (Albany State) Teacher, Burnt Hills-Ballston Lake High School, Burnt Hills, New York

SAMUEL W. KODIS, A.B. (California) Instr., San Diego State College

CLARENCE R. KROPP, B.S. (Great Falls Coll.) Teacher, Paris Gibson School, Great Falls, Montana

DARRYL E. KUHNS, Student, Miami University

DONALD L. LANG, B.S. (Great Falls Coll.) Teacher, Geraldine High School, Montana

PETER LEES, B.S. (New Zealand) Asst. Master, OneLunga High School, Auckland, New Zealand

HILBERT LEVITZ, B.A. (N. Carolina) Grad. Asst., Pennsylvania State University

PHILIP L. MATTSON, B.S. (California State) Computer-Programmer, Curtis Publishing Company, Philadelphia, Pennsylvania

VERNON A. MAZEMKE, B.A. (Wartburg Coll.) Grad. Student, Wartburg College

CARL M. METZLER, B.S. (Goshen Coll.) Instr., Goshen College

MRS. EMMA F. MIXON, M.S. (Louisiana State) Asst. Prof., Northeast State College

JACK D. MUNN, M.A. (Alabama) Asst. Prog., Mississippi Southern College

SAM NEWMAN, B.S. (Columbia) Mathematician, NAFEC, Atlantic City, New Jersey

HELEN O'DEA, M.A. (California) Asso. Prof., Northern Montana College

MARGARET A. PAPARILLO, M.A. (Columbia) Teacher, West Hempstead Junior-Senior High School, New York

BRUCE B. PETERSON, M.A. (Syracuse) Grad. Asst., Syracuse University

ROBERT F. PIERCE, Ed.M. (Boston) Editor, D. C. Heath & Company, Boston, Massachusetts

ELAINE E. POSTELNECK, B.A. (Barnard) Teacher, Sanford H. Calhoun High School, Merrick, New York

FREDERICK RENVYLE, A.B. (Harvard) Teacher, Abington Senior High School, Massachusetts

JOAN RICHARDSON, B.S. (Minnesota) Instr., Arizona State University

ROY M. ROBERTS, M.S. (Tennessee) Teacher, Maryville High School, Tennessee

ALLEN I. RUBENSTEIN, Student, College of the City of New York

JOHN M. SAADE, A.B. (Emory) Grad. Student, Emory University

THOMAS L. SAINDON, Student, Illinois Institute of Technology

HENRY J. SCHAFER, M.Ed. (Puget Sound) Teacher, Wilson High School, Tacoma, Washington

RALPH W. SCHREIBER, B.S. (Kansas State Coll.) Teacher, Harrison High School, Colorado Springs, Colorado

GEORGE S. SHAPIRO, Student, Forest Hills High School, New York

KENNETH L. SISSERSON, M.A. (Southern Methodist) Teacher, Monnig Junior High School, Fort Worth, Texas

RONALD J. SRODAWA, Student, Sewitt High School, Detroit, Michigan

MRS. VELLA STEWART, B.S. (Southwestern) Teacher, Cyril High School, Oklahoma

JOSEPH T. SUCHAR, B.S. (Kutztown State Coll.) Instr., Rider College

THURMAN L. TALLEY, M.S. (N. Mexico) Grad. Student, Florida State University
STEPHEN J. TEMPERO, Student, University of Nebraska
MICHAEL T. THAYER, Student, Walnut Hills High School, Cincinnati, Ohio
ROSEMARY C. TIGHE, M.A. (Hunter Coll.) Teacher, Forest Hills High School, New York
GARY G. TRESSEL, B.S. (Great Falls Coll.) Teacher, Granite County High School, Philipsburg, Montana
JEAN C. TURGEON, Ph.D. (Columbia) Lecturer, Sir George Williams University
MANUEL TAN UY, Student, Massachusetts Institute of Technology
ROBERT H. WALKER, M.E. (N. Dakota) Teacher, Lewis and Clark Junior High School, Billings, Montana
KENNETH L. WHIPKEY, M.A. (Kent State) Instr., Youngstown University
MRS. JOYCE W. WILLIAMS, Ph.D. (Illinois) Boxboro Depot Road, Littleton, Massachusetts
ROBERT E. WILLIAMS, B.S. (Missouri) Grad. Asst., Missouri University
P. K. WONG, M.S. (Carnegie I.T.) Instr., Carnegie Institute of Technology

THE FORTY-FIRST SUMMER MEETING OF THE ASSOCIATION

The Forty-first Summer Meeting of the Mathematical Association of America was held at Michigan State University, East Lansing, Michigan, from Monday, August 29 through Wednesday, August 31, 1960 in conjunction with summer meetings of the American Mathematical Society, the Society for Industrial and Applied Mathematics, the Pi Mu Epsilon Fraternity, and Mu Alpha Theta. There were registered 760 persons, including 568 members of the Association.

Sessions of the MAA were held on Monday morning and afternoon, on Tuesday morning and on Wednesday afternoon. All sessions were held in Anthony Hall Auditorium of Michigan State University. Presiding officers at the three Earle Raymond Hedrick Lectures were President C. B. Allendoerfer, First Vice-President A. S. Householder, and Second Vice-President Harley Flanders, at the session on special programs in the teaching of undergraduate mathematics Professor R. J. Wisner, at the session on the role of abstract and concrete approaches in the teaching of mathematics Professor J. C. Oxtoby, at the business meeting and lecture on Tuesday morning President Allendoerfer, and at the lecture on Wednesday afternoon Professor J. G. Herriot. The ninth series of Earle Raymond Hedrick Lectures were delivered by Professor Ivan Niven of the University of Oregon. The Program Committee for the meeting consisted of J. C. Oxtoby, Chairman; W. E. Briggs, J. G. Herriot, K. O. May, R. J. Wisner.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *Some Aspects of Diophantine Approximation*, Lecture I, by Professor Ivan Niven, University of Oregon.

The Hedrick Lectures by Professor Niven will be published by the Association at an early date.

Session on Special Programs in the Teaching of Undergraduate Mathematics

The Honors Program at Princeton, by Professor Albert W. Tucker, Princeton University.

A freshman honors course, started ten years ago by Emil Artin, has been gradually extended through sophomore and junior years (see Artin, *Calculus and Analytic Geometry* (C.U.P. 1957), M.A.A., Buffalo, also Nickerson, Spencer and Steenrod, *Advanced Calculus*, Van Nostrand, 1959). Only outstanding applicants, liberal arts or engineering, are accepted initially (top ten percent of freshman class, roughly) after careful screening via school records, CEEB scores, and interviews. Those with "advanced placement" calculus present a special problem; most go into an advanced division of freshman honors, a few into sophomore honors. Transfers into the honors program are rare, but transfers out occur at frequent intervals on instructor's advice or student's request. Each student is responsible for routine exercises on his own. Thirty of this year's forty-five mathematics majors, juniors and seniors, are products of the freshman-sophomore honors courses. The program demands much from students and staff, but is highly prized by both.

The New Program at Wesleyan, by Professor Robert A. Rosenbaum, Wesleyan University.

The College of Quantitative Studies is one of several experimental programs designed to minimize academic book-keeping, to emphasize independent work at an early stage of the student's career, and to encourage greater responsibility in the student for his own education. With its focus in the mathematics department, the College also draws on many other departments—its Directors of Studies now come from astronomy, economics, and political science as well as from mathematics. These Directors have the responsibility of supervising the students' choice of programs, of assessing satisfactory progress (in lieu of grades), of arranging for qualifying and comprehensive examinations, set by outside examiners, and of recommending candidates for the B.A.

Students enter the program at the beginning of the sophomore year. From the start, about one-quarter of their work is devoted to problems and projects, of a non-textbook sort. Some of the problems have an academic flavor, originating in such departments as physics, biology, and psychology; others come from business, industry, and government. It should be emphasized that some of the problems come from within mathematics itelf; and it is hoped to produce more future mathematicians, as well as graduates who can use mathematics effectively, than Wesleyan has produced in the past.

The Program at Dartmouth, by Professor John G. Kemeny, Dartmouth College.

After seven years of planning and experimentation, the new Dartmouth mathematics program is in operation. The department has attempted to strike a healthy balance between courses designed for the prospective mathematician, for the average student, and for men interested in the physical, biological and social sciences.

In the first two years the students may be placed into an honors program (somewhat similar to that described by A. W. Tucker), a regular sequence, or a special sequence for the non-physical-scientist. The former consists of a year of calculus followed by a course in logic-sets-probability-linear algebra, which is used for the introduction of functions of several variables. The special sequence combines introduction to the basic ideas of calculus with the more modern concepts mentioned. A lecture system has been employed successfully in the non-honors sections.

Upper class courses again correspond to the three tracks. Dartmouth's unusual three-term three-course system allows a maximum concentration on mathematics courses, with the possibility of completing a three-semester sequence in one year. This means that honors majors reach graduate courses in record time.

The liberal use of undergraduate research assistants both in the computation center and as personal assistants to staff members has resulted in a high degree of undergraduate interest in productive mathematical activity.

The Undergraduate Thesis Program at Reed, by Professor Lloyd B. Williams, Reed College.

The presentation of a thesis, written during the senior year is a requirement for the B.A. degree at Reed College. Practices vary somewhat from department to department within the College in administering this requirement, but generally the thesis is ". . . an independent project of research or of critical or creative work . . ." (quoted from the College Catalog).

The thesis requirement has been in effect since the first graduating class of the College in 1915. Since then, two hundred theses have been written by students majoring in mathematics. Professor F. L. Griffin has described the experience with this program in three articles in this MONTHLY, vol. 37, 1930, vol. 49, 1942, and vol. 58, 1951.

This paper considers trends in the program during the last ten years.

The Philips Visitor Program at Haverford, by Professor James O. Brooks, Haverford College.

The Philips Visitor program at Haverford College affords each year a one semester course in

some advanced undergraduate subject given by a leading mathematician or mathematicians of the country. This course makes an outstanding supplement to regular course offerings of the Haverford mathematics department. Students benefit not only from exposure to certain subject matter but also from contact with distinguished scientists and teachers.

The program began at Haverford with a course of lectures on the calculus of variations by Professor Richard Courant. Other Visitors appearing since in mathematics have included Professors Ross Beaumont, Arthur Coble, Charles Coulson, Ralph Fox, Paul Halmos, Mark Kac, Deane Montgomery, F. D. Murnaghan, John Oxtoby, Hans Rademacher, James Stoker, Robert Thrall, and Albert Tucker. This coming year Professors Marston Morse and Andre Weil will each give a series of lectures.

The program is endowed by a gift of a Haverford alumnus, William Pyle Philips.

Haverford students, especially math majors, take the Philips lectures plus an additional recitation session as a three semester hour course. The students write up notes on the lectures. The lectures will be published by Ginn and Co. in a series entitled the "Haverford Mathematical Monographs."

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Niven.

Session on the Role of Abstract and Concrete Approaches in the Teaching of Mathematics

Toward the Abstract: The Problem of Communications of Ideas, by Professor Arnold E. Ross, University of Notre Dame.

A continual and a rapid change has become the way of life of this generation and of uncountable generations to come.

Among the intellectual skills used in both science and technology to probe into the unknown, the art of "theorizing" is probably the most far reaching. Thus we are compelled by considerations of survival to bring the most powerful tool of exploration, viz., abstract thinking, within the grasp of our young. Since attitudes are now as vital as skills, now more than ever education must seek to develop intense curiosity, keenness of observation, a capacity for invention, and many other oft-referred-to virtues of the pioneering spirit.

The problem of communication of ideas, in particular of mathematical ideas, is one of the key questions before us. I propose that we consider this problem and that in doing this we subject to a painful reappraisal many of the ways in which we now use language as a tool of education.

What Price Abstraction? by Professor Mark Kac, Cornell University.

This paper will appear in this MONTHLY.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Niven.

Business Meeting of the Association.

Report of the Committee on the Undergraduate Program in Mathematics, by Professor R. Creighton Buck, University of Wisconsin.

Progress reports of the work of this Committee will appear from time to time in this MONTHLY. The first such report, namely that of the Teacher-Training Panel, will appear in the December issue.

Lecture: *Convergence Regions for Continued Fractions and Certain Other Infinite Processes*, by Professor Wolfgang J. Thron, University of Colorado.

The iterative definition of continued fractions is used as a starting point. Let $t_n(z) = a_n/(b_n+z)$ and let $T_n(z) = T_{n-1}(t_n(z))$, $T_1(z) = t_1(z)$. Then the continued fraction is defined as the sequence $\{T_n(0)\}$. Convergence criteria for this and for related infinite processes, where $t_n(z)$ is not neces-

sarily a linear fractional transformation, are derived. The criteria are of the convergence region type, that is a region G is determined such that if all $a_n \in G$ (it is now assumed that t_n depends on one parameter only which is called a_n) then the sequence $\{T_n(0)\}$ converges. The methods are based on a determination of value regions and corresponding element regions. Various methods of proof are discussed. Results can be obtained by using the Stieltjes-Vitali theorem or in certain cases by elementary methods involving a sequence of "nested sets" and an estimate of their diameters.

FOURTH SESSION OF THE ASSOCIATION

Lecture: *Fourier Transforms*, by Professor Jacob Korevaar, University of Wisconsin.

The Fourier transform $g = Tf$ is defined by the formula

$$g(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(u) e^{-ixu} du,$$

the conjugate Fourier transform $h = \overline{T}f$ by the analogous formula obtained on replacing $-i$ by i. The inversion theorem "if $g = Tf$ then $f = \overline{T}g$" and the derivative rule $TDf = \{ix\}Tf$ make the Fourier transformation an important tool of applied mathematics. Unfortunately the classical theory places severe restrictions on the growth of f as $x \to \pm\infty$. It is shown in this lecture how the theory of pansions makes it possible to use the Fourier transformation even when the functions involved grow as fast as $\exp(cx^2)$. Among the applications is a uniqueness theorem for the heat equation.

A pansion is a formal series $\sum c_k v_k$ of normalized Hermite functions. Pansions which are Hermite expansions of functions or distributions are identified with those functions or distributions. Within the class of pansions one can form the Fourier transform, the conjugate Fourier transform, the derivative, the product by $\{x\}$, etc. The definitions given agree with the classical definitions whenever the latter apply. For example, the Fourier transform of the pansion $\sum c_k v_k$ is the pansion $(\sum c_k Tv_k =) \sum (-i)^k c_k v_k$. The inversion and derivative rules now become valid without restriction. For details see Trans. Amer. Math. Soc., vol. 91, 1959, pp. 53–101.

SPECIAL SESSIONS OF THE ASSOCIATION

On Tuesday afternoon at 3:15 an open conference on High School Contests was held in Room 120, Physics-Mathematics Building, with 68 persons present. President Allendoerfer presided. The question of whether problems on "modern mathematics" should be included in future contests received particular attention. There was agreement that an attempt should be made to introduce such problems into the contest, but that not more than 5 out of 40 problems should be of this type in next year's contest.

On Wednesday evening at 7:15 a sound film in color on "Mathematical Induction" with Leon A. Henkin as lecturer and produced by the MAA's Committee on Production of Films was shown in Anthony Hall Auditorium. The film was made by Palmer Films, San Francisco, and consists of two parts, each 30 minutes long.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in Room 221, Physics-Mathematics Building of Michigan State University with thirty-two members present.

Professor W. S. Loud of the University of Minnesota was elected to the Board of Governors to fill the balance of the term of Professor J. M. H. Olmsted who has moved from the Minnesota Section.

Professor R. P. Bailey of the Naval Academy was elected to the Board of Governors to fill the balance of the term of Professor R. C. Yates who has moved from the Maryland-D.C.-Virginia Section.

The Board voted to invite Professor R. H. Bing of the University of Wisconsin to

deliver the tenth series of Earle Raymond Hedrick Lectures at the 1961 Summer Meeting.

The Board approved the following schedule of future meetings: Hotel Willard, Washington, D. C., January, 1961; Oklahoma State University, August 1961; Sheraton-Gibson Hotel, Cincinnati, Ohio, January 1962; University of British Columbia, August 1962; University of California, Berkeley, January 1963; University of Colorado, August 1963; University of Michigan, August 1964; Rutgers-The State University, New Brunswick, New Jersey, August 1966.

The Board acting in accordance with Article III, Section 5, of the By-Laws of the Association appointed Professor Harry M. Gehman Executive Director as of September 1, 1960, under terms and conditions of employment fixed by the Finance Committee.

The Board acting upon the recommendations of its Committee on Publications under the chairmanship of Professor Roy Dubisch approved the following:

I. The present Carus and Slaught Committees shall be abolished and, in their place, there shall be established a standing Committee on Publications. This committee shall have over-all responsibility for all publications of the Association including the MONTHLY, Carus, Slaught, and CUP publications. The editor of the MONTHLY shall be an *ex officio* member of the committee.

II. Two new publishing ventures shall be initiated as soon as possible.

A. The Association shall take over the publication and editorship of the *Mathematics Magazine* to be sold on a subscription basis with, possibly, a reduced rate to members of the Association and the National Council of Teachers of Mathematics. The mathematical level of the *Magazine* shall be below that of the MONTHLY but above that of the *Mathematics Teacher*. The editor of the *Magazine* shall be an *ex officio* member of the Committee on Publications.

B. A new series of publications, tentatively titled *MAA Studies in Mathematics*, shall be initiated to be published, if possible, in cooperation with a commercial publisher (as is now done for Carus) or, alternately, to be published by the American Mathematical Society. The first two volumes would be collections of papers, some reprints of older MONTHLY articles and some new manuscripts, of top quality exposition which have recently been collected by Professor R. V. Andree.

III. An editorial office shall be set up in Buffalo as soon as possible to handle the preparation of manuscripts for the printer, proof reading, reprint orders, etc.

IV. The publication of lists of new members in the MONTHLY shall be discontinued effective January 1961.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning with President Allendoerfer presiding. The Secretary reported that the membership of the Association was 9913, an increase of 15% since the corresponding date last year.

The Secretary then reported on the actions taken by the Board of Governors the previous evening and introduced Professor Harry M. Gehman as Executive Director whose appointment to the new position was received with great applause by the audience.

MEETING OF SECTION OFFICERS

A meeting of representatives of the Sections of the Association was held on Tuesday evening in Room 221, Physics-Mathematics Building. Approximately 48 persons were present representing 25 of the 27 Sections of the Association.

Professor Russel C. Phelps of the National Science Foundation reported on N.S.F. Programs for the support of science education projects. Professor J. G. Kemeny gave a report of the Panel on Teacher Training of the Committee on the Undergraduate Pro-

gram in Mathematics. Professor Roy Dubisch discussed the Reports of Section Meetings in the MONTHLY. The Section representatives voted in favor of continuing to print abstracts of talks delivered at section meetings, but on a voluntary basis, i.e., abstracts would not be demanded from speakers who prefer not to have them printed. A discussion of the "Role of Industrial Members in the Association" was opened by brief talks by Dr. H. O. Pollak of Bell Telephone Company and Dr. B. H. Mount, Jr., of Westinghouse Electric Corporation.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. The colloquium speaker was Professor S. S. Chern of the University of California, Berkeley, who spoke on "Geometrical Structures on Manifolds," and invited addresses were given by Professor Paul Halmos of the University of Chicago on "Recent Progress in Ergodic Theory" and Professor P. E. Conner of the University of Virginia on "Involutions and Equivariant Maps."

The Society for Industrial and Applied Mathematics had three sessions. The first, at which the von Neumann Lecture was presented by Professor L. V. Ahlfors of Harvard University, was held Wednesday afternoon at 1:30, the second Thursday evening at 8:00, and the third Friday afternoon at 2:40.

The Pi Mu Epsilon Fraternity held a luncheon and business meeting Tuesday noon. A session for seven 20-minute papers by students was held beginning at 2:45 P.M. Mu Alpha Theta, the National High School and Junior College Mathematics Club, held a luncheon meeting on Wednesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of W. E. Deskins, Chairman; H. L. Alder, J. S. Frame, F. Herzog, L. M. Kelly, M. L. Tomber, G. L. Walker, C. P. Wells, J. W. T. Youngs.

Registration headquarters was located in the lobby of Snyder Hall. Dormitory and cafeteria accommodations were provided by Michigan State University. The textbook exhibit and the Mathematical Sciences Employment Register were located in Snyder Hall near the registration desk.

On Monday evening there was an informal coffee hour at 8:00 P.M. in Phillips Hall, Lower Lounge. An informal tea was held Tuesday afternoon from 4:00 to 6:00 P.M. in Phillips Hall, Lower Lounge. A chicken barbecue was held Wednesday evening at 5:00 P.M. on the campus. The usual SIAM social evening was held on Thurdsay evening.

A resolution of thanks prepared by Professor W. T. Martin and adopted by the participating organizations expressed the most sincere appreciation to Michigan State University, to the local arrangements committee, and to all those on the University staff for the excellent arrangements which have been provided for the meeting. The resolution expressed the opinion of the participating organizations that the facilities provided at Michigan State have not been excelled within the memory of those attending, and that everyone will carry away the recollection of a most pleasant and profitable stay.

HENRY L. ALDER, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The annual spring meeting of the Oklahoma section of the Mathematical Association of America was held at the University of Oklahoma, Norman, Oklahoma, on April 16, 1960. Professor Katherine Mires, Chairman of the Section, presided during the meeting, which was devoted to the reading of mathematical papers of both research and expository nature. There were 56 persons present, including 47 members of the Association.

The following invited addresses of roughly twenty minutes each were presented during the morning session.

1. *Some properties of the von Sterneck or Ramanujan function*, by Professor C. A. Nicol, University of Oklahoma.

If k and n are integers such that $k \geqq 0$, $n > 0$, then $\phi(n)\mu(n/(k, n))/\phi(n/(k, n))$ is called the von Sterneck or Ramunjan function and is denoted by $\Phi(k, n)$. Here (k, n) is the G.C.D. of k and n, ϕ is the Euler totient, and μ is the Moebius function. Some arithmetic properties of the von Sterneck function are discussed as well as its appearance in some generating functions.

2. *Radial plane flows of a nonlinear viscous fluid*, by Professor J. A. Nickel, Oklahoma City University.

The viscous fluid considered is an incompressible isotropic Reiner-Rivlin fluid having a nonlinear stress function of the form $T = -pI + mD + nD^2$, where m and n are analytic functions of the scalar invariants of the rate of displacement matrix D and the dimensional parameters of the flow. If m, a generalization of the classical viscosity, is assumed to be independent of temperature and pressure in the flow, it is shown that a necessary and sufficient condition of a radial flow to exist is that m is a constant, and consequently, the velocity field is identical with that of the classical flow. Furthermore, the normal pressures are no longer hydrostatic, but depend upon the coefficient n in the stress matrix.

3. *Effects of correlation upon a variance testing procedure*, by Dr. Frank McFeely, Montana State College, Visiting Scholar at the University of Oklahoma.

4. *On ideals in integral quaternions*, by Professor W. A. Rutledge, University of Tulsa.

Using the definition given by Hurwitz, the (noncommutative) integral domain of quaternions is a principal ideal domain J. Similarity is defined by: a is similar to b if there exists c such that $caJ = cJ \cap bJ$, with $cJ + bJ = J$. This relation is studied in terms of norm, bound (ideal) and minimal containing ideal.

Papers presented during the afternoon session were:

5. *Green rectangles*, by Professor Arthur Bernhart, University of Oklahoma.

The numerical solution of the Laplace difference equation in two dimensions suggests the study of arrays in which each entry is the average of its four neighbors. To obtain factors for arbitrary boundary conditions a singularity is introduced at an interior point analogous to the Green's function.

6. *A problem in transforming hyperbolic to rectangular coordinates*, by Professor W. B. Garrison, University of Tulsa.

This paper concerns a problem encountered in radio navigation systems, in particular the system called LORAN, that makes use of two families of hyperbolas for positioning purposes.

7. *Existence of surfaces in a certain conformal correspondence*, by Professor T. K. Pan, University of Oklahoma.

Let v and $\bar{v}$ be two corresponding vector fields associated with two surfaces S and $\bar{S}$ respectively. Let the indicatric torsions of v and $\bar{v}$ along two corresponding curves C and $\bar{C}$ be equal (T. K. Pan, Torsion of a vector field, Proc. Amer. Math. Soc., vol. 7, p. 453). The existence of such surfaces is investigated. It is found that the general solution depends on one arbitrary function of two variables.

8. *Filters and ultrafilters*, by Mr. J. E. Allen, Oklahoma State University.

The ordered set of filters on a set E is inductive, and thus the theorem of Zorn implies that for every filter on a set E, there exists an ultrafilter finer than it. This theorem is used to obtain the main result of this paper: Every filter is the intersection of all the ultrafilters finer than it.

9. *A medieval antecedent of Descartes' graphing technique*, by Dr. T. M. Smith, University of Oklahoma, introduced by the Secretary.

A crude technique for generating graphic figures appeared in 14th-century Europe as a consequence of scholastic preoccupation with "the intension and remission of forms" and served as an aid to the comprehension of logical analyses of motion; it became a part of the kinematic mathematical-physical tradition culminating in Galileo's delineation of the law of free fall in the 17th century.

10. *The Boole table generalized*, by Dr. W. E. Stuermann, University of Tulsa.

In an earlier item in the Classroom Notes of this MONTHLY, the writer reported a graphical method for analyzing Boolean functions which were displayed in conjunctive or disjunctive normal form. The present paper shows how the Boole table may be used to analyze any Boolean function which can be constructed from the operations of sum, product, and complement, no matter how deeply and complexly the variables are overlaid by stacks of these operations.

11. *Markov chains as random input automata*, by Mr. Allen Davis, University of Oklahoma. (This paper has been accepted for publication in this MONTHLY.)

R. V. ANDREE, *Secretary*

PROPOSED AMENDMENTS TO THE BY-LAWS OF THE M.A.A.

At the annual business meeting of the Association to be held at the Willard Hotel in Washington, D. C., on Thursday, January 26, 1961, motions will be made to amend the By-Laws as follows:

1. That Article III, Section 8 (c) be amended by adding these two sentences: "A Governor who has moved permanently from the Section by which he was elected shall be considered to have ended his term of office on the Board. If the Governor has moved from the Section because he is no longer employed there, it shall be interpreted that he has moved permanently from the Section."

2. That a new section, to be numbered Article III, Section 13, be inserted in the By-Laws to read as follows:

"(a) There shall be an Executive Director who shall be a paid employee of the Association. He shall have charge of the central office of the Association, and shall carry out such other duties as may be assigned to him by the Board. He shall be responsible to the Board, and shall attend meetings of the Board, the Executive Committee, and the Finance Committee, but he shall not be *ex officio* a member of these bodies.

(b) The Executive Director shall be elected by the Board under terms and conditions of employment fixed by the Finance Committee."

HENRY L. ALDER, *Secretary*

THE EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the Annual Meeting at the Willard Hotel in Washington, D. C. on January 24, 25, and 26, 1961. The Register will be conducted from 9:00 A.M. to 5:00 P.M. on each of these three days.

The Employment Register Desk will be located in room 228. There is no charge for registering to either job applicants or to employers, except when the Late Registration Fee for employers is applicable. Provision will be made for anonymity of applicants upon request.

Job applicants and employers who wish to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for application forms and for position description forms, which must be completed and returned to Providence not later than January 4, 1961, in order to be included free of charge in the listings at the

meeting in Washington, D. C. Forms which arrive after this closing date, but before January 17, will be included in the listings at the meeting for a Late Registration Fee of $3.00, and will also be included in the printed listings, but not until ten days after the meeting. The printed listings will be available for distribution both during and after the meeting. The prices are as follows: Position descriptions, $2.00; listing of applicants, academic only, $5.00; comprehensive listing of applicants, academic, industrial, and government, $20.00.

It is essential that applicants and employers register at the Employment Register Desk promptly upon arrival at the meeting to facilitate the arrangement of appointments.

CARUS MONOGRAPH NUMBER 5

The above Monograph entitled "History of Mathematics in America before 1900" by D. E. Smith and Jekuthiel Ginsburg was published in 1934. For many years it has been out of print.

Recently the office of the Association has received a number of inquiries from members wishing to purchase Monograph 5. Anyone who owns a copy of this Monograph and wishes to sell it may notify the Buffalo office of the MAA, and prospective purchasers will be notified.

CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–31, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA

IOWA, Simpson College, Indianola, April 21, 1961.

KANSAS

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, December 3, 1960.

METROPOLITAN NEW YORK

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring. 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY

NORTHEASTERN, Wesleyan University, Middletown, Connecticut, November 26, 1960.

NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 26, 1960.

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 27–29, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

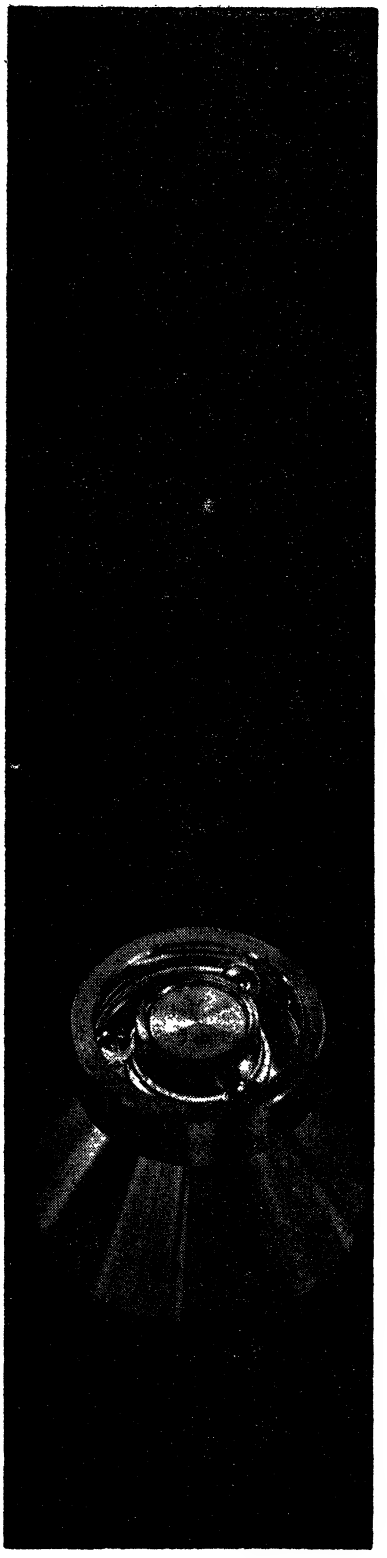
SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, April, 1961.

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.



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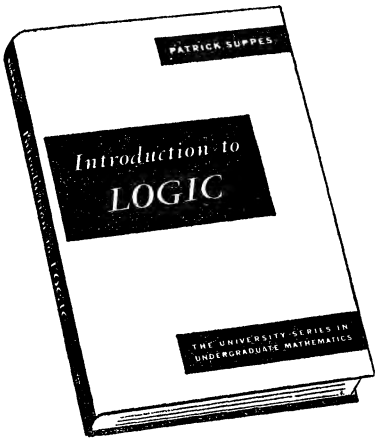
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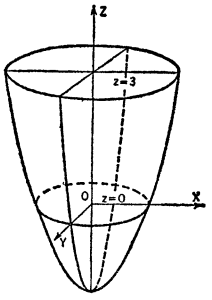
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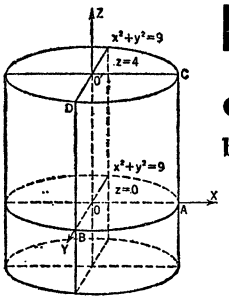
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Edited by HOWARD EVES, University of Maine, and E. P. STARKE, Rutgers, The State University

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BINOMIAL EXPANSIONS IN FACTORIAL POWERS*

LOUIS BRAND, University of Houston

1. Definitions. In the calculus of finite differences the factorial power $x^{(n)}$ plays a role analogous to that of x^n in the differential calculus. Yet at present there is not only a wide diversity of notations for the factorial power [1], but also a disagreement as to its definition when the index is negative. Moreover the notation $(ax+b)^{(n)}$ is sometimes used in a way inconsistent with the definition of $x^{(n)}$ and this usage leads to ambiguities which may be a fertile source of error. The way out of this confusion is to use a single definition applicable to all real values of the index. Moreover the alleged factorial powers $(ax+b)^{(n)}$ are readily converted into genuine factorial powers.

When n is a positive integer the factorial power $x^{(n)}$ is defined as the product of n factors

$$x^{(n)} = x(x-1)\cdots(x-n+1). \tag{1}$$

This definition leads to the functional equation

$$x^{(m+n)} = x^{(m)}(x-m)^{(n)} = x^{(n)}(x-n)^{(m)} \tag{A}$$

for factorial powers with positive index. Just as the functional equation $x^{m+n} = x^m x^n$ for ordinary powers is used to define the meaning of zero, negative, rational, and eventually all real exponents, we use (A) to obtain a definition of $x^{(n)}$ for all real n. We therefore postulate the truth of (A) for all real values of m and n.

When $m=0$, (A) becomes $x^{(n)} = x^{(0)}x^{(n)}$; hence if $x \neq 0$,

$$x^{(0)} = 1. \tag{2}$$

When $m=-n$, we have $x^{(0)} = x^{(-n)}(x+n)^{(n)}$ or $x^{(-n)} = 1/(x+n)^{(n)}$. Hence if n is a positive integer and $x \neq 0$, we have from (1)

$$x^{(-n)} = \frac{1}{(x+1)(x+2)\cdots(x+n)}. \tag{3}$$

Thus (1), (2) and (3) define factorial powers for all integral indices provided $x \neq 0$. We shall see that we can dispense with this last proviso.

Next let x be any real number and consider the function $f(x) = x^{(x)}$. Letting $m=1$, $n=x-1$ in (A) we have

$$f(x) = x^{(x)} = x(x-1)^{(x-1)} = xf(x-1).$$

But the gamma function $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ has $\Gamma(x+1) = x\Gamma(x)$ as functional equation. Hence when x is not a negative integer we define

* Presented to a joint meeting of the Mathematics and Physics Section of the Texas Academy of Sciences and the Texas Section of the Association, Austin, December 11, 1959.

$$x^{(x)} = \Gamma(x+1), \qquad x \neq -1, -2, \cdots. \tag{4}$$

When $x=0$ this gives $0^{(0)}=\Gamma(1)=1$; we may therefore remove the restriction $x\neq 0$ in definitions (2) and (3); thus

$$0^{(0)} = 1 \quad \text{and} \quad 0^{(-n)} = 1/n! \tag{5}$$

in sharp contrast with ordinary powers for which 0^0 is an indeterminate form and 0^{-n} has no meaning.

Again from (2) we have $x^{(x)}=x^{(n+x-n)}=x^{(n)}(x-n)^{(x-n)}$; using (4), we now give the general definitions of the factorial power [2]:

$$x^{(n)} = \frac{\Gamma(x+1)}{\Gamma(x+1-n)} \tag{6}$$

for all real values of x and n for which the gamma functions exist. This definition comprises all cases if we agree that, when the gamma function is infinite (for arguments $0, -1, -2, \cdots$),

$$x^{(n)} = \lim_{\epsilon\to 0} (x+\epsilon)^{(n)}. \tag{7}$$

Thus from (6), $(-1)^2=\Gamma(0)/\Gamma(-2)$ is not defined; but from (7)

$$(-1)^{(2)} = \lim_{\epsilon\to 0} (\epsilon-1)^{(2)} = \lim_{\epsilon\to 0} \frac{\Gamma(\epsilon)}{\Gamma(\epsilon-2)} = \lim_{\epsilon\to 0} (\epsilon-1)(\epsilon-2) = 2!$$

in agreement with (1).

An excellent occasion to use (6) arises in plotting the graph of the interesting continuous curve $y=0^{(x)}$—the "sidewinder."

The basic functional equation (A) for the factorial power follows at once from (6) and the identity:

$$\frac{\Gamma(x+1)}{\Gamma(x+1-m-n)} = \frac{\Gamma(x+1)}{\Gamma(x+1-m)} \frac{\Gamma(x-m+1)}{\Gamma(x-m+1-n)}. \tag{8}$$

By taking $n=-m$ in (A) we get the important special case:

$$x^{(m)}(x-m)^{(-m)} = x^{(-m)}(x+m)^{(m)} = 1. \tag{9}$$

2. Differences. The difference of a function $f(x)$ is defined as $\Delta f(x)=f(x+1)-f(x)$. The prime importance of the factorial power is due to the formula

$$\Delta(x+c)^{(n)} = n(x+c)^{(n-1)} \tag{B}$$

and the corresponding antidifference

$$\Delta^{-1}(x+c)^{(n)} = \frac{(x+c)^{(n+1)}}{n+1} + \omega \qquad (n \neq -1), \tag{C}$$

where ω is an arbitrary periodic function of period 1. The proof of (B) follows

at once from (6) and the functional equation $\Gamma(x+1)=x\Gamma(x)$:

$$\begin{aligned}\Delta(x+c)^{(n)} &= (x+c+1)^{(n)} - (x+c)^{(n)} \\ &= \frac{\Gamma(x+c+2)}{\Gamma(x+c+2-n)} - \frac{\Gamma(x+c+1)}{\Gamma(x+c+1-n)} \\ &= \frac{\Gamma(x+c+1)}{\Gamma(x+c+1-n)}\left(\frac{x+c+1}{x+c+1-n} - 1\right) \\ &= n\,\frac{\Gamma(x+c+1)}{\Gamma(x+c+2-n)} = n(x+c)^{(n-1)}.\end{aligned}$$

3. Diverse notations. We pass now to some matters of current usage relative to factorial powers.

First, a number of authors [3] define

$$x^{(-n)} = 1/\{x(x+1)\cdots(x+n-1)\}.$$

Although this definition preserves the difference relation (B), it is inadvisable because it does not conform to the functional equation (A) and requires a separate definition for negative indices. Others [4] avoid negative indices altogether or introduce special notations, thus restricting the generality of (A) and the difference formula (B).

A more serious defect [5] is to write the factorial *expression* (which we denote by an index in brackets)

$$(10)\qquad (ax+b)^{[n]} = (ax+b)\{a(x-1)+b\}\cdots\{a(x-n+1)+b\}$$

as a factorial power; whereas, in conformity with (1),

$$(ax+b)^{(n)} = (ax+b)(ax+b-1)\cdots(ax+b-n+1).$$

However $(ax+b)^{[n]}$ is readily expressed as a factorial power:

$$(11)\qquad (ax+b)^{[n]} = a^n(x+(b/a))^{(n)};$$

its difference then follows from (10):

$$(12)\qquad \Delta(ax+b)^{[n]} = a^n n(x+(b/a))^{(n-1)} = an(ax+b)^{[n-1]}.$$

4. Binomial expansions. By means of (6) the binomial coefficient

$$(13)\qquad \binom{x}{k} = \frac{x^{(k)}}{k^{(k)}} = \frac{\Gamma(x+1)}{\Gamma(x+1-k)\Gamma(k+1)}$$

is generalized to all values of x and k. Moreover this generalization preserves the relations

$$(14)\qquad \binom{x}{0} = \binom{x}{x} = 1,\quad \binom{x}{k} = \binom{x}{x-k},\quad \binom{x+1}{k} = \binom{x}{k} + \binom{x}{k-1}$$

originally suggested in the integral case by interpreting $\binom{x}{k}$ as the number of combinations of x things taken k at a time.

The generalized figurate numbers F_k^n defined by Frankel [6] are actually the coefficients in the expansion of $(1-x)^{-n}$:

$$(15) \qquad (1-x)^{-n} = \sum_{k=0}^{\infty} F_k^n x^k, \qquad |x| < 1.$$

Consequently for all integral values of n,

$$(16) \qquad F_k^n = (-1)^k \binom{-n}{k} = (-1)^k \frac{(-n)^{(k)}}{k!}.$$

The definition (6) may now be used to define F_k^n for all real n. The "criss-cross multiplication" of two sequences $a_k * b_k$ defined by Frankel in Section 5 is their Cauchy-product sequence; and the theorems in his Section 6 follow at once from this fact and equation (15). For example $(1-x)^{-m-n} = (1-x)^{-m}(1-x)^{-n}$ yields $F_k^{m+n} = F_k^m * F_k^n$.

The binomial theorem may be extended to factorial powers. Thus when n is a positive integer

$$(17) \qquad (x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(n-k)} y^{(k)},$$

a result known as Vandermonde's theorem. This identity, trivial for $n=1$, may be proved by induction.

A proof that reveals the essential connection of (17) with the binomial theorem for powers depends upon the identity $(1+t)^{x+y} = (1+t)^x(1+t)^y$; for if we replace the powers of $1+t$ by binomial series and equate the coefficients of t^n on sides, we have

$$\binom{x+y}{n} = \binom{x}{n}\binom{y}{0} + \binom{x}{n-1}\binom{y}{1} + \cdots + \binom{x}{0}\binom{y}{n}.$$

On multiplying this equation by $n!$ and using $n! = \binom{n}{k}(n-k)!k!$ we obtain (17). (See Frankel [6], Sec. 9.)

Under appropriate conditions the expansion (17) also applies when n is negative or fractional; in general, the series is then infinite and conditions for its convergence to $f(x)$ must be investigated. But unlike the ordinary binomial expansion, the series for $(x+y)^{(n)}$ will terminate when y is a positive integer. For example,

$$\begin{aligned}(\tfrac{1}{2}+2)^{(-1)} &= \sum_{k=0}^{2} \binom{-1}{k} (\tfrac{1}{2})^{(-1-k)} 2^{(k)} \\ &= (\tfrac{1}{2})^{(-1)} 2^{(0)} - (\tfrac{1}{2})^{(-2)} 2^{(1)} + (\tfrac{1}{2})^{(-3)} 2^{(2)} \\ &= \tfrac{2}{3} - \tfrac{8}{15} + \tfrac{16}{105} = \tfrac{2}{7},\end{aligned}$$

which is $(\frac{5}{2})^{(-1)}$. Moreover,

$$(\tfrac{1}{2}+2)^{(1/2)} = \sum_{k=0}^{2}\binom{\frac{1}{2}}{k}(\tfrac{1}{2})^{(1/2-k)}2^{(k)}$$
$$= (\tfrac{1}{2})^{(1/2)}2^{(0)} + \tfrac{1}{2}(\tfrac{1}{2})^{(-1/2)}2^{(1)} - \tfrac{1}{8}(\tfrac{1}{2})^{(-3/2)}2^{(2)}$$
$$= \Gamma(\tfrac{3}{2})(1+1-\tfrac{1}{8}) = \tfrac{15}{16}\sqrt{\pi},$$

which is $(\frac{5}{2})^{(\frac{1}{2})} = \Gamma(\frac{7}{2})/\Gamma(3)$. These examples suggest the

THEOREM. *Whenever a binomial expansion in factorial powers terminates, it is correct irrespective of the index.*

Proof. Let m be a positive integer; then from (A)

$$(x+m)^{(n)} = (x+m)^{(m+n-m)} = (x+m)^{(m)}x^{(n-m)}$$
$$= x^{(n-m)}(x+m-n+n)^{(m)}$$
$$= x^{(n-m)}\sum_{k=0}^{m}\binom{m}{k}(x+m-n)^{(m-k)}n^{(k)}.$$

Again from (A) we have

$$x^{n-m}(x+m-n)^{(m-k)} = x^{(n-m+m-k)} = x^{(n-k)};$$

hence

$$(x+m)^{(n)} = \sum_{k=0}^{m}\binom{m}{k}x^{(n-k)}n^{(k)} = \sum_{k=0}^{m}\binom{n}{k}x^{(n-k)}m^{(k)}$$

in view of (13). This proves the theorem.

The author wishes to thank the referee for suggesting a perusal of and reference to the paper of E. T. Frankel [6].

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EMBEDDING INCOMPLETE LATIN SQUARES

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An *incomplete* $n \times n$ *latin square* based on $1, \cdots, n$ is a square array of n rows and n columns such that a subset of the n^2 places are occupied by integers from the set $1, \cdots, n$ and such that no integer occurs more than once in the same row or column. We solve in this paper, the following problem. For each n, what is the smallest t such that any incomplete $n \times n$ latin square can be embedded in a $t \times t$ or smaller latin square. It is shown that, for $n \geqq 4$, $t = 2n$.

Applications of this result yield the following properties of loops. An incomplete loop containing n elements can be embedded in a loop containing $2n$ elements. An infinite loop defined by a finite set of generators and relations has a finite homomorphic image of order t, for every $t \geqq k$, where k is a positive integer determined by the generators and relations of the loop.

The above result on the embedding of incomplete latin squares is a simple consequence of the following theorem, due to H. J. Ryser [7], giving the conditions under which an $r \times s$ latin rectangle based on $1, \cdots, n$ may be extended to an $n \times n$ latin square based on $1, \cdots, n$.

THEOREM 1. *Let $N(i)$ be the number of times the integer i occurs in the latin rectangle. A necessary and sufficient condition that the $r \times s$ latin rectangle can be extended to an $n \times n$ latin square is that $N(i) \geqq r + s - n$, for $i = 1, \cdots, n$.*

We begin by exhibiting, for each $n \geqq 4$, an incomplete $n \times n$ latin square which cannot be embedded in a latin square of order less than $2n$. Consider the incomplete $n \times n$ latin square, which has, except for the 2nd and 3rd columns in the $(n-1)$th row and the 2nd column in the nth row, $i + j - 1 \pmod{n}$ as the element in the ith row and jth column. The element in the $(n-1)$th row and 2nd column is 1 and the other two places are unoccupied.

Assume that this incomplete latin square S is embedded in a $t \times t$ latin square T, based on $1, \cdots, t$. The unoccupied places in S cannot be occupied in T by elements in $\{1, \cdots, n\}$ and so $t \geqq n+1$. The element $n+1$ is in each of the first $n-2$ rows of T and hence T contains at least $n-2$ columns in addition to the n columns of S. That is, $t \geqq 2n-2$. Let S' be S with the unoccupied places filled in with the elements they occupy in T. Then S' may be considered as an $n \times n$ latin rectangle based on $1, \cdots, t$ and we may apply Theorem 1. If $t < 2n$, then $n + n - t > 0$ and so $N(i)$ is positive. Hence every integer in $\{1, \cdots, t\}$ occurs at least once in S'. Since S' contains at most two elements from $\{n+1, n+2, \cdots, t\}$ and $t \geqq 2n-2$, this is a contradiction if $n > 4$. If $n = 4$, we obtain a similar contradiction for $t = 7$. If $n = 4$ and $t = 6$, then $N(i) > 2$. But in this case, either both unoccupied places in S contain the same element in S' so that one element in $\{1, \cdots, 6\}$ does not occur in S' or the two places are occupied by 5, 6, each of which then occurs only once in S'. Again we have a contradiction. Thus S cannot be embedded in a latin square of order $< 2n$.

For the cases $n = 2$, 3, not covered in the above discussion, it is easy to

verify (i) an incomplete 2×2 latin square can be embedded in a 2×2 or 3×3 latin square and there is an incomplete 2×2 latin square which cannot be embedded in a 2×2 latin square, (ii) an incomplete 3×3 latin square can be embedded in a 3×3, 4×4 or 5×5 latin square and there is an incomplete 3×3 latin square which cannot be embedded in a 3×3 or 4×4 latin square.

We now prove that any incomplete $n\times n$ latin square S can be embedded in a $t\times t$ latin square T for any $t\geqq 2n$. Let A be a latin square of order $t-n$, where $t\geqq 2n$, based on the integers $n+1, n+2, \cdots, t$. Let S' be the $n\times n$ latin rectangle based on $1, \cdots, t$, obtained from S by filling in, for all i, j, an unoccupied place in the ith row and jth column of S with the element in the ith and jth column of A. Since $n+n-t\leqq 0$, the condition of Theorem 1 is satisfied and we conclude that S' can be extended to a $t\times t$ latin square based on $1, \cdots, t$. That is, S can be embedded in a $t\times t$ latin square.

THEOREM 2.* *For any n, an incomplete $n\times n$ latin square can be embedded in a $t\times t$ latin square, for any $t\geqq 2n$.*

There are immediate applications of Theorem 2 to the theory of loops and quasigroups. A *quasigroup* is a set closed with respect to a binary operation ab, such that for any pair of elements a, b, there are unique solutions of the equations $ax=b$, $ya=b$. A *loop* is a quasigroup with a unit element. An *incomplete quasigroup* is a set with a multiplication defined for some pairs such that if equations $ax=b$ or $ya=b$ have solutions, then these solutions are unique. An *incomplete loop* is an incomplete quasigroup with a unit element and the product of this unit with every element is defined. A full discussion of these ideas may be found in [1]. The multiplication tables of finite quasigroups and loops are latin squares and of finite incomplete quasigroups or loops are incomplete latin squares. Thus the following is an immediate consequence of Theorem 2.

COROLLARY. *An incomplete loop (quasigroup) containing n elements can be embedded in a loop (quasigroup) containing t elements, for any $t\geqq 2n$.*

In this deduction from Theorem 2, we cannot say in general that the elements of the incomplete loop I generate the loop L in which it is embedded. In the special cases, $t=2n$, $2n+1$ this is so (provided I is not a loop) for then the elements of I generate at least one element not in I and a loop of order $2n$ or $2n+1$ cannot contain a proper subloop of order greater than n. We can use these special cases to extend the corollary to the case where the incomplete loop does generate the loop in which it is embedded.

Let I be an incomplete loop containing n elements and with at least one product, say ab, undefined. Let I_p be the incomplete loop containing as elements the union of the set of elements in I and a set $\{c_1, \cdots, c_p\}$ of elements not in I. The multiplications defined in I_p consist of those defined in I and the products:

* I have recently been informed that a proof of this has been obtained independently by S. K. Stein.

$$ab = c_1, \qquad c_i^2 = c_{i+1}, \qquad ec_i = c_ie = c_i,$$

where e is the unit element of I.

Now I_p contains I and is generated by I. Hence any loop in which I_p is embedded and which is generated by I_p will also contain I and be generated by I. By the remarks following the above corollary, I_p can be embedded in a loop containing either $2(n+p)$ or $2(n+p)+1$ elements and I_p generates this loop.

A similar analysis gives corresponding results for quasigroups and so we may now state a stronger result than that of the first corollary.

SECOND COROLLARY TO THEOREM 2. *An incomplete loop (quasigroup) containing n elements can be embedded in a loop (quasigroup) of order t which is generated by the incomplete loop (quasigroup), for any* $t \geqq 2n$.

If L is an infinite loop defined by a finite set of generators and relations, then a finite set of generators and relations may be found for L such that these generators and relations form an incomplete loop I ([1], [3]). Furthermore, any embedding of I in a loop which I generates is a homomorphic image of L ([3]). Thus, from Corollary 2, we obtain the following theorem. (A similar theorem holds also for quasigroups.)

THEOREM 3. *An infinite loop defined by a finite set of generators and relations has a finite homomorphic image of order t, for any* $t \geqq k$, *where k is a positive integer determined by the generators and relations of the loop.*

This theorem may be compared with the example in [2] of a finitely generated and related infinite loop with uncountably many nonisomorphic homomorphic images of countably infinite order and with the example due to G. Higman [5] of a finitely generated and related group with no nontrivial finite homomorphic images.

We conclude by mentioning a few problems. Clearly some incomplete $n \times n$ latin squares can be embedded in latin squares of order less than $2n$. What are conditions for an incomplete $n \times n$ latin square to be embeddable in an $n \times n$ latin square? In particular, can an $n \times n$ incomplete latin square with at most $n-1$ places occupied be completed to an $n \times n$ latin square? It is easy to see that there are incomplete latin square with n places occupied which cannot be so completed. Can a pair of incomplete $n \times n$ latin squares which are orthogonal (insofar as the condition for orthogonality applies to the incomplete squares) be embedded respectively in a pair of $t \times t$ orthogonal latin squares and if so, what is the smallest t for each n? The ideas which are probably needed to attack these problems are discussed in detail in [6]. This paper also contains a proof of Theorem 1. A related, but presumably far more difficult, question concerns the embedding of partial planes in the sense of M. Hall [4]. Can any finite partial projective plane be embedded in a finite projective plane and if so, what is the minimal embedding? It is shown in [4] that any partial projective plane can

be embedded in a projective plane but the construction given there always results in an infinite containing plane.

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CONGRUENCES FOR THE NUMBER OF n-gons FORMED BY n LINES

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1. Let g_n denote the number of polygons of n sides (including degenerate cases) formed by a network of n lines. Robinson [3] showed that g_n satisfies the recurrence

$$g_{n+1} = ng_n + \tfrac{1}{2}n(n-1)g_{n-2} \qquad (n \geqq 2), \tag{1.1}$$

where $g_1 = g_2 = 0$, $g_3 = 1$; it is convenient to define $g_0 = 1$. The writer [1] showed that if m is an arbitrary positive integer then

$$g_{n+m} \equiv g_n g_m \pmod{m_0} \qquad (n \geqq 0), \tag{1.2}$$

where $m_0 = m$ or $\frac{1}{2}m$ according as m is odd or even; more generally it was proved that

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} g_{n+sm} g_{(r-s)m} \equiv 0 \pmod{m_0^{[(r+1)/2]}} \qquad (n \geqq 0); \tag{1.3}$$

where $[x]$ denotes the greatest integer $\leqq x$.

In the present note we obtain the following results. Let m_1 denote the largest odd divisor of m. Then first we have

$$g_m \equiv (-1)^m 2^{-m} \pmod{m_1}, \tag{1.4}$$

$$g_{n+m} \equiv (-1)^m 2^{-m} g_n \pmod{m_1}. \tag{1.5}$$

Next if we put

$$\Delta^r g_n = \sum_{s=0}^{r} (-1)^s \binom{r}{s} (-2)^{-(r-s)m} g_{n+sm}, \tag{1.6}$$

then we show that

$$\Delta^r g_n \equiv 0 \pmod{m_1^{[(r+1)/2]}}. \tag{1.7}$$

The last congruence may be compared with (1.3). The possibility of obtaining a congruence like (1.7) was suggested by some recent results concerning arithmetic properties of congruences of the second order [2].

2. It will be convenient to consider the polynomial $g_n(x)$ satisfying

$$g_{n+1}(x) = (x+n)g_n(x) + \tfrac{1}{2}n(n-1)g_{n-2}(x) \qquad (n \geqq 2), \tag{2.1}$$

where $g_0(x)=1$, $g_1(x)=x$, $g_2(x)=x^2+x$. Then $g_n(x)$ is of degree n and $g_n=g_n(0)$. It is easy to show that (2.1) implies

$$\sum_{n=0}^{\infty} g_n(x)(t^n/n!) = (1-t)^{-x-1/2} \exp(-\tfrac{1}{2}t - \tfrac{1}{4}t^2), \tag{2.2}$$

which yields

$$g_n(x) = \sum_{r+s+2t=n} (-1)^{s+t} 2^{-s-2t} \frac{n!}{r!s!t!} (x+\tfrac{1}{2})_r, \tag{2.3}$$

where $(x)_r = x(x+1)\cdots(x+r-1)$, $(x)_0=1$.

It is proved in [4] that if $u_n = 2(g_n/(n-1)!)$, then

$$\lim_{n\to\infty} (u_n^2/n) = 4/(e^{3/2}\pi). \tag{2.4}$$

In precisely the same way one can prove that

$$\lim_{n\to\infty} \frac{g_n(x)}{(x+\frac{1}{2})_n} = e^{-3/4} \qquad (x > -\tfrac{1}{2}). \tag{2.5}$$

Using Stirling's formula, (2.5) reduces to

$$\lim_{n\to\infty} \frac{n^{1/2-x} g_n(x)}{n!} = \frac{e^{-3/4}}{\Gamma(x+\frac{1}{2})} \qquad (x > -\tfrac{1}{2}). \tag{2.6}$$

For $x=0$, (2.6) agrees with (2.4).

3. Returning to (2.1) we suppose that x is either an indeterminate or a rational number that is integral (mod m_1), where m_1 is the largest odd divisor of m. Clearly (2.1) implies

$$\begin{aligned} g_{n+m+1}(x) &= (x+n+m)g_{n+m}(x) + \tfrac{1}{2}(n+m)(n+m-1)(g_{n+m-2}(x) \\ &\equiv (x+n)g_{n+m}(x) + \tfrac{1}{2}n(n-1)g_{n+m-2}(x) \pmod{m_1}, \end{aligned}$$

so that

$$\Delta g_{n+1}(x) \equiv (x+n)\Delta g_n(x) + \tfrac{1}{2}n(n-1)\Delta g_{n-2}(x) \pmod{m_1},$$

where

$$\Delta g_n(x) = g_m(x)g_n(x) - g_{n+m}(x). \tag{3.1}$$

Since

$$\begin{aligned}
\Delta g_0(x) &= g_m(x) - g_m(x) = 0,\\
\Delta g_1(x) &= xg_m(x) - g_{m+1}(x) \equiv 0,\\
\Delta g_2(x) &= (x^2+x)g_m(x) - g_{m+2}(x)\\
&\equiv -(x+1)g_{m+1}(x) + (x^2+x)g_m(x) \equiv 0,
\end{aligned}$$

it follows that

$$\Delta g_n(x) \equiv 0 \pmod{m_1} \qquad (n \geqq 0). \tag{3.2}$$

In the next place if we assume that x is a rational number that is integral (mod m_1), then by (2.3)

$$g_m(x) = \sum_{r+s+2t=m} (-1)^{s+t}2^{-s-2t}\frac{m!}{s!t!}\frac{(x+\frac{1}{2})_r}{r!};$$

every term on the right is divisible by m_1 except the term in which $r=0$, $s=m$, $t=0$. Therefore we get

$$g_m(x) \equiv (-1)^m 2^{-m} \pmod{m_1}. \tag{3.3}$$

Combining (3.2) with (3.3), it is clear that

$$g_{n+m}(x) \equiv (-1)^m 2^{-m} g_n(x) \pmod{m_1} \qquad (n \geqq 0). \tag{3.4}$$

For $x=0$, (3.3) and (3.4) reduce to (1.4) and (1.5), respectively.

4. We rewrite (2.1) as

$$xg_n(x) = g_{n+1}(x) - ng_n(x) - \tfrac{1}{2}n(n-1)g_{n-2}(x).$$

It follows that

$$x^k g_n(x) = \sum_{s=-2k}^{k} A_{ks}(n)g_{n+s}(x),$$

where the $A_{ks}(n)$ are polynomials in n with coefficients that are integral except for powers of two. Consequently

$$g_m(x)g_n(x) = \sum_{s=-2m}^{m} B_{ms}(n)g_{n+s}(x), \tag{4.1}$$

where the $B_{ms}(n)$ are polynomials in n with coefficients that are integral except

for powers of two. In (4.1) we think of m as fixed while $n=0, 1, 2, \cdots$; also we take $B_{ms}(n)=0$ $(s<-n)$.

We now rewrite (4.1) in the form

$$g_m(x)g_n(x) - g_{n+m}(x) = \sum_{s=-2m}^{m-1} B_{ms}(n)g_{n+s}(x) \tag{4.2}$$

and apply (3.2). Thus

$$\sum_{s=-2m}^{m-1} B_{ms}(n)g_{n+s}(x) \equiv 0 \pmod{m_1}. \tag{4.3}$$

We require the following

LEMMA. *Let* $u_0(x), u_1(x), \cdots, u_n(x)$ *denote a set of polynomials in* x *with integral coefficients and highest coefficient* $=1$; *also let* $\deg u_r(x)=r$ *for* $0\leqq r\leqq n$. *Assume that* $A_0, A_1, \cdots, A_n$ *are integers such that* $\sum_{s=0}^{n} A_s u_s(x)\equiv 0 \pmod{m}$; *then* $A_s\equiv 0 \pmod{m}$ $(0\leqq s\leqq n)$.

The proof of this lemma will be omitted. We remark that the lemma holds when m is odd and the various coefficients are integral except for powers of two. We also remark that when $u(x)$ is a polynomial with integral coefficients, the statement $u(x)\equiv 0 \pmod{m}$ means that each coefficient is divisible by m.

Applying the lemma to (4.3) we immediately obtain

$$B_{ms}(n) \equiv 0 \pmod{m_1} \qquad (-2m \leqq s \leqq m-1). \tag{4.4}$$

In the next place we define the operator

$$\Delta\phi_n(x) = g_m(x)\phi_n(x) - \phi_{n+m}(x)$$

and generally

$$\Delta^r\phi_n(x) = g_m(x)\Delta^{r-1}\phi_n(x) - \Delta^{r-1}\phi_{n+m}(x),$$

where $\phi_n(x)$ is an arbitrary function of n and x. It follows that

$$\Delta^r\phi_n(x) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} g_m^{r-s}(x)\phi_{n+sm}(x).$$

Applying Δ^{r-1} to both members of (4.2) we get

$$\Delta^r g_n(x) = \sum_{s=-2m}^{m-1} \Delta^{r-1}\{B_{ms}(n)g_{n+s}(x)\}. \tag{4.5}$$

In addition to Δ^r we also require the operator δ^r defined by

$$\delta^r\phi_n(x) = \sum_{s=0}^{r} (-1)^s \binom{r}{s} \phi_{n+sm}(x).$$

It is not difficult to verify that

$$\Delta^{r-1}\{B_{ms}(n)g_{n+s}(x)\} = \sum_{j=0}^{r-1}\binom{r-1}{j}\delta^j B_{ms}(n)\cdot\Delta^{r-1-j}g_{n+jm}(x).$$

Thus (4.5) becomes

$$\Delta^r g_n(x) = \sum_{s=-2m}^{m-1}\sum_{j=0}^{r-1}\binom{r-1}{j}\delta^j B_{ms}(n)\cdot\Delta^{r-1-j}g_{n+jm}(x). \tag{4.6}$$

We shall now prove that

$$\Delta^r g_n(x) \equiv 0 \pmod{m_1^{r_1}}, \tag{4.7}$$

where $r_1=[(r+1)/2]$, by an induction with respect to r. For $r=1$, (4.7) reduces to (3.2). We accordingly assume that (4.7) holds up to and including the value $r-1$. Since $B_{ms}(n)$ is a polynomial in n with coefficients integral (mod m_1), it follows that

$$\delta^j B_{ms}(n) \equiv 0 \pmod{m_1^j}. \tag{4.8}$$

Consider a typical term

$$C_{js} = \delta^j B_{ms}(n)\cdot\Delta^{r-1-j}g_{n+jm}(x) \tag{4.9}$$

in the right member of (4.6). For $j=0$, we use (4.4) to get

$$C_{0s} \equiv 0 \pmod{m_1^{[1+r/2]}} \tag{4.10}$$

by the inductive hypothesis. For $j\geqq 1$, we use (4.8) to get

$$C_{js} \equiv 0 \pmod{m_1^{j+[(r-j)/2]}}.$$

Since $1+[\frac{1}{2}r]\geqq[\frac{1}{2}(r+1)]$, $j+[\frac{1}{2}(r-j)]\geqq[\frac{1}{2}(r+1)](1\leqq j\leqq r)$, it follows from (4.6), (4.9) and (4.10) that

$$\Delta^r g_n(x) \equiv 0 \pmod{m_1^{[(r+1)/2]}}.$$

This completes the proof of (4.7).

If we now employ the identity

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}\lambda^{r-s}g_{n+ms}(x) = \sum_{s=0}^{r}\binom{r}{s}(\lambda - g_m(x))^s\Delta^s g_n(x)$$

with $\lambda=(-1)^m 2^{-m}$ and make use of (3.3), we get

$$\sum_{s=0}^{r}(-1)^s\binom{r}{s}(-2)^{-(r-s)m}g_{n+ms}(x) \equiv 0 \pmod{m_1^{r_1}}, \tag{4.11}$$

provided x is a rational number that is integral (mod m_1).

We may now state the following

THEOREM. *Let m be an arbitrary positive integer and m_1 the largest odd divisor of m. Then the polynomial $g_n(x)$ satisfies*

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} g_m^{r-s}(x) g_{n+sm}(x) \equiv 0 \pmod{m_1^{r_1}},$$

where $r \geqq 1$ and $r_1 = [(r+1)/2]$. Here x is either an indeterminate or a rational number that is integral (mod m_1). *Moreover in the latter case we have*

$$\sum_{s=0}^{r} (-1)^s \binom{r}{s} (-2)^{-(r-s)m} g_{n+ms}(x) \equiv 0 \pmod{m_1^{r_1}}. \tag{4.12}$$

For $x=0$, (4.12) evidently reduces to (1.7).

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BUSCHE-RAMANUJAN IDENTITIES

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1. Introduction. An arithmetic function $f(r)$ is called a linear function if for all positive integers r and s we have $f(rs)=f(r)f(s)$. $f(r)$ is called a quadratic function if there are two linear functions $f_1(r)$ and $f_2(r)$ such that for all positive integers r,

$$f(r) = \sum_{d|r} f_1(d) f_2(r/d). \tag{1}$$

A quadratic function $f(r)$ is multiplicative and so $f(1)=1$. A multiplicative function $f(r)$ is said to admit a Busche-Ramanujan identity if there is a multiplicative function $F(r)$ such that for all positive integers M and N,

$$f(MN) = \sum_{d|(M,N)} f(M/d) f(N/d) F(d).$$

All of this terminology is due to Vaidyanathaswamy [6], and he showed that the only multiplicative functions which admit Busche-Ramanujan identities are the quadratic functions ([6], Theorem XXXV).

Now, Ramanathan proved in [5] that a quadratic function $f(r)$ admits, in

fact, a Busche-Ramanujan identity of the form

$$f(MN) = \sum_{d \mid (M,N)} f(M/d)f(N/d)\mu(d)B(d), \tag{2}$$

where $B(r)$ is the linear function whose value at the prime p is $B(p) = f^2(p) - f(p^2)$. In Section 2 we shall give an exposition of the result of Vaidyanathaswamy mentioned above. As D. H. Lehmer remarked in his review (MR **6**, 58) of [**5**], a multiplicative function $f(r)$ is quadratic if for each prime p, $f(p^a)$ as a function of a, is given by a recurring series of the second order. The theorem of Section 2 makes this more precise. In Section 3 we generalize (2) and in Section 4 we discuss some examples of quadratic functions.

2. Vaidyanathaswamy's theorem. In this section we shall prove

THEOREM 1. *For a multiplicative function $f(r)$ the following statements are equivalent*:

(A) *$f(r)$ is a quadratic function.*

(B) *For each prime p and positive integer a*

$$f(p^{a+1}) = f(p)f(p^a) - f(p^{a-1})B(p), \tag{3}$$

where $B(p)$ is the function defined above.

(C) *$f(r)$ admits a Busche-Ramanujan identity.*

Proof. Assume that (A) is true. Then $f(r)$ is given by (1) and so if $f_1(p) = X = X(p)$ and $f_2(p) = Y = Y(p)$ we have for a nonnegative integer c

$$f(p^c) = \sum_{i=0}^{c} X^i Y^{c-i}. \tag{4}$$

Then $f(p) = X + Y$, $f(p^2) = X^2 + XY + Y^2$ and so $B(p) = XY$. Thus for a positive integer a the right-hand side of (3) is equal to

$$\begin{aligned} &(X + Y)\sum_{i=0}^{a} X^i Y^{a-i} - XY\sum_{i=0}^{a-1} X^i Y^{a-1-i} \\ &= \sum_{i=0}^{a} X^{i+1} Y^{a-i} + \sum_{i=0}^{a} X^i Y^{a+1-i} - \sum_{i=0}^{a-1} X^{i+1} Y^{a-i} \\ &= X^{a+1} + \sum_{i=0}^{a} X^i Y^{a+1-i} = \sum_{i=0}^{a+1} X^i Y^{a+1-i}, \end{aligned} \tag{5}$$

which is equal to $f(p^{a+1})$. Hence (B) is true.

Now assume that (B) is true and, for a prime p, let X and Y be the solutions of the equations $X + Y = f(p)$, $XY = B(p)$, that is,

$$X,\ Y = \tfrac{1}{2}[f(p) \pm \sqrt{\{f^2(p) - 4B(p)\}}].$$

Let $f_1(r)$ and $f_2(r)$ be the linear functions whose values at the prime p are, respectively, X and Y. Then, to show that $f(r)$ is quadratic, we shall show that

(4) holds for every positive integer c. By our choice of X and Y, (4) holds when $c=1$. Assume (4) for all $c \leqq a$. Then, using (3), we have by (5),

$$f(p^{a+1}) = (X+Y)\sum_{i=0}^{a} X^i Y^{a-i} - XY\sum_{i=0}^{a-1} X^i Y^{a-1-i} = \sum_{i=0}^{a+1} X^i Y^{a+1-i}.$$

Thus (A) is true.

Still assuming (B) to be true we shall show that $f(r)$ admits the identity (2). Now, if $(M_1N_1, M_2N_2)=1$, then $((M_1, N_1), (M_2, N_2))=1$ and $(M_1M_2, N_1N_2) = (M_1, N_1)(M_2, N_2)$. Thus, to prove (2) we need to do so only when M and N are powers of the same prime p. Let $M=p^a$ and $N=p^b$ and assume, without loss of generality, that $1 \leqq a \leqq b$. We wish to show that

$$\begin{aligned} f(p^{a+b}) &= \sum_{i=0}^{a} f(p^{a-i})f(p^{b-i})\mu(p^i)B(p^i) \\ &= f(p^a)f(p^b) - f(p^{a-1})f(p^{b-1})B(p). \end{aligned} \tag{6}$$

We shall prove this by induction on a: it is true for $a=1$ by (3). Thus if $a>1$,

$$f(p^a) = f(p)f(p^{a-1}) - f(p^{a-2})B(p), \qquad f(p^{b+1}) = f(p)f(p^b) - f(p^{b-1})B(p).$$

Hence, if we assume that (6) holds whenever a is replaced by an integer less than a, we have

$$\begin{aligned} f(p^{a+b}) &= f(p^{a-1+b+1}) \\ &= f(p^{a-1})[f(p)f(p^b) - f(p^{b-1})B(p)] - f(p^{a-2})f(p^b)B(p) \\ &= f(p^a)f(p^b) + f(p^{a-2})B(p)f(p^b) - f(p^{a-1})f(p^{b-1})B(p) - f(p^{a-2})f(p^b)B(p) \\ &= f(p^a)f(p^b) - f(p^{a-1})f(p^{b-1})B(p). \end{aligned}$$

Thus (C) is true.

Now assume (C), that is, for some multiplicative function $F(r)$,

$$f(MN) = \sum_{d \mid (M,N)} f(M/d)f(N/d)F(d).$$

If we now set $M=p^a$ and $N=p$ we have $f(p^{a+1}) = f(p)f(p^a) + f(p^{a-1})F(p)$. But, taking $a=1$ we see that $F(p) = -B(p)$, and so (B) is true.

This completes the proof of Theorem 1. Note that a quadratic function $f(r)$ is completely determined by the values of $f(p)$ and $f(p^2)$ for the various primes p.

3. A generalization of (2). Let $c_k(n, r)$ be the extended Ramanujan sum defined by Cohen in [1]. We then have

THEOREM 2. *If $f(r)$ is a quadratic function and $B(r)$ is the linear function defined above, then for any nonnegative integer n,*

$$\sum_{d \mid (M,N)} B(d)f(M/d)f(N/d)c_k(n, d) = \sum_{d \mid (M,N),\, d^k \mid n} d^k B(d)f(MN/d^2). \tag{7}$$

Proof. Using Cohen's evaluation of $c_k(n, r)$ ([1], Th. 2), we see that the left-hand side of (7) is equal to

$$\sum_{d|(M,N)} B(d)f(M/d)f(N/d) \sum_{e^k|(n,d^k)} e^k\mu(d/e)$$

$$= \sum_{e|(M,N),e^k|n} e^k \sum_{d|(M,N),d=\delta e} B(d)f(M/d)f(N/d)\mu(d/e)$$

$$= \sum_{e|(M,N),e^k|n} e^k B(e) \sum_{\delta|(M/e,N/e)} B(\delta)f((M/e)/\delta)f((N/e)/\delta)\mu(\delta)$$

and, using (2), we obtain the right-hand side of (7).

If s is a nonzero complex number we set

$$\phi_s(r) = \sum_{d|r} d^s\mu(r/d).$$

For a positive integer k, $\phi_k(r) = c_k(0, r)$, and the proof of the following identity is the same as the proof of (7):

$$\sum_{d|(M,N)} B(d)f(M/d)f(N/d)\phi_s(d) = \sum_{d|(M,N)} d^s B(d)f(MN/d^2). \tag{8}$$

The special case of (7) when $f(r) = \sigma_s(r) =$ the sum of the sth powers of the divisors of r was proved by the author in [4], Theorem 4. Cohen had previously obtained this result when $f(r) = \sigma_s(r)$ and $k = 1$ ([2], Th. 10).

4. Some quadratic functions. Let $\tau(r)$ be the arithmetic function of Ramanujan defined by ([3], Ch. X):

$$\sum_{r=1}^{\infty} \tau(r)x^r = x\prod_{r=1}^{\infty}(1 - x^r)^{24} \qquad (|x| < 1).$$

Mordell proved that $\tau(r)$ is multiplicative and that for every prime p and positive integer a,

$$\tau(p^{a+1}) = \tau(p)\tau(p^a) - p^{11}\tau(p^{a-1})$$

([3], Sec. 10.2). Then $\tau^2(p) - \tau(p^2) = p^{11}$ and so $\tau(r)$ is quadratic. This fact was noted by Ramanathan ([5], p. 114), as was the identity

$$\tau(MN) = \sum_{d|(M,N)} \tau(M/d)\tau(N/d)\mu(d)d^{11}.$$

From Theorem 2 we obtain

$$\sum_{d|(M,N)} d^{11}\tau(M/d)\tau(N/d)c_k(n, d) = \sum_{d|(M,N)d^k|n} d^{k+11}\tau(MN/d^2),$$

and from (8),

$$\sum_{d|(M,N)} d^{11}\tau(M/d)\tau(N/d)\phi_s(d) = \sum_{d|(M,N)} d^{s+11}\tau(MN/d^2).$$

If we set $M=N=r$ we obtain

$$\sum_{d|r} d^{11}\tau^2(r/d)\phi_s(d) = r^{s+11}\sum_{d|r}\tau(d^2)d^{s-11},$$

which yields on setting $s=-11$,

$$\sum_{d|r} d^{11}\tau^2(r/d)\phi_{-11}(d) = \sum_{d|r}\tau(d^2).$$

We also have for any prime p and positive integer a,

$$\tau(p^a) = 2^{-a}\sum_{i=0}^{a}[\tau(p)+\sqrt{\{\tau^2(p)-4p^{11}\}}]^i[\tau(p)-\sqrt{\{\tau^2(p)-4p^{11}\}}]^{a-i}.$$

In passing, we note that the Ramanujan hypothesis ([3], Sec. 10.7) states that $\tau(p)<2p^{11/2}$ for every prime p, so that the expression under the radical in the above formula is negative.

Let $\lambda(r)=(-1)^{\Omega(r)}$, where $\Omega(r)$ is the number of primes dividing r with repetitions counted. $\lambda(r)$ is a linear function and we consider the quadratic function $\beta_h(r)=\sum_{d|r} d^h\lambda(r/d)$. For this function, $B(r)=\lambda(r)r^h$. It was shown in [4], Section 6 that $\beta_h(r)=\sum_{de^2=r}\phi_h(d)$, and that $\beta_h(r)$ is the number of integers $a(\text{mod } r^h)$ such that the largest integral hth power which divides both a and r^h is a $2h$th power. From Theorem 2 we have

$$\sum_{d|(M,N)}\lambda(d)d^h\beta_h(M/d)\beta_h(N/d)c_k(n,d) = \sum_{d|(M,N)d^k|n}\lambda(d)d^{k+h}\beta_h(MN/d^2).$$

We can also write several other identities, such as

$$\sum_{d|r}\lambda(d)d^h\,\beta_h^2(r/d)\phi_{-h}(d) = \sum_{d|r}\lambda(d)\beta_h(d^2).$$

Note that $\beta_h(p^a)=\sum_{i=0}^{a}(-1)^{a-i}p^{hi}$.

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AN INTERLACING TRANSFORMATION PROBLEM

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Let n white chessmen, denoted by W, and n black chessmen, denoted by B, be lined up as follows.

$$WWW\cdots WWBB\cdots BBB. \tag{1}$$

We wish to realign them in the order

$$BWBW\cdots BW \tag{2}$$

in as few moves as possible, where a move consists of picking up any two adjacent chessmen and replacing them in any available position without changing their relative order. Thus, for $n=3$ and 4 we have the solutions

$$\begin{array}{ll} WWWBBB & WWWWBB\,BB \\ WBBBWW & W--WBB\,BB\,WW \\ WBB--WBW & WB\,B\,W--BB\,WW \\ BWBWBW & WB\,B\,WBWB--W \\ & B\,WBWBWB\,W. \end{array} \tag{3}$$

In each case we have a solution in n moves. We will prove that, for $n\geqq 3$, the chessmen can be realigned in n moves but not in fewer. The case $n=1$ is trivial and the puzzle has no solution if $n=2$, so we assume $n\geqq 3$.

First we prove that the puzzle cannot be solved in fewer than n moves. To do this we count the number of variations, WB and BW, similarly to the counting of changes of sign in Descarte's rule of signs. The starting alignment (1) has 1 variation, and (2) has $2n-1$. The first step in (3) for $n=3$ has 2 variations. The first part of a move consists of removing two adjacent chessmen, $X_1X_2X_3X_4$ becoming X_1--X_4. It cannot increase, and may decrease the number of variations. In fact, if X_1 and X_4 are of the same color, then X_1--X_4 presents no variation. If X_1 and X_4 are of different colors, then $X_1X_2X_3X_4$ presents at least 1 variation, while X_1--X_4 presents exactly 1.

The second part of a move consists of setting two chessmen back down, Y_1--Y_4 becoming $Y_1Y_2Y_3Y_4$, where Y_1 and Y_2 may possibly be blank spaces. Now $Y_1Y_2Y_3Y_4$ can present at most 3 variations and this can happen only if Y_1, Y_2, Y_3, Y_4 alternate in color. In this case Y_1--Y_4 presents 1 variation, and the increase in number of variations is 2. In all other cases $Y_1Y_2Y_3Y_4$ presents at most 2 variations and the increase is at most 2. Thus we see that each complete move can increase the total number of variations by no more than 2.

Consider the first move. It picks up two adjacent chessmen in (1) and sets them down at one end of the row. Clearly if a pair WW or BB is moved the number of variations is increased by at most 1, from 1 to 2. If the pair WB is moved, say to the right, we get

$$WWW \cdots W - - B \cdots BBBWB$$

which has 3 variations. But now it is easy to verify that the next move will yield an alignment that has at most 4 variations. Thus in all cases either the first or second move will increase the total number of variations by no more than 1. Therefore $n-1$ moves can increase the number of variations by at most $2(n-2)+1=2n-3$. Since (2) has $2n-1$ variations, an increase of $2n-2$ over (1), we see that $n-1$ moves are not enough.

Now we show that the puzzle can be solved in n moves. A solution for $n=3$ appears in (3). We will now assume $n \geqq 4$. It is convenient to number the positions, not the chessmen, in the following way:

$$w_1 w_2 w_3 \cdots w_{n-1} w_n b_n b_{n-1} \cdots b_3 b_2 b_1 b_0 b_{-1},$$

where positions b_0 and b_{-1} are empty at the start. We shall designate a move by writing down the symbol for the position of the left one of the pair of adjacent chessmen to be moved. In all our solutions the first pair to be moved will be put in $b_0 b_{-1}$. All later moves will put the pair into the positions left vacant by the preceding move. The solution (3) for $n=4$ is then w_2, b_4, b_1, w_1. This solution has the property that it uses only the positions originally occupied and the two adjacent ones to the right. Also, the final alignment leaves the first two positions on the left empty. Writing them out as in (3) one finds that

$$\begin{aligned} &w_2, b_3, w_5, b_1, w_1 && \text{for } n = 5, \\ &w_2, b_5, w_4, b_4, b_1, w_1 && \text{for } n = 6, \\ &w_2, b_4, w_5, b_5, w_7, b_1, w_1 && \text{for } n = 7 \end{aligned}$$

are solutions having the property described above.

Our result can now be obtained by mathematical induction. Suppose $n \geqq 8$ and that for $n-4$ white and $n-4$ black chessmen the puzzle has a solution in $n-4$ moves, with the above property. For n chessmen of each color we first make the two moves

$$\begin{aligned} &WWWWW \cdots WWBB \cdots BBBBB \\ &W - - WW \cdots WWBB \cdots BBBBBWW \\ &WB\,B\,WW \cdots WWBB \cdots B--BBWW. \end{aligned}$$

Now, leaving the four chessmen on the left and the four on the right in place, we can use the solution for $n-4$ to get

$$WBBW - - BWBW \cdots BWBWBBWW$$

in $n-4$ more moves. We finally make two more moves

$$\begin{aligned} &WBBWBWBWBW \cdots BWBWB--W \\ &\quad BWBWBWBW \cdots BWBWBWBW, \end{aligned}$$

thus solving the puzzle in $2+(n-4)+2=n$ moves. This completes the induction.

For $n \geqq 4$ we write $n=4m+k$, $k=0, 1, 2, 3$, and the solution we have developed can be written in three parts:

$$\text{I.} \quad w_2(b_4w_6)(b_8w_{10}) \cdots (b_{4m-4}w_{4m-2}).$$

$$\text{II.} \quad \begin{cases} b_n & \text{if } k = 0, \\ b_{n-2}w_n & \text{if } k = 1, \\ b_{n-1}w_{n-2}b_{n-2} & \text{if } k = 2, \\ b_{n-3}w_{n-2}b_{n-2}w_n & \text{if } k = 3. \end{cases}$$

$$\text{III.} \quad (b_{4m-3}w_{4m-3}) \cdots (b_9w_9)(b_5w_6)(b_1w_1),$$

where I consists of $2m-1$ moves, II of $k+1$, III of $2m$; a total of n moves.

ROTATING FRAMES OF REFERENCE AND THE VECTOR CONCEPT

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From the historical point of view, vector analysis is so closely related to developments in geometry that one can say that fundamentally the subject should be categorized under the heading of geometry. In particular, although it is not emphasized in elementary vector analysis courses, many interesting ideas relating to both geometry and physics follow from considerations of coordinate transformations.

This paper will investigate the classical expressions for velocity and acceleration associated with two reference frames, one in rotational motion with respect to the other; that is, the vector equations*

$$\mathbf{V} = \mathbf{V}_a + \omega \times \mathbf{r}, \tag{1}$$

$$\mathbf{A} = \mathbf{A}_a + 2[\omega \times \mathbf{V}_a] + (\delta\omega/\delta t) \times \mathbf{r} + [\omega \times (\omega \times \mathbf{r})], \tag{2}$$

where $\delta\omega/\delta t$ represents angular acceleration, with $\delta/\delta t$ being an "apparent" derivative (*i.e.*, one taken with the supposition that the basis vectors of the rotating system x^i are fixed or, in other words, taken from the viewpoint of someone in that system). The first and second bracketed expressions in (2) are usually called Coriolis acceleration and centripetal acceleration respectively.

The equations (1) and (2) will be obtained in a component form. It will be shown that under appropriate definition $\mathbf{V}$, $\mathbf{A}$ and ω are not of vector character. Finally $\mathbf{V}$ and $\mathbf{A}$ will be replaced by quantities which do have vector character and the nature of ω will be discussed.

* See Nathaniel Coburn, Vector and Tensor Analysis, New York, 1955.

In that which follows it is assumed that an inertial frame, that is one in which the Newton laws hold, exists. This frame of reference will be expressed in terms of rectangular cartesian coordinates $\bar{x}^i$. A frame in rotational motion to the inertial frame will be referred to rectangular cartesian coordinates x^i. The coordinate systems are then related by means of the transformation equations

$$\bar{x}^j = a_i^j x^i, \tag{3}$$

where the rotation matrix (a_i^j) is such that its elements are functions of t and satisfy the orthogonality conditions

$$\sum_{i=1}^{3} a_i^j a_i^k = \delta^{jk}, \qquad \sum_{j=1}^{3} a_i^j a_k^j = \delta_{ik}, \tag{4}$$

where $a = \det(a_i^j) = 1$ and $\delta_{ik} = \delta^{ik} = 1$ if $i = k$ and is 0 if $i \neq k$. In (3) the summation convention is used for the first time in this paper. That is, use of an index as both a subscript and superscript in a given term implies a summation over the range of the index.

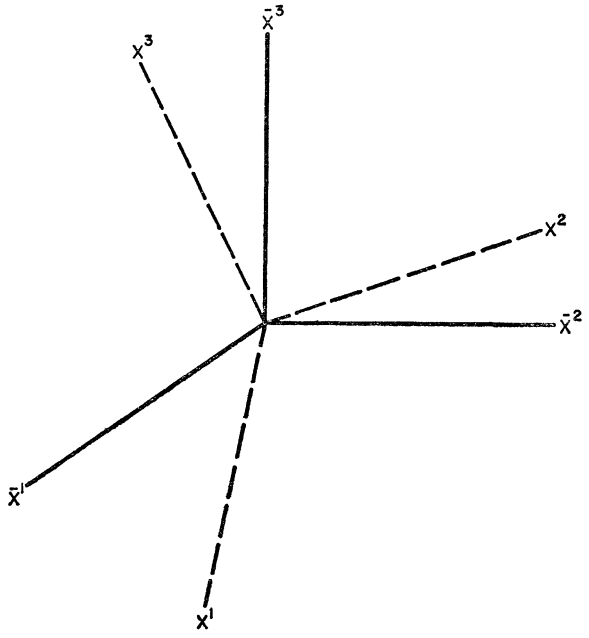


FIG. 1

Remarks which are made concerning the concept of vector or tensor are based upon the following definitions.

DEFINITION 1. *T is a 3-space vector means that it is represented by a set of three functions in each allowable coordinate system and that these functions are related as follows:** $\bar{T}^i = a^i_j T^j$.

DEFINITION 2. *An object is a 3-space tensor of order two means that it is represented by a set of 3^2 functions in each allowable coordinate system, the functions being related as follows:* $\bar{T}^{ij} = a^i_k a^j_p T^{kp}$.

Component expressions for velocity and acceleration can be obtained by straightforward differentiation of (3). After a slight rearrangement of terms one obtains†

$$\left(\frac{d\bar{x}^j}{dt} - a^j_i \frac{dx^i}{dt}\right) = \frac{da^j_i}{dt} x^i, \tag{5}$$

$$\left(\frac{d^2\bar{x}^j}{dt^2} - a^j_i \frac{d^2x^i}{dt^2}\right) = \frac{d^2a^j_i}{dt^2} x^i + 2\frac{da^j_i}{dt}\frac{dx^i}{dt}. \tag{6}$$

One immediately sees from examination of (5) and (6) that the components dx_i/dt and d^2x_i/dt^2 do not transform as components of vectors. The physical fact of rotation (equivalent to the mathematical fact that the a^j_i are functions of t) has destroyed the vector character of velocity and acceleration. The purpose of this paper is to introduce representations for velocity and acceleration such that their vector character is restored. However, let us first describe the way in which (5), (6) and (1), (2) are shown to be equivalent.

Let $(A^i_j) = (a^j_i)$ represent the reciprocal rotation matrix to (a^j_i); then corresponding to (4), $a^j_i A^j_k = \delta^i_k$, $a^j_i A^k_j = \delta^k_i$. The key to the relationship of the forms for velocity and correspondingly for acceleration is to notice that the expression

$$A^k_j \frac{da^j_i}{dt} = \sum_{j=1}^{3} a^j_k \frac{da^j_i}{dt}$$

is skew symmetric in k and i. This fact is obtained by differentiating $A^k_j a^j_i = \delta^k_i$ and using the relation $A^k_j = a^j_k$. From this point on the following notation will be used:

$$\omega_{ki} = \sum_{j=1}^{3} a^j_k \frac{da^j_i}{dt}. \tag{7}$$

The components ω_{ki} are commonly said to be components of an angular velocity tensor (they will simply be called angular velocity components without any

* In general one must distinguish between contravariant and covariant components of vectors or tensors, but for rectangular cartesian coordinate systems there is no difference. Hence for simplicity we just use the terms vector or tensor components.

† For a corresponding development, see J. L. Synge and A. Schild, Tensor Calculus, Toronto, 1956, p. 162.

reference to their tensor character in this article for reasons to be indicated later) and have an associated one-index system

$$\omega^p = \tfrac{1}{2}E^{pki}\omega_{ki}, \tag{8}$$

where $E^{pki}=e_{pki}=1$ or -1 according as p, k, i is an even or an odd permutation of 1, 2, 3; and $E^{pki}=e_{pki}=0$, otherwise. By using the property

$$E^{pki}e_{prs} = 2\delta_{rs}^{[ki]} = \delta_r^k\delta_s^i - \delta_r^i\delta_s^k, \tag{9}$$

one is able to solve (8) for ω_{ki} obtaining

$$\omega_{rs} = e_{prs}\omega^p. \tag{10}$$

Putting together (7) and (10) one has

$$A_j^k\frac{da_i^j}{dt} = \sum_{j=1}^{3} a_k^j\frac{da_i^j}{dt} = \omega_{ki} = e_{pki}\omega^p. \tag{11}$$

In order to bring about the desired comparison it is also necessary to represent $A_j^k(d^2a_i^j/dt^2)$ in terms of ω^p. The needed expression follows from (11) after multiplying by a_k^t, differentiating with respect to t and finally multiplying by A_t^q. One obtains

$$A_t^q\frac{d^2a_i^t}{dt^2} = e_{pki}\left(A_t^q\frac{da_k^t}{dt}\omega^p + \delta_k^q\frac{d\omega^p}{dt}\right). \tag{12}$$

A substitution according to (11) into the right-hand member of (12) and use of (9) produces the result

$$\sum_{i=1}^{3} a_q^t\frac{d^2a_i^t}{dt^2} = -\omega_p\omega^p\delta_{iq} + \omega_i\omega_q + e_{pqi}\frac{d\omega^p}{dt}.$$

Now if the relations (5) and (6) are each multiplied by A_j^k the substitutions (11) and (12) may be made. Having appropriately substituted, a sum and multiplication with a_k^q produces the results

$$\left(\frac{d\bar{x}^q}{dt} - a_k^q\frac{dx^k}{dt}\right) = \sum_k a_k^q e_{pki}\omega^p x^i, \tag{13}$$

$$\left(\frac{d^2x^q}{dt^2} - a_k^q\frac{d^2x^k}{dt^2}\right) = a_k^q[-(\omega_p\omega^p)x^k + (\omega_i x^i)\omega^k] + a_k^q e_{pki}\frac{d\omega^p}{dt}x^i + 2a_k^q e_{pki}\omega^p\frac{dx^i}{dt}. \tag{14}$$

The relations (13) and (14) may be compared directly with the components of (1) and (2).

It has been seen that the components of velocity and acceleration do not

have vector character under the transformations (3). Hence the last part of the paper will be devoted to the introduction of generalized concepts of velocity and acceleration which restore the vector character. In the process of development it will be observed that the components ω_{ij} behave as those of a linear connection, the simplest example of which is the so-called Christoffel connection. The motivation for the generalization to be introduced is given by (13) when that expression is put in the form

$$\frac{d\bar{x}^q}{dt} = a_k^q\left(\frac{dx^k}{dt} + \omega_{ki}x^i\right). \tag{15}$$

With (15) in mind the following definition is made.

DEFINITION 3. *Let W^k be the components of a vector; then the rotational derivative of W^k is defined as follows*:

$$\frac{DW^k}{dt} \equiv \frac{dW^k}{dt} + \omega_i^k W^i, \tag{16}$$

where $\omega_i^k = \delta^{kp}\omega_{pi}$.

The definition of rotational derivative of course does the job that is desired on the motivating relation (15); that is, since the $\bar{x}^i$ system is being thought of as stationary, $\bar{\omega}_i^k = 0$ and $D\bar{x}^q/dt = d\bar{x}^q/dt$. Hence (15) becomes $D\bar{x}^q/dt = a_k^q(Dx^k/dt)$. Note that the coordinates x^k are vector components when the set of transformations consist of rotations only. We are further encouraged by the fact that

$$\frac{D^2x^k}{dt^2} = \frac{D(Dx^k/dt)}{dt} = \frac{d(Dx^k/dt)}{dt} + \omega_p^k\frac{Dx^p}{dt}$$

can, by making use of Definition 3, be put in the form

$$\frac{D^2x^k}{dt^2} = \frac{d^2x^k}{dt^2} + 2\omega_i^k\frac{dx^i}{dt} + \left(\omega_p^k\omega_q^p + \frac{d\omega_q^k}{dt}\right)x^q. \tag{17}$$

For (17) is equivalent to the factor associated with a_k^q in (14) (in order to make the comparison the ω^k in (14) must be replaced according to (10) by ω_i^k).

In the following theorem it is shown that the rotation derivatives of 1st and 2nd order satisfy in general the appropriate transformation rule.

THEOREM 1. *Assume the existence of a Newtonian frame of reference. Let the frame be referred to coordinates $\bar{\bar{x}}^i$. Then the following transformation relations hold with respect to systems $\bar{x}^i$ and x^i*:

$$\text{(18)}\quad \text{(a)}\ \frac{D\bar{x}^p}{dt} = c_k^p\frac{Dx^k}{dt},\quad (b)\ \frac{D^2\bar{x}^p}{dt^2} = c_k^p\frac{D^2x^k}{dt^2},\quad \textit{where (c)}\ \bar{x}^p = c_k^p x^k,\ [c_k^p] \neq 0.$$

Proof. Since $\bar{\bar{x}}^j$ represents a Newtonian system one concludes from (15) and (16) that

$$(19)\qquad \text{(a)}\quad \frac{d\bar{\bar{x}}^i}{dt} = a_k^{\ j}\left(\frac{dx^k}{dt} + \omega_i^{\ k}x^i\right) = a_k^{\ j}\frac{Dx^k}{dt},$$

$$\text{(b)}\quad \frac{d\bar{\bar{x}}^i}{dt} = b_k^{\ j}\left(\frac{d\bar{x}^k}{dt} + \bar{\omega}_i^{\ k}\bar{x}^i\right) = b_k^{\ j}\frac{D\bar{x}^k}{dt},$$

where x^k and $\bar{x}^k$ are orthogonal systems related to $\bar{\bar{x}}^k$ by means of $\bar{\bar{x}}^i = a_k^i x^k$, $\bar{\bar{x}}^i = b_k^i \bar{x}^k$. From (19) (a), (b) it follows that

$$(20)\qquad b_k^{\ j}\frac{D\bar{x}^k}{dt} = a_k^{\ j}\frac{Dx^k}{dt}.$$

By multiplying and summing in (20) with B_j^p, where $B_j^p b_k^j = \delta_k^p$, we obtain

$$\frac{D\bar{x}^p}{dt} = B_j^p a_k^{\ j}\frac{Dx^k}{dt} = c_k^{\ p}\frac{Dx^k}{dt},$$

where $c_k^p = B_j^p a_k^j$. The relation (18)(b) is proved in an analogous manner.

It is of particular interest to find that the components ω_{ki} do not transform as tensor components but rather transform in a manner that reminds one of Christoffel symbols.* This result is produced by the next theorem.

THEOREM 2. *The components ω_{ki} satisfy the transformation rule*

$$\omega_{ki} = \frac{\partial \bar{x}^p}{\partial x^k}\frac{\partial \bar{x}^p}{\sigma x^i}\bar{\omega}_{pq} + \frac{\partial \bar{x}^q}{\partial x^k}\frac{d(\partial x^p/\partial x^i)}{dt}.$$

Proof. In our considerations the Newtonian frame of reference represented in terms of coordinates $\bar{\bar{x}}^i$ is a preferred frame of reference. In particular the angular velocity components ω_{ki} of any rotational frame are defined in terms of the rotation coefficients associating the coordinates describing the frame with those describing the Newtonian system. Symbolically if

$$(21)\qquad \bar{\bar{x}}^i = a_j^{\ i}x^j, \qquad \bar{\bar{x}}^i = b_j^{\ i}\bar{x}^j,$$

it follows by definition (see (7)) that

$$(22)\qquad \omega_{ki} = \sum_j a_k^{\ j}\frac{da_i^j}{dt}, \qquad \bar{\omega}_{ki} = \sum_j b_k^{\ j}\frac{db_i^j}{dt}.$$

The relations (22) can be expressed in terms of partial derivatives after noting from consideration of (21) that $\partial\bar{\bar{x}}^i/\partial x^k = a_k^i$, $\partial\bar{\bar{x}}^i/\partial\bar{x}^k = b_k^i$. Therefore, starting with the left-hand expression in (22) and for convenience omitting the summation sign (let the summation convention be extended to repeated indices whether in subscript-superscript combination or not), one obtains

* For a more general discussion of connections with a special reference to kinematics, see V. Hlavatý, Uber eine Art der Punktkonnexion, Math. Z. 1934, vol. 38, pp. 135–145.

$$\omega_{ki} = \frac{\partial \bar{\bar{x}}^j}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{\bar{x}}^j}{\partial x^i}\right) = \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^p}\frac{\partial \bar{x}^p}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^q}\frac{\partial \bar{x}^q}{\partial x^i}\right)$$

$$= \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^p}\frac{\partial \bar{x}^p}{\partial x^k}\left(\frac{d}{dt}\left(\frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^q}\right)\frac{\partial \bar{x}^q}{\partial x^i} + \frac{\partial \bar{\bar{x}}^j}{\partial \bar{x}^q}\frac{d}{dt}\left(\frac{\partial \bar{x}^q}{\partial x^i}\right)\right)$$

$$= \frac{\partial \bar{x}^p}{\partial x^k}\frac{\partial \bar{x}^q}{\partial x^i}\bar{\omega}_{pq} + \delta_{pq}\frac{\partial \bar{x}^p}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{x}^q}{\partial x^i}\right)$$

$$= \frac{\partial \bar{x}^p}{\partial x^k}\frac{\partial \bar{x}^q}{\partial x^i}\bar{\omega}_{pq} + \frac{\partial \bar{x}^p}{\partial x^k}\frac{d}{dt}\left(\frac{\partial \bar{x}^p}{\partial x^i}\right).$$

This completes the proof.

It is simply a matter of algebraic manipulation to show that

$$\bar{\omega}_{rs} = \frac{\partial x^k}{\partial \bar{x}^r}\frac{\partial x^i}{\partial \bar{x}^s}\omega_{ki} + \frac{\partial x^i}{\partial \bar{x}^r}\frac{d}{dt}\left(\frac{\partial x^i}{\partial \bar{x}^s}\right).$$

Of course one should not have expected the components ω_{ik} to have tensor character for they are defined for any system in terms of the coefficients relating that system to the $\bar{\bar{x}}^j$ system (*i.e.*, the coordinates of the preferred Newtonian frame). Hence by definition $\bar{\omega}_{ki} = \delta^j_k(d\delta^j_i/dt) = 0$. Since the symbols are zero in one system, tensor character would imply that they were zero in every system. This is contrary to physical experience.

RECOMMENDATIONS OF THE MATHEMATICAL ASSOCIATION OF AMERICA FOR THE TRAINING OF MATHEMATICS TEACHERS

The Committee on the Undergraduate Program in Mathematics (CUPM)* is a committee of the Mathematical Association of America and is supported in part by the National Science Foundation. The general purpose of this committee is to develop a broad program of improvement in the undergraduate mathematics curriculum of the nation's colleges and universities.

As part of its mandate, CUPM established a Panel on Teacher Training.† This panel was instructed to prepare for CUPM a set of recommendations of minimum standards for the training of teachers on all levels. The following report is the result of the work of the Panel on Teacher Training, and has re-

* Members of CUPM: R. C. Buck (*Chairman*), E. G. Begle, L. W. Cohen, W. T. Guy, Jr., R. D. James, J. L. Kelley, J. G. Kemeny, E. E. Moise, J. C. Moore, Frederick Mosteller, H. O. Pollak, G. B. Price, Patrick Suppes, Henry Van Engen, R. J. Walker, A. D. Wallace, R. J. Wisner (*Executive Director*).

† Members of the Panel on Teacher Training: J. G. Kemeny (*Chairman*), E. G. Begle, W. T. Guy, Jr., P. S. Jones, J. L. Kelley, B. E. Meserve, E. E. Moise, Rothwell Stephens, Henry Van Engen, R. C. Buck (*ex officio*), R. J. Wisner (*ex officio*).

ceived the endorsement of the Committee on the Undergraduate Program in Mathematics and of the Board of Governors of the Mathematical Association of America.

The Panel on Teacher Training has been further charged with the implementation of these recommendations and hopes to issue supplementary reports, as well as to hold various regional conferences, to make these minimum standards a reality.

The report consists of the following: General Recommendations, The Five Levels, Recommendations for the Five Levels, Summary of Recommendations, Curriculum-study Courses, Training of Supervisors, Sample Course Descriptions.

Further information and reprints of this report (not available before January 15, 1961) may be obtained by writing to the Executive Director: Professor Robert J. Wisner, Michigan State University Oakland, Rochester, Michigan.

GENERAL RECOMMENDATIONS

The purpose of this report is to present a preliminary outline of the panel's recommendations for the minimal college training program for teachers of mathematics. We have found it a most useful device to arrive at a classification of mathematics teachers which does not, as far as we know, depend on any present scheme of training teachers for their various tasks. The existing classifications seem to have arisen from a series of historical accidents and from fundamental psychological considerations. We hope this report reflects our feeling that we have made a serious attempt to classify teachers according to their position in an over-all sequential schedule of presenting the main ideas of mathematics.

For each classification presented, we give a recommendation as to the type and minimum amount of mathematics which should be taken by the student preparing for a career in teaching. Further, we spell out in some detail the *types* of courses—included as an Appendix—which we recommend to implement the goals described. The courses we describe which are specifically designed for prospective teachers should be taught by persons who are masters of their subject matter and who have, in addition, a knowledge of the problems which teachers face.

These sample courses are given solely for illustrative purposes to explain the type of courses and the levels of advancement desirable for prospective teachers. It should be clearly understood that different institutions will wish to exercise considerable freedom in implementing these recommendations, both as to the way topics are combined into courses and as to the exact choice of topics for individual courses.

There are several very sincere words of warning to be put forth in regard to the reading and interpretation of this report.

First, the classifications are to be taken in the rather loose fashion in which they are described. Their exact delineations will of course depend upon local

conditions of school and curricular organization. It should be noted that the various classifications overlap: this is done deliberately in an attempt to meet just such local conditions.

Second, the recommendations are not motivated by a desire to meet the demands of any special program of mathematics education; nor do the descriptions or outlines of courses to be taken by prospective teachers represent an attempt on the part of this committee to further the goals of any particular school curriculum planning organization. The recommendations are meant to be the minimum which should be required of teachers in any reasonable educational program, and the course descriptions are presented only to illustrate what is meant by the course titles.

Third, it is to be hoped that everyone recognizes good mathematics education to be a sequential experience. Thus, the teacher at any particular level should have an understanding of the mathematics which will confront the student in subsequent courses; and as a consequence, it is desirable that a teacher at a given level be prepared to teach at least some succeeding courses. Ideally, a person preparing for teaching should meet, in addition to the minimal requirements set forth here, as many of the requirements for the next level as his or her college program permits.

Fourth, this report is meant as a guide to the preparation of people who will be teaching any mathematics whatsoever. The suggestions apply, within each level, to all people who teach any mathematics. The recommendations do not in any sense exclude the teacher who is assigned classes scheduled primarily for students of low aptitude.

Fifth, every good teacher knows that mathematics must begin at the concrete level before it can proceed to the more technical or abstract formulation. Motivation for new concepts must be derived and later application of the theory to nature must be included. In each of the outlines to follow, it is assumed that topics will contain a judicious mixture of motivation, theory, and application. A purely abstract course for teachers would be madness, and a course in calculation with no theory would not be mathematics.

Sixth, the phrase "a course" occurs in several places in this report. For purposes of fixing ideas, this phrase is employed in the sense of a three-semester-hour presentation of the subject matter described, and it is not meant to exclude integrated programs or other curricular arrangements.

Finally, the reader should note that the training for Level I teaching is a separate program, while the curricula for the further levels form a cumulative sequence, in which each program is a continuation of the preceding one.

The committee benefited greatly from previous studies on teacher preparation, such as that of the Cooperative Committee on the Teaching of Science and Mathematics, a committee of the American Association for the Advancement of Science. The committee was also guided by discussions with a variety of professional organizations. It is pleased to note the considerable degree of agreement common to all proposals.

It should be emphasized that these recommendations are minimal in nature and that some institutions have already met and exceeded these recommendations. It is expected that as high school curricula are strengthened, these minimum recommendations will be revised.

THE FIVE LEVELS

I. Teachers of elementary school mathematics. This level consists of teachers confronted with the problem of presenting the elements of arithmetic and the associated material now commonly taught in grades K through 6. The committee recognizes that special pedagogical problems may be connected with grades K through 2, and so a special program may be appropriate for teachers of such grades.

II. Teachers of the elements of algebra and geometry. Included here are teachers who are assigned the task of giving introductory year courses in either algebra or geometry, or the less formal preliminary material in these fields. These introductory courses are now commonly taught in grades 7 through 10.

III. Teachers of high school mathematics. These teachers are qualified to teach a modern high school mathematics sequence* in grades 9 through 12.

IV. Teachers of the elements of calculus, linear algebra, probability, etc. This is a mixed level, consisting of teachers of advanced programs in high school, junior college teachers, and staff members employed by universities to teach in the first two years. These teachers should be qualified to present a modern two-year college mathematics program.

V. Teachers of college mathematics. These teachers should be qualified to teach all basic courses offered in a strong undergraduate college curriculum.

The levels having been presented, we are now ready to proceed to a description of our recommendations of the minimal college training requirements for entry into the teaching profession at each level.

RECOMMENDATIONS FOR LEVEL I

(Teachers of elementary school mathematics)

As a prerequisite for the college training of elementary school teachers, we recommend at least two years of college preparatory mathematics, consisting of a year of algebra and a year of geometry, or the same material in integrated courses. It must also be assured that these teachers are competent in the basic techniques of arithmetic. The exact length of the training program will depend on the strength of their preparation. For their college training, we recommend the equivalent of the following courses:

* Such sequences have been recommended by the Commission on Mathematics, the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, and others.

(A) A two-course sequence devoted to the structure of the real number system and its subsystems. (See course-sequence 1.)*
(B) A course devoted to the basic concepts of algebra. (See course 2.)
(C) A course in informal geometry. (See course 3.)

The material in these courses might, in a sense, duplicate material studied in high school by the prospective teacher, but we urge that this material be covered again, this time from a more sophisticated, college-level point of view.

Whether the material suggested in (A) above can be covered in one or two courses will clearly depend upon the previous preparation of the student.

We strongly recommend that at least 20 per cent of the Level I teachers in each school have stronger preparation in mathematics, comparable to Level II preparation but not necessarily including calculus. Such teachers would clearly strengthen the elementary program by their very presence within the school faculty. This additional preparation is certainly required for elementary teachers who are called upon to teach an introduction to algebra or geometry.

RECOMMENDATIONS FOR LEVEL II

(Teachers of the elements of algebra and geometry)

Prospective teachers should enter this program ready for a mathematics course at the level of a beginning course in analytic geometry and calculus (requiring a minimum of three years in college preparatory mathematics). It is recognized that many students will have to correct high school deficiencies in college. (However, such courses as trigonometry and college algebra should not count toward the fulfillment of minimum requirements at the college level.) Their college mathematics training should then include:

(A) Three courses in elementary analysis (including or pre-supposing the fundamentals of analytic geometry). (See course-sequence 4.)

This introduction to analysis should stress basic concepts. However, prospective teachers should be qualified to take more advanced mathematics courses requiring a year of the calculus, and hence calculus courses especially designed for teachers are normally not desirable.

(B) Four other courses: a course in abstract algebra, a course in geometry, a course in probability from a set-theoretic point of view, and one elective. One of these courses should contain an introduction to the language of logic and sets. (See courses 5–7.)

RECOMMENDATIONS FOR LEVEL III

(Teachers of high school mathematics)

Prospective teachers of mathematics beyond the elements of algebra and geometry should complete a major in mathematics and a minor in some field

* Sample courses, by numbers, are to be found in the Appendix.

in which a substantial amount of mathematics is used. This latter should be selected from areas in the physical sciences, biological sciences, and the social studies, but the minor should in each case be pursued to the extent that the student will have encountered substantial applications of mathematics.

The major in mathematics should include, in addition to the work listed under Level II, at least an additional course in each of algebra, geometry, and probability-statistics, and one more elective.

Thus, the minimum requirements for high school mathematics teachers should consist of the following:*

(A) Three courses in analysis. (See course-sequence 4.)
(B) Two courses in abstract algebra. (See course-sequence 5.)
(C) Two courses in geometry beyond analytic geometry. (See course-sequence 6.)
(D) Two courses in probability and statistics. (See course-sequence 7.)
(E) Two upper-class elective courses, *e.g.*, introduction to real variables, number theory, topology, history of mathematics, or numerical analysis (including use of high-speed computing machines).

One of these courses should contain an introduction to the language of logic and sets, which can be used in a variety of courses.

RECOMMENDATIONS FOR LEVEL IV

(Teachers of the elements of calculus, linear algebra, probability, etc.)

On this level we recommend a Master's degree with at least two-thirds of the courses being in mathematics, and for which an undergraduate program at least as strong as Level III training is a prerequisite. A teacher who has completed the recommendations for Level III should use the additional mathematics courses to acquire greater mathematical breadth.

Since these teachers will be called upon to teach calculus, we recommend that the program include the equivalent of at least two courses of theoretical analysis in the spirit of the theory of functions of real and complex variables.

It is important that universities have graduate programs available which can be entered with Level III preparation, recognizing that these students substitute greater breadth for lack of depth in analysis as compared with an ordinary B.A. with a major in mathematics. In other respects, graduate schools should have great freedom in designing the M.A. program for teachers.

RECOMMENDATIONS FOR LEVEL V

(College mathematics teachers)

We recognize the tremendous problems created by the shortage of qualified college mathematics teachers. A recommendation for the alleviation of this problem is now receiving serious attention.

* The requirements for Level II preparation have been included in this list.

SUMMARY OF RECOMMENDATIONS

Level	Description	Degree	High School Prerequisites	Minimum Number College Courses
I	Elementary School	B.A.	Two Years of College Preparatory Mathematics	4
II	Elements of Algebra and Geometry	B.A., Mathematics Minor	Preparation for Analytic Geometry and Calculus	7
III	High School	B.A., Mathematics Major	Preparation for Analytic Geometry and Calculus	11
IV	Elements of Calculus, Linear Algebra, Probability, etc.	M.A. in Mathematics	Preparation for Analytic Geometry and Calculus	18 (approx.)

BREAKDOWN BY SUBJECTS

Level	Numbers	Analysis	Algebra	Geometry*	Probability-Statistics	Elective
I	2		1	1		
II		2	1	2	1†	1§
III		2	2	3	2†	2§
IV‡		4	2	3	2	7

* Including analytic geometry.
† An introduction to the language of logic and sets should appear in some one course.
‡ The numbers in this row indicate the approximate number of courses.
§ Preferably from the areas specified.

CURRICULUM-STUDY COURSES

The above recommendations have dealt in detail with the subject-matter training of mathematics teachers. There are many other facets to the education of the scholarly, vigorous, and enthusiastic persons to whom we wish to entrust the education of our youth. One of these merits special mention by us. Effective mathematics teachers must be familiar with such items as:

(A) The objectives and content of the many proposals for change in our curriculum and texts.
(B) The techniques, relative merits, and roles of such teaching procedures as the inductive and deductive approaches to new ideas.
(C) The literature of mathematics and its teaching.

(D) The underlying ideas of elementary mathematics and the manner in which they may provide a rational basis for teaching, unless taken care of by mathematics courses especially designed for teachers.

(E) The chief applications which have given rise to various mathematical subjects. These applications will depend upon the level of mathematics to be taught and are an essential part of the equipment of all mathematics teachers.

Such topics are properly taught in so-called "methods" courses. We would like to stress that adequate teaching of these can be done only by persons who are well informed *both* as to the basic mathematical concepts and as to the nature of American public schools, and as to the concepts, problems, and literature of mathematics education. In particular, we do not feel that this can be done effectively at either the elementary or secondary level in the context of "general" methods courses, or by persons who have not had at least the training of Level IV.

TRAINING OF SUPERVISORS

There is a great need for providing adequately trained supervisors of mathematics, Grades K-12, for our public schools. At present, administrators find no ready supply of such individuals and, hence, are through necessity making appointments which are highly questionable, if not indefensible. For this reason, it is urgent to develop a program for supervisors and to seek adequate support for those individuals who have the desired qualities for supervision and the ability to benefit from advanced training. Such training would prepare the "leaders of teachers" in the local system, (A) to make sound judgments concerning mathematics programs for the schools, (B) to understand thoroughly the recommendations made by national committees, and (C) to enable schools better to articulate school mathematics with college mathematics.

Prerequsite to this program should be a regular Master's degree in mathematics or a Master's degree given as a result of participation in an Academic Year Institute. The program should consist of additional graduate courses in abstract algebra, analysis, and geometry, with courses selected from logic, statistics, theory of numbers, philosophy of education, history of education, history of mathematics, seminar courses on the program of the elementary school and secondary school mathematics, and additional elective courses in algebra, analysis, or geometry to provide some degree of concentration.

The committee feels that action must be taken to fill the need for supervisory personnel, and we recommend such action to the appropriate authorities.

APPENDIX: COURSE DESCRIPTIONS

We list below sample courses that might be used to fulfill the minimum requirements for Levels I through III, and the undergraduate requirements of Levels IV and V. These brief descriptions are included to clarify the meaning of course titles but are not intended as syllabi for actual courses. It must be

recognized that there are other equally good ways of combining various recommended topics, and colleges should be encouraged to work out detailed curricula to suit their own tastes and local conditions. However, the committee hopes that these very brief descriptions will help in indicating the types of courses desirable and the level of advancement.

Level I

1. *Algebraic structure of the number system* (2-course sequence). This is a study of the numbers used in elementary school—whole numbers, common fractions, decimal fractions, irrational numbers.

Emphasis should be on the basic concepts and techniques: properties of addition, multiplication, inverses, systems of numeration, and the number line. The techniques for computation with numbers should be derived from the properties and structure of the number system, and some attention should be paid to approximation. Some elementary number theory, including prime numbers, properties of even and odd numbers, and some arithmetic with congruences should be included.

2. *Algebra.* Basic ideas and structure of algebra, including equations, inequalities, positive and negative numbers, absolute value, graphing of truth sets of equations and inequalities, examples of other algebraic systems—definitely including finite ones—to emphasize the structure of algebra as well as simple concepts and language of sets.

3. *Intuitive foundations of geometry.* A study of space, plane, and line as sets of points, considering separation properties and simple closed curves; the triangle, rectangle, circle, sphere, and the other figures in the plane and space considered as sets of points with their properties developed intuitively; the concept of deduction and the beginning of deductive theory based on the properties that have been identified in the intuitive development; concepts of measurement in the plane and space, angle measurement, measurement of the circle, volumes of familiar solids; treatment of coordinate geometry through graphs of simple equations.

Levels II–V

4. *Analytic geometry and calculus* (3-course sequence). Approximately one-third of the sequence should be devoted to analytic geometry, taught either in coordination with calculus or after the calculus sequence. This should include the coordinate plane, functions, polar coordinates, the algebraic description of subsets of the plane—related to solutions of equations—and parametrically as the range of a function, change of coordinates, and brief treatment of conic sections.

The sequence should also give a thorough treatment of the calculus for functions of one variable, with stress on the basic ideas, but with adequate attention to manipulative skills. The course should introduce differentiation, integration, the rational, trigonometric, and exponential functions, as well as a brief treatment of series and some very elementary differential equations.

5. *Abstract Algebra* (2-course sequence). One course in this sequence constitutes an introduction to algebraic structures, such as groups, rings, fields, etc. The basic approach is to proceed from the concrete to the abstract. Use should be made of algebraic systems familiar to the student in order to motivate the abstract axioms. On the one hand, stress should be placed on rigorous algebraic proofs to convince the student that geometry is not the only area for axiomatic treatment. On the other hand, to keep the abstract procedure tied to the student's experience, various "concrete" applications should be given for theorems. Examples should be drawn from number systems, geometry, and other areas.

The other course should be devoted to linear algebra, restricted to real, finite-dimensional cases. This can be introduced by concrete manipulation of vectors and matrices, after which the student should be motivated to free himself from the accident of the choice of a basis. The student

should be taught the handling of vector equations and inequalities along with an intuitive introduction to linear programming and games. A good treatment of linear functions and transformations is needed, including a thorough understanding of the solution of m equations in n unknowns.

6. *Geometry* (2-course sequence). These recommendations have in general been based on the idea that advanced courses for teachers should be designed in such a way as to deepen understanding of the material which they will be teaching. In geometry, such a program involves special problems, because here some of the appropriate background material is not ordinarily thought of as being geometry at all, and much of it is not ordinarily taught on the undergraduate level.

The foundations of geometry, in the sense of Hilbert, is only one among many topics. Some further examples are as follows:

(1) Generalization of the idea of congruence to include rigid motions, that is, one-to-one correspondences preserving distances.
(2) A corresponding generalization of the idea of similarity.
(3) Enough measure theory to turn the familiar area and volume formulas into theorems, and to justify Cavalieri's Principle.
(4) "Pure analytic geometry," in which points, lines, and so on are defined and treated in terms of a coordinate system, without the use of any synthetic postulates at all. This is quite different from conventional analytic geometry in which the synthetic postulates are used in the very construction of coordinate systems. The "purely analytic" treatment can be used to give a consistency proof for the synthetic postulates.

These topics are given merely as illustrations of the sort of material that is needed. The choice of topics and the order of priority may require considerable study. The course might well take the form of a series of fairly long digressions from an outline of a high school course, with each advanced topic taken up at the point where it seems most relevant.

7. *Probability and statistics* (2-course sequence). The purpose of this sequence is to introduce the student to probability theory from a set-theoretic point of view, and to apply basic probability theory to problems of statistical inference.

The first course should be an introduction to random variables on a finite space. It must include motivation, axiomatic treatment of a measure on a finite space, and the proof of a few key theorems. There should be numerous applications from elementary statistics, stochastic processes, and everyday life.

In the second course more stress should be placed on stochastic processes, and probabilities on a continuous sample space should be treated. A substantial amount of time could be devoted to the development of principles of statistical inference.

Note: One of the course sequences, 4 through 7, should include an introduction to the language of logic and sets, so that these concepts may be used wherever appropriate. This introduction could be restricted to a brief treatment of the propositional calculus and of Boolean algebra, stressing the isomorphism between the two structures.

Electives (For Levels II–V)

8. *Introduction to real variables.*

9. *Number theory.*

10. *Elementary topology.*

11. *History of mathematics.*

12. *Numerical analysis, with the use of machines.*

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

THE DISTRIBUTION FUNCTION FOR EXTREME LUCK

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A coin is tossed repeatedly and at each step the ratio of the number of accumulated heads to tails, H/T, is recorded. Of course, the eventual value of this ratio is 1 (almost always) but before this value is approached, the ratio oscillates quite a bit and allows for a good deal of "luck," *i.e.* large values for H/T. Certain "proofs" of the existence of ESP are based on these large values, with the explanation that the eventual tapering off of H/T to 1 indicated merely that the subject had grown tired!

It is our purpose to investigate the "luck content" of a random sequence of coin tosses, that is to discover the probability distribution function for the derived random variable Max H/T. This random variable is rational valued and >1, almost always, and so the required information is contained in the following theorem.

THEOREM. *Let $a/b>1$ be a fraction in lowest terms, then*

$$\Pr\left[\operatorname{Max}\frac{H}{T}\leqq\frac{a}{b}\right]=\prod_{\rho}(1-\rho),$$

$$\Pr\left[\operatorname{Max}\frac{H}{T}=\frac{a}{b}\right]=\prod_{\rho}(1-\rho)+\tfrac{1}{2}\prod_{\rho}\left(1-\frac{1}{\rho}\right),$$

where the ρ are the zeros of $z^{a+b}-2z^b+1$ which lie in $|z|<1$.

Proof. Define, for each $n\geqq 0$, $p_n=\Pr[H\leqq(aT+n)/b \text{ always}]$.

Now, if we restrict $n\geqq b$ and consider the two alternatives, first toss is a H or first toss is a T, we obtain

$$p_n=\tfrac{1}{2}p_{n-b}+\tfrac{1}{2}p_{n+a},\qquad n\geqq b \tag{1}$$

and, by the same reasoning,

$$p_n=\tfrac{1}{2}p_{n+a}\qquad 0\leqq n<b. \tag{2}$$

Also, let us note that, by the strong law of large numbers, $p_n\to 1$ as $n\to\infty$.

If we introduce the generating function $f(x)=\sum_{n=0}^{\infty}p_nx^n$ we can state

$$f(x) \text{ is analytic for } |x|<1 \text{ and } (1-x)f(x)\to 1 \text{ as } x\to 1. \tag{3}$$

Now form $(1+x^{a+b}-2x^a)f(x)$ and consider the coefficient of x^n. For $n<a$ this is simply p_n; for $a\leqq n<a+b$, the coefficient is p_n-2p_{n-a} which is 0 by (2), and

for $n \geqq a+b$, the coefficient is $p_n+p_{n-a-b}-2p_{n-a}$ which is 0 by (1).

Thus we have

$$(1 + x^{a+b} - 2x^a)f(x) = \sum_{n<a} p_n x^n. \tag{4}$$

Now

$$(1 + x^{a+b} - 2x^a) = (1 - x) \prod_{\rho} (1 - \rho x) \prod_{\sigma} (1 - \sigma x),$$

where the ρ are the zeros of $z^{a+b}-2z^b+1$ which lie in $|z| < 1$, and the σ are the zeros of this same polynomial which lie in $|z| > 1$.

Applying Rouche's theorem to the equation $z^{a+b}+1=2z^b$, however, we see that the number of ρ is b. Hence the number of σ is $a-1$.

Since f is analytic for $|x| < 1$ by (3), we see that (4) implies that $\sum_{n<a} p_n x^n$ vanishes at all the points $1/\sigma$ and, since their number is $a-1$, we must have

$$\sum_{n<a} p_n x^n = C \prod (1 - \sigma x), \tag{5}$$

where C is a constant.

Combining (4) and (5) gives $(1-x)f(x)\cdot\prod(1-\rho x) = C$ and letting $x \to 1$, we obtain, by (3), $C=\prod(1-\rho)$ so that (5) becomes

$$\sum_{n<a} p_n x^n = \prod (1 - \rho) \prod (1 - \sigma x), \tag{6}$$

from which we read off

$$\Pr\left[\operatorname{Max} \frac{H}{T} \leqq \frac{a}{b}\right] = \Pr\left[H \leqq \frac{aT}{b} \text{ always}\right] = p_0 = \prod (1 - \rho) \tag{7}$$

and also $p_{a-1}=\prod(1-\rho)\prod(-\sigma)$. But since $-\prod(-\rho)\prod(-\sigma)=1$ this gives $p_{a-1} = -\prod(1-1/\rho)$. However,

$$\Pr\left[\frac{H}{T} < \frac{a}{b} \text{ always}\right] = \Pr\left[\text{first toss is } T\right]\cdot p_{a-1} = \tfrac{1}{2}p_{a-1}$$

and so, by the above and (7), we have

$$\Pr\left[\operatorname{Max} \frac{H}{T} = \frac{a}{b}\right] = \prod (1 - \rho) + \tfrac{1}{2} \prod \left(1 - \frac{1}{\rho}\right). \tag{8}$$

(7) and (8), however, are exactly what was to be proven. In the special case where a/b is an integer, *i.e.*, $b=1$, there is only one ρ and the result is

$$\Pr\left[\operatorname{Max} \frac{H}{T} \leqq a\right] = 1 - \rho, \quad \Pr\left[\operatorname{Max} \frac{H}{T} = a\right] = (1 - \rho)\left(1 - \frac{1}{2\rho}\right),$$

where ρ is the unique zero of $z^{a+1}-2z+1$ inside $|z| < 1$. The particular case $a=2$ leads to

$$\Pr\left[\operatorname{Max}\frac{H}{T} \leqq 2\right] = \frac{3-\sqrt{5}}{2} \quad (=.381966\cdots)$$

$$\Pr\left[\operatorname{Max}\frac{H}{T} = 2\right] = \frac{7-3\sqrt{5}}{4} \quad (=.072949\cdots).$$

Also

$$\Pr\left[\operatorname{Max}\frac{H}{T} \leqq 3\right] = .4563\cdots, \qquad \Pr\left[\operatorname{Max}\frac{H}{T} = 3\right] = .03668\cdots.$$

Certain odd by-products are contained in this theorem. First of all it follows that $\prod(1-\rho)>0$ and $\prod(1-\rho)<1$ (in fact $\prod(1-\rho)<\frac{1}{2}$). Also $\prod(1-1/\rho)<0$ and $\prod|\rho|>\frac{1}{2}$. Quite surprising is the fact that $\prod(1-\rho)$ is discontinuous as a function of a/b; in fact, from our theorem

$$\lim_{a/b\to 2}\prod(1-\rho) = \frac{\sqrt{5}-1}{4} \quad \text{while} \quad \prod(1-\rho) = \frac{3-\sqrt{5}}{2}$$

for $a/b=2$. Certain identities also follows, such as

$$\sum_{\text{all } a/b}\left[\prod(1-\rho)+\frac{1}{2}\prod\left(1-\frac{1}{\rho}\right)\right] = \frac{1}{2},$$

$$\sum_{a/b<2}\left[\prod(1-\rho)+\frac{1}{2}\prod\left(1-\frac{1}{\rho}\right)\right] = \frac{\sqrt{5}-1}{4}, \text{ etc.}$$

Finally, it is not at all independently obvious that $\prod(1-\rho)$ is an increasing function of a/b, a fact which our theorem insures.

NOTE ON THE NORMAL DEFORMATION OF A SURFACE

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1. Let

$$x_i = x_i(u^1, u^2, t) \tag{1}$$

represent a moving surface $\sum(t)$, where u^α are the curvilinear cordinates of the surface $\sum(t)$, t denotes time and x_i are allowable rectangular cartesian coordinates. Let $g_{\alpha\beta}$, $b_{\alpha\beta}$ be the coefficients of the first and second fundamental forms respectively. Let ν denote the unit surface normal and G denote the surface velocity in the direction of unit normal ν. In studying surfaces of discontinuity in continuum mechanics, Thomas [1] established a formula for the δ time derivative of surface normal ν, namely

$$\frac{\delta \nu_i}{\delta t} = -g^{\alpha\beta}G_{,\alpha}x_{i,\beta}. \tag{2}$$

He also observed that this formula should be derivable by purely geometric considerations in view of the expressions on the right hand side of (2). Using the same notation as he used, and the definition for δ time derivative as defined in his paper, this note presents a simple derivation of (2) and establishes the formulas for the δ time derivative for $g_{\alpha\beta}$ and $b_{\alpha\beta}$.

The author thanks the referee for pointing out that the three formulas derived in this note are special cases of the general equations of the deformation theory for V_m in V_n ([2], [3], [4]), where V denotes a Riemannian manifold. However, the elementary derivations of the present formulas do not require the heavy machinery of the general deformation theory and perhaps will help in the understanding of the general theory.

2. Derivation of the formula (2). It is assumed that the x_i given by (1) are twice continuously differentiable with respect to time t. Furthermore the differentiability of x_i with respect to u^α will be assumed according to necessity. Let $\delta t > 0$ be sufficiently small, then for any fixed time t, the moving surface $\sum(t+\delta t)$ at time $t+\delta t$ can be represented by

$$\hat{x}_i = x_i + Gv_i\delta t + O(\delta t)^2. \tag{3}$$

Differentiating this equation partially with respect to u^α, we have

$$\hat{x}_{i,\alpha} = x_{i,\alpha} + (G_{,\alpha}v_i + Gv_{i,\alpha})\delta t + O(\delta t)^2. \tag{4}$$

Let $\hat{v}_i$ denote the unit surface normal of $\sum(t+\delta t)$ at $\hat{x}_i$. Multiplying (4) by $\hat{v}_i$ and summing on the index i, we obtain

$$\hat{v}_i\hat{x}_{i,\alpha} = \hat{v}_i\{x_{i,\alpha} + (G_{,\alpha}v_i + Gv_{i,\alpha})\delta t + O(\delta t)^2\} = 0.$$

Also $v_ix_{i,\alpha} = 0$.

Subtracting the last equation from the preceding one, dividing the resulting equation by δt and letting δt approach zero, we find

$$\begin{aligned} \lim_{\delta t\to 0} \frac{\hat{v}_i - v_i}{\delta t} x_{i,\alpha} &= -\lim_{\delta t\to 0}\{\hat{v}_i(G_{,\alpha}v_i + Gv_{i,\alpha}) + O(\delta t)\}, \\ &= -v_i(G_{,\alpha}v_i + Gv_{i,\alpha}) = -G_{,\alpha}, \end{aligned} \tag{5}$$

in view of the fact that $\lim_{\delta t\to 0}\hat{v}_i = v_i$ and $v_iv_{i,\alpha} = 0$. By definition of δ time derivative, we can write (5) as

$$\frac{\delta v_i}{\delta t} x_{i,\alpha} = -G_{,\alpha}. \tag{6}$$

It is known [5] that

$$g^{\alpha\beta}x_{i,\alpha}x_{j,\beta} = \delta_{ij} - v_iv_j. \tag{7}$$

Multiplying both sides of (6) by $g^{\alpha\beta}x_{j,\beta}$ and summing on the index α, we obtain

$$\frac{\delta \nu_i}{\delta t} g^{\alpha\beta} x_{i,\alpha} x_{j,\beta} = - g^{\alpha\beta} G_{,\alpha} x_{j,\beta}.$$

Making use of (7), we can write this equation as

$$\frac{\delta \nu_i}{\delta t} (\delta_{ij} - \nu_i \nu_j) = - g^{\alpha\beta} G_{,\alpha} x_{j,\beta}.$$

That is

$$\frac{\delta \nu_i}{\delta t} = - g^{\alpha\beta} G_{,\alpha} x_{i,\beta}.$$

3. The δ time derivative of $g_{\alpha\beta}$ and $b_{\alpha\beta}$. Parallel to what has been done in Section 2, we can readily derive the formulas for δ time derivatives of $g_{\alpha\beta}$ and $b_{\alpha\beta}$ as follows:

Changing α to β in (4), we have

$$\hat{x}_{i,\beta} = x_{i,\beta} + (G_{,\beta} \nu_i + G \nu_{i,\beta}) \delta t + O(\delta t)^2. \tag{8}$$

Multiplying the corresponding sides of (4) and (8) and summing on the repeated index i, we obtain

$$\hat{x}_{i,\alpha} \hat{x}_{i,\beta} = x_{i,\alpha} x_{i,\beta} + (x_{i,\alpha} G \nu_{i,\beta} + x_{i,\beta} G \nu_{i,\alpha}) \delta t + O(\delta t)^2.$$

By definition of $\delta g_{\alpha\beta}/\delta t$, it follows that

$$\begin{aligned} \frac{\delta g_{\alpha\beta}}{\delta t} &= \lim_{\delta t \to 0} \frac{\hat{x}_{i,\alpha} \hat{x}_{i,\beta} - x_{i,\alpha} x_{i,\beta}}{\delta t} \\ &= \lim_{\delta t \to 0} \{ x_{i,\alpha} G \nu_{i,\beta} + x_{i,\beta} G \nu_{i,\alpha} + O(\delta t) \} \\ &= G (x_{i,\alpha} \nu_{i,\beta} + x_{i,\beta} \nu_{i,\alpha}) = - 2 b_{\alpha\beta} G, \end{aligned}$$

where we use $b_{\alpha\beta} = - x_{i,\alpha} \nu_{i,\beta}$.

For simplicity, we denote $\partial^2 x_i / \partial u^\alpha \partial u^\beta$ by $x_{i,\alpha\beta}$, $\partial^2 G / \partial u^\alpha \partial u^\beta$ by $G_{,\alpha\beta}$ and $\partial^2 \nu_i / \partial u^\alpha \partial u^\beta$ by $\nu_{i,\alpha\beta}$. Now differentiating (4) partially with respect to u^β, we get

$$\hat{x}_{i,\alpha\beta} = x_{i,\alpha\beta} + (G_{,\alpha\beta} \nu_i + G_{,\alpha} \nu_{i,\beta} + G_{,\beta} \nu_{i,\alpha} + G \nu_{i,\alpha\beta}) \delta t + O(\delta t)^2.$$

Multiplying both sides of this equation by $\hat{\nu}_i$ and summing on i, we find

$$\hat{x}_{i,\alpha\beta} \hat{\nu}_i = x_{i,\alpha\beta} \hat{\nu}_i + \hat{\nu}_i (G_{,\alpha\beta} \nu_i + G_{,\alpha} \nu_{i,\beta} + G_{,\beta} \nu_{i,\alpha} + G \nu_{i,\alpha\beta}) \delta t + O(\delta t)^2. \tag{9}$$

Since $b_{\alpha\beta} = x_{i,\alpha\beta} \nu_i$, we can write

$$\begin{aligned} \frac{\delta b_{\alpha\beta}}{\delta t} &= \lim_{\delta t \to 0} \frac{\hat{x}_{i,\alpha\beta} \hat{\nu}_i - x_{i,\alpha\beta} \nu_i}{\delta t} \\ &= \lim_{\delta t \to 0} \frac{(\hat{x}_{i,\alpha\beta} \hat{\nu}_i - x_{i\,\alpha\beta} \hat{\nu}_i) + (x_{i,\alpha\beta} \hat{\nu}_i - x_{i,\alpha\beta} \nu_i)}{\delta t} \end{aligned}$$

$$= \lim_{\delta t \to 0} \frac{\hat{x}_{i\,\alpha\beta}\hat{\nu}_i - x_{i,\alpha\beta}\hat{\nu}_i}{\delta t} + x_{i,\alpha\beta}\frac{\delta \nu_i}{\delta t}.$$

Using the relation (9), the last equation can be written as

$$\frac{\delta b_{\alpha\beta}}{\delta t} = \lim_{\delta t \to 0} \{\hat{\nu}_i(G_{,\alpha\beta\nu_i} + G_{,\alpha}\nu_{i,\beta} + G_{,\beta}\nu_{i,\alpha} + G\nu_{i,\alpha\beta}) + O(\delta t)\} + x_{i,\alpha\beta}\frac{\delta \nu_i}{\delta t}$$

$$= G_{,\alpha\beta} + G\nu_i\nu_{i,\alpha\beta} - x_{i,\alpha\beta}g^{\lambda\mu}G_{,\lambda}x_{i,\mu}.$$

Finally we note that the δ time differentiation satisfies the usual rules, so we can readily find an expression for $\delta g^{\alpha\beta}/\delta t$ by differentiating $g^{\alpha\beta}g_{\beta\gamma} = \delta^{\beta}_{\gamma}$ and $\delta\Omega/\delta t$ by differentiating $g^{\alpha\beta}b_{\alpha\beta} = \Omega$, where Ω is called mean curvature of surface.

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TWO REMARKS ON THE REGULAR REPRESENTATIONS OF AN ALGEBRA

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The note by A. J. Goldman [1] illustrates very well the essential disadvantage of too great adherence to the "basis" point of view in elementary algebra. The nonbasis approach (*e.g.* as in [2] p. 82) can be considerably clearer. If A is an algebra over k (which need not be finite dimensional) and θ is the left regular representation of A in the full algebra $\mathfrak{L}(A)$ of all linear transformations on the vector space $\langle A, +\rangle$ which sends an element $a \in A$ into $\theta(a) = U_a$, the transformation defined by $U_a(x) = ax$, then it is at once apparent that θ is a homomorphism of A onto a subalgebra $\overline{A} \subset \mathfrak{L}(A)$. Its kernel is then the ideal $J \subset A$ consisting of those elements a for which $U_a = 0$. Clearly, $U_a = 0$ if and only if $ax = 0$ for all $x \in A$; hence, θ is faithful [an isomorphism] if and only if annih$[A] = \{0\}$, the result obtained by Goldman.

A related question is that of giving an algebraic characterization for the image $\overline{A}$ within the full transformation algebra $\mathfrak{L}(A)$. It is clear that each of the operators U_a obeys the special identity:

$$(*) \qquad T(xy) = T(x)y \qquad \text{all } x, y \text{ in } A.$$

Moreover, the collection of all T that obey (*) forms a subalgebra $\mathfrak{A}$ of $\mathfrak{L}(A)$. Since $\mathfrak{A}$ contains the identity operator I and $\overline{A}$ is a left ideal in $\mathfrak{A}$, it is at once clear that $\overline{A}$ coincides with $\mathfrak{A}$ if and only if $\overline{A}$ contains I, and that this in turn is equivalent to A having some element that acts as a left unit. Of course, if

A has a two sided unit, then annih$[A]=\{0\}$ and A is isomorphic to $\overline{A}$ and hence to $\mathfrak{A}$. However, A can have a left unit, and still have annih$[A]\neq\{0\}$. When this happens, the *right* regular representation of A in $\mathfrak{L}(A)$ will be faithful, although the left is not. An example is:

$$A = \text{all}\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad x, y \in k.$$

When is the algebra $\mathfrak{A}$ properly larger than $\overline{A}$? If, for example, A is a left ideal in some algebra B, and b is any element of B, then $T(x)=bx$ defines a linear transformation on A which obeys the identity (*) but which need not arise as a multiplication operator U_a for some $a\in A$. It is natural to ask if this is typical.

THEOREM. *Given any algebra A with its associated algebra $\mathfrak{A}$, it is always possible to embed A as a left ideal in some algebra B so that every transformation T on A which obeys (*) arises by multiplication by an element of B.*

Proof. When θ is faithful, A is isomorphic to $\overline{A}$ and B can be chosen as $\mathfrak{A}$ itself. In the general case, a less pleasing solution can be obtained as follows. Let B consist of all finite sums of the form $T_0+a_1T_1+\cdots+a_rT_r$ where the $a_i\in A$ and $T_i\in\mathfrak{A}$. This defines addition in B. We define multiplication in B by:

$$(T_0)(aT) = T_0(a)T \qquad (aT)(T_0) = aTT_0 \qquad (a_1T_1)(a_2T_2) = a_1T_1(a_2)T_2.$$

It is easily seen that, because of (*), this is in fact an associative operation. B thus becomes an algebra; we embed A in B by identifying a with aI. A is then a left ideal in B, and any operator T on A that obeys (*) arises by multiplication by some element of B, namely T itself. (It would be interesting to have a more useful construction of a smaller and more tractable B.)

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NOTE ON A DIRECT LIMIT GROUP

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Associated naturally with a group G and an endomorphism f of G is the limit group of the constant direct sequence

$$G \xrightarrow{f} G \xrightarrow{f} \cdots \xrightarrow{f} G \xrightarrow{f} \cdots.$$

We denote this group by (G, f) and describe it for completeness. The elements of (G, f) are the indexed sequences of the form $(g_1, \cdots, g_n, \cdots)_j$ where $j\geqq 0$, $g_n\in G$ and $g_{n+1}=f(g_n)$ for all n. Two elements $x=(g_1, \cdots, g_n, \cdots)_j$ and $y=(g_1', \cdots, g_n', \cdots)_k$ are equal in (G, f) if and only if $g_{n+k}=g'_{n+j}$ for some

(and consequently all but a finite number of) n. Multiplication in (G, f) is defined by

$$xy = (h_1, \cdots, h_n, \cdots)_{j+k},$$

where $h_n = g_{n+k}g'_{n+j}$.

In this note we establish a strong relationship between G and (G, f) if G satisfies certain finiteness conditions. An example is given which indicates that without these finiteness conditions imposed on G the group (G, f) may be very loosely related to G. Finally, the groups (G, f) are described, where G is a finitely generated abelian group and f ranges over a wide class of endomorphisms of G.

Defining $G'_1 = G$ and inductively defining $G'_{i+1} = f(G'_i)$, we have

THEOREM 1. *If there exists an integer m such that $G'_{m+1} = G'_m$, then (G, f) is a homomorphic image of G.*

Proof. Clearly, the mapping h of G into (G,f), defined by $g \to (g, f(g), f^2(g), \cdots)_0$ is a homomorphism. In order to show that h is onto, let $x = (g_1, \cdots, g_n, \cdots)_j$ be an arbitrary element of (G, f). The relation $G'_m = G'_{m+1}$ implies that $G'_m = G'_{m+j}$. From the definition of the G'_i, we have $f^i(G) = G'_{i+1}$. Obviously, $g_m \in G'_m$. Thus

$$g_m \in G'_m = G'_{m+j} = f^{m+j-1}(G),$$

which implies that there exists $g \in G$ such that $f^{m+j-1}(g) = g_m$. It follows that $h(g) = x$, and the theorem is proved.

COROLLARY. *If G satisfies the minimal condition for subgroups, then (G, f) is a homomorphic image of G.*

THEOREM 2. *If G satisfies the minimal condition for subgroups and the maximal condition for normal subgroups (in particular, if G is finite), then (G, f) is a subgroup of G.*

Proof. From the proof of Theorem 1, h is a homomorphism of G onto (G, f). But x is in the kernel of h if and only if there exists an integer i such that $f^i(x) = e$ (the identity of G), that is, if and only if x is in the kernel of f^i for some i. Letting K_i denote the kernel of f^i, the kernel of h is $\bigcup_{i=1}^{\infty} K_i$. Since $K_i \subseteq K_{i+1}$, by the maximal condition $\bigcup_{i=1}^{\infty} K_i = K_n$ for some n. Thus (using equality in the sense of isomorphism) we have $(G, f) = G/K_n = f^n(G)$, a subgroup of G.

THEOREM 3. *If G is abelian and satisfies the hypothesis of Theorem 2, then (G, f) is a direct summand of G.*

Proof. Using the notation of the previous theorem, there exists an integer m such that $f^m(G) = f^{m+1}(G) = f^{2m}(G)$ and $K_m = K_{m+1}$. Since $f^m(G) \cap K_m = 0$, there is a subgroup H of G defined by $H = f^m(G) \oplus K_m$. If $g \in G$, there exists $g' \in G$ such that $f^{2m}(g') = f^m(g)$. Let $g = f^m(g') + x$. Then $f^m(x) = 0$, which implies that $g \in H$ and that $H = G$. But $(G, f) = G/K_m = f^m(G)$, a direct summand of G.

The following example shows that the above theorems do not hold in general (even if G satisfies the maximal condition for subgroups).

Example. If $G=\{g\}$ is the infinite cyclic group and if f is a nontrivial endomorphism of G defined by $f(g)=ng$, then (G, f) is the n-adic fractions, that is, the additive group of fractions of the form i/n^j, where $j\geqq 1$ and i is arbitrary. The isomorphism involved is: $i/n^j \rightarrow (ig, nig, n^2ig, \cdots)_{j-1}$.

DEFINITION. *An endomorphism f of the free abelian group F_r of rank r is regular if there exist a basis $g_1, \cdots, g_r$ and integers $n_1, \cdots, n_r$ such that $f^n(g_i)=n_ig_i$ $(i=1, 2, \cdots, r)$ for some positive integer n. And an endomorphism f of a finitely generated abelian group $G=T\oplus F$, where T is the torsion part of G, is regular if the summand F of G can be chosen such that f induces a regular endomorphism of F.*

Using the above example we now prove

THEOREM 4. *Let $G=T\oplus F_r$ be a finitely generated abelian group, where T is the torsion part of G and F_r is the free abelian group of rank r. There exists a regular endomorphism f of G such that $(G, f)=\overline{G}$ if and only if $\overline{G}=\overline{T}\oplus\overline{F}$, where $\overline{T}$ is a direct summand of T and $\overline{F}$ is the (abstract) direct sum of not more than r groups of n-adic fractions for various integers n.*

Proof. In view of the above example and the general fact that the direct product $(G_1, f_1)\otimes(G_2, f_2)$ is the same as $(G_1\otimes G_2, f_1\otimes f_2)$ the "if" part of the theorem is obvious. In order to prove the "only if" part, let f be a regular endomorphism of G. Then f induces an endomorphism of T which we will again denote by f. It is immediate that $\overline{T}=(T, f)$ is the torsion part of (G, f). By Theorem 3, $\overline{T}$ is a direct summand of T. Since $\overline{T}$ is finite, we know* that $\overline{T}$ is a direct summand of (G, f). Letting $(G, f)=\overline{T}\oplus\overline{F}$, by the natural identification, we obtain $\overline{F}=(G', f')$, where $G'=G/T$ and f' is the regular endomorphism of G', defined by: $f'(g+T)=f(g)+T$. Now letting $G'=\{g_1\}\oplus\{g_2\}\oplus\cdots\oplus\{g_r\}$ and $f'^n(g_i)=n_ig_i$, we have that the kernel K' of f'^n is generated by those elements g_i for which $n_i=0$. Thus if the subgroup H' is generated by those elements g_i for which $n_i\neq 0$, then we have that $G'=K'\oplus H'$. But in general it can be shown that $(G, f)=(G, f^n)$ for any positive integer n. Therefore $\overline{F}=(G', f')=(G', f'^n)=(H', f'^n)$, and it follows (from the above example) that $\overline{F}$ is the (abstract) direct sum of the n_i-adic fractions for those i for which $n_i\neq 0$.

ON THE FINAL DIGITS OF SQUARES

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If n digits are chosen at random, there is a certain probability, $P(n)$, that they are the final n digits of a perfect square. One might ask, "Can $P(n)$ be made as small as we please by taking n sufficiently large?" This seems plausible,

* See, for example, A. Kurosh, Theory of Groups, Vol. 1, New York, 1955, p. 202.

n	$R(10^n)$	$P(n)$
1	6	.60000
2	22	.22000
3	159	.15900
4	1044	.10440
5	9121	.09121
6	78132	.07813
7	748719	.07487
8	7161484	.07161
9	70800861	.07080
10	699869892	.06999

It is evident that the same approach can be used if numerals are written to a base m other than 10; the limiting probability, $P(\infty)$, can be expressed simply in terms of the primes that divide m. If m is divisible by, at most, r distinct primes, then $P(\infty) \geqq (\frac{1}{6})(\frac{3}{8})^{r-1}$, with equality only when $m=6$.

Reference

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THE ISOPERIMETRIC PROBLEM IN THE MINKOWSKI PLANE

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Let U be a strictly convex centrally symmetric closed curve bounding an area of π square units and with center at the origin in the euclidean plane. If P and Q are two points in the plane, we denote by $|PQ|_e$ the euclidean distance between P and Q. Let $2r$ be the length of the diameter of U parallel to the line PQ. Then the minkowskian distance $|PQ|$ between P and Q is defined as:

$$|PQ| = \frac{|PQ|_e}{r}.$$

It is easily verified that this defines a metric. The geometry obtained using this metric is called *minkowskian geometry*. See [1].

The curve U serves as the unit circle in this new geometry, and we denote by rU and call "a circle of radius r", any translate of a homothecy of U by the factor r.

It turns out that the "circle" is not in general a solution to the minkowskian isoperimetric problem; *i.e.*, among all closed curves of the same minkowskian perimeter, to find that enclosing the greatest area. It is well known that the solution to this problem is a curve homothetic to the "isoperimetrix" T, which is defined as follows: Let $r=r(\theta)$ be the polar equation of U. Then the isoperimetrix T is the closed convex curve with support function $p(\theta)=1/r(\theta+\frac{1}{2}\pi)$.

(In other words, T is the polar reciprocal of U, with respect to the euclidean unit circle, rotated through 90°.)

Consider for a moment a convex n-gon K in the euclidean plane. Let K^* be the n-gon which is circumscribed about the unit circle and whose sides are respectively parallel to the sides of K. Let L = perimeter of K, F = area enclosed by K, f = area enclosed by K^*. Then we have Lhuilier's inequality:

$$L^2 - 4fF \geqq 0.$$

A proof of this is given in [2], page 10. We show how that proof generalizes, virtually without change, to give the following.

THEOREM. *Let K be a convex n-gon in the minkowskian plane with unit circle U. Let K^* be the n-gon which is circumscribed about the isoperimetrix T and whose sides are respectively parallel to the sides of K. Let L = minkowskian perimeter of K, F = area of K, f = area of K^*, and r = radius of the largest anti-circle* (defined below) *contained in K. Then*

$$L^2 - 4fF \geqq 0$$

with equality holding if and only if K is circumscribed about an anti-circle of radius r.

Proof. We denote by sT and call "an anti-circle of radius s" any translate of a homothecy of T by the factor s.

We define now "the inner parallel polygon K_s to K at distance s." If b and c are two edges of K meeting in vertex A, then we may consider b and c as two support lines of an anti-circle of radius s: simply translate sT until it "fits" into the corner at A. Let A' be the center of this anti-circle sT. A' will be a vertex of a new polygon K_s whose other vertices are obtained by performing the same construction at the other vertices of K, using the same s. If we take s small enough, K_s will be a convex polygon inside and "parallel" to K and having the same number of sides as K. Let us call the ray AA' the "bisector" of the vertex A. Now let s increase steadily. At some stage, corresponding to a "critical" value s_0 of s, a pair of bisectors will meet. When this happens we obtain an inner parallel polygon K_{s_0} with fewer sides than K. We now use this polygon in place of K, building successive inner parallel polygons until another degeneracy occurs. This process continues until we are left with a point or a segment. We label the inner parallel polygons in the obvious fashion, letting $K_0 = K$, and K_r = the final point or segment. K_r is in fact the locus of the centers of the largest anti-circles contained in K.

Let $0 \leqq s_2 < s_1 \leqq r$ and let $s = s_1 - s_2$ be small enough so that K_{s_1} and K_{s_2} have the same number of sides. Let F_i = area enclosed by K_{s_i}, L_i = minkowskian perimeter of K_{s_i}, f_i = area enclosed by $K^*_{s_i}$, r_i = radius of largest anti-circle contained in K_{s_i} for $i = 1, 2$. One sees easily, keeping in mind the definition of sT, that:

$$F_2 = F_1 + L_1 s + f_1 s^2, \qquad L_2 = L_1 + 2f_1 s, \qquad r_2 = r_1 + s, \qquad f_2 = f_1.$$

Therefore:

$$L_2r_2 - F_2 - f_2r_2^2 = L_1r_1 - F_1 - f_1r_1^2.$$

Hence if we let F_s, L_s, r_s, and f_s be the area, length, and so forth corresponding to K_s, we see that $L_sr_s - F_s - f_sr_s^2$ is constant between the "critical" values of s, where the only change is that f_s increases, since the number of sides of K_s decreases. Thus $L_sr_s - F_s - f_sr_s^2$ is a decreasing function of s, $0 \leqq s \leqq r$, which is equal to $Lr - F - fr^2$ for $s=0$ and is equal to 0 for $s=r$. Hence $Lr - F - fr^2 \geqq 0$. This implies: $L^2 - 4fF \geqq 4f^2(L/2f - r)^2 \geqq 0$.

In case of equality $L = 2fr$, and one concludes easily that K is circumscribed about an anti-circle of radius r.

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CLASSROOM NOTES

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A SIMPLE PROOF OF A THEOREM CONCERNING REFLEXIVITY

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It is the purpose of this note to give a very simple proof of the following theorem:

THEOREM. *Let X be a Banach space and let its conjugate space X' be reflexive. Then X is reflexive.*

The proof makes use only of a corollary of the Hahn-Banach extension theorem.

Proof of the theorem. Let J be the canonical mapping of X into its second conjugate space X''. Since X is complete and J is a linear homeomorphism, JX is a closed subspace of X''. Suppose the theorem is false, *i.e.* $JX \neq X''$. Then there exist elements x''' in X''' and x'' in X'' such that $x'''(JX) = 0$ and $x'''(x'') \neq 0$. Since X' is reflexive, there exists an x' in X' such that $Jx' = x'''$ and $0 = x'''(JX) = JX(x') = x'X$. Hence $x' = 0$. But $0 \neq x'''(x'') = x''(x')$ which leads to a contradiction.

In order to interpret Theorem 2 geometrically, we differentiate (4) and apply Rolle's theorem to obtain

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{U}'(T) = 0, \qquad a < T < b \tag{5}$$

which, as is readily verified by expanding, is simply (3) written in vector form. The vector $\mathbf{U}'(T)$ lies along the tangent to C at a point S: $[x(T), y(T), z(T)]$ distinct from M and N (Fig. 3), and is perpendicular to the vector $\mathbf{A}\times\mathbf{B}$. But, $\mathbf{A}\times\mathbf{B}$ is perpendicular, in turn, to both $\mathbf{A}$ and $\mathbf{B}$. Consequently, $\mathbf{U}'(T)$ must, in general, be parallel to the plane of $\mathbf{A}$ and $\mathbf{B}$. In fact, by applying a similar analysis to the vectors $\mathbf{A}-\mathbf{K}$, $\mathbf{B}-\mathbf{K}$, and $\mathbf{U}-\mathbf{K}$, where $\mathbf{K}$ is an arbitrary constant vector, it can be shown that corresponding to any plane through the points M and N there is at least one point on C, distinct from M and N, where the tangent line to C is parallel to the given plane.

These observations reveal that even more general laws of the mean are generated when Rolle's theorem is applied to determinants of higher order than three, and that these determinants may be readily motivated by an analysis of the content of parallelotopes in n-dimensions.

Reference

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A SIMPLE PROOF OF THE ARITHMETIC MEAN, GEOMETRIC MEAN INEQUALITY

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The inequality in question is

$$A_n \equiv \frac{\alpha_1 + \cdots + \alpha_n}{n} \geqq (\alpha_1 \cdots \alpha_n)^{1/n} \equiv G_n, \tag{1}$$

where the α_i are all positive. This is true for $n=2$. Assume its truth for $n=m$. Then $A_m \geqq G_m$. Also

$$A \equiv \frac{\alpha_{m+1} + (m-1)A_{m+1}}{m} \geqq (\alpha_{m+1} A_{m+1}^{m-1})^{1/m} \equiv G.$$

Hence

$$A_{m+1} = \frac{A_m + A}{2} \geqq (A_m A)^{1/2} \geqq (G_m G)^{1/2} = (G_{m+1}^{m+1} A_{m+1}^{m-1})^{1/2m},$$

and so $A_{m+1} \geqq G_{m+1}$. The inequality (1) follows by induction.

This proof, which is a modification of the celebrated Cauchy proof, may not be new; but I have not come across it anywhere.

AN UNDERGRADUATE APPROACH TO A BOUNDARY VALUE PROBLEM

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It is unfortunate that advanced calculus texts usually treat boundary value problems in partial differential equations either superficially or not at all. Current activity in the field demonstrates that these problems are ideally suited to study with a variety of tools. The fact that they present, in situations of physical interest, applications of the techniques usually developed in a good course in advanced calculus suggests that a treatment more adequate than the usual one would be very appropriate to the immediate goals of the course. It would also give an inkling of one of the currently active fields of research.

As a specific illustration, consider the (vaguely defined) heat problem, of discovering a function $u(x, t)$ defined for $0 \leqq x \leqq 2\pi$ and $t \geqq 0$, with $u(x, 0) = f(x)$ (a prescribed function), $u(0, t) = u(2\pi, t)$, and $\partial^2 u/\partial x^2 = \partial u/\partial t$; this gives the heat distribution in an insulated ring. The methods used in analyzing this will involve a few standard facts about infinite series, differentiation under the integral, Schwarz' inequality, Parseval's equality for Fourier series, a simple criterion for the absolute convergence of such series, and Minkowski's (triangle) inequality for $p = 2$; the integral implied in these can be that of Riemann or Lebesgue.

A reasonable first step would be to develop the usual series solution in terms of the (complex) Fourier coefficients c_k of f,

$$u(x, t) = \sum_{-\infty}^{\infty} c_k e^{ikx} e^{-k^2 t}, \tag{I}$$

motivated, perhaps, as a sum of product solutions $X(x)T(t)$. The interesting part so often neglected is the study of this function, partly to find out in what sense it solves the original problem. The rest of the article concerns itself with properties of $u(x, t)$, under the assumption that $\int_0^{2\pi} f^2 dx < \infty$.

The solution given by (I) satisfies

$$\lim_{t \to 0} \int_0^{2\pi} | u(x, t) - f(x) |^2 dx = 0; \tag{1}$$

this shows in what sense $u(x, 0) = f(x)$. For the proof,

$$\int_0^{2\pi} | u - f |^2 = 2\pi \sum_{-\infty}^{\infty} | c_k |^2 [1 - e^{-k^2 t}]^2.$$

The fact that this sum converges to zero as $t \to 0$ can be shown directly, or by appeal to some convergence theorem.

(2) When $t > 0$, $u(x, t)$ is continuous, periodic in x, has continuous derivatives of all orders, and $\partial^2 u/\partial x^2 = \partial u/\partial t$; for the series in (I) can be differentiated indefinitely term-by-term, when $t > 0$. The fact that the equation is not neces-

sarily satisfied at $t=0$ is not surprising, since it is not known if f is even differentiable.

(3) There is only one solution with properties (1) and (2); in fact, not all the properties in (2) will be used in the proof of uniqueness. This proof is based on the intuitive feeling that the temperature in the whole ring, as given by some convenient measure, should steadily level off. To give it concisely, let u and v be solutions satisfying conditions (1) and (2), and $w=u-v$. Then

$$\left[\int_0^{2\pi} w(x,t)^2 dx\right]^{1/2} \leqq \left[\int_0^{2\pi} |u-f|^2\right]^{1/2} + \left[\int_0^{2\pi} |f-v|^2\right]^{1/2} \to 0$$

as $t\to 0$; thus w satisfies conditions (1) and (2) with initial value 0. Then for $t>0$

$$D_t \int_0^{2\pi} w^2(x,t)dx = 2\int_0^{2\pi} w\partial w/\partial t dx = 2\int_0^{2\pi} w\partial^2 w/\partial x^2 dx = -2\int_0^{2\pi} (\partial w/\partial x)^2 dx \leqq 0.$$

So as t decreases to 0, $\int w^2$ "increases" to zero; since it is nonnegative, of course it is always zero.

(4) This method can be extended to show the (intuitively more appealing) fact that $M(t)=\max_{0\leqq x\leqq 2\pi} |u(x,t)|$ is a nonincreasing function of t, for $t>0$, by means of the following well-known lemma, phrased for Riemann integration.

LEMMA. *If g is a continuous function on a finite interval $[a,b]$, then*

$$\max_{a\leqq x\leqq b} |g(x)| = \lim_{p\to\infty} \left(\int_a^b |g|^p\right)^{1/p}.$$

With this we have $M(t)=\lim_{n\to\infty}[\int_0^{2\pi} u(x,t)^{2n}dx]^{1/2n}$; but $[\int_0^{2\pi} u(x,t)^{2n}dx]^{1/2n}$ is a nonincreasing function of t, since

$$\begin{aligned} D_t \int_0^{2\pi} u(x,t)^{2n}dx &= 2n\int_0^{2\pi} u^{2n-1}\partial^2 u/\partial x^2 dx \\ &= -2n(2n-1)\int_0^{2\pi} u^{2n-2}(\partial u/\partial x)^2 dx \leqq 0. \end{aligned}$$

Since $M(t)$ is the limit of a sequence of nonincreasing functions of t, $M(t)$ is itself nonincreasing.

(5) The (unique) solution $u(x,t)$ is given by a kernel

$$K(x,y,t) = (1/2\pi)\sum_{-\infty}^{\infty} e^{ikx}e^{-iky}e^{-k^2t} \qquad (t>0),$$

through the formula

$$u(x,t) = \int_0^{2\pi} K(x,y,t)f(y)dy.$$

This is merely a restatement of (I), involving term-by-term integration. $K(x, x, t)\to\infty$ as $t\to 0$. When $t>0$, $K(x, y, t)$ has derivatives of all orders in x, y, and t, and $\partial^2K/\partial x^2=\partial K/\partial t=\partial^2K/\partial y^2$. $\int_0^{2\pi}K(x, y, t)dx=\int_0^{2\pi}K(x, y, t)dy=1$. $K(x, y, t)$ can also be written $L(x-y, t)=(1/2\pi)\sum_{-\infty}^{\infty}\exp\{ik(x-y)\}\exp(-k^2t)$.

(6) If $f(x)$ is the integral of a function f' (*i.e.* $f(x)=f(0)+\int_0^x f'(y)dy$), $f(2\pi)=f(0)$, and $\int_0^{2\pi}f'(y)^2dy<\infty$, then $f_t(x)=u(x, t)$ converges uniformly to f as $t\to 0$. Certainly $f(0)=f(2\pi)$ and $f(x)$ continuous are necessary conditions for uniform convergence to the initial value. The question whether these are sufficient is apparently more difficult than the one proposed here. To prove assertion (6), we show that if $u(x, t)$ solves the heat problem with initial value f, then $u_x=\partial u/\partial x$ solves it with initial value f'; for

$$u_x=(\partial/\partial x)\int_0^{2\pi}L(x-y, t)f(y)dy=\int_0^{2\pi}(\partial L/\partial x)fdy=-\int_0^{2\pi}(\partial L/\partial y)fdy$$

$$=\int_0^{2\pi}L(x-y, t)f'(y)dy.$$

So $\int_0^{2\pi}|u_x-f'|^2\to 0$ as $t\to 0$. Then

$$u(x, t)-f(x)=\int_0^x[u_x(y, t)-f'(y)]dy+u(0, t)-f(0).$$

Since $\int_0^{2\pi}f'(y)^2dy<\infty$, $\sum c_ke^{ikx}$ converges absolutely to f, and $u(0, t)-f(0)=\sum[\exp(-k^2t)-1]c_k$ converges to 0 as $t\to 0$. For the other term,

$$\left|\int_0^x(u_x-f')\right|\leqq\int_0^{2\pi}|u_x-f'|\leqq\left[2\pi\int_0^{2\pi}|u_x-f'|^2\right]^{1/2}\to 0 \text{ as } t\to 0.$$

(7) The kernel $K(x, y, t)$ is nonnegative. Suppose, on the contrary, that $K\leqq-\epsilon<0$ for some x_0, t_0, and all $|y-y_0|<\delta$. Pick η with $\eta<\delta$, $0<\eta<2\epsilon\delta/(\epsilon+N)$, where $N=\max_y|K(x_0, y, t_0)|$. Then define f of period 2π so that $f(y)=-1$ when $|y-y_0|<\delta-\eta$, f increases linearly from -1 to 1 in $[y_0+\delta-\eta, y_0+\delta]$, $f=1$ in $[y_0+\delta, y_0+2\pi-\delta]$, and f decreases linearly from 1 to -1 in

$$[y_0+2\pi-\delta, y_0+2\pi-\delta+\eta].$$

Then f satisfies the conditions in number (6), and if $u(x, t)$ is the corresponding solution of the heat problem, $M(t)=\sup_x|u(x, t)|$ is continuous at $t=0$, hence (by (4)) a nonincreasing function for $t\geqq 0$; this implies $M(t)\leqq M(0)=1$. But if E is any interval of length 2π containing $|y-y_0|<\delta$ in its interior,

$$u(x_0, t_0)=\int_E K(x_0, y, t_0)f(y)dy=\int_{|y-y_0|>\delta}K(x_0, y, t_0)dy$$

$$+\int_{|y-y_0|<\delta-\eta}-K(x_0, y, t_0)dy+\int_{\delta-\eta<|y-y_0|<\delta}K(x_0, y, t_0)f(y)dy.$$

Since $\int K=1$, while $K\leqq-\epsilon$ in $|y-y_0|<\delta$, the first integral on the right is $\geqq 1+2\epsilon\delta$; since $f=-1$ in $|y-y_0|<\delta$ and $K\leqq-\epsilon$ there, the second is $\geqq 2\epsilon(\delta-\eta)$; since $|f|\leqq 1$ and $|K|\leqq N$, the third is $\geqq -2N\eta$. Thus $u(x_0, t_0)\geqq 1+2\epsilon\delta+2\epsilon(\delta-\eta)-2\eta N$, which by the choice of η is >1. This contradicts $M(t_0)\leqq 1$.

Thus the positivity of a kernel for which $\int K=1$ is closely related to the nonincreasing nature of $M(t)$. This is illustrated further in the proof of the following property.

(8) If f is bounded, $M(t)$ is nonincreasing for $t\geqq 0$. By virtue of (4), it is only necessary to show that $M(t)\leqq M(0)=\sup|f|$; but since K is positive and $\int K=1$, $|u(x, t)|\leqq\int_0^{2\pi}K(x, y, t)|f(y)|dy\leqq M(0)$.

(9) $$\text{For } s>0,\ t>0,\quad \int_0^{2\pi}K(x, y, t)K(y, z, s)dy=K(x, z, t+s);$$

this can be verified by using the series for K. It shows that the operators K_t, $(K_tf(x)=\int_0^{2\pi}K(x, y, t)f(y)dy)$, form a semi-group. This may be far afield in some cases, but affords a handy way to introduce semi-groups and show their relation to Huygens principle. Property (1) shows that K_0 should be taken as the identity operator.

A PROPERTY OF THE BERNOULLI NUMBERS

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The Bernoulli numbers may be defined by the symbolic relation

$$(B+1)^n-B^n=0\ (n>1),\qquad B_0=1,$$

where after expansion the exponents are replaced by subscripts. By the Staudt-Clausen theorem, the denominator of B_{2n} contains the prime 2 to the first power. It is proved in [1] that

$$2B_{2n}\equiv 4n+1\pmod 8\qquad (n\geqq 2).$$

This result is due to Staudt. It is perhaps of interest in this connection to note that

(1) $$2B_{2n}\equiv 4n+1\pmod{16}\qquad (n>2),$$

a result due to Frobenius [2, p. 818].

To prove (1) we employ the Bernoulli polynomial of degree n

$$B_n(x)=\sum_{r=0}^{n}\binom{n}{r}B_{n-r}x^r$$

and the relations [3, pp. 18, 21]

$$B_n(x+1)-B_n(x)=nx^{n-1},\qquad B_n(1)=0\quad (n\geqq 1).$$

It follows that

$$B_{2n+1}(2) = B_{2n+1}(1) + (2n + 1) = 2n + 1 \qquad (n \geqq 1).$$

Then

$$2n + 1 = (2n + 1)2B_{2n} + \binom{2n+1}{3} 2^3 B_{2n-2} + \cdots,$$

so that

$$(2n + 1)(1 - 2B_{2n}) = \binom{2n+1}{3} 2^3 B_{2n-2} + \binom{2n+1}{5} 2^5 B_{2n-4} + \cdots$$

for $n > 2$. This evidently implies

$$1 - 2B_{2n} \equiv \tfrac{1}{3}n(2n - 1)2^3 B_{2n-2} \pmod{16}.$$

Since

$$2B_{2n} \equiv 1 \pmod 4 \qquad (n \geqq 2),$$

it follows that

$$1 - 2B_{2n} \equiv -4(2n^2 - n) \equiv -4n \pmod{16},$$

so that we have proved (1).

Since $B_4 = -1/30$, (1) does not hold for $n = 2$.

We remark that with a little more trouble one can prove [2, p. 821].

$$B_{2n} \equiv 1 - 12n \pmod{32} \qquad (n > 2)$$

and indeed more.

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THE PRINCIPAL VALUES OF arcsec x AND arccsc x

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The principal value of arcsec x for $x < -1$ is defined to be in the second quadrant in some texts and in the third quadrant in others. The superiority of the choice of the third quadrant is indicated by the theorem and corollary below.

As pointed out recently by C. O. Oakley [1], the evaluation $(1/a)\arccos(a/x)$ given in most integral tables for the integral on the left in (3) is incorrect for $x < -a$; and thus it is only with the third-quadrant choice of arcsec x for $x < -1$ that a correct evaluation without a "plus-or-minus" sign is obtained.

Note: The statement that y is in a certain quadrant will imply that for some fixed integer n, $\frac{1}{2}n\pi < y < \frac{1}{2}(n+1)\pi$.

THEOREM. *The relation*

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{(x^2 - 1)}} \tag{1}$$

is valid for $x < -1$ *if and only if* arcsec x *is chosen in the third quadrant, and is valid for* $x > 1$ *if and only if* arcsec x *is chosen in the first quadrant.*

Proof. With $|x| > 1$, for any determination of $y = \operatorname{arcsec} x$ we have $\cos y = 1/x$, so that

$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{dy}{dx} = \frac{1}{x^2 \sin y} = \frac{\delta(y)}{|x|\sqrt{(x^2 - 1)}}, \tag{2}$$

where $\delta(y) = -1$ if and only if $\sin y < 0$, $\delta(y) = 1$ if and only if $\sin y > 0$.

If $x < -1$, $|x| = -x$ and we see from (2) that if (1) is valid, then $\delta(y) = -1$. Hence $\sin y < 0$, and, since $\cos y = 1/x < 0$, y is in the third quadrant. The converse is proved in like manner.

If $x > 1$, a similar argument shows that (1) holds if and only if y is in the first quadrant. Hence the theorem is true.

A similar theorem holds for arccsc x, again involving the first and third quadrants.

COROLLARY. *The relation*

$$\int \frac{dx}{(x\sqrt{x^2 - a^2})} = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C, \qquad a > 0, \tag{3}$$

is valid for $x < -a$ *if and only if* arcsec(x/a) *is chosen in the third quadrant, and is valid for* $x > a$ *if and only if* arcsec(x/a) *is chosen in the first quadrant.*

Relation (3), which appears in many tables of integrals and in some calculus texts, becomes incorrect if arcsec(x/a) is taken in the second quadrant or in the fourth quadrant.

There is no question as to continuity when passing from negative to positive values of x, since the integrand is not real for $-a < x < a$.

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A PROBLEM ON ESCAPE VELOCITY

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A problem to be found in nearly every text on elementary differential equations was recently the subject of an interesting discussion in one of my classes. In the brief account which follows I omit all references to units, in order to save space.

The problem may be stated as follows: The earth attracts a mass m at a distance r from its center with a force proportional to m/r^2. At a distance of 4,000 miles from the center the force is $32m$. With what velocity must a projectile be fired (vertically) to escape the pull of gravity? (Neglect centrifugal effects and friction.)

I started by asking what the problem meant. The first reaction was that it meant what it said. I pointed out that the law of attraction we were assuming implied that nothing in space was free of the earth's pull. The next suggestion was that the projectile must travel far enough so that the attraction of some other celestial body became greater than that of the earth. I observed that if this were to be the interpretation, we would have to make specific assumptions as to what bodies were where, and how they were moving. This would go far beyond the original statement of the problem, and would almost certainly lead to a problem we could do nothing with. Then it was suggested that the projectile must reach any assigned distance from the earth. I asked whether the problem might not mean simply that the projectile must not return to earth, and was told that this was the same thing. No proof was offered.

We then turned our attention to the differential equation involved, which is of the form

$$\frac{d^2r}{dt^2} = \frac{k}{r^2},$$

where $k<0$. By the usual method this may be replaced by a first-order equation involving v and r, where v is the velocity. In this equation the variables are separable, and the solution is of the form

$$v = A/r + v_0^2 - B,$$

where v_0 is the initial velocity and A and B are positive constants. Their values do not matter here, since I am not after a numerical answer.

From this solution we can draw no obvious conclusions as to what happens as $t\to\infty$. It is true that if $v_0 \geqq B^{\frac{1}{2}}$, then since $v_0>0$, v never becomes zero and the projectile never returns. But $v_0=B^{\frac{1}{2}}$ may not be the minimum value which will do the trick. Thus we are back to the question of whether the projectile can approach a fixed distance from the earth as $t\to\infty$.

I pointed out that what we needed was a justification for our feeling that the

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attraction of the earth, acting on the projectile for long enough, would drag it back, unless the force approached zero as time passed. There was no suggestion as to how we might get this result, so I observed that $v=v_0+\int_0^t a(x)dx$, where $a(t)$ is the acceleration t seconds after the projectile is fired. It follows that if r were bounded above, $a(t)$ would have a negative upper bound and v would eventually become zero. Thus the projectile either returns to the earth or flies arbitrarily far from it, and the smallest initial velocity which will allow it to escape is $B^{\frac{1}{2}}$. Finally, I pointed out that the projectile could be fired so that it flew arbitrarily far from the earth and still be pulled back, which seemed to clash sharply with my students' intuitions. It might have helped to point out that this is equivalent to saying that if the projectile were taken to an arbitrary point in space and then released, it would fall back to earth. The trouble was certainly due partly to the fact that we were dealing with a two-body problem, and hence should have expected results which would offend our intuitions.

A careful reader will observe that there were opportunities, along the way, to discuss least upper and greatest lower bounds, the intermediate-value property of continuous functions, and the mean-value theorem for integrals.

NOTE ON DERIVING ANNUITY FORMULAS

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In the teaching of the mathematics of investment, an ingenious device enables the most general annuity formulas to be derived without reference to geometric progressions. The sequence of steps used is the concern of this note.

After introducing the definitions of, and formulas relating, nominal and effective rates of interest, it is unusual—but not unnatural—to introduce the present value of a perpetuity. Introduce the present value of the perpetuity by defining it to be the investment required to provide the payments of the perpetuity out of earned interest (that is, the investment must remain intact). With this definition, it is easy to explain that the present value of a given ordinary perpetuity is the same for every date on which this present value may be found. With this background, let A_∞ be the present value of the perpetuity, R be its periodic payment, and i be the interest rate per payment period. Then $A_\infty=R/i$. If the rate per conversion period of the given rate is identical with i, then the present value of an ordinary annuity certain of n payments of R may be found immediately as follows. (1) Let the date on which the present value of the annuity is desired be called *now*. Find the present value of a perpetuity of payment R *now*. This is $A_\infty=R/i$. (2) Find the present value of a perpetuity of payment R on a date n periods from *now*. This is again $A_\infty=R/i$. (3) Discount this latter present value for n periods. (4) Denote the present value of the annuity (the total of the discounted values of the payments) by A. (5) Using *now* as the comparison date, set up the equation of value:

$$A_\infty = A + A_\infty(1 + i)^{-n}.$$

Hence,

$$A = A_\infty - A_\infty(1+i)^{-n} = \frac{R}{i} - \frac{R}{i}(1+i)^{-n} = R\frac{1-(1+i)^{-n}}{i} = Ra_{\overline{n}|i},$$

if the definition

$$a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

is made.

If the rate per conversion period of the given rate is not identical with i, then the formula for the present value of a general annuity may be derived with equal ease. Suppose that for the nominal rate J, compounded M times per year, $i = J/M$ (remembering that i is the rate per payment period). Let j, compounded m times per year, represent the given interest rate. Then, if the rates are equivalent,

$$(1+J/M)^M = (1+j/m)^m,$$

and

$$i = J/M = (1+j/m)^{m/M} - 1.$$

Following the above steps for finding the present value of an annuity, with the understanding that n in step (2) above is the number of interest periods of the rate j, the present value of the annuity becomes

$$A = \frac{R}{i} - \frac{R}{i}(1+j/m)^{-n} = R\frac{1-(1+j/m)^{-n}}{i} = R\frac{1-(1+j/m)^{-n}}{(1+j/m)^{m/M}-1}.$$

If the term of the annuity is y years, if there are p payments per year (*i.e.*, if $M=p$), and if j is the given rate (compounded m times per year), the present value of an ordinary annuity certain is given by

$$A = R\frac{1-(1+j/m)^{-my}}{(1+j/m)^{m/p}-1}.$$

The formula for the sum may be obtained merely by accumulating the present value whether the annuity be simple or general.

EULER AND TANGENT NUMBERS AND THE EXPONENTIAL SHIFT

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While investigating certain congruences for Bernoulli numbers the author stumbled upon a special case of the recurrence relation (14) for tangent series coefficients, unaware that this relation appeared in Nörlund's memoir. Formulating the relation in the form (12) he succeeded in proving it by an exponential shift technique. This technique works equally well for the Euler numbers.

The exponential shift formula, familiar in the solution of linear differential equations with constant coefficients, may be written

$$f(D)e^{ax}g(x) = e^{ax}f(D+a)g(x) \tag{1}$$

where $f(D)$ denotes a polynomial in the differential operator $D=d/dx$.

Here we use the exponential shift to derive simple linear recurrence relations for the coefficients E_n (Euler numbers) and T_n (tangent numbers) in the power series expansions of sec x and x tan $(x/2)$. We write

$$\sec x = 1 + \sum_{n=1}^{\infty} E_n x^{2n}/(2n)!, \qquad \operatorname{sech} x = 1 + \sum_{n=1}^{\infty} (-1)^n E_n x^{2n}/(2n)!, \tag{2}$$

$$x\tan(x/2) = \sum_{n=1}^{\infty} T_n x^{2n}/(2n)!, \qquad x\tanh(x/2) = \sum_{n=1}^{\infty} (-1)^{n-1} T_n x^{2n}/(2n)!,$$

where the hyperbolic functions are obtained from the corresponding trigonometric functions by replacing x by ix.

First we note by formula (1) that

$$f(D)\cdot 1 = f(D)\cosh x \operatorname{sech} x = \tfrac{1}{2}[e^x f(D+1) + e^{-x} f(D-1)] \operatorname{sech} x.$$

In particular for $f(D) = (1-D^2)^n$, we find

$$1 = (1-D^2)^n \cdot 1 = \tfrac{1}{2}[e^x(-D^2-2D)^n + e^{-x}(-D^2+2D)^n] \operatorname{sech} x. \tag{3}$$

Evaluating (3) at $x=0$, and noting from (2) that $(-D^2)^m \operatorname{sech} x = E_m$ at $x=0$, we have

$$1 = \sum_{r=0}^{[n/2]} (-4)^r \binom{n}{2r} (-D^2)^{n-r} \operatorname{sech} x]_0 = \sum_{r=0}^{[n/2]} (-4)^r \binom{n}{2r} E_{n-r}.$$

Written as a linear recurrence relation this becomes

$$E_n = 1 + 4\binom{n}{2}E_{n-1} - 4^2\binom{n}{4}E_{n-2} + 4^3\binom{n}{6}E_{n-3} - \cdots;$$

$$E_1 = 1, \qquad E_4 = 1 + 4\binom{4}{2}E_3 - 4^2\binom{4}{4} \qquad E_2 = 1385, \tag{4}$$

$$E_2 = 1 + 4\binom{2}{2}E_1 = 5, \qquad E_5 = 1 + 4\binom{5}{2}E_4 - 4^2\binom{5}{4} \qquad E_3 = 50521$$

$$E_3 = 1 + 4\binom{3}{2}E_2 = 61, \qquad E_6 = 1 + 4\binom{6}{2}E_5 - 4^2\binom{6}{4}E_4 + 4^3E_3 = 2702765.$$

The Euler numbers E_n are positive odd integers ending alternately in the digits 1 and 5.

Next we make use of the identities

$$(5)\qquad x\tanh(x/2) = e^{x/2}x\operatorname{sech}(x/2) - x = x - e^{-x/2}x\operatorname{sech}(x/2).$$

Applying the operators $(D^2-D)^n$ and $(D^2+D)^n$ to (5) and using the exponential shift we find

$$(6)\qquad (D^2-D)^n x\tanh(x/2) = e^{x/2}(D^2-1/4)^n x\operatorname{sech} x/2 - (D^2-D)^n x,$$

$$(7)\qquad (D^2+D)^n x\tanh(x/2) = (D^2+D)^n x - e^{-x/2}(D^2-1/4)^n x\operatorname{sech} x/2.$$

Averaging (6) and (7), noting that $D^m x = \delta_{m1}$ for $m>0$, we have

$$\tfrac{1}{2}[(D^2-D)^n + (D^2+D)^n]x\tanh(x/2) = \delta_{n1} + \sinh(x/2)(D^2-1/4)^n x\operatorname{sech}(x/2).$$

Expanding the binomials, and evaluating derivatives at $x=0$ we have

$$\sum_{r=0}^{[n/2]}\binom{n}{2r} D^{2n-2r}x\tanh(x/2)\Big]_{x=0} = \delta_{n1} + 0,$$

$$\sum_{r=0}^{[n/2]}(-1)^r\binom{n}{2r}T_{n-r} = \delta_{n1}.$$

Written as a linear recurrence relation this becomes

$$T_n = \binom{n}{2}T_{n-1} - \binom{n}{4}T_{n-2} + \binom{n}{6}T_{n-3} - \cdots \quad \text{for } n>1; \quad T_1 = 1.$$

$$(8)\qquad \begin{aligned} T_2 &= \binom{2}{2}T_1 = 1, & T_5 &= \binom{5}{2}T_4 - \binom{5}{4}T_3 = 155,\\ T_3 &= \binom{3}{2}T_2 = 3, & T_6 &= \binom{6}{2}T_5 - \binom{6}{4}T_4 + \binom{6}{6}T_3 = 2073,\\ T_4 &= \binom{4}{2}T_3 - \binom{4}{4}T_2 = 17, & T_7 &= \binom{7}{2}T_6 - \binom{7}{4}T_5 + \binom{7}{6}T_4 = 38227. \end{aligned}$$

The coefficients B_n (Bernoulli numbers) and C_n in the cotangent and cosecant expansions

$$\frac{x}{2}\cot\frac{x}{2} = 1 - \sum_{n=1}^{\infty} B_n x^{2n}/(2n)!, \qquad x\csc x = 1 + \sum_{n=1}^{\infty} C_n x^{2n}/(2n)!,$$

are easily obtained from T_n by comparing coefficients of $x^{2n}/(2n)!$ in both members of the identities

$$\frac{x}{2}\tan\frac{x}{2} = \frac{x}{2}\cot\frac{x}{2} - x\cot x, \qquad x\csc x = \frac{x}{2}\tan\frac{x}{2} + \frac{x}{2}\cot\frac{x}{2}.$$

In fact we have

$$T_n/2 = (4^n-1)B_n; \qquad C_n = T_n/2 - B_n = (4^n-2)B_n.$$

Finally by another exponential shift we may write

$$\cosh x = D^{2n-1}\sinh x = D^{2n-1}\cosh x \tanh x$$
$$= \tfrac{1}{2}[e^x(D+1)^{2n-1} + e^{-x}(D-1)^{2n-1}]\tanh x.$$

Setting $x=0$ we find

$$1 = \sum_{r=0}^{n-1}\binom{2n-1}{2r} D^{2n-1-2r}\tanh x]_0 = \sum_{r=0}^{n-1}\binom{2n-1}{2r} t_{n-r}.$$

$$t_n = 1 - \binom{2n-1}{2} t_{n-1} - \binom{2n-1}{4} t_{n-2} - \cdots. \tag{9}$$

Thus we derive a recurrence relation for the coefficients t_n in the series

$$\tanh x = \sum_{n=1}^{\infty} t_n x^{2n-1}/(2n-1)! = \sum_{n=1}^{\infty} (-1)^{n-1} T_n (2x)^{2n-1}/(2n)!$$

The t_n are multiples of T_n/n, namely

$$t_1 = 1,\ t_2 = -2,\ t_3 = 16,\ t_4 = -272,\ t_5 = 7936,\ t_n = (-4)^{n-1}T_n/n.$$

Since t_n are integers, it follows that T_n is divisible by the highest odd factor of n.

Formulas equivalent to (9) and (8) are derived in a more complicated way in Nörlund's memoir on Bernoulli numbers (pp. 138 and 140), but formula (4) for the E_n requires only half as many terms as the recurrence relations for E_n given by Nörlund (pp. 133 and 136).

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CONDITIONS FOR THE INTEGRAND OF AN IMPROPER INTEGRAL TO BE BOUNDED OR TEND TO ZERO

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1. Introduction. In discussing the convergence of integrals of the form

$$\int_a^{\infty} f(t)dt = \lim_{x\to\infty}\int_a^x f(t)dt \tag{1}$$

(a is a constant and f is a real function Riemann integrable over every finite subinterval of $a \leqq t < \infty$) many books mention the similarity to the theory of infinite series. Some books also note, however, that even if $\int_a^{\infty} f_1$ exists we need not have $\lim_{t\to\infty} f(t) = 0$, in contrast to the situation for the general term of a convergent infinite series (*e.g.* [3], [4], [5]). A famous example of such behavior is the Fresnel integral $\int_0^{\infty}\cos t^2 dt$. (Cf. [3], p. 253.)

We wish to formulate some simple conditions which taken together with the existence of (1) (or, alternatively, the boundedness of $\int_a^x f$, $a \leqq x < \infty$) assure

that $\lim_{t\to\infty} f(t)=0$ (or, alternatively, that $f(t)$ is bounded on $a\leqq t<\infty$). Some applications of these conditions are given in Section 3.

This discussion was motivated by a conjecture suggested by an engineering problem concerning diodes. Namely, if $f(t)$ is a continuous function for which $\int_a^\infty f$ exists and if there is a constant $T>0$ such that

$$\lim_{t\to\infty}[f(t+T)-f(t)]=0,$$

then $\lim_{t\to\infty} f(t)=0$. Using the example given by Hardy ([5], p. 359) with $T=1$ one can see that this conjecture is false.

2. Results. A simple result which is well known is that if f is a monotone function and $\int_a^\infty f$ exists then $\lim_{t\to\infty} f(t)=0$. We wish to consider a different kind of condition. Given a function $f(t)$, defined for $a\leqq t<\infty$, and given $\delta>0$, $t_0\geqq a$ we define the *uniform oscillation* $\omega(f;\, t_0,\, \delta)$ by

$$\omega(f;t_0,\delta)=\sup_{t_1,t_2}[\,|f(t_1)-f(t_2)|\,:t_1,\,t_2\geqq t_0,\ |t_1-t_2|\leqq\delta]. \tag{2}$$

We then have the following necessary and sufficient condition.

THEOREM 1. *Suppose $\int_a^\infty f$ converges. Then $\lim_{t\to\infty} f(t)=0$ if, and only if,*

$$\lim_{\substack{t_0\to\infty\\ \delta\to 0}}\omega(f;\,t_0,\,\delta)=0. \tag{3}$$

Proof. i) If $\lim_{t\to\infty} f(t)=0$, then given $\epsilon>0$ there is $t_0(\epsilon)$ such that $|f(t)|\leqq\epsilon/2$ for $t\geqq t_0$. Hence, for $t_1,\, t_2\geqq t_0$

$$|f(t_1)-f(t_2)|\leqq|f(t_1)|+|f(t_2)|\leqq\epsilon.$$

Thus $\omega(f;\, t_0,\, \delta)\to 0$ as $t_0\to\infty$ and $\delta\to 0$.

ii) Suppose now that (3) holds and assume that $\lim_{t\to\infty} f(t)=0$ is false. Then there is a sequence $a_1,\cdots,a_k,\cdots$ with $a_k\uparrow\infty$, and a positive number b such that either $f(a_k)\geqq b$, $k=1,\,2,\cdots$ or $f(a_k)\leqq -b$, $k=1,\,2,\cdots$. It suffices to consider the possibility that $f(a_k)\geqq b$. Since $\omega\to 0$, there are δ and t_0 such that

$$|f(t_1)-f(t_2)|\leqq\tfrac{1}{2}b \tag{4}$$

whenever $t_1,\, t_2\geqq t_0$ and $|t_1-t_2|\leqq\delta$. Also since $\int_a^\infty f$ converges we can choose, by the Cauchy criterion, a number $R\geqq t_0$ so large that for any $P>Q>R$

$$\left|\int_Q^P f(t)dt\right|<b\delta. \tag{5}$$

Choose k so large that $a_k>R+\delta$. In view of (4) $f(t)\geqq b/2$ for $a_k-\delta\leqq t\leqq a_k+\delta$. Thus

$$\left|\int_{a_k-\delta}^{a_k+\delta} f(t)dt\right|\geqq(b/2)(2\delta)=b\delta,$$

contradicting (5) and proving the theorem. If $f(t)$ is continuous, condition (3) is equivalent to the uniform continuity of $f(t)$ for $a \leqq t < \infty$. Thus we have

COROLLARY 1. *If $f(t)$ is continuous on $a \leqq t < \infty$ and $\int_a^\infty f$ converges, then $\lim_{t\to\infty} f(t) = 0$ if, and only if, $f(t)$ is uniformly continuous on $a \leqq t < \infty$.*

Functions with bounded derivatives and functions almost periodic in the sense of Bohr [2] are two classes of functions uniformly continuous on $a \leqq t < \infty$. (We note as an aside that $\int_0^\infty \exp(it^p)dt$ exists if $p > 1$ and that $|\exp(it^p)| = 1$ for all t. Hence, $\exp(it^p)$ cannot be an almost periodic function.)

We now examine the case in which $\int_a^x f$ is bounded.

THEOREM 2. *Suppose $\left|\int_a^x f(t)dt\right| \leqq A$ for $a \leqq x < \infty$, where A is a constant, and that (3) holds. Then $f(t)$ is bounded on $a \leqq t < \infty$.*

Proof. Assume that f is not bounded. This implies there is a sequence $\{a_k\}$ such that $a_{k+1} > a_k + 1$ and such that $|f(a_k)| \geqq k$ $(k = 1, 2, \cdots)$. Let t_0 be chosen in such a way that

$$|f(t_1) - f(t_2)| \leqq 1$$

if $t_1, t_2 \geqq t_0$ and $|t_1 - t_2| \leqq 1$. Clearly, we can choose $a_1 \geqq t_0$. Then

$$\left|\int_a^{a_{k+1}} f(t)dt\right| \geqq \left|\int_{a_k}^{a_{k+1}} f(t)dt\right| - \left|\int_a^{a_k} f(t)dt\right| \geqq (k-1) - A \quad (k = 1, 2, \cdots).$$

This contradicts the hypothesis that $\int_a^x f$ is bounded and proves the theorem.

3. Applications. Using the above results we give some very simple proofs of known theorems (see [1, p. 117] and [6, p. 187], for instance) useful in the study of differential equations. See also [7].

COROLLARY 2. *If $f(t)$ and $f''(t)$ $(' = d/dt)$ are bounded for $a \leqq t < \infty$, then so is $f'(t)$. (If $f(t)$ also tends to a limit as $t \to \infty$, then $\lim_{t\to\infty} f'(t) = 0$).*

Proof. Since $f(t) = f(a) + \int_a^t f'$, one has that $\int_a^x f'$ is bounded (converges).

Since f' is uniformly continuous, it is bounded (tends to zero as $t \to \infty$).

COROLLARY 3. *If $\int_a^\infty f^2$ and $\int_a^\infty f''^2$ converge, then so does $\int_a^\infty f'^2$. Moreover $\lim_{t\to\infty} f(t) = 0$.*

Proof. From integration by parts

$$\int_a^x f(t)f''(t)dt = f(x)f'(x) - f(a)f'(a) - \int_a^x (f'(t))^2 dt. \tag{6}$$

Since $f^2 + f''^2 \geqq 2|ff''|$ the integral on the left hand side of (6) converges. Now either $\int_a^x f'^2$ converges or tends monotonically to infinity. If the latter holds, then $f \cdot f' \to \infty$ by (6). This is impossible, however, for it would imply $f^2 \to \infty$. Hence, $\int_a^\infty f'^2$ converges. Using (6) again we see that $\lim_{t\to\infty} f(t)f'(t)$ exists. Hence, by Corollary 1 $\lim_{t\to\infty} f^2(t) = 0$.

COROLLARY 4. *Let $g(x, y, z)$ be a continuous bounded function defined for x where $a \leqq x < \infty$ and for suitable domains in the y-z plane. If $f(t)$ is a bounded solution of*

$$f''(t) + g(t, f(t), f'(t)) = 0$$

and if $\int_a^\infty (f)^p$ converges for some $p \geqq 1$, then $\lim_{t\to\infty} f(t) = 0$.

Proof. Since $f''(t)$ is bounded, Corollary 2 implies $f'(t)$ is bounded and the result follows from Corollary 1.

An important special case occurs when $g(t, f, f') = \alpha(t)f(t) + \beta(t)$ when α and β are given bounded functions for $a \leqq t < \infty$.

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SIMPSON'S RULE FOR UNEQUALLY SPACED ORDINATES

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Every textbook in numerical analysis and almost every textbook in calculus derives or quotes Simpson's quadrature formula, viz.

$$\int_{-h}^{h} y \, dx = \tfrac{1}{3}h\{y_{-h} + 4y_0 + y_h\} \tag{1}$$

in terms of three equally spaced ordinates. The analogous formula in terms of unequally spaced ordinates is seldom, if ever, mentioned. We give a derivation of this formula in the case of three ordinates y_{-h}, y_0, and y_k, where the subscripts refer to the corresponding abscissae.

If we designate by $\Delta_b y_a$ and $\Delta^2_{c,b} y_a$ the divided differences $(y_b - y_a)/(b-a)$ and $(\Delta_c y_b - \Delta_b y_a)/(c-a)$ respectively, we see that

$$\Delta^2_{x,0}\, y_{-h} = \frac{\Delta_x y_0 - \Delta_0\, y_{-h}}{x + h}$$

$$= \frac{y_x}{(x+h)(x-0)} + \frac{y_0}{(0-x)(0+h)} + \frac{y_{-h}}{(-h-0)(-h-x)}$$

where x is a variable abscissa. If, furthermore, we assume a parabolic fit, second divided differences will, as usual, be constant and we may equate $\Delta^2_{x,0} y_{-h}$ to $\Delta^2_{x,0} y_{-h}$, where k is an abscissa such that y_k is known. We then have

$$\frac{y_x}{(x+h)x} - \frac{y_0}{xh} + \frac{y_{-h}}{h(x+h)} = \Delta^2_{k,0}\, y_{-h}$$

or

$$y_x = -\frac{x}{h}\, y_{-h} + \frac{(x+h)}{h}\, y_0 + (x+h)x\, \Delta^2_{k,0}\, y_{-h}$$
$$= y_{-h} + (x+h)\, \Delta_0\, y_{-h} + (x+h)x\, \Delta^2_{k,0}\, y_{-h}.$$

This is Newton's formula for divided differences, as far as second differences. Rearranging in powers of x, we obtain

$$y_x = \left(y_{-h} + h\, \Delta_0\, y_{-h}\right) + \left(\Delta_0\, y_{-h} + h\, \Delta^2_{k,0}\, y_{-h}\right)x + \left(\Delta^2_{k,0}\, y_{-h}\right)x^2.$$

We now integrate between a and c and get

$$\int_{-h}^{k} y_x dx = (h+k)\left[\left(y_{-h} + h\, \Delta_0\, y_{-h}\right) + \left(\Delta_0\, y_{-h} + h\Delta^2_{k,0}\, y_{-h}\right)\tfrac{1}{2}(k-h) + \left(\Delta^2_{k,0}\, y_{-h}\right)\tfrac{1}{3}(k^2 - kh + h^2)\right]$$

(2)

$$= \tfrac{1}{6}(h+k)\left[6y_{-h} + 3(h+k)\, \Delta_0\, y_{-h} + (2k^2 + kh - h^2)\, \Delta^2_{k,0}\, y_{-h}\right]$$

$$= \tfrac{1}{6}(h+k)\left[\left\{2 - \frac{k}{h}\right\} y_{-h} + \frac{(h+k)^2}{hk}\, y_0 + \left\{2 - \frac{h}{k}\right\} y_k\right].$$

This result is interesting in certain special cases. If $k=h$, formula (2) reduces to formula (1). If one interval is double the other, *e.g.* $k=2h$, only two of the ordinates are used in the quadrature, viz.

$$\int_{-h}^{2h} y_x dx = \tfrac{3}{4}h(3y_0 + y_{2h}).$$

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D.C.

AN EXPERIMENT WITH A LARGE CALCULUS CLASS

DORIS S. STOCKTON, University of Massachusetts

Acknowledging that increased enrollments and teacher shortages are forcing universities to abandon small classes, the author conducted an experiment with a large section of beginning calculus at the University of Massachusetts in the academic year 1959–60. Each semester one section containing about ninety students and one section containing about thirty students were taught. An attempt was made to organize and regiment the sections in such a way that the large section would be taught as effectively as the small section. It was hoped that the organization would effect a saving in teaching manpower and that the plan could be readily adapted to accommodate several hundred students if television were used.

There were four hour-examinations and one final examination. Except for insignificant modifications both sections took the same examinations. Homework was done on standardized paper and was collected each day. The instructor graded all examinations and checked the homework personally. An undergraduate assistant checked attendance and helped proctor.

There were two help sessions a week for these sections. In addition the students were encouraged to write questions on their homework papers. All such questions were answered either in writing or in class.

Each student's record was kept on an IBM mark-sensing card, and the weighted averages were computed on an IBM 650. For comparison purposes the records were also kept in a conventional grade book and the weighted averages were computed "by hand." If there were several hundred students involved, it is believed that the IBM machine would save at least six hours per hundred students for the instructor each semester.

A careful comparison of the small section with the large section was kept throughout each semester. The differences in the averages between the two sections were relatively slight.

At the end of each semester the students in the large section were asked to check one of the alternatives in the statement: If I had been enrolled in a section containing thirty students, having the same instructor, and using the same text, I think I would have learned more, less, the same about the material in this course. The students in the small section were asked the same question with the number 30 replaced by the number 20. The first semester about 50% of the students in both sections thought they would have learned more in smaller sections. The second semester only 23% of the students in the large section and

19% of the students in the small section thought they would have learned more in smaller sections.

In view of the negligible differences in the averages, it appears that the students in the large sections learned just about as much as those in the small sections. If we assume that the two groups of students were of about the same ability, the results indicate that with the scheme used here a section of ninety students can be taught beginning calculus as effectively as a section of thirty students.

It also appears that the plan used can save teacher-hours. In the experiment 118 students were taught with only the aid of an undergraduate assistant working three hours a week. This is equivalent to handling four normal-sized sections. The instructor estimated that it took less time than it would have taken to teach four separate normal-sized sections and slightly more time than it would have taken to teach three separate normal-sized sections. Another thirty students might have been placed in the large section.

It is apparent that the plan used here could be readily adapted for a televised lecture and a class of several hundred students. It would only be necessary to use machine-scored examinations and to have a staff of graduate assistants to handle the help sessions and the homework and to record the grades on mark-sensing cards. The assistants could daily present the lecturer with a list of the more common questions appearing on the homework so as to retain a personal link between lecturer and student. So many students ask the same questions that the problem of providing answers to all the questions is not as great as it may at first seem.

A SUMMER INSTITUTE IN MATHEMATICS FOR ARITHMETIC TEACHERS AND SUPERVISORS

JAMES R. SMART, San Jose State College

Forty teachers and supervisors of elementary school arithmetic, representing twenty-five states, were participants in a National Science Foundation Institute in Mathematics held at San Jose State College from June 20 to July 29, 1960. The Institute members included twenty-four elementary teachers, four teaching principals, four principals, six supervisors or consultants, and two college teachers of arithmetic.

The objectives were to help elementary teachers and supervisors to become subject matter specialists in arithmetic and to better understand how arithmetic relates to secondary mathematics; to improve the participants' understanding of the newer topics in current experimental programs in elementary school mathematics; to provide an opportunity to observe a well-trained teacher presenting new topics to a group of elementary pupils; and to stimulate discussion of current problems and trends in elementary school arithmetic.

All participants registered for three mathematics classes. "The Expanding World of Arithmetic" was a study of some of the newer topics which may become part of the arithmetic curriculum, including inequalities, integral ex-

ponents, introductory set theory, elementary number theory and congruence, numbers to other bases, and probability.

"Foundations of Arithmetic" was a careful examination of arithmetic as a science and as a branch of elementary mathematics. The algorisms of arithmetic were explained in terms of basic properties of number systems. Modern approaches to topics such as per cent, measurement, equal ratios, and equations were justified by referring back to postulates and definitions.

The third course was designed around a demonstration class of above-average fifth and sixth grade pupils. All of the topics from "The Expanding World of Arithmetic" were introduced in demonstration lessons, as well as some number laws and equal ratios. Participants helped evaluate the results and were encouraged to prepare similar materials for their own use.

Three informal activities supplemented the Institute classwork: informal problem sessions at which students could secure help with homework, a materials laboratory for aiding in the development of concrete or written materials for use in the arithmetic classroom, and ten seminars with guest speakers or student discussions about arithmetic education.

A NOTE ON ANALYTIC GEOMETRY

I. A. BARNETT, University of Cincinnati

The trend today is to give the student only a smattering of analytic geometry in the unified analytics-calculus course. The equations of the straight line and circle are introduced for the purpose of illustrating some examples of the calculus. The student leaves such a course with but a meager knowledge of analytic geometry and with little idea of its importance in the stream of mathematics. He is rarely introduced to the idea of families of curves so necessary to an understanding of differential equations. He is seldom acquainted with the parametric method used in solving locus problems. Few students know how to find the coordinates of the vertices and of the foci of a conic, even when the axes of the conic are parallel to the coordinate axes.

Until such time as the high schools of our country are able to give their students a thorough preparation in mathematics up to the calculus, the colleges should return to a one-semester course in analytics. The colleges must be flexible enough to accept the existing state of high school mathematics preparation and not shut their eyes to it. In this transition period the colleges must compensate for this lack of preparation in the basic subjects and not proceed to push badly prepared students through more and more advanced work.

The high schools should find a place in their curricula for a one-semester course in analytic geometry. Analytics is the logical course to show how algebra and geometry are combined, and to introduce the students to mathematics beyond what we inherited from the Greeks and Arabs. Such a course would serve (1) as a terminal course for those who have had $1\frac{1}{2}$ years of algebra and 1 year of plane geometry; and (2) as a prerequisite course for those who plan to start the calculus in the freshman year of college.

NEW COOPERATIVE MATHEMATICS TESTS

BENJAMIN SHIMBERG, Educational Testing Service, Princeton, N. J.

Teachers who have been wondering when standardized tests would reflect the significant changes that are taking place in mathematics instruction will be interested in knowing that the Cooperative Test Division of The Educational Testing Service is currently developing eleven new examinations geared specially to a modern curriculum in mathematics.

The Cooperative Test Division development program got underway more than a year ago when an advisory committee of practicing mathematicians and mathematics teachers was set up to advise and review the formulation of the program.

The program includes the development of eleven new tests aimed at students from the 7th grade through the second year in college. For junior high school students through the 9th grade, there will be an examination covering arithmetic reasoning and computation as well as the conceptual and structural aspects of arithmetic.

For students who have participated in the experimental junior high school program of the School Mathematics Study Group, there will be an examination based on the unit "The Structure of the Number System." This examination may also be given to students who have taken three or four years of high school mathematics and to seniors in teachers' colleges who are bound for the elementary grades or for mathematics teaching.

The other examinations in preparation are in algebra, geometry, trigonometry, analytic geometry, and calculus. There are to be three separate algebra examinations: beginning, intermediate, and advanced or college algebra.

Two levels of both geometry and calculus are planned. The lower level of each of these tests is intended to test adequately the student's knowledge of basic facts and techniques in the subject. The upper level will not only cover more factual knowledge, but will also attempt to measure the student's depth of understanding of what he has learned. All of these examinations are designed with consideration of the requirements of both traditional courses and those which have incorporated newer approaches.

The Advisory Committee for the new Cooperative Mathematics Tests includes: E. G. Begle of Yale University and the School Mathematics Study Group; Donovan A. Johnson, University of Minnesota; John G. Kemeny, Dartmouth College; Kenneth O. May, Carleton College; John R. Mayor, American Association for the Advancement of Science; Robert A. Rosenbaum, Wesleyan University; Eugene Smith, Wilmington, Delaware, Public Schools, and Wayne State University, Detroit; Max A. Sobel, Montclair State Teachers College; Ben A. Sueltz, State Teachers College, Cortland, New York; Henry Van Engen, University of Wisconsin.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This Department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1441. *Proposed by Bart Park, Michigan College of Mining and Technology*

A resident of Los Vegas, on his deathbed, held the following conversation with a friend who liked to solve problems.

"Over the past few years I have saved every silver dollar that came into my hands. When I reached one hundred I tied them in a bag. The first three hundred accumulated pretty fast and I hoped to reach a thousand, but I didn't make it. The several bags, each containing one hundred dollars, are in the closet of the next room. What I want you to do is visit my minor son on his next and each succeeding birthday and give him the number of dollars which equals his age. I figure that on your last trip you will have just used up all of the money."

"That's interesting," said his friend. "Before I see the bags let me try to figure out how many trips will be required." After an interval the friend said, "I'll have to know your son's age." He was told. Then he said, "I now know how many trips will be required." How many will he have to make?

E 1442. *Proposed by Andrzej Makowski, Warsaw, Poland*

Prove that the equation $x^2+36=y^5$ has no solution in integers x and y.

E 1443. *Proposed by N. A. Court, University of Oklahoma*

If two of the four lines joining the vertices of a tetrahedron to the orthocenters of the opposite faces are coplanar, the remaining two lines are also coplanar.

E 1444. *Proposed by Lincoln Teng, Willow Run Laboratories, University of Michigan*

Let A and B, two $n\times n$ matrices, be such that A and $AB-BA$ commute. Show that $n(A^kB-BA^k)=k(A^nB-BA^n)A^{k-n}$ for any integer $k>n$.

E 1445. *Proposed by M. S. Klamkin, AVCO Research and Advanced Development*

A number n is defined as *almost perfect* if $\sum_{d|n} d=2n\pm 1$. Are there any other almost perfect numbers besides numbers of the form $n=2^m$?

SOLUTIONS

Cyclic Polygons

E 1411 [1960, 378]. *Proposed by H. Y. Shee, Hualien, Taiwan, China*

Let $ABCDE$ be any pentagon inscribed in a circle and let P, Q, R, S, T be intersections of the diagonals such that P and Q lie on AC, Q and R on BD, R and S on CE, S and T on DA. Prove that

$$\frac{AB\cdot BC\cdot CD\cdot DE\cdot EA}{AC\cdot BD\cdot CE\cdot DA\cdot EB} = \frac{AP\cdot BQ\cdot CR\cdot DS\cdot ET}{CP\cdot DQ\cdot ER\cdot AS\cdot BT}.$$

Generalize.

Solution by the proposer. From the two pairs of similar triangles APB, CPE and BCP, AEP we have

$$AB/CE = BP/CP \quad \text{and} \quad EA/CB = AP/BP.$$

Combining the above two equations we get $(EA\cdot AB)/(CE\cdot CB) = AP/CP$. In a similar way we find

$$(AB\cdot BC)/(DA/DC) = BQ/DQ, \qquad (BC\cdot CD)/(EB\cdot ED) = CR/ER,$$
$$(CD\cdot DE)/(AC\cdot AE) = DS/AS, \qquad (DE\cdot EA)/(BD\cdot BA) = ET/BT.$$

The desired result now follows by multiplying together the corresponding members of the last five equations.

In the same way we can establish the following generalization: Let $A_1A_2\cdots A_n$ $(n\geqq 5)$ be any cyclic n-gon and let P_k be the intersection of A_kA_{k+2} and $A_{k+1}A_{k-1}$, where the subscripts are reduced modulo n to numbers of the set $\{1, 2, \cdots, n\}$. Then

$$\prod_{k=1}^{n} A_kA_{k+1}/A_kA_{k+3} = \prod_{k=1}^{n} A_kP_k/A_{k+2}P_k.$$

Other interesting generalizations are possible.

Also solved by Richard Cottle, Michael Goldberg, L. D. Goldstone, Ku Yi Hui, D. C. B. Marsh, C. L. Sterling, and Lawrence Zalcman.

Editorial Note. Essentially the relation of this problem was given by Ku Yi Hui in the May 1959 issue of *The Mathematics Student Journal*. A number of the solvers incorrectly generalized to $\prod A_kA_{k+1}/A_kA_{k+2} = \prod A_kP_k/A_{k+2}P_k$.

A Divergent Series

E 1412 [1960, 378]. *Proposed by H. E. Chrestenson, Reed College*

Let $\{m_k\}$, $k\geqq 0$, be a strictly increasing sequence of integers with $m_0=0$. Under what conditions does the series $\sum_{k=1}^{\infty}(m_k-m_{k-1})/m_k$ converge?

I. *Solution by W. C. Waterhouse, Harvard University.* It never converges. For any n there is an n' such that $m_{n'}>2m_n$; then

$$\sum_{k=n+1}^{n'} (m_k - m_{k-1})/m_k > \sum_{k=n+1}^{n'} (m_k - m_{k-1})/m_{n'} = 1 - m_n/m_{n'} > 1/2.$$

Hence by Cauchy's criterion the series diverges.

II. *Solution by R. H. Breusch, Amherst College.* If $0<a_k<1$, then $\sum a_k$ converges if and only if $\prod(1-a_k)$ converges. Now

$$\prod_{k=2}^{n} [1 - (m_k - m_{k-1})/m_k] = \prod_{k=2}^{n} m_{k-1}/m_k = m_1/m_n.$$

Since $\{m_k\}$ is a strictly increasing sequence of integers, $\lim_{n\to\infty} m_n = \infty$. Thus the product diverges, and therefore the concerned series diverges. So the answer is: never.

III. *Solution by J. L. Brown, Jr., Ordnance Research Laboratory, Pennsylvania State University.* More generally it is known (see Knopp, *Theory and Application of Infinite Series*, p. 308) that for

$$0 < p_0 \leqq p_1 \leqq \cdots, \quad p_n \to +\infty,$$

the series $\sum_{n=1}^{\infty} (p_n - p_{n-1})/p_n^{\alpha}$ converges when $\alpha>1$ and diverges when $\alpha\leqq 1$. Thus the proposed series is always divergent.

IV. *Solution by Albert Wilansky, Lehigh University.* Never! A known theorem is that if $a_n \geqq 0$ and $\sum a_n = \infty$ and $m_n = \sum_{k=1}^{n} a_k$, then $\sum a_n/m_n = \infty$. In the present case the m_n are integers, so $\lim m_n$ does not exist, *i.e.* $\sum a_n = \infty$.

If the proposer deletes the word integer we have the result that the series converges if and only if $\lim m_n$ exists.

Also solved by Steve Andrea, Donald Batman, Fred Gilman, Todd Gitlin, A. R. Hyde, Shigeru Ishii and Joel Levy (jointly), R. B. Kirchner, Donald Knuth, T. V. Lakshminarasimhan, D. C. B. Marsh, C. S. Ogilvy, Jack Silver, Allan Trojan, David Zeitlin, and the proposer.

Other pertinent references are Bromwich, *Theory of Infinite Series*, p. 36 and Hobson, *Theory of Functions of a Real Variable*, vol. 2, p. 13. The proposer pointed out that the problem arose in connection with the solution of Problem 113, Chapter I, *Aufgaben und Lehrsätze aus der Analysis*, vol 1, Pólya and Szegö, and provides a solution of that problem somewhat different from the one given by the authors.

Some Special Integers

E 1413 [1960, 378]. *Proposed by B. K. Harrison, University of California, Los Alamos Scientific Laboratory*

Consider an integer, written in the usual form to the base r as

$$N = a_0 a_1 a_2 \cdots a_n, \qquad 0 \leqq a_i < r.$$

We require that $r \geqq n+1$. Find all N, for any n, such that a_k is equal to the number of times that k appears in the sequence $a_0, a_1, a_2, \cdots, a_n$.

Solution by Seth Catlin, University of Oregon. For such an $N = a_0 a_1 a_2 \cdots a_n$, we clearly must have $\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} i a_i = n+1$. In particular,

$$a_0 = \sum_{i=2}^{n} (i-1)a_i.$$

This implies that if $a_0=1$, then $a_2=1$, $a_3=\cdots=a_n=0$, and if $a_0=2$, then $a_2=2$, $a_3=\cdots=a_n=0$. (Note that $a_0=2$ and $a_3=1$ would force $a_2=0$, contrary to $a_0=2$.) The only solutions in these cases are then seen to be: 1210, 2020, 21200.

If $a_0=t>2$, then $a_t\geqq 1$, and

$$t-(t-1)a_t = \sum_{t\neq i=2}^{n} (i-1)a_i \geqq 0.$$

Since $a_t>1$ implies $t-(t-1)a_t<0$, we must have $a_t=1$ and $t-(t-1)a_t=1$. So in this case the only solution is $a_0=t$, $a_1=2$, $a_2=a_t=1$, and $a_i=0$ $(i=3,\cdots,t-1,t+1,\cdots,n)$, where $n=t+3$.

Also solved by J. L. Botsford, T. C. Brown, Donald Knuth, D. C. B. Marsh, Allan Trojan, and the proposer.

Some of these solutions were incomplete.

An Integral Inequality

E 1414 [1960, 378]. *Proposed by George Glauberman, Polytechnic Institute of Brooklyn*

Suppose g is a nonnegative and integrable function on $[0, 1]$. Let $A_r=\int_0^1[g(x)]^r dx$ for r a nonnegative integer. Prove that $A_r\geqq A_1^r$.

I. *Solution by J. W. Ellis and J. H. Wahab, Louisiana State University in New Orleans.* Since $g\geqq 0$, $g^k(x)\leqq g^k(y)$ if and only if $g(x)\leqq g(y)$. Therefore, if S is the unit square,

$$\begin{aligned} 0 &\leqq \iint_S [g^k(x)-g^k(y)][g(x)-g(y)]dS \\ &= 2\iint_S g^k(x)g(x)dS - 2\iint_S g^k(x)g(y)dS \\ &= 2A_{k+1} - 2A_kA_1. \end{aligned}$$

Hence $A_1^k\leqq A_k$ implies $A_1^{k+1}\leqq A_kA_1\leqq A_{k+1}$. The case $r=1$ is trivial.

II. *Solution by Albert Wilansky, Lehigh University.* We give a far more general result, namely $\{(b-a)^{-1}\int_a^b g^r\}^{1/r}$ is an increasing function of r for $r>0$. The problem follows by taking $b=1$, $a=0$, and restricting r to be an integer.

Let $r>s>0$. In Hölder's inequality,

$$\int uv \leqq \left\{\int u^p\right\}^{1/p}\left\{\int v^q\right\}^{1/q},$$

let $v=1$, $u=f^s$, $p=r/s$. Then $1/q=1-s/r$ and the inequality becomes

$$\int f^s \leqq \left\{\int f^r\right\}^{s/r} (b-a)^{1-s/r}.$$

Hence

$$\left\{(b-a)^{-1}\int f^s\right\}^{1/s} \leqq \left\{(b-a)^{-1}\int f^r\right\}^{1/r}.$$

Also solved by Steve Andrea, Donald Batman, Ralph Beals, R. C. Bohuslou, D. A. Breault, J. L. Brown, Jr., Richard Cottle, J. B. Garner, Michael Goldberg, W. C. Guenther, J. C. Hickman, Vern Hoggatt, Shigeru Ishii and Joel Levy (jointly), G. J. Janusz, R. B. Kelman, Donald Knuth, T. V. Lakshminarasimhan, J. F. Leetch, Joe Lipman, D. C. B. Marsh, C. S. Ogilvy, J. E. Potter, C. H. Sampson, Jack Silver, T. H. Slook, W. C. Teachout, Jr., Allan Trojan, Václev Vítek, W. C. Waterhouse, David Zeitlin, and the proposer.

A Snowblower Problem

E 1415 [1960, 378]. *Proposed by Michael Herschorn, McGill University*

It began snowing in the morning; at noon a snowblower set out to clear a street of constant width. Later on a second blower followed, and, still later, a third. Each blower cleared at a constant rate (volume per unit of time), the spacing between blowers remaining constant with the first separation twice the second. At any given time during the day the rate of snowfall was the same throughout the town. A total of four inches had fallen when the second blower started out, and another four inches fell before the third began on the route. How deep was the snow at noon?

Solution by D. C. B. Marsh, Colorado School of Mines. If the lesser separation between blowers is k linear units, then when the first blower has traversed a distance of $3k$, the second and third have gone k and 0 respectively. Should the depth of snow confronting Blower 1 at location x be labelled $d(x)$, then the snow-depths confronting Blowers 2 and 3 are respectively $d(x)-d(x-2k)$ and $d(x)-d(x-k)$. Since the street is of constant width and the spacing is maintained (so the speeds are equal), the snow-depths confronting each blower are proportional to the blowers' rates of clearing. For any two locations k units apart we then have

$$\frac{d(x)}{d(x-k)} = \frac{d(x)-d(x-2k)}{d(x-k)-d(x-3k)} = \frac{d(x)-d(x-k)}{d(x-k)-d(x-2k)},$$

which readily yield $d(x)d(x-2k)=d^2(x-k)$. The given data show that $d(2k)=4$ and $d(3k)=8$. Letting $x=3k$ we find $d(k)=2$, and then letting $x=2k$ we find $d(0)=1$. Therefore the snow was *one inch* deep at noon.

Also solved by A. N. Aheart, Robert Bart, E. W. Brown, A. R. Hyde, Philip Smedley, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4935. *Proposed by H. Schwerdtfeger, McGill University*

Prove that if c is a rational number such that the equation $x^m = c$ is irreducible over the rational field, then the determinant

$$\begin{vmatrix} a_0 & ca_{m-1} & ca_{m-2} & \cdots & ca_2 & ca_1 \\ a_1 & a_0 & ca_{m-1} & \cdots & ca_3 & ca_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m-2} & a_{m-3} & a_{m-4} & \cdots & a_0 & ca_{m-1} \\ a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_1 & a_0 \end{vmatrix}$$

is different from zero for all sets of m rational numbers $a_0, a_1, \cdots, a_{m-1}$ which are not all equal to zero.

4936. *Proposed by R. C. Buck, University of Wisconsin*

Let A be an algebra of functions defined on a space X and taking values in an arbitrary (not necessarily commutative) field k. Suppose that A separates X: given x and y in X, $x \neq y$, there is $f \in A$ with $f(x) \neq f(y)$. How close does A come to being dense in the algebra of all functions on X to k? Given a finite set $S \subset X$, and a function g on X, is there $f \in A$ so that $f(x) = g(x)$ for all $x \in S$?

4937. *Proposed by G. P. Sturm, Jr., Oklahoma State University*

Prove that the matrix product of any two third order magic squares is a doubly symmetric matrix. A magic square is defined as a square array of n^2 elements with the property that the sum of the elements of any line (both diagonals included) is a constant; double symmetry is defined as symmetry in both diagonals.

4938. *Proposed by R. W. McChesney, Albion College*

It is known that every real number $x \in [0, 1)$ has at most two representations of the form $x = \sum_{m=1}^{\infty} a_m N^{-m}$, where N is any integer greater than 1 and each a_i is one of the set $0, 1, \cdots, N-1$. Now consider a representation of the form

$x=\sum_{m=1}^{\infty} a_m b^{-m}$, where $x\in[0, 1)$, and a_i is one of 1, 2, $\cdots$, $n-1$; n an integer greater than 1 and $n-1\leqq b<n$.

(a) Show that an x can exist having more than two such representations.

(b) Prove that at most two representations of x exist if

$$\tfrac{1}{2}[n-1+\sqrt{(n^2+2n-3)}]\leqq b\leqq n.$$

4939. *Proposed by C. N. Campopiano, Radio Corp. of America, Camden, N. J.*

Let E be a topological space. Let p be the collection of all those (and only those) subsets A of E such that $A\cap\overline{B}\subset\overline{A\cap B}$ for every $B\subset E$. ($\overline{X}$ is the closure of X for $X\subset E$.) The following facts are known: if A is an open subset of E, then $A\in p$; p is nonempty; there exist topological spaces having non-open subsets which are not in p. (See Bourbaki, *General Topology*, Book III, Chap. I, p. 16, Proposition 3 and p. 18, Exercise 6.)

Prove that, if $A\in p$, then A is open; thus, $A\in p$ if and only if A is open.

4940. *Proposed by M. S. Klamkin, AVCO Research, and D. J. Newman, Brown University*

Problem no. 151 in the "Scottish" book of problems due to Wavre poses the question of the existence of a harmonic function defined in a region containing a cube in its interior such that it vanishes on all its edges. Show that such a function $\neq 0$ exists for any number of dimensions.

SOLUTIONS

A Diophantine Equation

4666 [1955, 734; 1957, 54]. *Proposed by R. Venkatachalam Iyer, Trivandrum, India*

If $T_p=p(p+1)/2$, solve in integers the equation

$$\frac{1}{T_x}+\frac{1}{T_y}=\frac{1}{T_z}.$$

II. *Solution by the proposer.* In this MONTHLY, January 1957, solutions were obtained making the special assumptions, $x=y$, $x=y\pm 1$, and the solution was labelled "partial" since no reason was advanced why x and y might not be otherwise related. However, the earlier solution is complete as the following argument will show.

By the substitutions $a=2x+1$, $b=2y+1$, $c=2z+1$, the given equation becomes

$$(a^2-1)^{-1}+(b^2-1)^{-1}=(c^2-1)^{-1}. \tag{1}$$

This is easily put into the form

$$(ab-c^2)^2=c^4+\{(a-b)^2-2\}c^2+1.$$

The right side is a quadratic expression in c^2 and so will be a perfect square if and only if $\{(a-b)^2-2\}^2-4=0$, whence $a-b=\pm 2$ or 0. Thus solutions can exist under these two conditions only with the result that the special cases considered earlier constitute, in fact, the complete solution.

In both cases (1) reduces to a Pell's equation of the form $m^2-2n^2=-1$ for which all solutions are known.

It is interesting to note that if m, n satisfy the condition, $m^2-2n^2=-1$, then $x=2n-1$, $y=2n+1$, $z=m$, $w=n$ satisfy

$$(x^2-1)^{-1}+(y^2-1)^{-1}+(z^2-1)^{-1}=(w^2-1)^{-1}.$$

Ring Whose Additive Group Is the Group of Reals

4884 [1960, 87]. *Proposed by Hans Schneider, Queen's University, Belfast*

It is known that a ring whose additive group is the group of rationals (mod 1) has trivial multiplication. (C. Chevalley, *Fundamental Concepts of Algebra*, Example on chapter 3.) Find a nontrivial ring whose additive group is the group of reals (mod 1).

Solution by N. J. Fine, Institute for Advanced Study. Let f be a nonzero solution of the equation $f(x+y)=f(x)+f(y)$ such that $f(1)=0$ and $f(x)$ is rational for all x. [This may be done by choosing a basis $\{b_\alpha\}$ for the reals over the rationals, letting $f(b_\alpha)$ be rational for each α, subject to the condition $f(1)=0$. Let b_1, b_2 be basis elements not involved in the representation of 1, and define $f(b_1)$, $f(b_2)$ so that $f(b_1)\cdot f(b_2)\not\equiv 0 \pmod 1$.]

Define $x \circ y=f(x)\cdot f(y) \pmod 1$. The distributive laws are easily verified for the reals (mod 1) with $(+, \circ)$ as the operations, and the associative law follows from $(x \circ y)\circ z=0=x \circ (y \circ z)$, since f is zero at all rational numbers. Now if $x=b_1$, $y=b_2$, then $x \circ y=f(b_1)f(b_2)\not\equiv 0 \pmod 1$.

Also solved by D. J. Newman, Joseph Rotman, and the proposer.

Cardinality of a Hausdorff Space

4886 [1960, 87]. *Proposed by Isaac Namioka, Cornell University*

Let X be a Hausdorff (T_2) topological space and let Y be its dense subset. Prove that $\kappa(X)\leqq 2^{2^{\kappa(Y)}}$, where $\kappa(A)$ denotes the cardinality of the set A. Show by an example that this inequality cannot be improved even when X is compact. Is this assertion true for T_1-spaces?

Solution by J. V. Whittaker, University of British Columbia. For each $x\in X$, let U_x be the family of open subsets of X which contain x. We define a mapping θ of X into 2^{2^Y} by setting $\theta(x)=\{Y\cap U: U\in U_x\}$. To show that θ is one-to-one, let x, $y\in X$, $x\neq y$. We can find $U\in U_x$, $V\in U_y$ such that $U\cap V=\varnothing$, whence $\varnothing\neq U\cap Y\neq V\cap Y\neq\varnothing$, and $\theta(x)\neq\theta(y)$. Hence, $\kappa(X)\leqq 2^{2^{\kappa(Y)}}$. To show that equality can occur, let I be the closed unit interval and $X=2^I$ with the product topology. Then X is compact and separable. If we let Y be a countable dense

subset, we obviously have $\kappa(X)=2^{2^{\kappa(Y)}}$. Now let X be any set with $\kappa(X)>\kappa(2^I)$, and let the closed subsets of X consist of X and all finite subsets. Then X is T_1 but not T_2. If $Y\subset X$ is countable, then Y is dense in this topology, but $\kappa(X)>2^{2^{\kappa(Y)}}$.

Also solved by Meyer Jerison, P. S. Landweber, Stephen Love, and the proposer. E. Hewitt solved the problem earlier in *A remark on density characters*, Bull. Amer. Math. Soc. 52 (1946), pp. 641–643, which see also for references to earlier work by Pospisil.

How to Tie up a Smooth Sphere

4887 [1960, 187]. *Proposed by Z. A. Melzak, McGill University*

What is the shortest length of infinitely strong, infinitely thin and perfectly inextensible string, with which a smooth sphere of radius 1 can be tied up so that the resulting parcel can be picked up and carried by the string without any danger of slipping apart?

Solution by Michael Goldberg, Washington, D. C. If two nodes are joined by only two strings, a circle is formed. This cannot confine the sphere. If two nodes are joined by three semicircular strings, the sphere is confined in unstable equilibrium since the strings can slide on the smooth sphere to make a nonconfining circle.

If one node is replaced by a small equilateral triangle, each of the three other strings is less than a semicircle and also the sum of two of them and a side of the triangle is less than a great circle, so that they cannot slip to make a full great circle. Therefore, the length of string needed is $3\pi+\epsilon$ where ϵ is as small as we please. There is no shortest length.

A second solution, unsigned, was received.

Related Infinite Series

4888 [1960, 187]. *Proposed by Emil Grosswald, University of Pennsylvania*

Determine the values of α for which the following series converge

$$S_1(\alpha)=\sum_{m=0}^{\infty}\frac{(m+1)^{m-1}}{m!}\alpha^m, \qquad S_2(\alpha)=\sum_{m=0}^{\infty}\frac{(m+1)^{m-1}}{m!}\alpha^{m-1}.$$

Show also for these values of α, $S_2=\log S_1$.

I. *Solution by Robert Weinstock, Oberlin College.* Clearly both series converge for the same values of α. The ratio test gives e^{-1} as the common radius of convergence. Application of Stirling's approximation reveals that $[(m+1)^{m-1}/m!e^m]$ is asymptotic to $(e/\sqrt{2\pi})m^{-3/2}$, so that both series converge for $|\alpha|\leqq e^{-1}$.

Both $S_1(\alpha)$ and $S_2(\alpha)$ are evaluated by Pólya-Szegö: *Aufgaben und Lehrsätze*, abschn. III, problems 209, 210. Indeed, if β is the solution of $\beta e^{-\beta}=\alpha$ that reduces to zero for $\alpha=0$, then $S_1(\alpha)=e^{\beta}$ and $S_2(\alpha)=\beta$. Thus $S_2=\log S_1$, as required.

II. *Solution by S. H. Greene, Moorestown, N. J.* Referring to problem 4868 [1960, 704] we may write

$$S_1(\alpha) = f(1, \alpha) = e^{\alpha S_1(\alpha)}, \qquad S_2(\alpha) = \alpha S_1(\alpha),$$

so that $\log S_1(\alpha) = \alpha S_1(\alpha) = S_2(\alpha)$.

Also solved by Robert Breusch, L. Carlitz, Jane Evans, N. J. Fine, M. S. Klamkin, Yoshio Matsuoka, M. B. Ritterman, E. L. Stout, Alan Wilson, and the proposer.

A Generalization of Wilson's Theorem

4892 [1960, 188]. *Proposed by Andrew Kraus, Boulder, Colorado*

Show that if p is an odd prime and $k \geqq 1$, then

$$(p^{k-1} + 1) \cdots (p^k - 1) + p^{p^{k-1}-k} \equiv 0 \pmod{p^{p^{k-1}}}.$$

Note that for $k = 1$ this is Wilson's theorem.

Solution by Harley Flanders, University of California, Berkeley. We set

$$A_k = (p^{k-1} + 1) \cdots (p^k - 1) = B_k C_k,$$

where B_k is the product of those factors divisible by p and C_k is the product of the remaining factors. Thus

$$B_k = (p^{k-1} + p)(p^{k-1} + 2p) \cdots (p^{k-1} + g_k p),$$

where $g_k = p^{k-1} - p^{k-2} - 1$, $(k \geqq 2)$, hence $B_k = p^{g_k} A_{k-1}$. Next, we let D_k denote the product of all whole numbers less than p^k which are not divisible by p. Clearly $C_k = D_k / D_{k-1}$, and so

$$A_k = p^{g_k} A_{k-1} D_k / D_{k-1}.$$

By recursion we have

$$A_k = p^{p^{k-1}-k} D_k,$$

since $A_1 = D_1$ and $g_2 + \cdots + g_k = p^{k-1} - k$.

The numbers $1, 2, \cdots, p^k - 1$ prime to p may be paired according to multiplicative inverses modulo p^k: $xx' \equiv 1 \pmod{p^k}$. Only 1 and $p^k - 1$ are their own inverses. Hence

$$D_k \equiv -1 \pmod{p^k}, \qquad A_k \equiv -p^{p^{k-1}-k} \pmod{p^{p^{k-1}}}.$$

Also solved by Robert Breusch, L. Carlitz, George Glauberman, Peter Greiner, Emma Lehmer, J. E. Vinson, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Intermediate Algebra for Colleges. (2nd ed.) By Joseph B. Rosenbach, E. A. Whitman, Bruce E. Meserve and Philip M. Whitman. Ginn, Boston, Mass., 1960. xxxi+315 pp. $5.00.

This is a second edition of a popular textbook. It is characterized by clear exposition of elementary concepts and techniques—mostly the latter. The usual material is presented with many illustrations, some "warnings" against usual pitfalls and directives for the solution of problems. The theory of exponents is presented with more than usual care and the solution of linear and quadratic equations is treated algebraically and graphically. Some of the things in this book are not to the reviewer's taste; one of his pet peeves is the treatment usually accorded determinants in elementary algebra books. Pictures of two- and three-rowed determinants are drawn and the crisscross method of evaluation is given. Not one of the basic properties of determinants is mentioned, with the result that the average student gets the impression that a determinant is a complicated formula, something like b^2-4ac, only worse. The division transformation is mentioned neither for integers nor for polynomials; it is reduced to a recipe for obtaining the answer. Why "for Colleges" is in the title is not clear; there is nothing in the book that a high school student should not be expected to understand.

The book is highly teachable, although—in view of all the post-sputnik soul searching—the choice of topics does not seem to meet today's needs. It is high time we stop writing algebraic cookbooks and start developing our young students' reasoning ability and imagination.

M. S. KNEBELMAN
Washington State University

College Arithmetic. By W. I. Layton. Wiley, New York, 1959. ix+200 pp. $3.50.

This textbook presents a review of traditional elementary school mathematics, through the eighth grade. It is written in a style suitable for persons of college age, and is agreeably concise. It is essentially a "cookbook" survey. In mathematical soundness it is on a par with many weak elementary school texts. There are sufficiently many exercises, with answers to odds supplied, others available separately.

ROBERT L. SWAIN
Rutgers, The State University

Finite Markov Chains. By J. G. Kemeny and J. L. Snell. Van Nostrand, Princeton, N. J., 1960. viii+210 pp. $5.00.

The authors have succeeded in presenting this important subject from a fresh and exciting point of view. A number of new and valuable theorems are included. Interest builds steadily as one peruses the book. "Peruse" is used advisedly; the writing leaves nothing to be desired in clarity, in simplicity in the proof, in accuracy of detail, or in organization. While the authors may have failed to treat the material so that it is accessible to most undergraduates, their avowed purpose to judge from the preface, it is certainly true that any undergraduate with sufficient background in probability and matrix theory to follow the first chapter (Prerequisites) will certainly experience no difficulty later on.

There are some minor flaws. These include capricious punctuation, lack of references (a few are included near the end of the book), little attention to historical development such as the omission of any reference to Ellis' work on the problem of the gambler's ruin, stating theorems in terms of weak convergence when strong convergence can be proved, overburdening the text with examples, and so on. But these are at worst venial sins and by no means impair the general excellence of this individualistic, interesting treatment of Markov chains.

K. A. Bush
University of Idaho

Advanced Calculus. By H. K. Nickerson, D. C. Spencer and N. E. Steenrod. Van Nostrand, Princeton, N. J., 1959. 540 pp. $6.50.

Vector analysis was invented by physicists for physicists. The natural beauty of this discipline lies in the semiphysical manner in which its theorems are "derived" or "proved." These terms are put in quotes because the derivations and proofs which are customarily presented lack mathematical rigor, which is largely compensated by invoking physical inspection and intuition. However, all these proofs can be made mathematically exact by supplementary additional mathematical arguments, but this makes these "proofs" often quite awkward, which is the reason for not doing the latter in the first place.

The authors of these lecture notes, which were designed for an Honors Course in Advanced Calculus at Princeton University, strip this discipline in their "proposition-proof-remark" treatment entirely of its natural beauty by putting it into an abstract algebraic setting. (For a detailed account of the content see the review in the Bulletin of the American Mathematical Society, vol. 66, Number 3, p. 148.) By doing this, they gain mathematical rigor of presentation and a rather elegant unification of the entire field, but lose almost everything that makes vector analysis appealing as a discipline to those with an inclination towards physics.

Hans Sagan
University of Idaho

Analytic Trigonometry. By Paul S. Mostert. Prentice-Hall, Englewood Cliffs, N. J., 1960. x+166 pp. $3.95.

This textbook represents a novel departure from the traditional treatment of elementary trigonometry. The author has aimed both at a "modern" analytical approach and sufficient flexibility to make the book available as a supporting text for students of the calculus. To achieve these objectives Dr. Mostert has sandwiched a brief and rather abstract trigonometry text for beginners between a chapter on functions and analytic geometry and five disparate appendices on topics mostly useless for beginners and inadequate for the advanced student.

J. M. FELD
Queens College

Fundamentals of Mathematics. By E. P. Vance. Addison Wesley, Reading, Mass., 1960. x+413+56 pp. $7.50.

The reader familiar with Vance's *Unified Algebra and Trigonometry* (reviewed this MONTHLY, vol. 63, 1956, pp. 54–56) will be familiar with about seventy-five percent of the book under review since Professor Vance has taken the entire contents of his earlier text without appreciable alteration to be the backbone of the present text. There has been some rearrangement; the bulk of the chapter on identities in the circular functions has been moved into the initial chapter on that topic and the sections on progressions have been located in various other chapters. The additional material, listed by chapter titles, is: Linear Functions (largely introductory analytic geometry); Functions, Limits, and Continuity; Derivatives; Applications of Differentiation; Conic Sections; and Integration. Also, in appropriate chapters there have been added sections on the differentiation of the circular and exponential functions and their inverses.

In treatment, the added material might be described as an "improved standard version" of introductory calculus and analytic geometry. There is no material on sets, logic, or language such as is found in many recent books of similar title. One finds out when it is said that one variable y is a function of another variable x, but no definition of a function. However, derivatives are developed with less recourse to variables than has been common; there are no Δy's and the value of the derivative of f at x is denoted $Df(x)$ or $f'(x)$. Thus, while f, Df, f' do not appear, neither does du, dv, etc. The chapter introducing limits appears to be especially clearly and carefully developed (although, the "proposition from plane geometry" alluded to on p. 198, while reasonable—even "obvious"—will not be recalled explicitly by either students or instructor for good reason).

The user should not fail to note that starred exercises are part of the development of the text.

R. N. BRADT
The University of Kansas

Introduction to Higher Mathematics for the General Reader. By Constance Reid. Crowell, New York, 1960. 184 pp. $3.50.

Mrs. Reid has managed to pack an amazing number of interesting and rather deep results from various branches of mathematics into this small volume written for the "general reader." For the most part, she succeeds in explaining rather complicated mathematical ideas in language which should be understandable to the intelligent layman. The topics touched on range from the theory of numbers through group theory, dimension theory, quaternions, non-Euclidean geometries, topology, analytic geometry and calculus to truth tables and Turing machines. A few statements intended for the layman will shock the mathematician—for example, "Zero, being nothing" Some awkward constructions and quite a few obscure passages may tend to alienate the reader who is concerned about literary style. In spite of these flaws, one must admire this attempt to make a great many of the most beautiful achievements of mathematics accessible to the general reader.

JEAN M. CALLOWAY
Carleton College

Basic Concepts of Elementary Mathematics. By William L. Schaaf. Wiley, New York, 1960. 386 pp. $5.50.

This could have been an excellent text for introducing prospective teachers to the foundations of logic, arithmetic, geometry, and algebra. Instead, it is a powerful argument in favor of the thesis that such a text should be written cooperatively by specialists in education and in mathematics. The author is a Professor of Education and, certainly, the text contains a great many clear explanations, with good illustrative examples; all presented from a viewpoint which should be both interesting and useful to a secondary school teacher. Just as certainly, much of this viewpoint would have been lost if the author had been a mathematician. Unfortunately, there are so many statements which are mathematically objectionable that the text might do more harm than good. The following four examples may suffice: Page 23: " . . . equality and inequality are transitive [relations]." Page 117: "What [the principle of mathematical induction] means, essentially, is this: if whatever holds true for n also holds true for $n+1$, then it holds true for all n." Page 149: " . . . an infinite set can never be exhausted by removing one element at a time, nor even by repeated removals of subsets, however large." Page 237: " . . . if an arbitrary segment is applied to the diagonal and to the side of a square, there would always be a remainder in the case of the diagonal, even with an infinite number of repetitions."

B. H. ARNOLD
Oregon State College

An Introduction to the Calculus of Finite Differences and Difference Equations. By K. S. Miller. Holt, Rinehart and Winston, New York, 1960. viii+167 pp. $4.50.

Four chapters are entitled "The Calculus of Finite Differences," "Infinite Products," "Bernoulli Numbers and Polynomials," and "Linear Difference Equations in the Real Domain." Exercises at the respective chapter endings number 110, 51, 23, and 24; few have answers; several request the proofs of results utilized within the chapter.

The main emphasis of the author is the parallelism between his subject and the more familiar discipline which the student has probably already learned: analogy between the difference calculus and the differential calculus, between the sum calculus and the integral calculus, between infinite products and infinite series, between difference equations and differential equations. The comparisons are pertinent, although the more traditional material is not demanded as a prerequisite. Besides the items mentioned in the chapter titles, some topics which receive attention are factorial polynomials, Stirling numbers, the gamma and beta functions, the Euler-Maclaurin formula, asymptotic expansions, Stirling's formula. Formal manipulation with operators is very limited.

The preface says that the "mathematical difficulty of the material increases throughout the book." Fully as appropriate would be the warning that the mathematical background needed by the reader increases throughout the book. In the second chapter about half a page is wasted establishing the continuity of the exponential function ["wasted," because it is difficult to imagine any reader plunged so far into the book and so unaware of the basic properties of the elementary functions]; by contrast, in the third chapter in a repeated two-fold infinite series with both positive terms and negative terms, the order of summation is interchanged with the flip of not a single eyelash.

There are numerous evidences of too little thought devoted by the author to his project. In the proof of one theorem (page 43), the necessity and the sufficiency are interchanged. An infinite product can converge uniformly on an interval even though it fails to converge at every point of the interval. The treatment accorded to polynomials needs much modernization. In one development (page 71), the author motivates by explaining the desirable consequences "if we can show that" a certain relation identified as (16) is true; later on, the reader finds "To verify . . . (16), we have . . . "; but (16) never is established! [Perhaps this is fortunate, since (16) is false.] On page 58, we learn that $\Gamma(1)=1$; on page 75, three full lines are required to establish the fact that $\log \Gamma(1)=0$.

In brief, the worth of the basic theme of analogy has been unfortunately marred by the hurried atmosphere of the output. The publishers chose for subscripts, superscripts, and such symbols a very small-sized type especially designed to provoke eye-strain.

R. A. Good
University of Maryland, University of Oklahoma

Mathematical Statistics. By J. N. Kapur and H. C. Saxena. Chand, Delhi, India, 1960. 400 pp. Rupees 10.00.

The first impression one has on opening this book is one of dismay. Compared with the glossy and opulent appearance of typical American texts, this volume with its greyish thin paper, narrow margins, and crowded print looks like a shabby poor relation. However, it costs only the equivalent of two American dollars and it gives a great deal of value for the money.

The book covers practically all the mathematical statistics taught in Indian universities up to the level of the Master's degree. Readers are supposed to be familiar with the calculus up to the B.A. standard (roughly two years of calculus here), and a couple of brief appendices deal with Gamma and Beta Functions and with the Calculus of Finite Differences. The topics are fairly traditional, including frequency distributions, probability, moments and cumulants, discrete and continuous probability distributions, curve fitting, correlation and regression, exact sampling distributions, tests of significance, estimation and the testing of hypotheses, and sequential analysis, ending with a few pages on distribution-free methods.

From the point of view of a teacher of statistics, the main value of the book lies in the collection of exercises, many of which are taken from Indian examination papers. These exercises are of different degrees of difficulty, some providing simple and direct applications of theory and some presenting a challenge to the brighter students. Anyone looking for problems which are easier than those in Kendall's two big volumes might do worse than use this collection.

Students attempting to read the book might be put off by the rather numerous misprints. They should also be warned that the English is by no means a model. The word "data" is consistently used as if it were singular and "dice" as if it were both singular and plural, and sentences like "Such variation is variously called as dispersion, spread, scatter, . . . " or "The first thing that has been dealt in Statistics is how to condense the data . . . ," occur frequently.

The diagram on page 108 labelled "Cauchy's Curve" is obviously a triangular distribution. The authors are occasionally a little too forthright in their views, as for example in stating that "The mode is ill-defined, indeterminate and indefinite It is not capable of further mathematical treatment." This is not true of many common continuous distributions. With some exercise of caution, however, students in a second course in statistics could learn a great deal from the many illustrative examples worked out in detail.

E. S. KEEPING
The University of Alberta

Special Functions. By E. D. Rainville. Macmillan, New York, 1960. 365 pp. $11.75.

From the staggering wealth of material on special functions, the author has extracted the pertinent material on fifty of these special functions—those most commonly in use—and has emerged with a modest volume which should go far in implementing courses on special functions in colleges and universities. This is no mere compendium of formulae, but a systematic treatment of the special functions singled out for attention together with rigorous proofs of the various properties of these functions. Good sets of exercises make this book particularly attractive as a textbook in a course whose prerequisites should be advanced calculus (including a formal treatment of functions of a complex variable) and differential equations.

One will find here the basic classical properties of these special functions, together with the more recent developments such as the Boas and Buck generating functions for polynomials, the Sheffer classification of polynomial sets, Brafman's generating functions for the various classical orthogonal polynomials, and Sister Celine's technique for deriving pure recurrence relations.

ARTHUR E. DANESE
Union College

Differential Equations (2nd ed.). By R. P. Agnew. McGraw-Hill, New York, 1960. ix+485 pp. $7.50.

This book is a complete revision of the one by the author on the same subject published in 1942. It should prove to be a valuable reference work, as it contains much material not ordinarily found in a first text on differential equations, such as chapters on Laplace transforms, bending of beams, series expansions of operators and the Picard method of approximation. Though the author states that a knowledge of the calculus is all that is needed by students who use this as a text, it is doubtful that a student with just this background, who does not have an exceedingly avid love for mathematics, would be at all happy with this work. To appreciate it, one must have a good background in physics and either have at hand a knowledge of much advanced mathematics or be willing to learn such mathematics in addition to the theory of differential equations. However, the exceptional student with the appropriate background and willingness to work would have learned a great deal about the theory of differential equations after studying this text, but might be weak in ability to mechanically solve a differential equation.

RALPH G. SANGER
Kansas State University

Elementary Analysis, A Modern Approach. By H. C. Trimble and Fred W. Lott, Jr. Prentice-Hall, Englewood Cliffs, N. J. 1960. xii+621 pp. $6.95.

This carefully written text is designed to prepare the entering freshman for a course in calculus by increasing his understanding of what mathematics really is. In addition to the usual topics of the freshman year, the idea of vector is introduced, there is a section on matrices, and in an optional chapter a simple mathematical structure is developed and the ideas of group and field introduced. Set-theoretic language and symbolism are used throughout (no set theory as such is included), and functional relationships are presented as sets of ordered pairs in a Cartesian product space. In addition, the structure of the set of real numbers, as developed in the first two chapters, continues to be a unifying idea that is constantly emphasized.

The style is simple and informal, the treatment thorough, and calculated to keep the student thinking. The many problems and exercises are designed to increase familiarity or understanding, seldom manipulative skill, and answers are given only when the authors feel they are necessary to let the student know whether he is on the right track; a few more might be given.

The reviewer particularly liked the reduction of separate "topics" to a minimum (*e.g.*, both types of progressions are developed naturally in connection with linear and exponential functions, respectively), and also the inclusion of certain "exploratory" sections, such as those in which signed, rational, and complex numbers are represented as sets of ordered pairs and their algebraic structures developed.

The material is designed to fill two four-hour semesters, but if it is to be followed by an "integrated" calculus course, it appears possible to cut it to one such course without sacrificing its most important features. The reviewer believes that a good high school teacher, well-trained in traditional mathematics, could teach the first six chapters of this text (roughly half the material) to senior high school students, with profit to all concerned. It would be fun to try.

BESS E. ALLEN
Wayne State University

An Introduction to Modern Mathematics. By Robert W. Sloan. Prentice-Hall, Englewood Cliffs, N. J., 1960. xi+73 pp. $3.75.

This small volume (only seventy-three pages) provides a readable introduction to many of the up-to-the-minute advances in modern mathematics, including equational and symbolic logic, miniature axiomatic systems, truth sets, open sentences, and functions. The author has shown imagination in his choice of clever examples, has included sufficient exposition of the basic principles of elementary mathematics, and has left out much that is not necessary for a student seeking an introduction to modern mathematics.

The reviewer has found this book particularly useful for two groups of peo-

ple, namely, high school students who are interested in modern mathematics but do not hear of it in their classes, and college freshmen and sophomores who have come through traditional courses in algebra and trigonometry and desire to fill in gaps for themselves. Although the book was used in an independent-study manner with both of these groups, it could be used to an advantage by high school teachers in developing supplementary units in modern mathematics, particularly at the senior level.

VIRGINIA CARLTON
Centenary College

Algebra and Trigonometry. By Edward A. Cameron. Holt, Rinehart and Winston, New York, 1960. xi+290 pp. $5.00.

This textbook gives a modern treatment of algebra and trigonometry which is adequate in content and simple rigor for students who will continue in mathematics. Brief axiomatic treatments of real numbers, sets, and functions, with complex numbers introduced as ordered pairs, enhance the discussions of the conventional algebraic material. The trigonometry is presented in two chapters using general angles leading to all the standard methods and identities. Included are ample lists of exercises in each section, diagrams, graphs, demonstrations, tables, answers to odd-numbered exercises, and an index.

DAVID B. DEKKER
University of Washington

Axiomatic Set Theory. By Paul Bernays and Abraham A. Fraenkel. *Studies in Logic and The Foundations of Mathematics.* Edited by L. E. J. Brouwer, E. W. Beth and A. Heyting. North Holland Publishing Company, Amsterdam, 1958. viii+226 pp. $6.00.

Set theory has come to play a very important role in what is called "modern mathematics." Modern texts are increasingly written from a set-theoretic standpoint, but the approach is usually intuitive or, in fact, often quite naive. Introductory textbooks leave the impression that although set theory is basic to mathematics it is not itself a rigorous mathematical theory. Moreover, the exciting years when mathematics was in a stage of crisis because of the antinomies that resulted from the naive concept of a set have apparently been forgotten. It is not widely known that set-theory has many interesting problems and is a mathematical system in its own right. *Axiomatic Set Theory* tries to avoid the antinomies on the one hand, to serve as a true basis for classical mathematics on the other, and to develop its own results as well. The attempts to provide an axiomatic set theory serve to make of axiomatic set theory the "missing link" between logic and mathematics.

The present book is divided into two main parts. Professor Fraenkel has written an historical introduction of some 35 pages, while Professor Bernays presents a formal system of axiomatic set theory with indications of the method

of obtaining cardinal arithmetics and abstract algebras in Chapter VII. The book presupposes an acquaintance with the problems of axiomatics. Unfortunately the rather clumsy English tends to mar an otherwise valuable book.

LOUIS O. KATTSOFF
Boston College

BRIEF MENTION

A Collection of Mathematical Problems. By S. M. Ulam. Interscience, New York, 1960. xiii+150 pp. $5.00.

This collection of unsolved problems in various fields of mathematics is truly an outstanding contribution. The first 114 pages are devoted to various mathematical problems which will try the soul of any mathematician, pure or applied. The last 29 pages are devoted to the important changes which modern high speed computers are making in mathematical research, in particular with their use as a heuristic aid in enlarging the experimental background from which new mathematical theory is projected. This book is highly recommended for the private library of every mathematician who has research-level students as well as for mathematical libraries and computer laboratories.

Proceedings of the International Congress of Mathematicians 1958. Edited by J. A. Todd. Cambridge University Press, New York, 1960. lxiv+573 pp. $12.50.

This is another book which research mathematicians will welcome. The official record of the Edinburgh International Congress contains the complete texts of the 17 one-hour and the 33 half-hour addresses, as well as a list of the short communications.

Mathematical Methods and Theory in Games, Programming, and Economics, Volumes I and II. By Samuel Karlin. Addison-Wesley, Reading, Mass., 1959. x+433 and xi+386 pp. $10.75 each.

Volume I is devoted to matrix games, programming, and mathematical economics while Volume II considers the theory of infinite games. There is a good deal of hard, solid mathematics in these two volumes as well as a wealth of illustrative material. Mathematicians who tend to be a bit snobbish about the use of mathematics in the social sciences will do well to examine this two-volume work.

Monograph 17, Theory of Value. By Gerard Debreu. Wiley, New York, 1959. xii+114 pp. $4.75.

This axiomatic analysis of economic equilibrium is another publication by the Cowles Foundation for Research in Economics at Yale University. The first chapter presents an interesting treatment of some of the mathematical background needed by today's economists (sets, functions and correspondences, preordering, limits, continuity, vectors, and fixed points). The authors feel that these topics provide a background which permit an analysis which is both more natural and more general than that provided through traditional calculus courses.

The Special Theory of Relativity. By J. Aharoni. Oxford University Press, New York, 1959. viii+285 pp. $7.20.

A text using tensors and spinors to explain physical theory which satisfy the theory of relativity which is developed from its foundations. No pre-knowledge of relativity, tensor analysis or group theory is assumed.

Theory of Mechanical Vibration. By Kin N. Tong. Wiley, New York, 1960. xii+348 pp. $9.75.

If you wish to see matrices and eigenvalue problems in practical use, this is the book for you. The major problems, however, are still stated in terms of partial differential equations for elasticity, structural damping and mechanical vibration in general.

Dimensions, Units, and Numbers in the Teaching of Physical Sciences. By Renée G. Ford and Ralph E. Cullman. Bureau of Publications, Teachers College, Columbia University, New York, 1959. 49 pp. $1.00.

This is a possible source book for teachers, particularly on the high school level, who wish to give a touch of dimension analysis in their courses. Presumably, it will be of more use to physics and chemistry teachers than mathematics teachers, but merits mention.

Advanced Engineering Mathematics (2nd ed.). By C. R. Wylie, Jr. McGraw-Hill, New York, 1960. xi+696 pp. $9.00.

Teachers of engineering mathematics should welcome this enlarged and revised edition of Wylie's book. The main changes in the subject matter include the addition of a chapter on determinants and matrices, and one on finite differences. However, other chapters have also been expanded and the section on fluid mechanics omitted.

German-English Mathematics Dictionary. By Charles Hyman. Interlanguage Dictionaries Publishing Corp., New York, 1960. 129 pp. $8.00.

Readers may well be somewhat disappointed in this book, although it is certainly an attempt to meet a serious need. Your reviewer has found the list in the back of the new James and James Mathematical Dictionary more helpful.

Symbolic Logic. By Clarence Irving Lewis and Cooper Harold Langford. Dover, New York, 1960. 518 pp. $2.00.

A republication of the authors' 1932 and earlier works.

Theory of Probability. By William Burnside. Dover, New York, 1960. 106 pp. $1.00.

A republication of the 1928 edition of Burnside's work during the previous decade.

Guide to Government Grants and Contracts, Federal Agencies Financing Research, Document 14. Social Legislation Information Service, Washington, D. C., 1960. 28 pp. $1.00.

Theory of Differential Equations. By A. R. Forsyth. Dover, New York, 1960. Set of three volumes. xiii+344, x+534, and xx+596. $15.00 set.

This six volume work bound as three volumes contains Forsyth's treatises originally published about the turn of the century.

The First Six Million Prime Numbers. By C. L. Baker and F. J. Gruenberger. The Microcard Foundation, Madison, Wis., 1959. $35.00.

This is an interesting and somewhat unique presentation of a large volume of tabular material. Using an IBM 704, the first six million prime numbers have been computed and tabulated on 4,800 pages with 1,250 numbers to a page. On each line of the page the first prime is displayed explicitly, the remaining 24 show only their terminal 3 digits. 39 pages of tabulation are then reduced photographically and placed on one side of a 3×5 card with another 39 tabulation pages on the other side. In this fashion the entire 4,800 pages are reduced to 62 cards which are legible using a 5 power magnifying glass. Since a table of large prime numbers is not apt to be consulted too frequently, this technique may well

be the answer to the librarian's storage problem. Presumably, if the price could be dropped to one third of its present level, many mathematicians would be interested in securing copies for their own libraries.

The Gentle Art of Mathematics. By Dan Pedoe. Macmillan, New York, 1959. 143 pp. $3.50.

Another enjoyable collection of 19th century recreational mathematics suitable for light reading.

String Figures, and Other Monographs. By W. W. R. Ball, J. Petersen, H. S. Carslaw and F. Cajori. Chelsea, New York, 1960. 528 pp. $3.95.

This is actually a collection of four small books published under one cover: *String Figures* by Ball, *Methods and Theories for the Solution of Problems of Geometrical Construction* by Petersen, *Non-Euclidean Plane Geometry and Trigonometry* by Carslaw, and *A History of the Logarithmic Slide Rule* by Cajori.

How to Organize and Conduct Mathematics Clubs, Bulletin Number 13. By W. W. Sawyer. Department of School Services and Publications, Wesleyan University Press, Education Center, Columbus 16, Ohio. *Free.*

Dr. Sawyer's gifted pen gives considerable encouragement to those wishing to organize high school mathematics clubs. The National High School and Junior College Mathematics Club, Mu Alpha Theta, which is sponsored by the Mathematical Association of America, is credited with being "an extremely helpful organization" in this pamphlet, but the bulletin should prove interesting and useful for any high school teacher of mathematics. Our sincere congratulations to Dr. Sawyer and to Wesleyan University Press for making this bulletin available without charge.

Five Little Stories. By William W. Strader. National Council of Teachers of Mathematics, Washington, D. C., 1960. 16 pp. 50¢.

Possibly a little elementary even for high school students, but if even one student becomes interested in a mathematical problem through this book, your fifty cents is well spent.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Leo Moser, University of Alberta, Edmonton, Alberta, Canada, has been elected a Fellow of the Royal Society of Canada.

Dr. Katharine E. O'Brien, Head of the Department of Mathematics at Deering High School, Portland, Maine, received an honorary degree of Doctor of Science in Education from the University of Maine, August, 1960.

Professor J. B. Rosser, Cornell University, has been appointed Chairman of the Division of Mathematics of the National Academy of Sciences—National Research

Council, to succeed Professor S. S. Wilks of Princeton University.

Arizona State University: Associate Professors E. D. Nering, University of Arizona, and Simon Green, Assumption University of Windsor, Windsor, Ontario, Canada, have been appointed Professors; Associate Professor B. M. Ingersoll, Lamar State College of Technology, has been appointed Associate Professor; Mr. R. W. Sanders, Lockheed Aircraft Corporation, Palo Alto, California, has been appointed Assistant Professor.

Auburn University: Dr. Emilie Haynsworth, Bureau of Standards, Washington, D. C., has been appointed Associate Professor; Mr. W. D. Ray, Arizona State University, has been appointed Instructor; Associate Professors R. W. Ball and L. P. Burton have been promoted to Professors; Professor W. W. Wilson has retired as a member of the Mathematics Staff and is teaching at Millsaps College this year.

Boston College: Dr. G. G. Bilodeau, Sylvania Electronics Systems, Waltham, Massachusetts, has been appointed Assistant Professor; Mr. P. J. Sally, Jr., Brandeis University, has been appointed Instructor; Associate Professor J. A. Sullivan, University of Notre Dame, has been appointed Professor.

Bowling Green State University: Dr. R. N. Townsend, San Jose State College, has been appointed Assistant Professor; Mr. J. W. Ridge, West Virginia University, has been appointed Intern Instructor.

Brigham Young University: Mr. Kenneth Larsen, University of California, Los Angeles, has been appointed Assistant Professor; Mr. Gurcharin Gill, University of Utah, Mr. Richard Davidson, and Mr. Burton Gee, Murray Hill High School, Provo, Utah, have been appointed Instructors.

Butler University: Mr. Robert Lamberson, Purdue University, has been appointed Instructor; Assistant Professor Juna L. Beal retired June, 1960.

Carnegie Institute of Technology: Dr. R. N. Pederson, Massachusetts Institute of Technology, has been appointed Assistant Professor; Dr. Michael Branson, University College, London, England, has been appointed Research Mathematician; Assistant Professors R. A. MacCamy, A. D. Martin, R. A. Moore and A. F. Strehler have been promoted to Associate Professors; Associate Professors Walter Noll and Alan Perlis have been promoted to Professors.

Case Institute of Technology: Mr. Martin Levy, Sacramento State College, has been appointed part-time Instructor; Assistant Professor Bayard Rankin has been promoted to Associate Professor; Professor R. A. Clark has been appointed Acting Head of the Department of Mathematics; Professor Max Morris was given the title Professor Emeritus in July, 1960, but still continues his teaching duties.

Colorado State University: Professor F. A. Graybill, Oklahoma State University, has been appointed Professor; Mr. M. S. Blevins and Mr. K. J. Whitcomb have been promoted to Assistant Professors; Assistant Professors W. C. Butler, L. C. Hayward, R. H. Niemann, and J. M. Staley have been promoted to Associate Professors; Associate Professor M. L. Madison has been promoted to Professor.

Dartmouth College: Professor L. A. Henkin, University of California, Berkeley, has been appointed Visiting Professor; Dr. D. L. Kreider, Massachusetts Institute of Technology, has been appointed Assistant Professor; Mr. R. I. Ritchie, Princeton University, has been appointed Resident Instructor; Assistant Professor Mirkil has been promoted to Associate Professor.

Grinnell College: Professor Emeritus E. S. Allen, Iowa State University, has been appointed Visiting Professor; Associate Professor L. E. Pursell has been appointed Chairman of the Department of Mathematics.

Georgetown University: Mr. M. R. Mullen and Mr. J. R. Hock, Catholic University, have been appointed Instructors; Assistant Professor A. K. Aziz has been awarded a research fellowship sponsored jointly by the National Academy of Science and the United States Naval Ordnance Laboratory and will be on leave of absence for 1960–61.

Hampton Institute: Miss Geraldine C. Darden and Mr. S. A. Phelps have been appointed Instructors.

Kansas State University: Miss Beatrice L. Hagen, Pennsylvania State University, has been appointed Temporary Associate Professor; Assistant Professor R. L. Yates, University of Houston, and Dr. J. W. Meux, University of Florida, have been appointed Assistant Professors; Mr. L. W. Manuel, University of Illinois, Mr. M. H. Miller, Iowa State University and Mr. J. A. Smith, Ball State Teachers College, have been appointed Instructors; Professor R. W. Babock has retired with the title Professor Emeritus.

McMaster University: Assistant Professor G. O. Sabidussi, Tulane University, has been appointed Associate Professor; Mr. J. D. Kenworthy, Oxford University, Oxford, England, has been appointed Assistant Professor; Mr. I. Z. Chorneko, University of Alberta, Edmonton, Alberta, Canada, and Mr. H. L. Jackson, University of Illinois, have been appointed Lecturers; Assistant Professor W. J. McCallion has been promoted to Associate Professor.

New Mexico State University: Dr. J. M. Irwin and Dr. E. D. Gaughan, University of Kansas, have been appointed Assistant Professors; Mr. Louis Child retired June, 1960.

Northwestern University: Dr. Hsin Chu, University of Michigan, Assistant Professor D. S. Greenstein, University of Michigan, and Dr. A. I. Weinzweig, University of California, Berkeley, have been appointed Assistant Professors; Drs. R. J. Crittenden, Wellesley College, Michael Geraghty, Northwestern University, and Edward Kobayashi, University of Washington, have been appointed Visiting Assistant Professors; Dr. Kenneth Mount, University of California, Berkeley, has been appointed Instructor; Associate Professor Hidehiko Yamabe, University of Minnesota, has been appointed Professor; Associate Professors Teruhisa Matsusaka and Daniel Zelinsky have been promoted to Professors; Professor H. T. Davis retired September 1, 1960 and now is at Southwest Foundation for Research and Education, San Antonio, Texas.

St. Lawrence University: Mr. J. J. Kinney, Instructor, has been promoted to Assistant Professor; Associate Professor F. C. Warner has been promoted to Professor.

South Dakota School of Mines and Technology: Assistant Professor D. C. Benson, Chico State College, has been appointed Associate Professor; Professor L. C. Barrett has been appointed Head of the Department of Mathematics.

State University of Iowa: Assistant Professors D. W. Wall, University of North Carolina, and Hiroshi Uehara, University of Southern California, have been appointed Associate Professors; Dr. Motoyoshi Sakuma, Tokushima University, Tokushima City, Japan, has been appointed Visiting Lecturer for the academic year 1960–61; Mr. J. C. Hickman has been appointed Instructor; Assistant Professor S. K. Berberian has been promoted to Associate Professor.

State University of South Dakota: Mr. Robert Walter and Mr. Robert Schwabauer, University of Nebraska, have been appointed Instructors.

Stevens Institute of Technology: Mr. J. F. Manogue, City College of New York, has been appointed Assistant Professor; Professor Ainsley Diamond, Webb Institute, has been appointed Professor; Associate Professor N. J. Rose has been promoted to Professor and Head of the Department of Mathematics; Assistant Professors P. D. Ritger and Lawrence Goldman have been promoted to Associate Professors.

United States Military Academy: Associate Professor J. S. B. Dick has been promoted to Professor and appointed Deputy Head of the Department of Mathematics; Assistant Professors G. W. Bixby and R. E. Plett have been appointed Associate Professors.

United States Naval Academy: Mr. E. P. Berger and Mr. D. L. Muench have been appointed Assistant Professors; Associate Professors J. M. Home and J. F. Paydon have been promoted to Professors; Assistant Professor Richard Molloy has been promoted to Associate Professor.

University of Alaska: Mr. E. J. Gauss, University of California, Los Angeles, has been

appointed Assistant Professor; Mr. Donval Simpson, Appalachian State Teachers College, has been appointed Instructor.

University of Alabama: Professor R. W. Bagley, Mississippi Southern College, has been appointed Associate Professor; Dr. W. L. Wilson, Jr., North American Aviation Corporation, Los Angeles, California, has been appointed Assistant Professor; Mrs. Ayrlene M. Jones is on leave this year as Associate Professor at Millsaps College; Associate Professor H. C. Filgo, Jr., is on leave this year as Visiting Professor at the University of Georgia.

University of Alberta: Professor S. M. Shah, Northwestern University, has been appointed Visiting Professor; Dr. W. F. Sharp, Atomic Energy of Canada, Limited, Chalk River, Ontario, Canada, has been appointed Associate Professor; Dr. R. G. Sinclair, Massachusetts Institute of Technology, has been appointed Assistant Professor; Associate Professor Leo Moser has been promoted to Professor.

University of Buffalo: Mr. R. J. Benice, Harvard University, Dr. Judith R. Blankfield, University of Illinois, and Mr. William Young, Pennsylvania State University, have been appointed Instructors.

University of California, Davis: Dr. D. A. Pope, University of Minnesota, has been appointed Associate Professor and will direct activities at the computing Center; Dr. Kurt Kreith, University of California, Berkeley, has been appointed Assistant Professor; Assistant Professor Takayuki Tamura, Tokushima University, Tokushima City, Japan, has been appointed Lecturer.

University of California, Los Angeles: Professor S. T. Hu, Wayne State University, and Associate Professor H. A. Dye, University of Iowa, have been appointed Professors; Associate Professors R. M. Redheffer and E. G. Straus have been promoted to Professors.

University of Hawaii: Mrs. Shirley O. Hockett, Cornell University, and Mrs. Elaine La V. Tathum, University of Kansas, have been appointed Instructors; Assistant Professor Hiroshi Yamuchi has been promoted to Associate Professor.

University of Michigan: Associate Professor H. H. Schaefer, Washington State University, has been appointed Associate Professor; Professors Lamberto Cesari, University of Wisconsin, Mathematics Research Center, and L. J. Savage, University of Chicago, have been appointed Professors; Drs. A. J. Cohn, Harvard University, L. C. Eggan, University of Oregon, and C. N. Lee have been appointed Instructors; Drs. D. R. Hughes, University of Chicago and G. J. Minty, University of Washington, have been appointed Assistant Professors; Dr. W. M. Kincaid has been promoted to Assistant Professor; Assistant Professors D. G. Higman, R. K. Ritt, A. L. Shields, C. J. Titus and Oscar Wesler have been promoted to Associate Professors; Associate Professor W. J. Leveque has been promoted to Professor; Professor H. C. Carver retired September, 1960.

University of Montreal: Dr. Aubert Daigneault, Royal Military College, Kingston, Ontario, Canada, has been appointed Assistant Professor; Associate Professor Jacques St-Pierre has been promoted to Professor.

University of New Mexico: Assistant Professors James Abbott and Judah Rosenblatt, Purdue University, and Heinz Renggli, Rutgers University, have been appointed Associate Professors; Mr. Edwin Wagner, University of Nevada, has been appointed Instructor; Associate Professor F. C. Gentry has been promoted to Professor.

University of Oklahoma: Dr. G. M. Ewing, Fort Sill, Oklahoma, has been appointed Professor; Professor H. V. Huneke, University of Wichita, has been appointed Associate Professor; Dr. J. C. Mathews, Iowa State University, has been appointed Assistant Professor.

University of Ottawa: Dr. G. J. van der Maas, Kansas State College, has been appointed Associate Professor; Dr. D. B. J. Tomiuk, Fordham University, has been appointed Assistant Professor.

University of Saskatchewan: Associate Professor A. P. Guinand, University of Alberta,

has been appointed Professor; Assistant Professor Naoki Kimura, University of Washington, and Dr. C. L. Kaller, Purdue University, have been appointed Assistant Professors; Assistant Professor S. A. Husain, University of Seattle, has been appointed Special Lecturer; Mr. Lucio Artiaga, Nova Scotia Tumour Clinic, Halifax, and Mr. W. W. Zwirner, University of Saskatchewan, have been appointed Instructors.

University of Washington: Dr. T. K. Boehme, California Institute of Technology, and Mrs. Jean Butler, University of Arizona, have been appointed Instructors; Dr. P. L. Crawley, California Institute of Technology, has been appointed Research Assistant Professor; Drs. Ludwig Danzer, Munich Institute of Technology, Munich, Germany, Branko Grunbaum, University of Kansas, and Veeravalli Varadajan, Princeton University, have been appointed Visiting Assistant Professors; Dr. Anne Morel, Institute for Advanced Study, has been appointed Associate Professor; Dr. Gerald Paulick, University of Chicago, and Dr. Kenneth Ross have been appointed Research Instructors; Assistant Professor Ronald Pyke, Columbia University, and Dr. R. W. Richardson, Jr., Princeton University, have been appointed Assistant Professors; Dr. Jack Segal, University of Georgia, has been appointed Instructor; Dr. J. L. Selfridge, International Business Machines, Yorktown Heights, New York and Professor Mavek Fisz, University of Warsaw, Poland, have been appointed Visiting Associate Professors; Assistant Professors J. M. G. Fell, R. K. Getoor, and J. P. Jans, have been promoted to Associate Professors; Associate Professors E. A. Michael and R. S. Pierce have been promoted to Professors; Drs. Kathleen B. O'Keefe and W. B. Woolf have been promoted to Assistant Professors. On leave for 1960–61 are: Professor Z. W. Birnbaum, Guggenheim Fellowship; Associate Professor J. M. G. Fell, N. S. F. Grant, Harvard University; Professor E. A. Hewitt, N.S.F. Senior Post-Doctoral Fellowship; Associate Professor J. P. Jans, Air Force Research Associateship, Princeton University; Professor E. A. Michael, Guggenheim Fellowship, Institute for Advanced Study; and Assistant Professor J. H. Walter, Visiting Assistant Professor, University of Chicago.

University of Western Ontario: Dr. James Talman, American University, Beirut, Lebanon, has been appointed Assistant Professor; Mr. C. M. Glennie, Yale University, has been appointed Instructor; Dr. J. H. Blackwell has been appointed Lecturer.

Utah State University: Dr. N. C. Hunsaker, Head of the Department of Mathematics, has returned from his leave at the University of Illinois; Professor V. H. Tringey retired in June, 1960.

Vanderbilt University: Dr. G. F. Clanton, University of Minnesota, has been appointed Assistant Professor; Assistant Professors B. F. Bryant and J. R. Wesson have been promoted to Associate Professors.

Woman's College, University of North Carolina: Mrs. Ruby B. Smith, Asheboro High School, Asheboro, North Carolina and Mr. J. D. Jones, University of North Carolina, have been appointed Instructors; Professor Helen Barton, Head of the Department of Mathematics, retired July 1, 1960, with the title Professor Emeritus and has accepted a part-time teaching appointment for this year.

Mr. Walter Abramowitz, Control Instrument Company, Incorporated, Brooklyn, New York, has accepted a position as Research Engineer with the Grumman Aircraft Engineering Corporation, Bethpage, Long Island, New York.

Mrs. Sarabeth T. Barnes, University of Minnesota, has accepted a position as Instructor at Montana State College.

Assistant Professor E. H. Batho, University of Rochester, has been appointed Associate Professor at the University of New Hampshire.

Assistant Professor P. R. Beesack, McMaster University, Hamilton, Ontario, Canada, has been appointed Associate Professor at Carleton University, Ottawa, Canada.

Dr. L. N. Bidwell, University of Pennsylvania, has been appointed Assistant Professor at Haverford College.

Miss Anne E. M. Bode, University of Saskatchewan, Saskatoon, Saskatchewan, Canada, has been appointed Instructor at Huron College.

Professor L. H. Bunyan, Rutgers University, has been appointed Professor at the College of William and Mary in Norfolk.

Associate Professor J. H. H. Chalk, McMaster University, Hamilton, Ontario, Canada, has been appointed Associate Professor at the University of Toronto, Toronto, Ontario, Canada.

Dr. Roger Chalkley, University of Cincinnati, has been appointed Assistant Professor at Knox College.

Mr. V. G. Child, University of Kansas, has been appointed Teacher, Department of Army, Washington, D. C.

Professor Esther Comegys, University of Maine, has been appointed Lecturer at Wellesley College.

Mr. C. R. Deeter, University of Kansas, has been appointed Assistant Professor at Texas Christian University.

Associate Professor F. C. DeSua, Northern Illinois University, has been appointed Professor and Chairman of the Department of Mathematics at Simmons College.

Assistant Professor E. J. Downie, Colgate University, has been promoted to Associate Professor.

Mr. Herbert Farkas, Brandeis University, has been appointed Instructor at Illinois Institute of Technology.

Dr. R. W. Heath, Woman's College, University of North Carolina, has been appointed Assistant Professor at the University of Georgia.

Associate Professor G. K. Horton, University of Alberta, Edmonton, Alberta, Canada, has been appointed Professor of Theoretical Physics at Rutgers University.

Associate Professor C. W. Huff, Auburn University, has been appointed Professor at Winthrop College.

Miss Rora F. Iacobacci, University of Pennsylvania, has been appointed Lecturer at Hunter College.

Mr. J. J. Kim, Eastern New Mexico University, has accepted a position as Actuarial Student with the Aetna Life Insurance Company, Hartford, Connecticut.

Assistant Professor R. M. Kozelka, Williams College, has been promoted to Associate Professor.

Dr. H. P. Kuang, Agricultural & Technical College of North Carolina, has been appointed Professor at the North Dakota Agricultural College.

Assistant Professor A. E. Livingston, University of Washington, has been appointed Chairman of the Department of Mathematics at Montana State University.

Dr. F. P. Palermo, University of Michigan, has accepted a position as Staff Mathematician with the International Business Machines Corporation, Yorktown Heights, New York.

Mr. D. L. Persico, Lockheed Aircraft Company, Burbank, California, has accepted a position as Research Engineer with Rocketdyne Incorporated, Division of North American Aviation, Canoga Park, California.

Professor R. S. Phillips, University of California, Los Angeles, has been appointed Professor at Stanford University.

Mr. P. E. Reed, Geneva Public High Schools, has been appointed Instructor at Hobart College.

Dr. James Reid, University of Washington, has been appointed Instructor at Syracuse University.

Mr. L. H. Rhodes, II, Holloman Air Force Base, New Mexico, has accepted a position with the Sales Support Division of Remington Rand UNIVAC, Dallas, Texas.

Mr. G. R. Rising, Greece Olympia High School, Rochester, New York, has been ap-

pointed Mathematics Coordinator for the Norwalk Public Schools, Norwalk, Connecticut.

Dr. D. J. Schaefer, Ohio State University, has accepted a position as Research Engineer with the Lockheed Aircraft Company, Santa Cruz, California.

Dr. S. A. Schelkunoff, Bell Telephone Laboratories, Murray Hill, New Jersey, has been appointed Professor of Electrical Engineering at Columbia University.

Dr. Tetsundo Sekiguchi, Oklahoma State University, has been appointed Assistant Professor at the University of Arkansas.

Dr. Richard Sheer, University of New Mexico, has been appointed Assistant Professor of Philosophy at South Dakota State College.

Dr. D. R. Shreve, Oklahoma State University, has joined the staff of North Carolina State College as Head of the Computing Laboratory.

Sister Marie Blanche, Notre Dame University, has been appointed Chairman of the Department of Mathematics at Providence High School, Chicago, Illinois.

Sister Mary Canisia, Holy Name Academy, Chicago, Illinois, has been appointed Vice-President and Chairman of the Department of Mathematics at DeLourdes College.

Professor M. F. Smiley, State University of Iowa, has been appointed Professor at the University of California, Riverside.

Mr. R. K. Smith, Space Technology Laboratories, Los Angeles, California, has accepted a position as Research Specialist with the Boeing Airplane Company, Seattle, Washington.

Professor H. P. Thielman, Iowa State University, has accepted a position as Mathematician, Technical Staff, with the Land-Air, Incorporated, Pacific Missile Range, Point Mugu, California.

Assistant Professor W. C. Turner, Baylor University, has been appointed Assistant Professor at the College of William and Mary.

Mr. George Van Zwalenberg, Bowling Green State University, has been appointed Lecturer at Calvin College.

Associate Professor W. L. Fields, Hampton Institute, died May 29, 1960. He was a member of the Association for nineteen years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor Henry L. Alder, Secretary, announces that the following 44 persons have been elected to membership by the Board of Governors on applications duly certified.

ROGER C. ALLEN, Student, Oklahoma State University

DAVID H. ALSPAUGH, Student, Southwest Missouri State

JAMES P. ANDERSON, M.S. (S. Carolina) Asst. Prof., College of Charleston

CIRIL ARKO, Diploma (Ljubljana) Teacher, LaJunta High School, Colorado

MRS. BARBARA C. BLACK, B.S. (Florida A & M) Instr., Florida A and M University

JOHN E. BUCHANAN, B.S. (Purdue) Grad. Student, Purdue University

ROBERT A. BUTTON, B.S. (Washington) Instr., Olympic Junior College

VINCENT A. CIMINERA, M.S. (St. Johns) Jr. Operations Research Analyst, I.B.M. Corp., Owego

LIONEL E. CLARKE, Ph.D. (Cambridge) Senior Lecturer, University College of Ghana

JAMES M. DICKEY, III, Student, University of Redlands

MRS. DOROTHY FISH, A.B. (Rockford Coll.) Teacher, Pandora Gilboa High School, Ohio

ANDREW E. FORD, JR. B.A. (Wesleyan) Editor, John Wiley & Sons, Inc.
MARVIN GOODMAN, M.S. (New York) Teacher, Jamaica High School, Queens, New York
COLLIN J. HIGHTOWER, B.S. (Arkansas) Grad. Asst., Tulane University
JOHN W. HOOKER, M.A. (Minnesota) Asst. Prof., State University of New York, College of Education at Oswego
LEONARD R. HOWELL, JR., M.S. (Emory) Instr., U. S. Air Force Academy, Colorado
STEVEN L. HUNT, Student, Michigan State University, Oakland
GARY DON JOHNSON, B.S. (Stanford) Grad. Student, University of California, Los Angeles
BJARNE JUNGE, B.Sc. (California) Asso. Engr., I.B.M. Corp., San Jose
WILLIAM M. KANTOR, Student, Brooklyn College
BARRY KREEL, B.S. (Witwatersrand) Master, Fletcher High School, Gwelo, Southern Rhodesia
RICHARD W. LAMBERT, B.S. (Oregon State Coll.) Teacher, David Douglas High School, Portland, Oregon
MELVYA S. LYNN, M.A. (California) Grad. Research Mathematician, University of California, Los Angeles
MRS. JUNE R. MALAN, M.S. (Kansas) Teacher, Harrisburg Township High School, Illinois
KEITH E. MORRIS, A.A. (Long Beach Coll.) Electronic Research Mech., Autonetics
RAYMOND J. NADEAU, B.Sc. (Alberta) Vice-Principal, Barrhead Senior High School, Alberta
VANCE D. NORUM, M.S. (Wisconsin) Member, Tech. Staff, Space Technology Labs.
EDGAR M. PALMER, M.S. (Trinity Coll.) Teaching Fellow, University of Michigan
JON EARL PETERSEN, Student, Stanford University
JOHN A. PETERSON, M.A. (Montana State) Asst. Prof., Montana State University
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THE CONFERENCE BOARD AND THE WASHINGTON OFFICE

The Conference Board of the Mathematical Sciences had its origin in the War Policy Committee,* which was appointed by the American Mathematical Society and the Mathematical Association of America at the end of 1942 to deal with some of their common problems which arose out of World War II. The War Policy Committee was supported by a grant from the Rockefeller Foundation. The war over, the War Policy Committee was discharged in November, 1945. The American Mathematical Society immediately took the lead in the formation of the Mathematical Policy Committee, usually known simply as the Policy Committee.† This Committee grew and eventually had six of the mathematical organizations as its members.

In 1958 the Policy Committee was developed into the Conference Organization of the Mathematical Sciences, and a constitution and by-laws were drawn up. In December, 1958, the Mathematical Association of America received a grant from the Carnegie Corporation of New York for the establishment of a Washington Office. At its Salt Lake City meeting in 1959 the Association recommended that the Washington office be established by the Conference Organization with the Carnegie grant. The Conference Organization accepted the responsibility for establishing the Washington Office with the Carnegie grant, and on February 25, 1960, the Conference Organization was incorporated in the District of Columbia with the new name Conference Board of the Mathematical Sciences. G. Baley Price, who had been appointed the first Executive Secretary, opened the Washington office on July 1, 1960.

The Conference Board of the Mathematical Sciences has six member organizations and no individual members. The six member organizations are the American Mathematical Society, the Association for Symbolic Logic, the Institute of Mathematical Statistics, the Mathematical Association of America, the National Council of Teachers

* The history of the War Policy Committee is sketched in the following references: this MONTHLY, vol. 50 1943, pp. 138, 205, 466, 593; vol. 51 1944, pp. 112–115, 549; vol. 52 1945, p. 115. Bulletin of the American Mathematical Society, vol. 49, 1943, p. 199.

† The first reference to the formation of the Policy Committee occurs in the report of "The Twenty-Ninth Annual Meeting of the Association," this MONTHLY, vol. 53 1946, p. 178.

of Mathematics, and the Society for Industrial and Applied Mathematics.

The Washington Office will not be involved in any way in the operation of the activities of the member organizations, and it is not expected that any of the activities of the member organizations will be transferred to the Washington Office or to the Conference Board.

The Washington Office will gather information about events and developments, especially those in Washington, which concern mathematics, and will relay this information to those member organizations, mathematicians, and others who may wish to receive it.

The Washington Office will supply information and help on mathematical matters as the opportunity arises. Furthermore, in carrying out this function, the Washington Office will arrange for assistance from the member organizations and individual mathematicians as the situation demands and opportunity permits. Member organizations may request the assistance of the Washington Office.

From time to time the Washington Office will manage or operate special projects which are compatible with the purposes and functions of the Conference Board and which will serve the common interests of the member organizations. These projects may be supported by contracts or grants from foundations, government agencies, or other organizations.

The management of the Conference Board is vested in its Council, which consists of two representatives from each of the member organizations and of six representatives at large. The officers of the Conference Board are the following: Chairman, S. S. Wilks; Secretary, J. R. Mayor; Treasurer, A. E. Meder, Jr.; Executive Secretary, G. B. Price.

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

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CALENDAR OF FUTURE MEETINGS

Forty-fourth Annual Meeting, Willard Hotel, Washington, D. C., January 25–27, 1961.

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–31, 1961.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1961.

IOWA, Simpson College, Indianola, April 14, 1961.

KANSAS, Ottawa University, April 15, 1961.

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 17–18, 1961.

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MICHIGAN, Wayne State University, Detroit, March 25, 1961.

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NEW JERSEY

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NORTHERN CALIFORNIA, San Jose State College, January 14, 1961.

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, April 28–29, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, April, 1961.

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.

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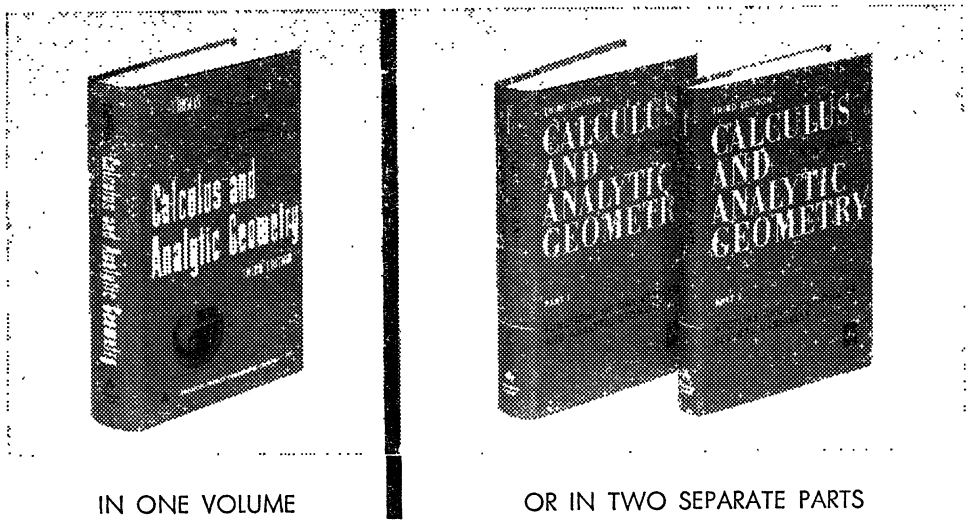
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